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On Generalized Hoops, Homomorphic Images of Residuated Lattices, and (G)BL-Algebras

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On generalized hoops, homomorphic images of residuated lattices, and (G)BL-algebras

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In memory of Franco Montagna

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Abstract Right-residuated binars and right-divisible residuated binars are defined as precursors of generalized hoops, followed by some results and open problems about these partially ordered algebras. Next we show that all complete homomorphic images of a complete residuated lattice A can be constructed easily on certain definable subsets of A. Applying these observations to the algebras of Hajek's Basic Logic (BL-algebras), we give an effective description of the HS-poset of finite subdirectly irreducible BL-algebras. The lattice of finitely generated BL-varieties can be obtained from this HS-poset by constructing the lattice of downward closed sets. These results are extended to bounded generalized BL-algebras using poset products and the duality between complete perfect Heyting algebras and partially ordered sets.

We also prove that the number of finite generalized BL-algebras with n join-irreducible elements is, up to isomorphism, the same as the number of preorders on an n-element set, hence the same as the number of closure algebras (i.e. S4-modal algebras) with 2^n elements. This result gives rise to a faithful functor from the category of finite GBL-algebras to the category of finite closure algebras that is full on objects, providing a novel connection between some substructural logics and classical modal logic. Finally we show how generic *satisfaction modulo theories solvers* (SMT-solvers) can be used to obtain practical decision procedures for propositional Basic Logic and many of its extensions.

1 Residuated binars and generalized hoops

We begin by considering structures with a simpler signature than residuated lattices. The aim of this section is to focus on the right-divisibility axiom in the setting of right-residuated structures, and without further assumptions such as associativity or commutativity.

A *right-residuated binar* is of the form $(A, \leq, \cdot, /)$ where $(A, \leq)$ is a partially ordered set, $\cdot$ is a binary operation on A and $/$ is its right residual. This means that for all $x, y, z \in A$

$$xy \leq z \iff x \leq z/y.$$

It follows that $\cdot$ is order-preserving in the left argument since if $x \leq y$ then $yz \leq yz$ implies $y \leq yz/z$, hence $x \leq yz/z$, which is equivalent to $xz \leq yz$. A similar derivation show that $/$ is order-preserving in the left argument. A *left-residuated binar* is of the form $(A, \leq, \cdot, \backslash)$ and satisfies $xy \leq z \iff y \leq x\backslash z$. Finally, $(A, \leq, \cdot, \backslash, /)$ is a *residuated binar* if $(A, \leq, \cdot, /)$ is a right-residuated binar and $(A, \leq, \cdot, \backslash)$ is a left-residuated binar. We use the convention that $\cdot$ has higher priority than $/$ and $\backslash$, so x/yz is read as $x/(yz)$. Note that the logic of residuated binars is given by the non-associative Lambek calculus (see e.g. [12]). The universal theory of residuated binars is decidable since Farulewski [11] proves the finite embeddability for this class of partially ordered algebras.

The theory becomes considerably more algebraic if $\leq$ is definable by an equation. Recall that a *right natural preorder* is given by the *right-divisibility axiom*:

$$x \succeq y \iff \exists u(x = uy).$$

In any monoid $\succeq$ is a preorder but here, instead of assuming associativity, we use a (version of) this axiom to define a subclass of right-residuated binars in which $\leq$ is definable. We first note that even in the general setting of a right-residuated binar, the existential quantifier can be eliminated.

Lemma 1 *The following are equivalent in any right-residuated binar.*

(i) *For all* x, y $(x \leq y \iff \exists u(x = uy))$
(ii) *For all* x, y $(x \leq y \iff x = (x/y)y)$.
(iii) *The identities* $(y/y)x = x$ *and* $(y/x)x = (x/y)y$ *hold.*

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Proof (i) implies (ii): Suppose $x \le y \iff \exists u(x = uy)$ holds. Assuming $x \le y$ one obtains $uy = x \le x$ for some u, hence $u \le x/y$. Since $\cdot$ is order preserving in the left argument, we have $x = uy \le (x/y)y$. The reverse inequality $(x/y)y \le x$ holds in any right-residuated binar, so we conclude that $x \le y$ implies $x = (x/y)y$.

Conversely, if $x = (x/y)y$ holds, then $\exists u(x = uy)$, whence the first condition implies $x \le y$.

Clearly (ii) implies (i) since we can take $u = x/y$.

(ii) implies (iii): Assume that $x \le y \iff x = (x/y)y$ for all x, y. Since $x \le x$, we get $x = (x/x)x$. We always have $x \le xy/y$, hence $xy \le (xy/y)y$ holds. The reverse inequality is also true in general, so $xy = (xy/y)y$. From the assumption it follows that $xy \le y$. Therefore we have $x \le y/y$ as an identity, hence $x/x \le y/y$. Interchanging x, y proves $x/x = y/y$. Multiplying by x on the right we get $x = (x/x)x = (y/y)x$.

For the second identity, since $(x/y)y \le x$, we use the assumption with x replaced by $(x/y)y$ and y replaced by x to get $(x/y)y = ((x/y)y/x)x$. As in the proof of the first identity, we have $xy \le y$. Dividing and multiplying by z on both sides gives the identity $(xy/z)z \le (y/z)z$. Now replace x by x/y and z by x to see that $((x/y)y/x)x \le (y/x)x$. It follows that $(x/y)y \le (y/x)x$, and interchanging x, y proves the identity.

Finally we show (iii) implies (ii). Assume the identities $(y/y)x = x$ and $(y/x)x = (x/y)y$ hold, and let $x \le y$. Then $(y/y)x \le y$, hence $y/y \le y/x$. Multiplying by x on the right and using the second identity we get $x = (y/y)x \le (y/x)x = (x/y)y$. The reverse inequality follows from right-residuation, whence $x = (x/y)y$.

Again, assume the two identities of (iii) holds, and let $x = (x/y)y$. By right-residuation we have $(y/x)x \le y$, so we deduce $(x/y)y \le y$ from the second identity. Since we started with $x = (x/y)y$, we conclude that $x \le y$. $\square$

A *right-divisible residuated binar* is a right-residuated binar that satisfies the identities in Lemma 1(iii). Note that y/y is a left identity for $\cdot$, and it is the top element in any right-divisible residuated binar, as shown in the proof of (ii)$\Rightarrow$(iii). Hence we can expand the signature of such binars with an element 1 to obtain the following definition of the quasivariety of divisible residuated binars.

A *right-divisible unital residuated binar* is a residuated binar $(A, \le, \cdot, 1, /)$ such that the three identities $x/x = 1$, $1x = x$ and $(y/x)x = (x/y)y$ hold. The third identity is called *right-div* in proofs below. The partial order is definable by $x \le y \iff x = (x/y)y$ and the left-unit 1 is the top element in this poset. Note that $(x/y)y$ is a lower bound for any pair of elements x, y and we always have $1 \le 1/x$. Moreover, $x \le y \iff 1x \le y \iff 1 \le y/x$, so we obtain the following result.

