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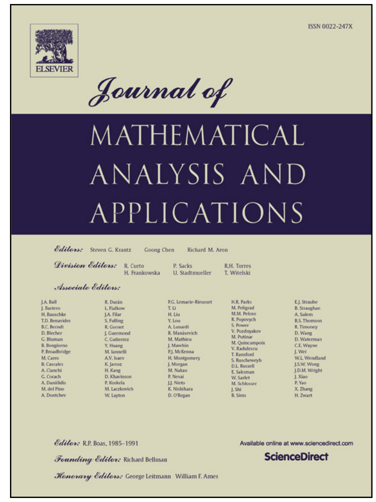

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The Stationary Phase Method for Real Analytic Geometry

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Abstract

We prove that the existence of isolated solutions of systems of equations of analytical functions on compact real domains in $\mathbb{R}^p$, is equivalent to the convergence of the phase of a suitable complex phase integral $I(h)$ for $h \to \infty$. As an application, we then use this result to prove that the problem of establishing the irrationality of the value of an analytic function $F(x)$ at a point x_0 can be rephrased in terms of a similar phase convergence.

1 Introduction

Real algebraic geometry, has developed relatively late its own techniques and ideas to mirror, in part, the extensive theoretical development of complex algebraic geometry [2], thanks in particular to generalizations of great impact such as the theory of Nash manifolds [12]. However, a general tool that can encompass problems on a very large class of transcendental functions is lacking, and in this paper we suggest that complex phase integrals and

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the stationary phase method could provide a powerful global approach for the study of solutions of systems of real analytic equations. In particular, the main result of this paper shows that the existence of isolated solutions of a system of analytical equations $\mathbf{F}(x) = 0$ over a compact $\mathcal{B}$ in $\mathbb{R}^p$ is equivalent, under suitable conditions, to the existence of the limit of the phase of a complex phase integral $I(h)$ for h that goes to infinity.

We then approach another, apparently unrelated question: establishing the irrationality of special numbers. As it is well known, this question, together with the more general question on the transcendence of numbers, has a very long and rich history, see for example [7]. However, the abundance of open problems in the field, especially about the irrationality of convergent power series and special numbers [5, 6, 8], suggests the need for further ideas specifically tailored to analytical functions. In Section 3 we propose that a new viewpoint is possible, that transforms the problem of the irrationality of $F(x_0)$, with F analytical, into the geometric problem of finding zeros of a systems of equations on a four dimensional open, bounded domain. This problem is then phrased in terms of the phase integral method developed in Section 2 for geometric problems. Some of these results were announced, without proofs, in [11].

2 Geometric Phase integrals

The main result of this section, Theorem 2.1, follows as an application and specialization to real analytic geometry of the method of stationary phase [13, 4, 3]. Applications of the stationary phase method for an analytic study of convex geometry can be found already in [9] (for example Theorem 7.7.16 therein), here we focus on the more basic problem of establishing the existence of real solutions of systems of analytical equations. More particularly, consider p real analytic functions $F_1, \dots, F_p$ defined on a compact set $\mathcal{B} \subset \mathbb{R}^p$, and the vector function $\mathbf{F}(x) = [F_1(x), ..., F_p(x)]$. We can construct the asso-

ciated function $L(x) = \sum_{i=1}^{p} F_i(x)^2$, which is such that $L(x) = 0$ if and only if $F_i(x) = 0$ for all i. Moreover, it is immediate to see that every point such that $L(x) = 0$ is also a critical point for $L(x)$. The relations between critical points of $L(x)$ and solutions of the system of equations $\mathbf{F}(x) = 0$ can be made more compelling by building a suitable phase integral, whose asymptotic behavior depends on the existence of solutions to the system itself. Indeed the following theorem holds:

Theorem 2.1. *Let $\mathbf{F}(x) = [F_1(x), ..., F_p(x)]$ be an analytical, vectorial function defined on a compact domain $\mathcal{B} \subset \mathbb{R}^p$, let $L(x) = \sum_{i=1}^{p} F_i(x)^2$ have only isolated critical points in $\mathcal{B}$, and let $\mathcal{A}$ be a closed interval in $\mathbb{R}$ that does not contain the origin. Consider the integral*

$$I(h) = \int_{\mathcal{A}} \int_{\mathcal{B}} e^{ihL(x)y^2} dxdy, \;\; y \in \mathcal{A} \subset \mathbb{R}, \;\; x \in \mathcal{B} \subset \mathbb{R}^p, \tag{1}$$

and denote by $\phi(I(h))$ the phase of $I(h)$. Then the system $\mathbf{F}(x) = 0$ has a solution in $\mathcal{B}$ if and only if $\phi(I(h))$, the complex phase of $I(h)$, has a limit for h going to infinity.

