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TOPOLOGICAL DUALITY AND LATTICE EXPANSIONS PART II: LATTICE EXPANSIONS WITH QUASIOPERATORS

M. ANDREW MOSHIER AND PETER JIPSEN

1. INTRODUCTION

Lattices have many applications in mathematics and logic, in which they occur together with additional operations. For example, in applications of Hilbert spaces, one is often concerned with the lattice of closed subspaces of a fixed space. This lattice is not distributive, but there is an operation taking a given subspace to its orthogonal subspace. More generally, *ortholattices* are lattices with a unary operation $(-)^\dagger$ that is involutive ($a = a^{\dagger\dagger}$), sends finite joins to meets and for which a and $a^\dagger$ are complements. *Bounded modal lattices* $(L, \vee, \wedge, 0, 1, \Diamond, \Box)$ are models of (not necessarily distributive) modal logic, where $\Diamond$ and $\Box$ are unary operations that preserve finite join and finite meet, respectively, and represent *possible* and *necessary*. *Bounded lattice-ordered monoids* are bounded lattices with an associative binary operation $\cdot$ and an identity element 1. In these examples it is postulated that the additional operations "preserve structure" in various different senses. Orthocomplementation sends finite joins to meets (and finite meets to joins). The modal operators preserve finite joins and finite meets, respectively. Similarly, the monoid operation distribute over finite joins. *Bounded residuated lattices* are bounded lattice-ordered monoids with two further operations $\backslash$, $/$ that interact with $\cdot$ via the universally quantified *residuation law*:

$$x \cdot y \leq z \qquad \Leftrightarrow \qquad x \leq z/y \qquad \Leftrightarrow \qquad y \leq x\backslash z.$$

This law implies $\backslash$ is join reversing (i.e. sends joins to meets) in the first argument and meet preserving in the second, whereas $/$ is meet preserving in the first and join reversing in the second argument.

These examples illustrate that the additional operations on lattices can preserve structure in a variety of ways. Each one, however, is join reversing or meet preserving in each argument, or dually is meet reversing or join preserving in each argument. Such operations are called quasioperators, and we will use the example of bounded residuated lattices to illustrate the general case.

The main objective of this paper (the second of two parts) is to show that quasioperators can be dealt with smoothly in the topological duality established in Part I. Similar operators have been discussed by [Har97], [HD97], and in the setting of canonical extensions and generalized Kripke frames by [DGP05], [Geh06].

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Each quasioperator $f : L^n \to L$ has an associated monotonicity type $\varepsilon \in \{1, \partial\}^{n+1}$ which determines whether f is join or meet preserving or reversing in each argument. Here L^∂ denotes the order-dual of L, and $L^1 = L$. The value of ε_i is chosen so that f will be join preserving in each argument when considered as a map from $\prod_{i=0}^{n-1} L^{\varepsilon_i}$ to L^{ε_n}. For example the operation $\backslash$ has monotonicity type $(1, \partial, \partial)$. In a bounded modal lattice, the "possible" operator has monotonicity type $(1, 1)$, whereas the "necessary" operator has type (∂, ∂).

2. Summary of Part I

In Part I, [JM], we prove duality theorems for bounded lattices involving the following notions. We refer the reader to Part I for proofs of all results in this section.

The category **Lat** consists of bounded lattices and bounded lattice homomorphisms. Taking the meet semilattice reducts of lattices yields a larger category $\mathbf{Lat}_{\wedge,1}$ of lattices and meet semilattice homomorphisms. Also, the category **SLat** consists of meet semilattices and meet semilattice homomorphisms.

In a T_0 topological space X, the *specialization order* on X is defined by $x \sqsubseteq_X y$ if and only if every neighborhood of x is also a neighborhood of y. Indeed, the T_0 axiom says exactly that this is a partial order; the T_1 axiom says that it is trivial. A *saturated set* is an upper set with respect to specialization. Alternatively, because of how the specialization order is defined, saturated sets are characterized as the intersections of opens. A *filter in* X is a saturated subset $F \subseteq X$ that is also downward directed, i. e., it is non-empty and for any $x, y \in F$, there exists $z \in F$ so that $z \sqsubseteq x$ and $z \sqsubseteq y$.

Define the following collections of subsets of X.

- $\mathsf{K}(X)$: the collection of compact saturated subsets of X.
- $\mathsf{O}(X)$: the collection of open subsets of X.
- $\mathsf{F}(X)$: the collection of filters of X.

Intersections of these are denoted by concatenation, e.g., $\mathsf{OF}(X) = \mathsf{O}(X) \cap \mathsf{F}(X)$. In particular, OF, KO and KOF will be important.

In any topological space X, a filter in X is compact if and only if it is a principal filter. So the collection $\mathsf{KOF}(X)$ consists of certain principal filters. Letting $\uparrow x$ denote the upper set (equivalently, filter) generated by x, define

- $\mathsf{Fin}(X) = \{x \in X \mid \uparrow x \in \mathsf{KOF}(X)\}$.

So there is an order reversing bijection between $\mathsf{KOF}(X)$ and $\mathsf{Fin}(X)$.

For set $A \subseteq X$, define the *F-saturation of A* by

$$\mathsf{fsat}(A) = \bigcap \{F \in \mathsf{OF}(X) \mid A \subseteq F\}$$

Say that a set is *F-saturated* if it is its own F-saturation. Clearly, in any topological space, the F-saturated sets form a complete lattice in which meets are formed by taking intersections and joins are formed by $\bigsqcup_i S_i = \mathsf{fsat}(\bigcup_i S_i)$. We let $\mathsf{FSat}(X)$ denote this complete lattice.