Lemma 2 *In a right-divisible unital binar the partial order is down-directed and the identity $1/x = 1$ holds. The order is also definable by $x \le y \iff y/x = 1$.*

Reflexivity and antisymmetry of $\le$ can be deduced from the three identities, but transitivity and the residuation property do not follow from them. Hence the class of right-divisible unital residuated binars is a quasivariety, defined by the three identities, transitivity of $\le$ and the residuation implications. It is not known if this class can be defined by identities alone, or whether there is a decision procedure for the (in)equational theory.

We now show that adding one more identity, produces an interesting subvariety. In the arithmetic of real numbers (or in any field) the following equation is fundamental to the simplification of double fractions:

$$\frac{\frac{x}{y}}{z} = \frac{1}{z} \cdot \frac{x}{y} = \frac{x}{zy}.$$

In a right-residuated binar this equation is called the *right hoop identity*: $(x/y)/z = x/zy$.

Lemma 3 *In a right-divisible unital residuated binar the right hoop identity $x/yz = (x/z)/y$ implies $x(yz) = (xy)z$, $x1 = x$ and $x/1 = x$.*

Proof $x(yz) = 1(x(yz))$ (left unital)
$= [(xy)z/(xy)z](x(yz))$ since $1 = x/x$
$= [((xy)z/z)/xy](x(yz))$ (right hoop id.)
$= [(((xy)z/z)/y)/x](x(yz))$ (right hoop id.)
$= [((xy)z/yz)/x](x(yz))$ (right hoop id.)
$= [(xy)z/x(yz)](x(yz))$ (right hoop id.)
$= [x(yz)/(xy)z]((xy)z)$ by right-div
$=$ reverse steps to get $= (xy)z$.
Now $x \le 1$ implies $x = (x/1)1$, hence $x1 = ((x/1)1)1 = (x/1)(11) = (x/1)1 = x$. Finally $x/1 = (x/1)1 = (1/x)x = 1x = x$. $\square$

A *right generalized hoop* is an algebra $(A, \cdot, 1, /)$ that satisfies the identities $x/x = 1$, $1x = x$, $(x/y)y = (y/x)x$ and $x/(yz) = (x/z)/y$. We also define the term-operation $x \wedge y = (x/y)y$ and a binary relation $\le$ by $x \le y \iff x = x \wedge y$. The next lemma shows that $\wedge$ is a semilattice operation, hence $\le$ is a partial order on A. Moreover, A is right-residuated with respect to this order and the left-unit 1 is the top element. Algebras with this latter property are said to be *integral*.

Lemma 4 *Let A be a right generalized hoop. Then*

(i) *the term $x \wedge y = (x/y)y$ is idempotent, commutative and associative,*
(ii) *$\le$ is a partial order and $\wedge$ is a meet-semilattice operation with respect to $\le$,*
(iii) *$x \le y \iff y/x = 1$ for all $x, y \in A$,*
(iv) *$xy \le z \iff x \le z/y$ for all $x, y, z \in A$, and*
(v) *$x \le 1$ for all $x \in A$, i.e., A is integral.*

Proof (i) The idempotence follows from the first two identities, and commutativity follows from the third. For associativity we calculate $(x\wedge y)\wedge z = (((x/y)y)/z)z$ by definition

$= (z/(x/y)y)(x/y)y$ by right-div

$= ((z/y)/(x/y))(x/y)y$ (right hoop id.)

$= ((x/y)/(z/y))(z/y)y$ by assoc. and right-div

$= (x/(z/y)y)(z/y)y$ (right hoop id.)

$= x \wedge (z \wedge y) = x \wedge (y \wedge z)$

(ii) Reflexivity, antisymmetry and transitivity of $\leq$ and the observation that $x \wedge y$ is the greatest lower bound of x, y follow in the standard way from (i).

(iii) $x \leq y$ is equivalent to $x = (x/y)y$ hence $y/x = y/((x/y)y) = y/((y/x)x) = (y/x)/(y/x) = 1$, where the third equality uses the right hoop identity. Conversely, if $y/x = 1$ then $(x/y)y = (y/x)x = 1x = x$ and we conclude $x \leq y$.

(iv) From $xy \leq z$ we deduce $z/xy = 1$ by (iii). Hence $(x/(z/y))(z/y) = ((z/y)/x)x = (z/xy)x = 1x = x$, or equivalently $x \leq z/y$. Conversely, if $x \leq z/y$ then $x = (x/(z/y))(z/y) = (z/xy)x$, so $xy = (z/xy)xy = (xy/z)z$ which is equivalent to $xy \leq z$.

(v) Since $xy \leq xy$, (iv) implies $x \leq xy/y$. Multiplying by y gives $xy \leq (xy/y)y$, and the reverse inequality also holds by (iv). Hence $xy = (xy/y)y$, or equivalently $xy \leq y$. A final application of (iv) produces $x \leq y/y = 1$. □

In particular, the above lemma shows that a right generalized hoop is a right-divisible meet semilattice-ordered integral residuated monoid, although the monoid operation need not be order-preserving in the right argument (see e.g. the 4-element right generalized hoop at the end of this section). Adding the identity $x(y \wedge z) \wedge xy = x(y \wedge z)$ would be a way to ensure this property holds as well. It is an interesting question whether right generalized hoops (with or without the additional identity) have a decidable equational theory.

A class of right-residuated monoids that has been studied previously is the quasivariety of *porrims* (short for *partially ordered right-residuated integral monoids*), see e.g. [5,22]. However in these algebras the monoid operation is order-preserving in both arguments, so results about porrims do not automatically apply to right generalized hoops.

A *generalized hoop* is an algebra $(A, \cdot, 1, \backslash, /)$ such that $(A, \cdot, 1, /)$ is a right generalized hoop, $(A, \cdot, 1, \backslash)$ is a left generalized hoop (defined by the mirror-image identities of a right generalized hoop) and both these algebras have the same meet operation, i. e., the identity $(x/y)y = y(y\backslash x)$ holds. Generalized hoops were first studied by Bosbach [6,7] and the name *hoop* was introduced by Büchi and Owen [1975]. Generalized hoops are also called *pseudo hoops* in the literature on residuated structures. By the preceding lemma, they are indeed left- and right-residuated. Botur, Dvurečenskij and Kowalski [8] prove that generalized hoops are congruence distributive.

In a residuated binar, the residuation property implies that $\cdot$ distributes over any existing joins in each argument. However, this is not true for meets. The following result was proved by N. Galatos for GBL-algebras (defined below) but already holds for generalized hoops.

Theorem 5 *In any generalized hoop $(x \wedge y)z = xz \wedge yz$ and $x(y \wedge z) = xy \wedge xz$.*

Proof From $xz \leq xz$ it follows that $x \leq xz/z$, hence $xz \leq (xz/z)z$. Likewise, from $xz/z \leq xz/z$ we deduce $(xz/z)z \leq xz$, therefore $xz = (xz/z)z$. Note that $(x \wedge y)z \leq xz \wedge yz$ always holds since $\cdot$ is order-preserving. To complete the proof, we calculate:

$xz \wedge yz = (xz/yz)yz$ by definition

$= ((xz/z)/y)yz$ (right hoop id.)