Proof. The integration in x in the integral in (1) can be written, for $h \to \infty$, with respect to the critical points of $L(x)$ in $\mathcal{B}$, using standard stationary phase approximation methods [4, 13].

We can consider separately the critical points such that $L(x) = 0$, and those for which $L(x) \neq 0$, and we have:

$$\begin{aligned} \lim_{h\to\infty} I(h) = & \int_{y\in\mathcal{A}} \sum_{L(x_i)=0} (\frac{2\pi}{h})^{\frac{p}{2}} \frac{1}{y^p(\det H(x_i))^{1/2}} e^{i\frac{\pi}{4}\sigma_i} dy + \\ & \int_{y\in\mathcal{A}} \sum_{L(x_j)\neq 0} (\frac{2\pi}{h})^{\frac{p}{2}} \frac{1}{y^p(\det H(x_j))^{1/2}} e^{ihL(x_j)y^2 + i\frac{\pi}{4}\sigma_j} dy \end{aligned} \tag{2}$$

where $H(x_*)$ denotes the Hessian matrix of $L(x)$ evaluated at x_*, and σ_* denotes the signature of $H(x_*)$.

We are assuming here that there is at least one critical point with Hessian different from zero, but very similar arguments to those we present here can be deduced without this restriction, at the cost of a more complicated argument that depends on higher derivatives of L. Since the function L is analytical, so is the system of equations whose solution defines its critical points, and if we assume all such solutions are isolated, they are in finite number over a compact set (see for example [10], page 180). We will now simplify the representation of $I(h)$, when h is large, by considering separately the two summands in (2). We begin with the first summand, which we will call $I_1(h)$:

$$\begin{aligned}\lim_{h\to\infty} I_1(h) := \lim_{h\to\infty}\int_{y\in\mathcal{A}}\sum_{L(x_i)=0}(\frac{2\pi}{h})^{\frac{p}{2}}\frac{1}{y^p(\det H(x_i))^{1/2}}e^{i\frac{\pi}{4}\sigma_i}dy = \\ \lim_{h\to\infty}\sum_{L(x_i)=0}(\frac{2\pi}{h})^{\frac{p}{2}}\frac{1}{(\det H(x_i))^{1/2}}e^{i\frac{\pi}{4}\sigma_i}S\end{aligned} \tag{3}$$

where $S=\int_{\mathcal{A}}\frac{1}{y^p}dy$. Since the sum in the expression is a finite sum, and by factoring out $\frac{1}{h^{p/2}}$, one immediately sees that the phase of $I_1(h)$ is independent of h and only depends on the critical points x_i's, or more exactly, on their signatures σ_i.

Let us now analyze $I_2(h)$, namely the second summand in (2):

$$I_2(h) := \int_{y\in\mathcal{A}}\sum_{L(x_j)\neq 0}(\frac{2\pi}{h})^{\frac{p}{2}}\frac{1}{y^p(\det H(x_j))^{1/2}}e^{ihL(x_j)y^2+i\frac{\pi}{4}\sigma_j}dy \tag{4}$$

We note first of all that each integral

$$\int_{y\in\mathcal{A}}(\frac{2\pi}{h})^{\frac{p}{2}}\frac{1}{y^p(\det H(x_j))^{1/2}}e^{ihL(x_j)y^2+i\frac{\pi}{4}\sigma_j}dy \tag{5}$$

can be written as

$$(\frac{2\pi}{h})^{\frac{p}{2}} \frac{1}{(\det H(x_j))^{1/2}} e^{i\frac{\pi}{4}\sigma_j} \int_{y\in\mathcal{A}} \frac{1}{y^p} e^{ihL(x_j)y^2} dy, \tag{6}$$

and it is therefore a phase integral in y computed over an interval that does not include the only critical point ($y = 0$). Such integral decreases at least like $O(\frac{1}{hL(x_j)})$, the leading contribution from the boundary points of $\mathcal{A}$ ([4], page 52; [13], page 488). Therefore

$$\lim_{h\to\infty} I_2(h) = \lim_{h\to\infty} (2\pi)^{\frac{p}{2}} \sum_{L(x_j)\neq 0} \frac{1}{(\det H(x_j))^{1/2}} e^{i\frac{\pi}{4}\sigma_j} O(\frac{1}{h^{p/2+1}L(x_j)}). \tag{7}$$