Consider the following properties of a topological space X:

(1) X is sober;
(2) $\mathsf{KO}(X)$ is closed under finite intersection and forms a basis for the topology on X;
(3) $\mathsf{OF}(X)$ is closed under finite intersection and forms a basis for the topology on X;
(4) $\mathsf{fsat}(U)$ is open whenever U is open.

A space satisfying (1) and (2) is called a *spectral space*. Spectral spaces are the spaces that Stone identified as the duals of distributive lattices. A space satisfying (1), (2) and (3) is called a *semilattice space* (*SL* space). A space satisfying (1), (2), (3) and (4) is called a *bounded lattice space* (*BL* space).

For sets $A, B \subseteq X$, say that A is way below B (written $A \ll B$) if every open cover of B contains a finite subcover of B. In particular, $A \ll A$ holds if and only if A is compact in the usual sense. A function $f\colon X \to Y$ between topological spaces is *spectral* if f is continuous and f^{-1} preserves the way below relation on open sets. In the case open subsets of spectral spaces, $U \ll V$ holds if and only if there is a compact open K so that $U \subseteq K \subseteq V$. So a function between spectral spaces is spectral if and only if f^{-1} sends compact opens in Y to compact opens in X.

For *SL* spaces X and Y, a function $f\colon X \to Y$ is called F*-continuous* if it is spectral and $\mathsf{fsat}(f^{-1}(U)) \subseteq f^{-1}(\mathsf{fsat}(U))$ holds for all opens $U \subseteq Y$. Furthermore, f is F*-stable* if it is spectral and $\mathsf{fsat}(f^{-1}(U)) = f^{-1}(\mathsf{fsat}(U))$ for all opens $U \subseteq Y$.

Clearly, F-continuous maps and F-stable maps compose, so we have three categories:

- **SL** – *SL* spaces and F-continuous maps;
- $\mathbf{BL}_c$ – the full subcategory of **SL** consisting of *BL* spaces;
- **BL** – the subcategory of $\mathbf{BL}_c$ consisting of *BL* spaces and F-stable maps.

The following results are summarized from Part I.

Lemma 2.1. *In a SL space X, the set $\mathsf{KOF}(X)$ is closed under finite intersection. For an F-continuous map $f\colon X \to Y$, $f^{-1}(K) \in \mathsf{KOF}(X)$ whenever $K \in \mathsf{KOF}(Y)$. If X is a BL space, then $\mathsf{KOF}(X)$ is a sublattice of the complete lattice $\mathsf{FSat}(X)$. Moreover, if X and Y are BL spaces and $f\colon X \to Y$ is F-stable, then f^{-1} also preserves joins of compact open filters.*

This lemma tells us that KOF extends to a contravariant functor $\mathsf{KOF} : \mathbf{SL} \Rightarrow \mathbf{SLat}$ via $\mathsf{KOF}(f) = f^{-1}$. The lemma also says that KOF restricts and co-restricts to contravariant functors $\mathsf{KOF} : \mathbf{BL}_c \Rightarrow \mathbf{Lat}_{\wedge,1}$ and $\mathsf{KOF} : \mathbf{BL} \Rightarrow \mathbf{Lat}$.

For a semilattice L, let $\mathsf{Filt}(L)$ be the space of filters in L. The topology on $\mathsf{Filt}(L)$ is generated by the basic opens

$$\varphi_a = \{F \in \mathsf{Filt}(L) \mid a \in F\}$$

for each $a \in L$.

Lemma 2.2. *For any meet semilattice L, $\mathsf{Filt}(L)$ is an SL space. For any meet semilattice homomorphism $h\colon L \to M$, h^{-1} preserves filters and h^{-1} is F-continuous as a map*

$\mathsf{Filt}(M) \to \mathsf{Filt}(L)$. *If X is a lattice, then* $\mathsf{Filt}(L)$ *is a BL space. Moreover, if L and M are lattices and $h\colon L \to M$ is a lattice homomorphism, then h^{-1} is F-stable.*

Thus Filt is a contravariant functor **SLat** $\Rightarrow$ **SL** that restricts and co-restricts to $\mathbf{Lat}_{\wedge,1} \Rightarrow \mathbf{BL}_c$ and to **Lat** $\Rightarrow$ **BL**.

Theorem 2.3. *The functors* KOF *and* Filt *determine dual equivalences:*

- $\mathbf{SLat} \equiv \mathbf{SL}^{op}$
- $\mathbf{Lat}_{\wedge,1} \equiv \mathbf{BL}_c^{op}$
- $\mathbf{Lat} \equiv \mathbf{BL}^{op}$

Although the details of the proof are found in [JM], we will need explicit definitions for the unit and co-unit of the adjunction. For lattices L, one checks that $a \mapsto \varphi_a$ is the required natural isomorphism $L \to \mathsf{KOF}(\mathsf{Filt}(L))$. For *BL* spaces X, the natural homeomorphism $X \to \mathsf{Filt}(\mathsf{KOF}(X))$ is given by

$$\theta_x = \{K \in \mathsf{KOF}(X) \mid x \in K\}$$

A complete lattice C is a *completion* of a lattice L if L is a sublattice of C (more generally, L is embedded in C). L is *lattice dense* in C if

$$\mathsf{Meets}_C(\mathsf{Joins}_C(L)) = C = \mathsf{Joins}_C(\mathsf{Meets}_C(L)),$$

where

$$\begin{aligned} \mathsf{Meets}_C(A) &= \{\bigwedge A' \mid A' \subseteq A\} \\ \mathsf{Joins}_C(A) &= \{\bigvee A' \mid A' \subseteq A\} \end{aligned}$$

Furthermore L is *lattice compact* in C if for all $U, V \subseteq L$, $\bigwedge_C U \leq \bigvee_C V$ implies there exist finite $U_0 \subseteq U$, $V_0 \subseteq V$ for which $\bigwedge U_0 \leq \bigvee V_0$.