$= (y/((xz)/z))(xz/z)z$ by assoc. and divisibility

$= (y/((xz)/z))xz$ by the derived identity

$\leq (y/x)xz = (y \wedge x)z$ since $x \leq (xz)/z$. The second identity is proved using the left generalized hoop axioms. □

In the last step we made use of the implication $x \leq y \Rightarrow z/y \leq z/x$ which holds in all residuated binars. It is interesting to note that this result requires that $\cdot$ is order-preserving in the right argument. Indeed, the distribution of $\cdot$ over $\wedge$ from the right fails in the following 4-element right generalized hoop. Let $R = (\{a, b\}, \cdot)$ be the unordered 2-element right-zero semigroup, which means $aa = ba = a$ and $ab = bb = b$. Extend R to $R_{01} = (\{0 < a, b < 1\}, \cdot)$ such that 1 is an identity element as well as the top element, and $0x = x0 = 0$ is the least element. Adding a zero and/or an identity preserves associativity, so R_{01} is a partially-ordered monoid. The operation tables for this algebra are

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	b	a
b	0	a	b	b
1	0	a	b	1

$/$	0	a	b	1
0	1	0	0	0
a	1	1	0	a
b	1	0	1	b
1	1	1	1	1

and it is easy to check that R_{01} is a right generalized hoop. However, $(a \wedge b)a = (a/b)ba = 0ba = 0$ while $aa \wedge ba = a \wedge a = (a/a)a = 1a = a$. Note that the monoid operation fails to be order-preserving in the right argument since $a \leq 1$ but $a = ba \not\leq b1 = b$.

2 Homomorphic images of residuated lattices, hoops and GBL-algebras

In this section we point out that for finite residuated lattices there is a simple and efficient way to construct all homomorphic images. Rather than using the usual universal algebraic quotient construction, the universe of the homomorphic image is a specific subset of the residuated lattice, with operations "relativized" to this subset. We first mention some standard results that

can be found, e.g., in [12]. Recall that a residuated lattice is of the form $(A, \wedge, \vee, \cdot, 1, \backslash, /)$ such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid, and $\backslash, /$ are the left- and right-residuals of the monoid operation. A congruence relation θ of such an algebra is determined by the congruence class $[1]_\theta$, and for a finite residuated lattice this congruence class has a smallest element c. It is easy to see that such an element is always a negative central idempotent, which means that $c \le 1$, $cc = c$ and $cx = xc$ for all $x \in A$. The set of all negative central idempotents of a residuated lattice A is denoted by I_A. This set is a join-subsemilattice of A, and a distributive lattice when $\cdot$ is used as meet operation. In fact, in the finite case, $(I_A, \cdot, \vee)$ is dually isomorphic to the congruence lattice of A [12, p. 198]. For an element $c \in A$ we define $Ac = \{xc : x \in A\}$, and operations $u \wedge_c v = (u \wedge v)c$, $u/_c v = (u/v)c$, $u\backslash_c v = (u\backslash v)c$.

Theorem 6 *Let A be a residuated lattice and $c \in I_A$. Then $A_c = (Ac, \wedge_c, \vee, \cdot, c, \backslash_c, /_c)$ is a residuated lattice and the map $h : A \to Ac$ given by $h(x) = xc$ is a surjective homomorphism onto A_c. If θ is the kernel of h then xc is the smallest element of $[x]_\theta$.*

Proof Observe that Ac is closed under the operations: $xc \vee yc = (x \vee y)c$ and $(xc)(yc) = xyc$ are both in Ac, and for the other operations the same holds by construction. The map h is clearly surjective, so it suffices to check that it is a homomorphism, then the homomorphic image will be a residuated lattice since homomorphisms preserve identities. Distributivity of $\cdot$ over $\vee$ shows that h preserves $\vee$, centrality and associativity imply that h preserves $\cdot$, $h(x) \wedge_c h(y) = (xc \wedge_c yc) = (xc \wedge yc)c \le (x \wedge y)c = h(x \wedge y)$ since $c \le 1$, while $(x \wedge y)c \le xc$ and $(x \wedge y)c \le yc$ imply $(x \wedge y)c \le (xc \wedge yc)c$. In any residuated lattice $(x/y)y \le x$, so $(x/y)yz \le xz$ and therefore $x/y \le xz/yz$. In particular, $(x/y)c \le (xc/yc)c$, which proves $h(x/y) \le h(x)/_c h(y)$. For the opposite inequality we have $(xc/yc)yc \le xc \le x$, hence by centrality and idempotence $(xc/yc)c \le (x/y)c$. $\Box$

The theorem works for arbitrary residuated lattices. However in general it does not construct all homomorphic images, only those where the 1-congruence class of the kernel (and hence every congruence class) has a smallest element.

Corollary 7 *Let A be a finite (or complete) residuated lattice and B any (complete) homomorphic image of A. Then B is isomorphic to A_c where c is the smallest negative central idempotent of A that is mapped by the homomorphism to 1 in B.*

Commutative generalized hoops are called *hoops*. In this case $x/y = y\backslash x$ and this operation is usually written as $y \to x$. As we saw in the previous section, generalized hoops are meet-semilattice-ordered algebras. *Integral generalized Basic Logic algebras*, or IGBL-algebras for short, are lattice-ordered generalized hoops, i.e., generalized hoops $(A, \wedge, \cdot, 1, \backslash, /)$ expanded with a join operation $\vee$ such that $(A, \wedge, \vee)$ is a lattice. Alternatively they can be defined as residuated lattices that satisfy the identity $x \wedge y = (x/y)y = y(y\backslash x)$, or equivalently satisfy the quasiequations $x \le y \implies x = (x/y)y = y(y\backslash x)$.

Theorem 8 *Let A be an IGBL-algebra with a central idempotent element $c \in A$. Then A_c is isomorphic to the principal ideal $\downarrow c$, hence $\wedge_c = \wedge$ and the map $h(x) = xc$ does not identify any elements of this ideal.*

Proof By the preceding quasiequation, if $x \le c$ then $x = (x/c)c$, and therefore $x \in A_c$. Also, $h(x) = xc = (x/c)cc = x$, so $h\restriction_{\downarrow c}$ is the identity map. $\Box$

Hajek's Basic Logic algebras (BL-algebras) are defined as IGBL-algebras that satisfy the identities $xy = yx$ and $(x \to y) \vee (y \to x) = 1$ (prelinearity) and have a new constant 0 that denotes the bottom element. The prelinearity property implies that subdirectly irreducible BL-algebras are linearly ordered. Generalized BL-algebras (or GBL-algebras) are just divisible residuated lattices, but still retain many of the properties of BL-algebras. For example they have distributive lattice reducts, the fusion operation distributes over the meet operation, and in the n-potent case they are integral and commutative [18]. Subdirectly irreducible (I)GBL-algebras are, in general, not linearly ordered, but in the finite case they have a well-understood structure theory based on the poset product construction [18,19], see Section 4 below.

It is easy to see that all finite generalized hoops are reducts of integral GBL-algebras, since a finite meet semilattice with a top element is a lattice. Moreover, finite GBL-algebras are commutative [17], hence finite generalized hoops are in fact hoops. The preceding theorem also applies to generalized hoops, and in the finite setting it describes all homomorphic images.