Recall we are in the generic case where there is at least one point x_j with $\det H(x_j) \neq 0$, and that there are finitely many critical points, and therefore also finitely many critical points for which $L(x_j) \neq 0$. This last observation allows us to conclude that all the values $L(x_j)$ can be bounded away from 0, and the entire sum above can be estimated as

$$\lim_{h\to\infty} I_2(h) = O(\frac{1}{h^{p/2+1}}). \tag{8}$$

This is a negligible quantity with respect to $I_1(h) \sim \frac{1}{h^{p/2}}$. We can conclude that if $L(x) = 0$ for at least a specific x_j, then the limit for $h \to \infty$ of $I(h) = I_1(h) + I_2(h)$ has constant phase. If on the other hand there are no values for which $L(x) = 0$, the phase will not converge: this is easy to see when we have at least a critical point x_j with $L(x_j) \neq 0$ and $\det H(x_j) \neq 0$, since in that case the term $e^{ihL(x_j)y^2}$ in $I_2(h)$ will continue to change phase as h goes to infinity.

Note that if the critical points such that $L \neq 0$ have $\det H = 0$, we would need to look at higher order asymptotic terms, but, since the number of critical points is finite, we could still look at the highest order, dominant critical points, whose phase is dependent on $e^{ihL(x_j)y^2}$ ([13] page 483).

Suppose instead that there are no critical points at all, then the integral in (1) is dominated by the evaluation of some derived phase integral on the boundary of $\mathcal{A} \times \mathcal{B}$; more precisely, it is true that (adapted from [13], page 488):

$$I(h) \sim -\frac{i}{h} \int_{\partial(A \times B)} G e^{ihL(x)y^2} da \tag{9}$$

where $\partial(\mathcal{A} \times \mathcal{B})$ is the boundary of $\mathcal{A} \times \mathcal{B}$, da is a suitable measure on the boundary, and G is a multiplier function dependent on $L(x)y^2$.

Now, $\mathcal{A} \times \mathcal{B}$ is an hypercube, and a recursive application of the result in (9), to lower and lower dimensional boundaries of its hyperfaces, will reduce the asymptotic evaluation of $I(h)$ to a sum of suitable multiples of evaluations of $e^{ihL(x)y^2}$ at the vertexes of the hypercube. None of these values is independent of h, since we assumed there are no critical points of L on $\mathcal{A} \times \mathcal{B}$, and therefore $L(x)y^2 \neq 0$ everywhere. This implies that $\lim_{h\to\infty} \phi(I(h))$ does not exist when there are no critical points on $\mathcal{A} \times \mathcal{B}$. □

Remark 2.2. Strictly speaking, the proof relied on considering the special (but generic) setting with at least one of the critical points with Hessian different from zero. While, as we pointed out, this restriction can be avoided, it is important to note that such a setting is sufficient to prove our main result in Section 3.

Because of its value in establishing the existence of solutions of systems of real analytical equations, we will call the integrals in (1) *geometric phase integrals*. Similarly, we will call $L(x)$ the *geometric Lagrangian* associated to $\mathbf{F}(x) = 0$, in analogy to the Lagrangian functions used in defining path and field integrals [1].

3 An application to Irrationality Tests

We will now apply the general setting of Section 2 to a more complex case that involves infinitely many critical points, but that is such that the relative contributions of each critical point can be controlled.