A completion C is a *canonical extension of* L if L is lattice dense and lattice compact in C. The existence and uniqueness of a canonical extension is due to Gehrke and Harding [GH01]. In [JM] it is proved in the following topological form.

Theorem 2.4. *For a BL space X,* $\mathsf{FSat}(X)$ *is a canonical extension of* $\mathsf{KOF}(X)$.

Corollary 2.5. *Every lattice has a canonical extension, unique up to isomorphism.*

3. The Opposite Lattice

The construction of a *BL* space from a lattice L can be performed on the order opposite lattice L^∂, yielding $\mathsf{Filt}(L^\partial) = \mathsf{Idl}(L)$. So $\mathsf{KOF}(\mathsf{Idl}(L))$ is isomorphic to L^∂. This is essentially the duality theorem (on objects) that is developed in [GHK+80]. However open filters in $\mathsf{Filt}(L)$ correspond to ideals of L. This leads to a direct construction of a space X' for which $\mathsf{KOF}(X)^\partial \simeq \mathsf{KOF}(X')$.

For *SL* space X, define a topology on $\mathsf{OF}(X)$ generated by opens

$$\psi_x = \{F \in \mathsf{OF}(X) \mid x \in F\}.$$

We take $\mathsf{OF}(X)$ to be this topological space. Notice that the co-unit θ of the dual equivalence $\mathsf{KOF} \dashv \mathsf{Filt}$ is almost identical to ψ. Specifically, $\theta_x = \psi_x \cap \mathsf{KOF}(X)$.

The results below make use of the following technical observation from [JM].

Lemma 3.1. *In a topological space X, let $F_1, \dots, F_m$ be pairwise incomparable filters. Then $F_1 \cup \cdots \cup F_m$ is compact if and only if each F_i is a principle filter.*

Lemma 3.2. *Let X be an SL space. The defining sub-basis of the topology on $\mathsf{OF}(X)$ is closed under finite intersection, hence is a basis. The specialization order is inclusion. Moreover, ψ_x is compact if and only if $x \in \mathsf{Fin}(X)$.*

Proof. Evidently, $\psi_x \cap \psi_y = \psi_{x \sqcap y}$ and $\psi_\top = \mathsf{OF}(X)$, where $\top$ denotes the maximal element of X. Obviously, if $x \in F \setminus G$, then $F \not\sqsubseteq G$ because $F \in \psi_x$, but $G \notin \psi_x$. On the other hand, if $F \subseteq G$, then $F \sqsubseteq G$ because the basic opens ψ_x are defined by membership.

Suppose ψ_x is compact. Obviously each ψ_x is an open filter in $\mathsf{OF}(X)$, so by Lemma 3.1 ψ_x is principal. That is, there exists $G \in \mathsf{OF}(X)$ so that for all $F \in \mathsf{OF}(X)$, $G \subseteq F$ if and only if $x \in F$. In particular, $x \in G$, so $\uparrow x \subseteq G$. Suppose $x \not\sqsubseteq y$. Then there is an open filter F containing x, but not y. Hence $y \notin G$. That is, $G = \uparrow x$ and it follows that $x \in \mathsf{Fin}(X)$. Conversely, if $x \in \mathsf{Fin}(X)$, then apparently $x \in F$ if and only if $\uparrow x \subseteq F$. So ψ_x is a principal open filter. $\square$

Lemma 3.3. *For an SL space X, if $\mathsf{OF}(X)$ is a spectral space, then X is a BL space.*

Proof. By Theorem 3.2 of [JM] it suffices to check that $\mathsf{fsat}(\uparrow x \cup \uparrow y)$ is open whenever $x, y \in \mathsf{Fin}(X)$. In that case, ψ_x and ψ_y are compact open filters in $\mathsf{OF}(X)$. Since $\mathsf{OF}(X)$ is spectral, $\psi_x \cap \psi_y = \psi_{x \sqcap y}$ is also a compact open filter. Hence $x \sqcap y \in \mathsf{Fin}(X)$ and $\mathsf{fsat}(\uparrow x \cup \uparrow y) = \uparrow(x \sqcap y)$ is open. $\square$

Theorem 3.4. *For any BL space X, $\mathsf{OF}(X)$ is a BL space and $\mathsf{KOF}(X)^\partial \simeq \mathsf{KOF}(\mathsf{OF}(X))$. Also $\mathsf{OF}(\mathsf{OF}(X))$ is homeomorphic to X.*

Proof. A compact open $K \subseteq \mathsf{OF}(X)$ is a finite union of basic opens ψ_x, which can be chosen to be pairwise incomparable. So by Lemma 3.1, each ψ_x is principal, hence is compact. In other words, $K = \psi_{x_1} \cup \cdots \cup \psi_{x_m}$ where $x_1, \dots, x_m \in \mathsf{Fin}(X)$. An intersection of two such compact opens is thus a finite union of basic opens of the form $\psi_{x_i \sqcap y_j}$, where each x_i and y_j is finite. Since X is a *BL* space, $x_i \sqcap y_j$ is also finite. Likewise, $\mathsf{OF}(X)$ itself is an open filter in $\mathsf{OF}(X)$. To see that $\mathsf{OF}(X)$ is sober, consider a completely prime filter P of $\mathsf{O}(\mathsf{OF}(X))$. Then the set $F_P = \{x \in X \mid \psi_x \in P\}$ is a filter, and it is open since if $\bigsqcup^\uparrow D \in F_P$ then $\psi_{\bigsqcup^\uparrow D} \in P$, i.e. $\bigcup_{d \in D} \psi_d \in P$. So by complete primality $\psi_{d_0} \in P$ for some $d_0 \in D$, whence $d_0 \in F_P$. It follows that $F_P \in \mathsf{OF}(X)$ and by definition ψ_x is a basic open neighborhood of F_P precisely when $x \in F_P$, i.e. when $\psi_x \in F_P$. Therefore $\mathsf{OF}(X)$ is sober and hence spectral. Clearly $\mathsf{OF}(X)$ is a semilattice and $\mathsf{OF}(\mathsf{OF}(X))$ is a basis. Moreover, the greatest element $\top$ of X is finite. So $\{\top\}$ is the smallest element of $\mathsf{OF}(X)$. It follows from Theorem 2.5 of [JM] that $\mathsf{OF}(X)$ is an *SL* space.