3 Finitely generated varieties of BL-algebras

In this section we give a description of finite subdirectly irreducible BL-algebras and use it to calculate the first few levels of the HS-poset of small BL-algebras. The observations are known, but we recall them here in preparation for extending them to generalized BL-algebras.

It is well known that subdirectly irreducible BL-algebras are linearly ordered (this is more generally true for commutative prelinear residuated lattices). In the finite case this means that they are simply n-element chains.

The *ordinal sum* $A \oplus B$ of two integral residuated lattices (or posets) A, B is defined by taking the disjoint union of A, B, then identifying the two units $1_A = 1_B = 1$ and extending the partial order to all elements so that every (non-unit) element of A is less than every element of B. The operation $\cdot$ is extended to $A \oplus B$ by $ab = ba = a$ and the residuals are extended by $a\backslash b = b/a = 1$ and $b\backslash a = a/b = a$ for all $a \in A - \{1\}$ and $b \in B$.

The *finite MV-chain with* $n+1$ elements is the BL-algebra $MV_n = (\{0 = c_n < \cdots < c_1 < c_0 = 1\}, \wedge, \vee, \cdot, \to, 1, 0)$ where $c_i \cdot c_j = c_{\min(i+j,n)}$ and $c_i \to c_j = c_{\max(j-i,0)}$. Every finite subdirectly irreducible BL-algebra A is an ordinal sum of finite MV-chains, hence the structure of A is completely determined by the idempotent elements in the chain. The top and bottom of the chain are always idempotent, so if A has n elements then there are 2^{n-2} choices for the idempotent elements, and therefore 2^{n-2} nonisomorphic subdirectly irreducible BL-algebras. We will denote each of these algebras by $B_{a_1a_2\ldots a_m}$ where $a_1, a_2, \ldots, a_m$ is a list of positive integers, m is the number of join-irreducible idempotent elements and a_i is one greater than the number of non-idempotent elements between the ith idempotent element and the $(i+1)$th idempotent element in the chain, counting from the bottom. Note that with this definition we have $B_{a_1a_2\ldots a_m} = MV_{a_1} \oplus MV_{a_2} \oplus \cdots \oplus MV_{a_m}$, hence $B_{11\ldots 1}$ (with n 1's in the subscript) is the $(n+1)$-element linear Heyting algebra, $B_n = MV_n$, B_1 is the 2-element Boolean algebra, and we use B_0 to denote the trivial algebra. The length of the chain is always $a_1 + a_2 + \cdots + a_m + 1$.

Theorem 9 *Let A, B be finite subdirectly irreducible BL-algebras. Every subalgebra of A is subdirectly irreducible, and if B is a homomorphic image of A then B is isomorphic to a subalgebra of A. Hence $B \in HS(A)$ if and only if $B \in S(A)$.*

Proof As mentioned before, subdirectly irreducible BL-algebras are chains, so let $A = B_{a_1a_2\ldots a_m}$ and $B = B_{b_1b_2\ldots b_k}$ denote the two finite BL-chains. Conversely any finite linearly-ordered residuated lattice is subdirectly irreducible, since it has a largest negative central idempotent < 1. Hence every subalgebra of A is subdirectly irreducible.

Let $h : A \to B$ be a homomorphism. Then h maps idempotent elements to idempotent elements, and by Theorem 8 h maps the principal filter above an idempotent of A to 1_B, and is injective on the complement of this filter. Hence h is uniquely determined by an order-preserving surjection $\hat{h} : \{1 < \cdots < k\} \to \{1 < \cdots < m'\}$ for some $m' \le m$ such that for all $i \in \{1, \ldots, m'\}$, if $j = \min \hat{h}^{-1}[\{i\}]$ then $a_i | b_j$. In this case define $h_i : MV_{a_i} \to MV_{b_j}$ by $h_i(c_\ell) = c_{n_i\ell}$ where $n_i = \frac{b_j}{a_i}$. Given such a map $\hat{h}$, the homomorphism h is defined by $h(1) = 1$ and $h(x) = 1$ if $x \in MV_{a_i}$ for $i > m'$, while $h(x) = h_i(x)$ for $i \le m'$ where . Thus the MV components of A that correspond to numbers above m' are collapsed to the top element of B and for $i \le m'$ the ith component of A is embedded into the least component j of B such that $\hat{h}(j) = i$. The homomorphic image $h[A]$ is isomorphic to a subalgebra of A since for $i \le m'$ the ith ordinal sum components of A and $h[A]$ are both isomorphic to MV_{a_i}. □

Note that the map $\hat{h}$ is a special case of the weight preserving p-morphisms in [3].

A variety $\mathcal{V}$ of algebras of finite similarity type is said to be finitely generated if $\mathcal{V} = HSP(\mathcal{K})$ for some finite set $\mathcal{K}$ of finite algebras. If, in addition, $\mathcal{V}$ is congruence distributive then by Jónsson's Lemma [20] the subdirectly irreducible members of $\mathcal{V}$ are all contained in $HS(\mathcal{K})$, hence there are only finitely many such members. In particular, for two finite subdirectly irreducible algebras A, B of the same type, $HSP(A) \subseteq HSP(B)$ if and only if $A \in HS(B)$, and we write $A \le_{HS} B$ in case the latter relation holds.

Since $HSHS = HS$, the relation $\le_{HS}$ is a partial order on isomorphism classes of finite subdirectly irreducible algebras. Since any variety is determined by it subdirectly irreducible members, the lattice of finitely generated subvarieties is isomorphic to the lattice of finite downsets of this partial order.

The preceding result simplifies calculating the $\le_{HS}$ partial order relation between subdirectly irreducible BL-algebras. Komori gave a complete description of the lattice of subvarieties of MV-algebras, showing that it is countable and that the $\le_{HS}$ poset of finite subdirectly irreducible MV-algebras is isomorphic to the divisibility lattice $\mathbb{D} = (\mathbb{N}\setminus\{0\}, |)$, with $MV_m \le_{HS} MV_n$ if and only if $m|n$. Here we describe the $\le_{HS}$ poset for finite subdirectly irreducible BL-algebras. As observed previously, these algebras are chains determined by finite sequences of positive integers.

Theorem 10 *The $\le_{HS}$ poset of finite s.i. BL-algebras is isomorphic to $\mathbb{D}^* = \bigcup_{n=0}^{\infty} \mathbb{D}^n$ with the order on $\mathbb{D}^*$ extending the pointwise divisibility order on each component by $(a_1, \ldots, a_m) \le (b_1, \ldots, b_n)$ if and only if there exists an order-preserving injection $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $f(1) = 1$ and $a_i | b_{f(i)}$ for all $i \in \{1, \ldots, m\}$.*

The order relation $(a_1, \ldots, a_m) \le (b_1, \ldots, b_n)$ is a covering relation if and only if either

- *$m = n$ and*

  $$(b_1, \ldots, b_n) = (a_1, \ldots, a_{i-1}, pa_i, a_{i+1}, \ldots, a_n)$$

  for some prime p and some unique $i \le n$, or
- *$m + 1 = n$ and*

  $$(b_1, \ldots, b_n) = (a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_m)$$

  for some $i \in \{2, \ldots, n\}$.