Suppose we want to know whether $F(x_0)$ is irrational. The system of equations

$$\begin{aligned}
&F(x) - \alpha = 0, \;\; x \in [x_0 - \delta, \; x_0 + \delta] \\
&x - x_0 = 0, \quad \alpha \in F([x_0 - \delta, x_0 + \delta]) \\
&\sin\frac{\pi}{m} = \sin\frac{\pi}{n} = 0, \quad m, n \in (0, \, 1] \\
&\alpha m - n = 0
\end{aligned} \tag{10}$$

has a solution if and only if $F(X_0)$ is a rational number. We can adapt the stationary phase integral analysis performed in Section 2 to be of relevance in this case. We build to this purpose the geometric Lagrangian function:

$$L(x, \alpha, m, n) = (F(x) - \alpha)^2 + (x - x_0)^2 + \sin^2\frac{\pi}{m} + \sin^2\frac{\pi}{n} + (\alpha m - n)^2 \tag{11}$$

Again, $L(x, \alpha, m, n) = 0$ if and only if the previous system has a zero solution, and we may ask whether the limit for $h \to \infty$ of the phase of the following integral has any relation to the rationality of $F(x_0)$:

$$I_L(h) = \int_{y \in \mathcal{A}} \int_{\omega \in \Omega_\delta} e^{ihL(\omega)y^2} d\omega dy, \;\; 0 \notin \mathcal{A} \tag{12}$$

where $\omega = (x, \alpha, m, n)$ and we have denoted by Ω_δ the cartesian product of the domains allowed for each of the components of ω in (10).

The main complication, with respect to the similar setting in Section 2,

is the existence of infinitely many critical points, every time there is at least one point such that $L(\omega) = 0$. Indeed the critical points of $L(\omega)$ are solutions of the system:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 2(F(x) - \alpha)\frac{dF(x)}{dx} + 2(x - x_0) = 0 \\
\frac{\partial L}{\partial \alpha} &= -2(F(x) - \alpha) + 2(\alpha m - n) = 0 \\
\frac{\partial L}{\partial m} &= 2\sin\frac{\pi}{m}\cos\frac{\pi}{m}(-\frac{\pi}{m^2}) + 2(\alpha m - n)\alpha = 0 \\
\frac{\partial L}{\partial n} &= 2\sin\frac{\pi}{n}\cos\frac{\pi}{n}(-\frac{\pi}{n^2}) - 2(\alpha m - n) = 0.
\end{aligned} \tag{13}$$

We can see that if $\omega_0 = (x_0, \alpha_0, m_0, n_0)$ is a solution of $L(\omega_0) = 0$, then it is also a critical point of L. However, also $\omega_i = (x_0, \alpha_0, m_i, n_i)$ will be a zero and a critical point of L, where $m_i = \frac{m_0}{i}$ and $n_i = \frac{n_0}{i}$, i any integer (this can be seen by simple substitution in $\alpha m - n = 0$, assuming $\alpha_0 m_0 - n_0 = 0$). Note that all critical points with $L(\omega) = 0$ need to have $x = x_0$ and $\alpha = \alpha_0 = F(x_0)$.

To overcome this proliferation of critical points, our argument will assume that we work in the limit when the domain approaches zero in the variables m, n. We will need also to control the decay of the determinant of the Hessian in the asymptotic expression used to prove Theorem 2.1. Regarding the first issue, we cut the domain of m and n as $m \in [M, 1]$ and $n \in [N, 1]$ with $0 < M, N < 1$ and consider the compact domain

$$\Omega_\delta(M, N) := [x_0 - \delta,\ x_0 + \delta] \times F([x_0 - \delta, x_0 + \delta]) \times [M, 1] \times [N, 1]. \tag{14}$$

The main conclusion of our analysis can be stated as a theorem:

Theorem 3.1. *Let $F(x)$ be an analytical function in the interval $[x_0 - \delta, x_0 + \delta]$, with δ sufficiently small, and assume $F'(x_0) \neq 0$. Consider the following*

phase integral, obtained by restricting $I_L(h)$ to the domain $\Omega_\delta(M,N)$:

$$I_L(h,M,N) := \int_{y\in\mathcal{A}}\int_{\omega\in\Omega_\delta(M,N)} e^{ihL(\omega)y^2}d\omega dy,\ 0\notin\mathcal{A}, \tag{15}$$

where L is defined in (11). Let $\phi(I_L(h,M,N))$ be the complex phase of $I_L(h,M,N)$. Then $F(x_0)$ is a rational number if and only if the following limit converges:

$$\lim_{M,N\to 0}\lim_{h\to\infty}\phi(I_L(h,M,N)). \tag{16}$$

Proof. We start our proof with a simple analysis of the dimensionality, in x and α, of the set of solutions of the first equation of the systems in (13), that define the critical points. We will eventually prove that for δ small enough all critical points in Ω_δ are isolated. To achieve this goal, we note that, for δ sufficiently small, we can control the norm of the function $(F(x)-\alpha)-F'(x)(x-x_0)$, in Ω_δ; indeed we have

$$\begin{aligned}
&|(F(x)-\alpha)-F'(x)(x-x_0)| = |(F(x)-(F(x_0)+\epsilon_1))-(F'(x_0)+\epsilon_2)(x-x_0)| = \\
&|(F(x)-F(x_0))-F'(x_0)(x-x_0)-\epsilon_1-\epsilon_2(x-x_0)| \leq \\
&|(F(x)-F(x_0))-F'(x_0)(x-x_0)|+|\epsilon_1|+|\epsilon_2(x-x_0)| \leq \\
&|(F(x)-F(x_0))-F'(x_0)(x-x_0)|+|\epsilon_1|+|\epsilon_2\delta| \leq \\
&|\epsilon_3|+|\epsilon_1|+|\epsilon_2\delta|
\end{aligned} \tag{17}$$

where we used the fact that the derivative of $F(x)$ is well defined and continuous in a neighborhood of x_0, and ϵ_t, $t=1,2,3$, can be made as small as necessary by choosing δ small enough. We can interpret this result by saying that the vectors $(F(x)-\alpha, x-x_0)$ and $(1,-F'(x))$ are almost orthogonal for all (x,α) in Ω_δ, whenever δ is sufficiently small. Now the equation $2(F(x)-\alpha)\frac{dF(x)}{dx}+2(x-x_0)=0$ in (13) is equivalent to saying that

$(F(x)-\alpha, x-x_0)$ and $(F'(x), 1)$ are orthogonal, for some choice of (x, α) in Ω_δ. Together with the previous calculations, this implies, for two dimensional vectors, that $(F'(x), 1)$ and $(1, -F'(x))$ should be almost parallel; however, for the choice of (x, α) made above, these vectors are themselves orthogonal, and we therefore conclude there is no solution of $2(F(x)-\alpha)\frac{dF(x)}{dx} + 2(x-x_0) = 0$, unless $(F(x)-\alpha, x-x_0) = (0, 0)$, in which case $x = x_0$ and $\alpha = F(x_0)$. Note that this argument depends on the assumption $F'(x_0) \neq 0$ otherwise we would not be able to infer $\alpha = F(x_0)$ from $x = x_0$, in the first equation of (13).

We deduce moreover, from the whole set of equations in system (13), that critical points with $x = x_0$ and $\alpha = F(x_0)$, if they exists, are bound to have $\alpha m - n = 0$, $2\sin\frac{\pi}{m}\cos\frac{\pi}{m}(-\frac{\pi}{m^2}) = 0$, and $2\sin\frac{\pi}{n}\cos\frac{\pi}{n}(-\frac{\pi}{n^2}) = 0$. Therefore they are all isolated points, in finite number on all compacts $\Omega_\delta(M, N)$ and they either satisfy $\sin\frac{\pi}{m} = 0$ and $\sin\frac{\pi}{n} = 0$ (and therefore $L(\omega) = 0$), or they are such that $\cos\frac{\pi}{m} = 0$ and/or $\cos\frac{\pi}{n} = 0$. Since critical points are isolated and finitely many in $\Omega_\delta(M, N)$, for any $0 < M, N < 1$, we are in the position of applying Theorem 2.1 in the rest of our argument.