To see that it is in fact a *BL* space, consider some $\Psi = \bigcup_{x \in A} \psi_x$. Evidently, this is contained in $\psi_{\bigsqcup A}$. On the other hand, suppose Ψ is a filter and consider $F \in \psi_{\bigsqcup A}$. That is, $\bigsqcup A \in F$. Because F is open in X, there exist $x_1, \ldots, x_m \in A$ so that $x_1 \sqcup \cdots \sqcup x_m \in F$. And so there exist $a_1, \ldots, a_m \in \mathsf{Fin}(X)$ so that $a_i \sqsubseteq x_i$ and $a_1 \sqcup \cdots \sqcup a_m \in F$. Hence $\Psi = \psi_{\bigsqcup A}$. It follows that $\mathsf{KOF}(X) = \mathsf{Fin}(\mathsf{OF}(X))$ and the map $x \mapsto \psi_x$ is a bijection from X to $\mathsf{OF}(\mathsf{OF}(X))$.

In $\mathsf{OF}(\mathsf{OF}(X))$, the basic opens are the sets

$$\Psi_F = \{\psi_x \in \mathsf{OF}(\mathsf{OF}(X)) \mid F \in \psi_x\} = \{\psi_x \in \mathsf{OF}(\mathsf{OF}(X)) \mid x \in F\}$$

for $F \in \mathsf{OF}(X)$. So the bijection ψ is open and continuous. □

4. Products of *BL* spaces

Categorically, a co-product of lattices is dual to a product of *BL* spaces (in the category of *BL* spaces). So we know such products exist. Moreover, they are crucial to applications to quasioperators.

Lemma 4.1. *Let $\{X_\alpha\}_{\alpha \in I}$ be a family of SL spaces. In the product space, a set A is an open filter if and only if $A = \pi_{\alpha_0}^{-1}(F_0) \cap \cdots \cap \pi_{\alpha_{m-1}}^{-1}(F_{m-1})$ where $\{\alpha_0, \ldots, \alpha_{m-1}\}$ is finite, and for each $i < m$, $F_i \in \mathsf{OF}(X_{\alpha_i})$.*

Proof. Specialization in a product space is determined coordinate-wise. Evidently, $\pi_\alpha^{-1}(F)$ is an open filter for open any filter $F \subseteq X_\alpha$. And since the product space is a semilattice, finite intersections of open filters are open filters.

Suppose $F \subseteq \prod_\alpha X_\alpha$ is an open filter. Since projection maps are open maps, $\pi_\alpha(F)$ is an open filter. Hence for any (finite) set of indices $\{\alpha_0, \ldots, \alpha_{m-1}\}$, we have $F \subseteq \pi_{\alpha_0}^{-1}(\pi_{\alpha_0}(F_0)) \cap \cdots \cap \pi_{\alpha_{m-1}}^{-1}(\pi_{\alpha_{m-1}}(F_{m-1}))$. Choose $\{\alpha_0, \ldots, \alpha_{m-1}\}$ so that $\pi_\beta(F) = X_\beta$ for every index $\beta \notin \{a_0, \ldots, \alpha_{m-1}\}$. Suppose $x \notin F$. Then for some $i < m$, $\pi_{\alpha_i}(x) \notin \pi_{\alpha_i}(F)$. So $x \notin \pi_{\alpha_0}^{-1}(\pi_{\alpha_0}(F_0)) \cap \cdots \cap \pi_{\alpha_{m-1}}^{-1}(\pi_{\alpha_{m-1}}(F_{m-1}))$. □

Lemma 4.2. *The topological product of SL (BL) spaces is an SL (resp., BL) space, and the projections are F-continuous.*

Proof. The topological product of spectral spaces is spectral and the projections are specrtal. Lemma 4.1 implies that open filters in the product are closed under finite intersection. A sub-basic open $\pi_\alpha^{-1}(U)$ for open $U \subseteq X_\alpha$ is a union of open filters of the form $\pi_\alpha^{-1}(F)$. So a basic open in the product space is a union of open filters. Since specialization in a product space is determined coordinate-wise, the product is a meet semilattice. For an open filter $F \subseteq X_\alpha$, $\pi_\alpha^{-1}(F)$ is an open filter in the product space. So the projection maps are F-continuous. Moreover, for an open filter $F' \subseteq \prod_\alpha X_\alpha$, $\pi_\alpha(F')$ is a filter in X_α. It is open because projections are open maps. So $\pi_\alpha^{-1}(\mathsf{fsat}(U)) \subseteq \mathsf{fsat}(\pi_\alpha^{-1}(U))$ for any open $U \subseteq X_\alpha$. Thus the projections are F-continuous.

If the component spaces are *BL* spaces, then $\pi_\alpha^{-1}(U) \subseteq \pi_\alpha^{-1}(\mathsf{fsat}(U))$, and the latter is an open filter. So π_α is F-stable. □

5. MIRRORED *BL* SPACES

The relation between a lattice and its order opposite is represented in *BL* spaces by a space X and its "opposite" OF. This hides the underlying symmetry in the lattices themselves. In this section we develop a symmetrical representation of *BL* spaces paired with their opposites. This is a useful step toward connecting Hartung's duality theory and ours.