Proof To establish the first part we need to show that $A = B_{a_1,\ldots,a_m}$ is (isomorphic to) a subalgebra of $B = B_{b_1,\ldots,b_n}$ iff a function f with the stated properties exists. Assume $h : A \to B$ is an embedding. Then h sends idempotent to idempotents, so let $f : \{1, \ldots m\} \to \{1, \ldots, n\}$ be defined by $f(i) = j$ if the ith idempotent of A is sent by h to the jth idempotent of B (numbered from bottom to top). Since BL-algebra homomorphisms preserve the bottom element, we have $f(1) = 1$, and since h is an embedding, the ith MV-component of A is embedded in the $f(i)$th MV-component of B, whence $a_i | b_{f(i)}$.

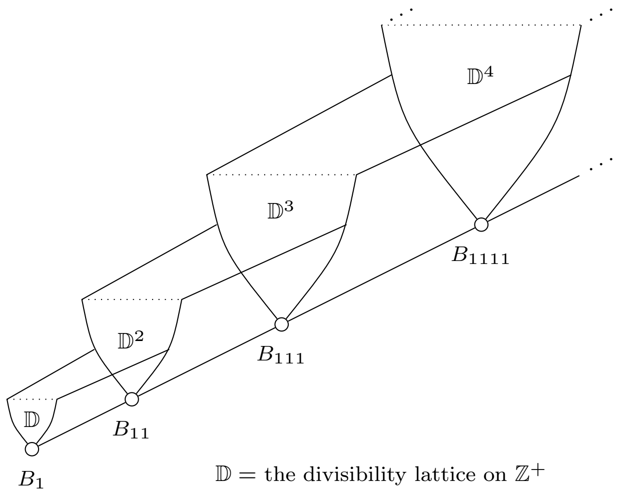

Fig. 1 Outline of the HS-poset of s. i. BL-algebras

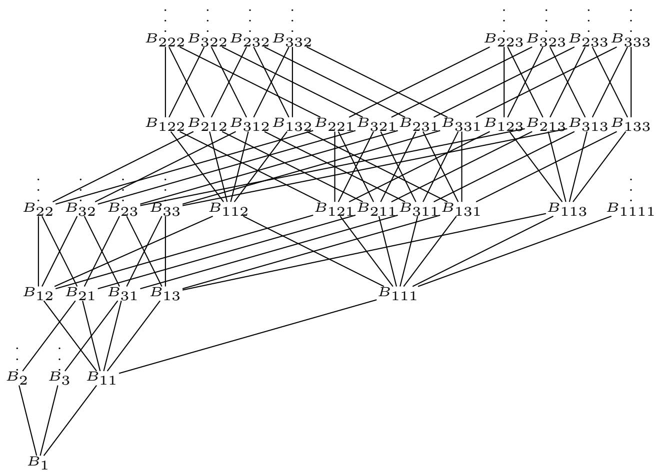

Fig. 2 Bottom of the HS-poset of s. i. BL-algebras

Conversely, if f is such an order-preserving function then one can construct an embedding $h : A \to B$ from the union of the embeddings $h_i : C_i \to D_{f(i)}$ where C_i, D_j are the ith and jth MV-components of A and B respectively.

The second part follows by observing that the conditions capture the two possible ways for A to be isomorphic to a maximal proper subalgebra of B. $\square$

A schematic diagram of the HS-poset is shown in Figure 1 followed by some detail of the bottom of this poset in Figure 2. The lattice of finitely generated BL-varieties is isomorphic to the lattice of downsets of this poset.

4 The lattice of finitely generated varieties of bounded GBL-algebras

We now extend the results about finitely-generated varieties of BL-algebras to bounded integral GBL-algebras. In this paper we use the adjective "bounded" to mean that the lattice-ordered algebra has a least element 0 that is also a constant operation of the algebra. Recall that finite GBL-algebras are poset products of finite simple *Wajsberg hoops* (= 0-free reducts of MV-algebras) [18]. Similarly finite bounded GBL-algebras are poset products of finite simple MV-algebras. Hence they are integral, commutative, and we can construct the HS-poset of finite bounded subdirectly irreducible GBL-algebras by analyzing homomorphisms between poset products of simple MV-algebras. The results do not depend on the divisibility law, so we first consider the more general setting of a poset product of bounded integral simple commutative residuated lattices.

Let P be a poset. The (dual) poset product of a family $\{L_i : i \in P\}$ of bounded integral residuated lattices is defined on a subset of the cartesian product by

$$\prod_{\mathbf{P}} L_i = \{f \in \prod_{i\in P} L_i : \forall i > j \in P\,(f(i){=}0 \text{ or } f(j){=}1)\}.$$

The operations $\wedge, \vee, \cdot$ are defined pointwise and the bounds are the constant functions $\mathbf{0}, \mathbf{1}$. The residuals are given by

$$(f\backslash g)(i) = \begin{cases} f(i)\backslash g(i) & \text{if } f(j) \le g(j) \text{ for all } j < i \\ 0 & \text{otherwise} \end{cases}$$

$$(g/f)(i) = \begin{cases} g(i)/f(i) & \text{if } f(j) \le g(j) \text{ for all } j < i \\ 0 & \text{otherwise.} \end{cases}$$

It is shown in [18] that a (dual) poset product (called poset sum in that paper) of bounded residuated lattices is again a bounded residuated lattice, and if the factors are divisible, so is the poset product. Hence a poset product of bounded GBL-algebras is also a bounded GBL-algebra. Note that if the poset P is linear then the poset product is an ordinal sum of the factors. If the poset is an antichain, then the poset product is the direct product. If the factors L_i are 2-element Boolean algebras, then the poset product is a Heyting algebra.

Recall that an algebra is simple if it only has two congruence relations. As in Section 2, an element c in a monoid is *central* if it commutes with every element of the monoid, and it is *idempotent* if $cc = c$. For any central idempotent $c \le 1$ in a residuated lattice, the principal filter $\uparrow c$ is a normal filter and hence determines a congruence of the residuated lattice. Since finite GBL-algebras are commutative, every element is central. Let $\mathcal{S}$ be the class of all bounded commutative integral simple residuated lattices, where we denote the bounds by $0, 1$ and assume that 1 is the monoid identity. By simplicity $0, 1$ are the only idempotents of each member of $\mathcal{S}$. It follows that the only idempotents in a poset product of members of $\mathcal{S}$ will be the functions with range $\{0, 1\}$. In addition, the set of idempotents is a Heyting subalgebra of the poset product (i.e., $f \cdot g = f \wedge g$ for idempotents).

For a complete lattice A we let $J(A)$ denote the poset of all completely join-irreducible elements, with the order induced by A. An element $j \in J(A)$ has

a unique lower cover denoted j_*. A lattice is *join-perfect* if every element is the join of completely join-irreducible elements. For Heyting algebras, it is the case that join-perfect implies the dual notion of meet-perfect, hence they are simply called perfect.