To conclude the proof of the theorem, we need the following estimate: suppose α_0 is rational and that m_0, n_0 are the largest values such that $L(x_0, \alpha_0, m_0, n_0) = 0$, then

$$\det H(x_0, \alpha_0, m_i, n_i) \sim C\frac{i^8}{m_0^8} \tag{18}$$

when i goes to infinity, and where $m_i = \frac{m_0}{i}$, $n_i = \frac{n_0}{i}$, i positive integer and C is a positive number bigger than 1. Indeed, remembering that, for critical points $\omega_i = (x_0, \alpha_0, m_i, n_i)$ with $L(\omega_i) = 0$, we have $\alpha_0 m_i - n_i = 0$, $\sin\frac{\pi}{m_i} = 0$, $\sin\frac{\pi}{n_i} = 0$ (and therefore $\cos\frac{\pi}{n_i} = 1$, $\cos\frac{\pi}{m_i} = 1$), we can write the Hessian matrix of $L(\omega)$ evaluated at such critical points as:

$$H(\omega_i) = \begin{pmatrix} 2F'(x_0)^2 + 2 & -2F'(x_0) & 0 & 0 \\ -2F'(x_0) & 2 + 2m_i^2 & 2\alpha_0 m_i & -2m_i \\ 0 & 2\alpha_0 m_i & 2\frac{\pi^2}{m_i^4} + 2\alpha_0^2 & -2\alpha_0 \\ 0 & -2m_i & -2\alpha_0 & 2\frac{\pi^2}{n_i^4} + 2 \end{pmatrix}. \tag{19}$$

Using again the fact that, for these critical points, $\alpha_0 m_i = n_i$, the evaluation of the determinant gives:

$$\det H(\omega_i) = (4F'(x_0)^2 + 4m_i^4 + 4)\big(4(\frac{\pi^2}{m_i^4} + \alpha_0^2)(\frac{\pi^2}{\alpha_0^4 m_i^4} + 1) - 4\alpha_0^2\big) + 4(F'(x_0)^2 + 2)\alpha_0^2 m_i^2(-\frac{\pi^2}{\alpha_0^4 m_i^4} - 1 + 8) - 16(F'(x_0)^2 + 1)m_i^2(\frac{\pi^2}{m_i^4} + \alpha_0^2). \tag{20}$$

By recalling $m_i = \frac{m_0}{i}$ with $i = 1, 2, 3...$, if we let $i \to \infty$ (i.e. $m_i \to 0$), the leading term of the determinant will be:

$$\det H(\omega_0) \sim 16\frac{\pi^2}{m_i^4}\frac{\pi^2}{\alpha_0^4 m_i^4} = \frac{16\pi^4}{\alpha_0}\frac{i^8}{m_0^8} \tag{21}$$

which is exactly the estimate in (18), with $C = \frac{16\pi^4}{\alpha_0}$. This being the case, we can be assured that there is a i_T such that for $i > i_T$ the Hessian $H(x_0, \alpha_0, m_i, n_i)$ has nonzero (positive) determinant, and therefore the quadratic asymptotic approximation used in Theorem 2.1 holds for all $i > i_T$.

Also, note that, for $i < i_T$ any critical point such that $H(x_0, \alpha_0, m_i, n_i) = 0$ will depend from h, in the asymptotic expansion, as $\frac{1}{h^{j+2}}$ for some integer $j > 0$ that depends from the order of the zero, while all critical points with $H(x_0, \alpha_0, m_i, n_i) \neq 0$ depend from h as $\frac{1}{h^2}$ ([13], page 480). This implies that we can neglect critical points that have Hessian equal to zero, when h goes to infinity, since the asymptotic relation in (18) assures us that there are infinitely many dominant critical points with non-zero determinant of the

Hessian in Ω_δ, and therefore at least one of them for M, N sufficiently small. Therefore we have:

$$\lim_{M,N\to 0}\lim_{h\to\infty}\phi(I_L(h,M,N)) = \lim_{M,N\to 0}\lim_{h\to\infty}\int_{y\in A}\sum_{\substack{L(\omega_i)=0\\ \det H(\omega_i)\neq 0\\ \omega_i\in\Omega_\delta(M,N)}}(\frac{2\pi}{h})^2\frac{1}{y^4(\det H(\omega_i))^{1/2}}e^{i\frac{\pi}{4}\sigma_i} \tag{22}$$

where we have used the results from Theorem 2.1, the fact that $p = 4$, and neglected already the (finitely many) critical point for which $L(\omega) \neq 0$, or those for which $L(\omega_i) = 0$ and $\det H(\omega_i) = 0$.