Suppose that we have two *SL* spaces X and X' and a homeomorphism $i{:}X \simeq \mathsf{OF}(X')$. Notice that we have borrowed the notation from L^∂, denoting the order opposite of L. But here X and X' are not assumed to have the same underlying set. Per Lemma 3.3, the homeomorphism means that for the corresponding lattices, $\mathsf{KOF}(X)^\partial \simeq \mathsf{KOF}(X')$. In other words, the triple (X, X', i) is a representation of the lattice ($\mathsf{KOF}(X)$) which explicitly accounts for the fact that a lattice is essentially two semilattices on the same underlying set that are "glued together" properly. Because $\mathsf{OF}(X')$ is a collection of subsets of X', a homeomorphism i is concretely given by a binary relation between X and X'.

Lemma 5.1. *Suppose X and X' are SL spaces and $R \subseteq X \times X'$ satisfies the following:*

(1) *R is open in the product topology;*
(2) *xRy_1 and xRy_2 implies $xR(y_1 \sqcap y_2)$;*
(3) *x_1Ry and x_2Ry implies $(x_1 \sqcap x_2)Ry$;*
(4) *for any $F \in \mathsf{OF}(X)$, there exists $y \in X'$ so that $x \in F \leftrightarrow xRy$; and*
(5) *for any $G \in \mathsf{OF}(X')$, there exists $x \in X$ so that $y \in G \leftrightarrow xRy$.*

Then the map $x \mapsto R[x]$ is a homeomorphism from X to $\mathsf{OF}(X')$. So X and X' are BL spaces representing order opposite lattices.

Proof. From (1) it follows that $R[x]$ $(= \{y \in X' \mid xRy\})$ is open, hence and upper set, and together with (2) we have $R[x] \in \mathsf{OF}(X')$ for all $x \in X$. Moreover, $x \mapsto R[x]$ is a continuous map from X to $\mathsf{OF}(X')$. From (5), the map is onto. Suppose $x \not\sqsubseteq x'$. Then there is an open filter F so that $x \in F$ and $x' \notin F$. By (4), there is a $y \in X'$ so that xRy and $\neg(x'Ry)$. So the map is one-to-one. It remains to check that it is open. Since open filters in X form a basis, it suffices to check that $R[F]$ is open in $\mathsf{OF}(X')$ for each $F \in \mathsf{OF}(X)$. By (4), let y be such that for all $x \in X$, $x \in F \leftrightarrow xRy$. Then immediately, $R[x] \in \psi_y$ for all $x \in F$. For the inclusion $\psi_y \subseteq R[F]$, consider $G \in \psi_y$. By (5), let x be such $R[x] = G$. In particular, xRy, so $x \in F$, hence $G = R[x] \in R_F$. □

Call a triple (X, X', R) consisting of two *SL* spaces and a binary relation satisfying the conditions in the lemma a *mirrored BL space*. We will refer to R as a *mirror relation*. Obviously, since the conditions on mirror relations are symmetric, if R is a mirror relation from X to X', then the converse relation, denoted by $\breve{R}$, is a mirror relation from X' to X. Since $x \mapsto R[x]$ is a homeomorphism (when co-restricted to $\mathsf{OF}(X')$), we write $R^*(F)$ for the unique x for which $F = R[x]$. Because both $\breve{R}$ and R^* play a role in the following, the reader will need to keep this distinction in mind. To spell things out, for a mirror relation $R \subseteq X \times X'$, we have the following related notions:

- $R[-]$, the homeomorphism $X \to \mathsf{OF}(X')$;
- $\breve{R}[-]$, the homeomorphism $X' \to \mathsf{OF}(X)$;
- $R^*(-)$, the homeomorphism $\mathsf{OF}(X') \to X$; and
- $\breve{R}^*(-)$, the homeomorphism $\mathsf{OF}(X) \to X'$.

It is immediately clear that for a *BL* space X, the triple $(X, \mathsf{OF}(X), \in)$ is a symmetric *BL* space, which naturally can be called the *mirroring of* X.

We are headed for a category equivalence between *BL* spaces and mirrored *BL* spaces. But for applications to lattice expansions, we can also consider other "structure preserving" maps:

- An F-continuous map $f\colon X \to Y$ corresponds to a meet preserving map between lattices.
- An F-continuous map $f\colon X' \to Y'$ corresponds to a join preserving map between lattices.
- An F-continuous map $f\colon X \to Y'$ corresponds to a map that sends joins to meets.
- An F-continuous map $f\colon X' \to Y$ corresponds to a map that sends meets to joins.

Evidently, a pair of suitably compatible F-continuous maps will correspond to an F-stable map from X to Y. The next lemma characterizes this compatibility.

Lemma 5.2. *Suppose (X, X', R) and (Y, Y', S) are symmetric BL spaces, and $f\colon X \to Y$ and $f'\colon X' \to Y'$ are F-continuous maps satisfying:*

(1) *$f'^{-1}(S[y]) \subseteq R[x]$ if and only if $y \sqsubseteq f(x)$; and*
(2) *$f^{-1}(\breve{S}[y']) \subseteq \breve{R}[x']$ if and only if $y' \sqsubseteq f'(x')$.*

Then both f and f' are F-stable. Moreover, if $g\colon X \to Y$ is F-stable, then there is a unique F-continuous map $h\colon X' \to Y'$ so that the pair (g, h) satisfies conditions (1) and (2).