For posets P, Q a *p-morphism* is a map $q : Q \to P$ such that $q[\downarrow a] = \downarrow q(a)$, where $\downarrow x$ is the principal downset $\{y : y \leq x\}$. For a set $X \subseteq P$ the downset $\downarrow X = \bigcup_{x \in X} \downarrow x$, and the family of all downsets of P is $D(P) = \{\downarrow X : X \subseteq P\}$. The sets $\uparrow x$ and $\uparrow X$ are defined dually. For a complete homomorphism $h : A \to B$ define $J(h) : J(B) \to J(A)$ by $J(h)(j) = \bigwedge h^{-1}[\uparrow j]$. It follows from the completeness of h that $h^{-1}[\uparrow j]$ is a principal upset, and if A is a complete perfect Heyting algebra then the meet is in $J(A)$ and $J(h)$ is a p-morphism.

It is easy to see that $(D(P), \cap, \cup, \to, \emptyset, P)$ is a complete perfect Heyting algebra with $X \to Y = P \setminus \uparrow (X \setminus Y)$. Given a p-morphism $q : Q \to P$, define a map $D(q) : D(P) \to D(Q)$ by $D(q)(X) = q^{-1}[X]$. Then $D(q)$ is a complete Heyting algebra morphism. Moreover J : cpHA $\to$ pPos and D : pPos $\to$ cpHA are functors and $D(J(A)) \cong A$ and $J(D(P)) \cong P$, hence the category cpHA of complete perfect Heyting algebras with complete homomorphisms is dually equivalent to the category pPos of posets with p-morphisms (this duality extends Tarski's duality between complete and atomic Boolean algebras with complete homomorphisms and the category of sets). Note that the complete perfect Heyting algebra $D(P)$ can also be constructed as a poset product $\prod_{\mathbf{P}} \mathbf{2}$ where $\mathbf{2} = \{0, 1\}$ is the two-element Boolean algebra. It is well known that a Heyting algebra is subdirectly irreducible if and only if the top element 1 is completely join-irreducible, or equivalently if the dual poset has a top element.

The next result effectively extends this duality to certain poset products.

Theorem 11 (i) *Suppose $A = \prod_{\mathbf{P}} C_i$ is a poset product of a family of simple integral bounded commutative residuated lattices, and let I_A be the set of idempotents of A. For each $i \in P$ define the function $\hat{i} : P \to C_i$ by $\hat{i}(k) = 1$ if $k \leq i$ and $\hat{i}(k) = 0$ otherwise. Then I_A is a complete perfect Heyting subalgebra of A and the map $i \mapsto \hat{i}$ is an isomorphism from P to $J(I_A)$.*

(ii) *The residuated lattice $A = \prod_{\mathbf{P}} C_i$ is subdirectly irreducible if and only if $\mathbf{P}$ has a top element.*

(iii) *Suppose $B = \prod_{\mathbf{Q}} D_j$ is also a poset product with $D_j \in \mathcal{S}$ and $h : A \to B$ is a homomorphism such that h restricted to I_A is a complete Heyting algebra homomorphism. Then $h\restriction_{I_A}$ maps into I_B, and h is uniquely determined by a p-morphism $\bar{h} : Q \to P$ and by the maps $h_j : A_{\bar{h}(j)} \to B_j$ where $A_i = [\hat{i}_*, \hat{i}] \cong C_i$, $B_j = [\hat{j}_*, \hat{j}] \cong D_j$ and $h_j(f) = (h(f) \wedge \hat{j}) \vee \hat{j}_*$.*

(iv) *Now assume C_i, D_j are complete for all $i \in P, j \in Q$. Given a p-morphism $\bar{h} : Q \to P$ and complete homomorphisms $h_j : C_{\bar{h}(j)} \to D_j$ define a map $h : A \to B$ by $h(f)(j) = h_j(f(\bar{h}(j)))$. Then A, B are complete, h is a complete homomorphism, and every complete homomorphism from A to B can be obtained in this way.*

Proof (i) I_A is the subposet of functions in $A = \prod_{\mathbf{P}} C_i$ with range $\{0, 1\}$, and is isomorphic to $D(P)$, which is a complete perfect Heyting algebra. For all $i \in P$, the function $\hat{i}$ is in $J(I_A)$ since it corresponds to the principal ideal $\downarrow i$ of P and hence to a completely join-irreducible element of $D(P)$.

(ii) This is similar to the proof that Heyting algebras are subdirectly irreducible if and only if they have a unique co-atom, using the observation that in a simple bounded commutative residuated lattices any element $a < 1$ generates a normal filter that is equal to the whole algebra.

(iii) Let h have the stated properties. Then h restricted to I_A maps into I_B since homomorphisms send idempotents to idempotents. Being a complete homomorphism, $h\restriction_{I_A}$ is determined by its dual p-morphism from $J(I_B)$ to $J(I_A)$, and these posets are isomorphic to Q and P respectively via the map $\hat{i} \mapsto i$. The factors of the poset product A can be obtained as intervals between a completely join-irreducible $\hat{i} \in I_A$ and its unique lower cover in I_A (note that $\hat{i}$ need not be completely join-irreducible in A).

(iv) This result is a generalization of the observation that a cartesian product of complete lattices is complete, and that if the factors are simple then complete homomorphisms between two such cartesian products can be built from families of complete homomorphisms between the factors. □

Since every finite bounded GBL-algebra is a poset product of finite simple MV-algebras, the preceding theorem simplifies the calculation of the HS-poset of finite bounded subdirectly irreducible GBL-algebras. The bottom part of the HS-poset for finite subdirectly irreducible Heyting algebras is shown in Figure 3, where the algebras are represented by their dual posets of join-irreducibles. For finite bounded GBL-algebras, the HS-posets extends this one by noting that if A is a subalgebra of B then I_A is a Heyting subalgebra of I_B and all MV-components of A have a size that divides the size of the corresponding component of B, and if A is a homomorphic image of B then I_A is a homomorphic image of I_B and each MV-component of B is either collapsed or mapped isomorphically to the corresponding component of A.

Recall that a preorder $(P, \preceq)$ is a set P with a reflexive transitive relation $\preceq$, and by the usual quotient construction using the equivalence relation $p \sim q \iff (p \preceq q$ and $q \preceq p)$ one obtains a poset $(P/{\sim}, \leq)$ where $p/{\sim} \leq q/{\sim} \iff p \preceq q$. Conversely, given a poset $\mathbf{Q} = (Q, \leq)$ and a family of disjoint nonempty sets $\{P_i : i \in Q\}$ define the preorder $P = \bigcup_{i \in Q} P_i$ and $\preceq$ on P by $p \preceq q \iff \exists i, j \in Q\, (p \in P_i, q \in P_j$ and $i \leq j)$. Then $(P, \preceq)$ is a preorder and $(P/{\sim}, \leq)$ is isomorphic to $\mathbf{Q}$.