Consider now the partial sums:

$$\theta_{M,N} := \sum_{\substack{L(\omega_i)=0\\ \det H(\omega_i)\neq 0\\ \omega_i\in\Omega_\delta(M,N)}}\frac{(2\pi)^2}{(\det H(\omega_i))^{1/2}}e^{i\frac{\pi}{4}\sigma_i}; \tag{23}$$

then

$$\lim_{M,N\to 0}\lim_{h\to\infty}\phi(I_L(h,M,N)) = \lim_{M,N\to 0}\lim_{h\to\infty}\phi\big(\int_{y\in A}\frac{1}{h^2}\frac{1}{y^4}\theta_{M,N}dy)\big) = \lim_{M,N\to 0}\lim_{h\to\infty}\phi\big(\frac{1}{h^2}S\theta_{M,N}\big) = \lim_{M,N\to 0}\phi(\theta_{M,N}) \tag{24}$$

where $S = \int_{y\in\mathcal{A}}\frac{1}{y^4}dy$. Now, since $\det H(x_0,\alpha_0,m_i,n_i) \sim C\frac{i^8}{m_0^8}$, when i goes to infinity, we can argue that the following series converges:

$$\theta = \sum_{\substack{L(\omega_i)=0\\ \det H(\omega_i)\neq 0\\ \omega_i\in\Omega_\delta}}\frac{(2\pi)^2}{(\det H(\omega_i))^{1/2}}e^{i\frac{\pi}{4}\sigma_i}. \tag{25}$$

Indeed, the convergence of the this series can be reduced to the conver-

gence of its absolute value

$$\sum_{\substack{L(\omega_i)=0\\ \det H(\omega_i)\neq 0\\ \omega_i\in\Omega_\delta}} \frac{(2\pi)^2}{(\det H(\omega_i))^{1/2}} \tag{26}$$

and, by comparison with the convergent series $\sum_i \frac{1}{i^4}$, we obtain

$$\lim_{i\to\infty} \frac{(2\pi)^2}{(\det H(\omega_i))^{1/2}} \Big/ \frac{1}{i^4} = \lim_{i\to\infty} \frac{(2\pi)^2}{\sqrt{C}(i^8/m_0^8)^{1/2}} \Big/ \frac{1}{i^4} = (2\pi)^2 m_0^4/\sqrt{C}. \tag{27}$$

Since the limit of the quotient above is nonzero, the series in (26) converges, and θ in (25) is well defined. The convergence of the series defining θ allows us one final limiting argument, i.e.,

$$\lim_{M,N\to 0} \lim_{h\to\infty} \phi(I_L(h, M, N)) = \lim_{M,N\to 0} \phi(\theta_{M,N}) = \phi(\theta). \tag{28}$$

This last equality completes the proof of the Theorem. □

Remark 3.2. The convergence of the series defining θ in (25) is intimately related to the estimate in Eq. (18). The existence of this estimate depends on the fact that we use the equations $\sin\frac{\pi}{m} = 0$, $\sin\frac{\pi}{n} = 0$, on a bounded domain, to force the rationality of $F(x_0)$ (via the additional equation $\alpha m - n = 0$). Such convergence would not hold if rationality was enforced via the equations $\sin \pi m = 0$, $\sin \pi n = 0$ on an unbounded domain. Note also that the phase integral in (12) depends *functionally* on $F(x)$, so that the local behavior of $F(x)$ for $x \sim x_0$ becomes relevant for the irrationality of $F(x_0)$.

Remark 3.3. Our choice of the particular dependence of the geometric Lagrangians from the variable y is not the only one that would establish the results in Theorems 2.1 and 3.1, even though it is probably the simplest. Alternatively, one could look at the geometric Lagrangian $L(\omega)\exp(y) + y^3$ whose critical points are only those associated to $L(\omega) = 0$, removing the necessity of the careful estimate of the contribution of critical points with

$L(\omega) \neq 0$. However, this more complicated geometric Lagrangian leads always to degenerate critical points in the stationary phase asymptotic approximation and therefore to a more intricate proof of the two theorems.

Ultimately, our approach suggests that analytical techniques and ideas from the asymptotic and non-perturbative study of complex phase integrals are relevant to problems of real analytic geometry, as well as to problems about the irrationality of point-wise evaluation of analytical functions.

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