Proof. Consider an open $U \subseteq Y$, and element $x \in f^{-1}(\mathsf{fsat}(U))$. We need to show that for any $F \in \mathsf{OF}(X)$, if $f^{-1}(U) \subseteq F$ then $x \in F$. The open U is a union of open filters. Because S is a mirror relation, for a suitable choice of $V \subseteq Y'$, we have $U = \bigcup_{v \in V} \breve{S}[v]$. So we may fix $v' \in V$ for which $f(x)Sv'$. Now consider any $x' \in X'$ for which $f^{-1}(U) \subseteq \breve{R}[x']$. By (2), $v' \sqsubseteq f'(x')$, so $f(x)Sf'(x')$. By (1), xRx' holds as required for F-stability. The proof for f' is symmetric.

Uniqueness: Suppose g is F-stable, h and h' are F-continuous, and the pairs (g, h) and (g, h') satisfy (1) and (2). Suppose $h(x') \neq h'(x')$ for some fixed $x' \in X'$. Then there is an open filter in Y' separating these. Without loss of generality, suppose $y \in Y$ is such that $ySh(x')$ and $\neg(ySh'(x'))$.

By (2), for every $y' \in Y'$ such that ySy', there exists $x \in X$, so that $f(x)Sy'$ and not xRx'. In particular, since $ySh(x')$ holds, (1) implies that there exists x so that $g(x)Sh'(x')$, which then implies xRx', contradicting the choice of x'.

Existence: Suppose $g\colon X \to Y$ is F-stable. For $x' \in X'$, define the following:

$$\begin{aligned} D_{x'} &= \{y' \in Y' \mid g^{-1}(\breve{S}[y']) \subseteq \breve{R}[x']\} \\ h(x') &= \bigsqcup D_{x'} \end{aligned}$$

The map $h\colon X' \to Y'$ is well defined because Y' is a complete lattice in its specialization order. We make the following observations.

(1) $D_{x'}$ is directed because g is F-stable.
(2) $h(x') \in D_{x'}$ because the maps $\breve{S}[-]$ and $\breve{R}[-]$ are homeomorphisms and a directed union of open filters is an open filter.
(3) h satisfies (2) by construction.
(4) For a filter $y' \in Y'$, $h^{-1}(\uparrow y') = \uparrow\{\breve{R}^*(g^{-1}(\breve{S}[y']))\}$ because $y' \sqsubseteq h(x')$ if and only if $g^{-1}(\breve{S}[y']) \subseteq \breve{R}[x']$.
(5) Because of the previous observation, h is F-continuous. That is, consider $y' \in \mathsf{Fin}(Y')$. Then $\breve{S}[y'] \in \mathsf{KOF}(Y)$, so $g^{-1}(\breve{S}[y']) \in \mathsf{KOF}(X)$, hence $\breve{R}^*(g^{-1}(\breve{S}[y'])) \in \mathsf{Fin}(X')$.

Fix $x \in X$ and $y \in Y$. Because $S[y] = \bigcup\{\uparrow z' \mid ySz'\}$, we have $h^{-1}(S[y]) = \uparrow\{\breve{R}^*(g^{-1}(\breve{S}[z'])) \mid ySz'\}$. And since $R[x]$ is an upper set, we have that

$$\begin{aligned} h^{-1}(S[y]) \subseteq R[x] \quad &\Leftrightarrow \quad \{\breve{R}^*(g^{-1}(\breve{S}[z'])) \mid ySz'\} \subseteq R[x] \\ &\Leftrightarrow \quad g(x) \in \breve{S}[z'] \text{ for all } z' \in S[y] \\ &\Leftrightarrow \quad S[y] \subseteq S[g(x)] \\ &\Leftrightarrow \quad y \sqsubseteq g(x). \end{aligned}$$

□

Theorem 5.3. *The category of BL spaces and F-stable maps is equivalent to the category of mirrored BL spaces and pairs of maps (f, f') satisfying the compatibility conditions of Lemma 5.2.*

Proof. Evidently, the construction $(X, X', R) \mapsto X$, and $(f, f') \mapsto f$ is functorial. Likewise, $X \mapsto (X, \mathsf{OF}(X), \in)$ extends to a functor. The composition in one direction is the identity on the category of *BL* spaces. In the other direction it is a natural isomorphism because X' is homeomorphic via R to $\mathsf{OF}(X)$. □

6. Lattices with quasioperators

The duality for lattices can be smoothly extended to handle n-ary quasioperators. Our treatment is simplified by considering mirrored *BL* spaces.

Recall from the introduction that each quasioperator $f : L^n \to L$ has an associated monotonicity type $\varepsilon \in \{1, \partial\}^{n+1}$ which determines whether f is join or meet preserving or reversing in each argument. Here L^{ε_i} denotes the order-dual of L if $\varepsilon_i = \partial$, and $L^1 = L$. The value of ε_i is chosen so that f will be join preserving in each argument when considered as a map from $\prod_{i=0}^{n-1} L^{\varepsilon_i}$ to L^{ε_n}.

Before we consider the general n-ary case, consider the simplest case of a unary quasioperator $j\colon L^1 \to L^1$, and a mirrored *BL* space (X, X', R) for which $L \simeq \mathsf{KOF}(X)$. Since j is *join preserving* it corresponds to a meet preserving map $L^\partial \to L^\partial$. Hence its dual is an F-continuous map $X' \to X'$. According to our duality theory, this relation is contravariant.

This suggests that a general topological representation of n-ary quasioperators will need to account for this contravariance. It also suggests that a small generalization will be helpful. Namely, we can look at maps $j\colon L_0 \times \cdots \times L_{n-1} \to L_n$ that preserve finite joins in each argument separately, and in which the lattices L_i are not assumed to be otherwise related. The point is that such a map j is *not* a morphism in the category of lattices, or even the category of join semilattice reducts of lattices. We refer to such maps between lattices as *join distributive* maps.