A *closure algebra* $(B, \vee, \neg, 0, \Diamond)$ (also called an S4-modal algebra) is a Boolean algebra $(B, \vee, \neg, 0)$ with

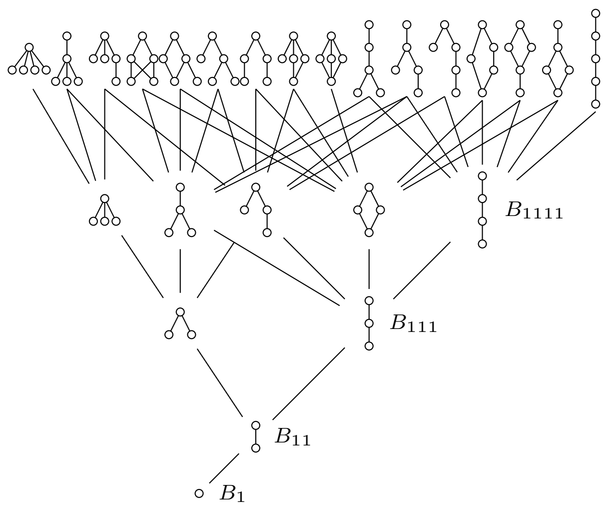

Fig. 3 Bottom of HS-poset of s. i. Heyting algebra duals

a unary operation $\Diamond$ that satisfies $\Diamond 0 = 0$, $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$ and $x \leq \Diamond x = \Diamond\Diamond x$. It is well known from modal logic that the duals of finite closure algebras are preorders defined on the atoms of the Boolean algebra.

The last result in this section provides a novel connection between divisible substructural logics and classical modal logic. It follows from the above remarks, together with the observation that a finite simple MV-algebra is determined by its cardinality.

Theorem 12 (i) *For a finite bounded GBL-algebra A, let $\prod_{\mathbf{Q}} A_i$ be a poset product of simple MV-algebras that is isomorphic to A, and let $(P, \preceq)$ be the preorder constructed from $\mathbf{Q}$ and the family of sets $\{J(A_i) : i \in Q\}$. Then $(P/{\sim}, \leq) \cong (J(I_A), \leq)$ and the preorder $(P, \preceq)$ uniquely determines A.*

(ii) *The number of (bounded) GBL-algebras with n join-irreducible elements is (up to isomorphism) equal to the number of preorders on an n-element set (up to isomorphism), hence the same as the number of closure algebras with 2^n elements.*

(iii) *Let F be the map that sends a finite bounded GBL-algebra A to the closure algebra $(\mathcal{P}(P), \cup, \neg, \emptyset, \Diamond)$, where $\Diamond X = \{y \in P : y \preceq x$ for some $x \in X\}$ and $(P, \preceq)$ is the preorder from* (i). *For a homomorphism $h : A \to A'$ let $F(h) : \mathcal{P}(P) \to \mathcal{P}(P')$ be given by $F(h)(X) = J(h)^{-1}[X]$. Then F is a faithful functor from the category of finite bounded GBL-algebras to the category of finite closure algebras that is full on objects.*

The number of homomorphisms between GBL-algebras is, in general, less than the number of homomorphisms between the corresponding closure algebras, since simple MV-algebras are rigid and there is at most one homomorphism between two simple MV-algebras, while simple closure algebras (i.e., monadic algebras, also known as S5-modal algebras) with 2^n elements have $n!$ automorphisms, hence there are many homomorphisms between two simple closure algebras with more than 2 elements.

5 Deciding propositional Basic Logic with SMT-solvers

Basic Logic was introduced by Hájek [16] to provide a unified approach to fuzzy logics, and judging by its rapid adoption in the research community, it has enjoyed considerable success in this regard. One of the reasons is that while it is a very general logic, it has elegant semantics with respect to the real unit interval, which allow for practical applications and tools over a suitably broad range. In particular, for propositional Basic Logic it is decidable whether a formula is a tautology, while for generalized Basic Logic this is still an open problem. Here we present an implementation of a decision procedure for propositional Basic Logic by encoding it into the Satisfiability Modulo Theories (SMT) framework. This method is based on an interpretation of Lukasiewicz logic, Gödel logic and product logic into SMT [4]. Ultimately these ideas go back to Mundici's result [21] that satisfiablity for Lukasiewicz logic is NP-complete, and Hähnle's translation from Lukasiewicz logic to integer linear programming [14,15]. In the current setting the translation to SMT is very simple, and since there are several efficient SMT-solvers available, this is an effective and flexible ways of implementing a decision procedure for propositional basic logic.

Boolean satisfiability solvers (SAT-solvers) are programs that take a classical propositional formula (often restricted to conjunctive normal form) as input and search for an assignment of truth values to the variables such that the formula is true, or report that no such assignment exists. Satisfiability modulo theories solvers (SMT-solvers) are generalizations of SAT-solvers that take as input a formula of typed first-order logic with equality (perhaps restricted to be quantifier-free), and determine if there is an assignment into a specific model (such as $\mathbb{R}$ or $\mathbb{Z}$) under which the formula is true. The "modulo theories" in the name of SMT-solvers refers to the theory of the model in which satisfiability is tested. E.g. a formula such $0 < x + y < 10$ & $x + x - y - y = 1$ would be satisfiable in $\mathbb{R}$ but not in $\mathbb{Z}$.

Applying SMT-solvers to decide propositional formulas in Lukasiewicz logic or Gödel logic is straight forward, as shown in [4]. We take an algebraic view, and implement decision procedures for prelinear Heyting algebras, abelian lattice-ordered groups, MV-algebras and BL-algebras. Recall that $(A, \wedge, \vee, \to, 1, 0)$ is a *Heyting algebra* if $(A, \wedge, \vee, 1, 0)$ is a bounded distributive lattice and $x \wedge y \leq z \iff y \leq x \to z$ for all $x, y, z \in A$. It is *prelinear* if the identity $(x \to y) \vee (y \to x) = 1$ holds, in which case the subdirectly irreducible models are linearly ordered. Prelinear Heyting algebras are the algebraic semantics of Gödel logic, and a propositional formula φ of Gödel logic is a tautology precisely when the equation $\varphi = 1$ is an identity of

prelinear Heyting algebras. The same correspondence holds for Lukasiewicz logic and MV-algebras.

An *abelian lattice-ordered group* is of the form $(A, \wedge, \vee, +, -, 0)$ where $(A, \wedge, \vee)$ is a (necessarily distributive) lattice, $(A, +, -, 0)$ is an abelian group and $+$ is order-preserving in both arguments. The variety of abelian lattice-ordered groups is generated by the model $(\mathbb{Z}, \min, \max, +, -, 0)$ as well as by $(\mathbb{R}, \min, \max, +, -, 0)$.

A MV-algebra is given by $(A, \wedge, \vee, \cdot, \neg, 1, 0)$ where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a commutative monoid, $\cdot$ is order-preserving in both arguments, $\neg\neg x = x$, $0 = \neg 1$ and $x \cdot y \leq z \iff y \leq \neg x \vee z$. This definition of MV-algebras emphasizes that they are residuated lattices, though they are often defined equationally using the dual operation $x \oplus y = \neg(\neg x \cdot \neg y)$.