Again, the unary case is instructive. Consider mirrored *BL* spaces (X, X', R) and (Y, Y', S) and, again for simplicity, an F-continuous map $f\colon Y' \to X'$. Define the map $\hat{f}\colon X \to Y$ by

$$\hat{f}(x) = \breve{S}^*(f^{-1}(R[x]))$$

Note that this is well defined precisely because F-continuity of f guarantees that $f^{-1}(R[x])$ is an open filter in Y'. There is no reason that $\hat{f}$ should be F-continuous, but it does have some useful properties, which can be read from the characterization found in [GHK$^+$03] Chapter 4 of the maps on arithmetic lattices that correspond to join preserving maps on lattices.

In a *BL* space, define a binary relation $\ll_X \subseteq X \times X$ by $x_0 \ll_X x_1$ if and only if there exists an open filter $F \in \mathsf{OF}(X)$ so that $x_1 \in F$ and for all $G \in \mathsf{OF}(X)$, $x_0 \in G$ implies $F \subseteq G$. As usual, we omit the subscript whenever possible. In a mirrored *BL* space (X, X', R), $x_0 \ll_X x_1$ is obviously equivalent to there being some $x' \in X'$ so that $x_1 R x'$ and for all $x'' \in X'$, $x_0 R x''$ implies $x' \sqsubseteq x''$. Evidently, $x \ll x$ holds if and only if $x \in \mathsf{Fin}(X)$. Say that a function $f\colon X \to Y$ between *BL* spaces is *strongly continuous* if it is continuous and it preserves $\ll$.

Lemma 6.1. *Let (X, X', R) and (Y, Y', S) be mirrored BL spaces and $f\colon Y' \to X'$ be F-continuous. Then the map $\hat{f}$ defined above satisfies the following.*

- *$\hat{f}$ is strongly continuous; and*
- *$\hat{f}$ preserves finite meets.*

Moreover, for any strongly continuous meet preserving $g\colon X \to Y$, there is a unique F-continuous $f\colon Y' \to X'$ so that $g = \hat{f}$.

Proof. Since f^{-1} is Scott continuous as a map from $\mathsf{OF}(X')$ to $\mathsf{OF}(Y')$, $\hat{f}$ is a composite of continuous functions. Likewise, f^{-1} preserves finite intersections. The other two maps are homeomorphisms, and so preserve all specialization structure. If $x_0 \ll_X x_1$, then every open cover of $R[x_1]$ has a finite subcover of $R[x_0]$. Because f is spectral, every open cover of $f^{-1}(R[x_1])$ has a finite subcover of $f^{-1}(R[x_0])$. The open filter $f^{-1}(R[x_1])$

is a directed union of compact open filters, so for some $y' \in Y'$, $f^{-1}(R[x_0]) \subseteq \uparrow y' \subseteq f^{-1}(R[x_1])$.

Suppose g satisfies the conditions. Define $\breve{g}(y') = \breve{R}^*(g^{-1}(\breve{S}[y']))$. Because g is continuous and satisfies (2), $g^{-1}(\breve{S}[y'])$ is an open filter. So $\breve{g}$ is well defined. Consider $x' \in \mathsf{Fin}(X')$. Then $x' \ll x'$, so there exists x for which $\breve{R}[x'] = \uparrow x$. Hence x is also finite and $x' \sqsubseteq \breve{g}(y')$ if and only if $g(x)Sy'$. So $\breve{g}$ is F-continuous. Finally, $\widehat{\breve{g}}(x) = \breve{S}^*[\breve{g}^{-1}(\breve{R}[x])]$, and $xR\breve{g}(y')$ and only if $g(x)Sy'$. So $\widehat{\breve{g}}(x) = g(x)$. The analogous argument shows that for F-continuous $f\colon Y' \to X'$, $f = \breve{\hat{f}}$. $\square$

These lemmas suggest how to represent a join distributive function of higher arity directly.

Theorem 6.2. *Let $X_0, \ldots, X_n$ be BL spaces. The join distributive maps*

$$j\colon \mathsf{KOF}(X_0) \times \cdots \times \mathsf{KOF}(X_{n-1}) \to \mathsf{KOF}(X_n)$$

are bijective with maps $f\colon X_0 \times \cdots \times X_{n-1} \to X_n$ satisfying

(1) *f is strongly continuous in the product topology; and*
(2) *f preserves finite meets in each argument.*

Proof. The product space $X_0 \times \cdots \times X_{n-1}$ is a *BL* space, and by Lemma 4.1, the $\ll$ relation on the product is determined coordinate-wise.

Suppose f satisfies the listed conditions. We define a map from $j_f\colon \mathsf{Fsat}(X_0) \times \cdots \times \mathsf{FSat}(X_{n-1}) \to \mathsf{FSat}(X_n)$ as

$$j_f(F_0, \ldots, F_{n-1}) = \bigcap\{G \in \mathsf{OF}(X_n) \mid F_0 \times \cdots \times F_{n-1} \subseteq f^{-1}(G)\}$$

Since $x \ll x$ holds if and only if $x \in \mathsf{Fin}(X)$ in any *BL* space, j_f restricted to compact open filters in all arguments co-restricts to compact open filters in X_n. Moreover, with fixed $x_1 \in \mathsf{Fin}(X), \ldots, x_{n-1} \in \mathsf{Fin}(X_{n-1})$, the map $x \mapsto f(x, x_1, \ldots, x_{n-1})$ satisfies the conditions of Lemma 6.1, and likewise for all other argument positions. So j_f restricts and co-restricts to a join distributive map.