The input for SMT-solvers is usually written in a standard language called SMT-LIB2. The input for deciding MV-identities is given below and can be used with a variety of solvers, such as CVC4, Z3, SMTinterpol, opensmt, etc. For the algebraic operations we use standard LaTeX names for the symbols. Any semicolon and all following characters up to the end of each line are optional comments. The SMT-LIB2 language has a syntax similar to LISP, so expressions are lists of tokens separated by spaces and enclosed in parentheses. The first token is usually a command or function name, and the remaining tokens are inputs for the function. E.g. (ite (< x y) x y) is the if-then-else function applied to a boolean test and producing (in this case) the smaller of the two values as output. The full syntax is defined at `www.smtlib.org`.

The first line of the code is a descriptive comment and the second line selects *quantifier-free linear real arithmetic* (QF_LRA) as the theory used by the SMT-solver. The next 7 lines define the MV-operations on the unit interval by $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$, $x \oplus y = (x + y) \wedge 1$, $x \cdot y = (x + y - 1) \vee 0$, $\neg x = 1 - x$, $x \to y = (1 - x + y) \wedge 1$, and $x \leftrightarrow y = (x \to y) \wedge (y \to x)$. The lines that start with "declare-const" define two real variables x, y and restrict their values to the interval $[0, 1]$. The third last line asserts the formula that is to be checked, followed by a comment showing the formula in standard notation. The last line asks the SMT-solver to check if the formula $\varphi < 1$ is satisfiable, in which case the formula φ is not a tautology. To test if an equation $s = t$ is an identity, one would check the formula $s \leftrightarrow t$, adding more "declare-const" lines if the formula contains more than 2 variables. Other commands like (get-model), (push 1), (pop 1) can be used to get information about a specific assignment where the formula does not evaluate to 1, or to assert, check and remove several formulas in a single file.

; Testing MV formulas in SMT
(set-logic QF_LRA)
(define-fun wedge ((x Real) (y Real)) Real
(ite (> x y) y x)); $x \wedge y$
(define-fun vee ((x Real) (y Real)) Real
(ite (> x y) x y)); $x \vee y$
(define-fun oplus ((x Real) (y Real)) Real
(wedge (+ x y) 1)); $x \oplus y$
(define-fun cdot ((x Real) (y Real)) Real
(vee (- (+ x y) 1) 0)); $x \cdot y$
(define-fun neg ((x Real)) Real (- 1 x)); $\neg x = 1 - x$
(define-fun to ((x Real) (y Real)) Real
(wedge 1 (- (+ 1 y) x))); $x \to y$
(define-fun leftrightarrow ((x Real) (y Real)) Real
(wedge (to x y) (to y x)))
(declare-const x Real)
(assert (<= 0 x)) (assert (<= x 1))
(declare-const y Real)
(assert (<= 0 y)) (assert (<= y 1))
(assert (< (to (vee (cdot x x) (cdot y y))
(cdot (vee x y) (vee x y))) 1))
; test if $(x^2 \vee y^2) \to (x \vee y)^2 < 1$ is satisfiable
(check-sat)

Checking equations in prelinear Heyting algebras is a matter of deleting the definitions for oplus and cdot, and replacing the next two lines by:
(define-fun neg ((x Real)) Real (ite (= x 0) 1 0)); $\neg x$
(define-fun to ((x Real) (y Real)) Real (ite (<= x y) 1 y)); $x \to y$

An abelian ℓ-group inequation $s \leq t$ can be expressed directly using the operations $+, -, 0$ of the logic QF_LRA, and the SMT-solver is used to check if $s > t$ is satisfiable. The assertions that restrict variables to the unit interval have to be removed in this case. For equations $s = t$ one checks if $s > t$ or $t > s$ is satisfiable, i.e., (assert (or (< s t) (< t s))) in SMT-LIB2 syntax. A similar approach can be used to check (in)equations in the negative cone of $\mathbb{R}$ by defining $x \cdot y = (x + y) \wedge 0$. By using a translation with an extra variable z as in [13] one can also check (in)equations in the negative cone of $\mathbb{R}$ with a new bottom element, which is equivalent to checking propositional formulas in *product logic*. This is an improvement over the suggestion in [4] to use full real arithmetic for product logic, since implementations of *linear* real arithmetic in SMT-solvers are currently more efficient.

To decide propositional basic logic with an SMT-solver requires the following result of [2] (see also [1]).

Theorem 13 *Let $A_n = \bigoplus_{i=0}^{n} [0, 1]$ be the ordinal sum of $n + 1$ unit-interval MV-algebras, and let $\mathcal{V}_n$ be the variety generated by all n-generated BL-algebras. Then $\mathcal{V}_n = HSP(A_n)$, hence an n-variable BL-identity holds in A_n if and only if it holds in all BL-algebras.*

By constructing the algebra A_n of the above result within the SMT language, one obtains an effective means of checking n-variable BL-identities. The universe for A_n is taken to be the interval $[0, n + 1]$. The

definition of fusion and implication are

$$x \cdot y = \begin{cases} \max(x+y-1-\lfloor y \rfloor, \lfloor x \rfloor) & \text{if } \lfloor x \rfloor = \lfloor y \rfloor \\ \min(x,y) & \text{otherwise} \end{cases}$$

$$x \to y = \begin{cases} n+1 & \text{if } x \leq y \\ y & \text{if } \lfloor y \rfloor < \lfloor x \rfloor \\ \min(1+y-x+\lfloor x \rfloor, 1+\lfloor y \rfloor) & \text{otherwise} \end{cases}$$

Some SMT-solvers can express the floor function $\lfloor x \rfloor$ in which case one can use the above definitions as given. However the the floor function is not part of the standard SMT-LIB2 language. Instead we give here a straightforward SMT-LIB2 implementation of these operations that uses $n+1$ cases. So the length of the formula grows linearly with respect to n. For $n = 1$ and $n = 2$, here are the formulas that can be used to check 1-variable and 2-variable BL-identities.

$n = 1$:
(define-fun cdot ((x Real) (y Real)) Real (ite (and (< x 1) (< y 1)) (vee (- (+ x y) 1) 0) (ite (and (>= x 1) (>= y 1)) (vee (- (+ x y) 2) 1) (wedge x y))))
(define-fun to ((x Real) (y Real)) Real (ite (<= x y) 2 (ite (and (>= x 1) (< y 1)) y (wedge 1 (- (+ 1 y) x)))))

$n = 2$:
(define-fun cdot ((x Real) (y Real)) Real (ite (and (< x 1) (< y 1)) (vee (- (+ x y) 1) 0) (ite (and (>= x 1) (< x 2) (>= y 1) (< y 2)) (vee (- (+ x y) 2) 1) (ite (and (>= x 2) (>= y 2)) (vee (- (+ x y) 3) 2) (wedge x y)))))
(define-fun to ((x Real) (y Real)) Real (ite (<= x y) 3 (ite (and (< x 1) (< y 1)) (+ (- 1 x) y) (ite (and (<= 1 x) (< x 2) (<= 1 y) (< y 2)) (+ (- 2 x) y) (ite (and (<= 2 x) (<= 2 y)) (+ (- 3 x) y) y)))))

For larger values of n such formulas can be generated algorithmically. A program has been written in Python that takes a BL-formula written in LaTeX as input, counts the number of distinct variables, translates the formula to SMT-LIB2, generates the code of the operations for this number of variables, submits this code to the CVC4 SMT-solver and finally indicates whether the formula is a BL-tautology.

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