Suppose $j\colon L_0 \times \cdots \times L_{n-1} \to L_n$ is a join distributive map on lattices. Define $f_j\colon \mathsf{Filt}(L_0) \times \cdots \times \mathsf{Filt}(L_{n-1}) \to \mathsf{Filt}(L_n)$ by

$$f_j(F_0, \ldots, F_{n-1}) = \bigcap\{G \in \mathsf{Filt}(L_n) \mid F_0 \times \cdots \times F_{n-1} \subseteq j^{-1}(G)\}$$

To check continuity, it suffices to check that f_j preserves directed unions in each argument separately. But $(\bigcup_\alpha F_\alpha) \times F_1 \times \cdots \times F_{n-1} = \bigcup_\alpha (F_\alpha \times F_1 \times \cdots \times F_{n-1})$. So $y \notin f_j(F_\alpha, F_1, \ldots, F_{n-1})$ for all α implies that for all α, there is some G_α for which $y \notin G_\alpha$ and $F_\alpha \times F_1 \times \cdots \times F_{n-1} \subseteq j^{-1}(G_\alpha)$. Taking the intersection $\bigcap_\alpha G_\alpha$ provides a witness that $y \notin f_j(\bigcup_\alpha F_\alpha, F_1, \ldots, F_{n-1})$.

Consider filters satisfying $(F_0 \cap F_0') \times F_1 \times \cdots \times F_{n-1} \subseteq j^{-1}(G)$. Then $y \notin G$ implies that for all $a_0 \in F_0$, $a_0' \in F_0'$, $a_1 \in F_1$,..., $a_{n-1} \in F_{n-1}$, we have $j(a_0 \vee a_0', a_1, \ldots, a_{n-1}) \neq y$. So $y \notin f_j(F_0, \ldots, F_{n-1}) \cap f_j(F_0', \ldots, F_{n-1})$. In other words, $f_j(F_0, \ldots, F_{n-1}) \cap f_j(F_0', \ldots, F_{n-1}) \subseteq G$. But obviously, $(F_0 \cap F_0') \times F_1 \times \cdots \times$

$F_{n-1} \subseteq j^{-1}(j_f(F_0, \ldots, F_{n-1}))$. Thus f_j preserves meets in the first argument, and all other arguments separately be the same argument.

In $\mathsf{Filt}(L_i)$, the relation $\ll$ is especially simple: $F \ll G$ holds if and only if $F \subseteq {\uparrow}a \subseteq G$ for some $a \in L_i$. So clearly, if $F_i \ll G_i$ holds for each $i < n-1$, there are elements $a_i \in L_i$ witnessing this. Obviously, $j_f(F_0, \ldots, F_{n-1}) \ll j_f(G_0, \ldots, G_{n-1})$ is witnessed by $j(a_0, \ldots, a_{n-1})$.

Finally, let $j\colon L_0 \times \cdots \times L_{n-1} \to L_n$ be join distributive. Consider $a_0 \in L_0$, $\ldots$, $a_{n-1} \in L_{n-1}$ and $F \in \mathsf{Filt}(L_n)$. Then $j(a_0, \ldots, a_{n-1}) \in F$ if and only if $f_j({\uparrow}a_0, \ldots, {\uparrow}a_{n-1}) \subseteq F$, if and only if $F \in j_{f_j}(\varphi_{a_0}, \ldots, \varphi_{a_{n-1}})$. Likewise, let $f\colon X_0 \times \cdots \times X_{n-1} \to X_n$ be strongly continuous and preserve finite meets in each argument. Consider $x_0 \in X_0, \ldots, x_{n-1} \in X_{n-1}$ and $F \in \mathsf{KOF}(X_n)$. Then $f(x_0, \ldots, x_{n-1}) \in F$ if and only if $j_f({\uparrow}x_0, \ldots, {\uparrow}x_{n-1}) \subseteq F$ if and only if $F \in f_{j_f}(\theta_{x_0}, \ldots, \theta_{x_{n-1}})$. Since $a \mapsto \varphi_a$ is the natural isomorphism $L \to \mathsf{KOF}(\mathsf{Filt}(L))$ and $x \mapsto \theta_x$, the natural homeomorphism $X \to \mathsf{Filt}(\mathsf{KOF}(X))$, for the dual equivalence, these show that the construction $f \mapsto j_f$ is the desired bijection. $\square$

Finally, we are in a position to represent quasioperators on a lattice. A given lattice L is represented by a mirrored *BL* space (X, X', R), i.e., $L^1 = L \cong \mathsf{KOF}(X)$ and $L^\partial \cong \mathsf{KOF}(X')$. For notational convenience, we also define $X_1 = X$ and $X_\partial = X'$. For a fixed monotonicity type $\varepsilon \in \{1, \partial\}^{n+1}$, a quasioperator $j\colon L^n \to L$ of monotonicity ε is therefore a join distributive map $j\colon L^{\varepsilon_0} \times \cdots \times L^{\varepsilon_{n-1}} \to L^{\varepsilon_n}$. And this is uniquely represented by a strongly continuous function $f\colon X_{\varepsilon_0} \times \cdots \times X_{\varepsilon_{n-1}} \to X_{\varepsilon_n}$ that preserves meets in each argument.

7. Conclusions

The results of this paper show that the duality between **Lat** and **BL** developed in [JM] can be extended to a duality between lattices with quasioperators and mirrored *BL* spaces with strongly continuous functions that are meet-preserving in each argument. The dual objects in this treatment are constructed within a natural topological framework, providing connections with other areas of research, such as domain theory and positive modal logic, as well as applications of these result to specific varieties of lattices with quasioperators, such as modal lattices, lattice-ordered monoids and residuated lattices.

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