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Scaled Global Operators and Fueter Variables on Non-zero Scaled Hypercomplex Numbers

Daniel Alpay, Ilwoo Cho and Mihaela Vajiac* 

Abstract. In this paper we describe the rise of global operators in the scaled quaternionic case, an important extension from the quaternionic case to the family of scaled hypercomplex numbers \mathbb{H}_t , $t \in \mathbb{R}^*$, of which the $\mathbb{H}_{-1} = \mathbb{H}$ is the space of quaternions and \mathbb{H}_1 is the space of split quaternions. We also describe the scaled Fueter-type variables associated to these operators, developing a coherent theory in this field. We use these types of variables to build different types of function spaces on \mathbb{H}_t . Counterparts of the Hardy space and of the Arveson space are also introduced and studied in the present setting. The two different adjoints in the scaled hypercomplex numbers lead to two parallel cases in each instance. Finally we introduce and study the notion of rational function.

Mathematics Subject Classification. 30G35, 46E22.

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1. Introduction and the Algebra of Scaled Hypercomplex Numbers

In the case of \mathbb{H} , the skew-field of quaternions, generated on the real numbers by $1, i, j, k$ with the usual Cayley table, a global operator G was defined in [28] as:

$$G_q = (x_1^2 + x_2^2 + x_3^2) \frac{\partial}{\partial x_0} + (ix_1 + jx_2 + kx_3) \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right). \quad (1.1)$$

In [15], the related operator,

$$V_q = \frac{\partial}{\partial x_0} - \frac{1}{ix_1 + jx_2 + kx_3} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right), \quad (1.2)$$

(which we still name “the global operator”) was studied extensively by two of the co-authors (together with Kamal Diki). In particular new Fueter-type variables associated to V_q were found and analyzed, and Fueter-type expansions were given for smooth functions in its kernel.

In recent works [3–5, 7, 8], the space of quaternions was extended to a family of spaces indexed by the real numbers, and, for $t \in \mathbb{R}$, these papers

introduced and studied the scaled families of rings \mathcal{H}_2^t of $\mathbb{C}^{2 \times 2}$ matrices, with elements of the form:

$$q_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}, t \in \mathbb{R}. \quad (1.3)$$

When $t = -1$ one obtains the matrix representation of the skew field of quaternions, while the case $t = 1$ corresponds to the matrix representation of the ring of split-quaternions. In the latter case ($t = 1$), hyperbolic numbers correspond to real a and b .

In the present paper we first extend the results of [15] to the scaled family of rings mentioned above, interpolating from the space of quaternions to the one of split quaternions. Then we study counterparts of the classical Hardy space in the present setting. An important new point with respect to the previous work [15] is the appearance of Krein spaces (as in [17]).

We now review the definitions and main results of the scaled-hypercomplex algebras introduced in [3–5, 7, 8]. We start with the first definition of the space \mathcal{H}_2^t as above, with two equivalent characterizations to follow:

Definition 1.1. Let $t \in \mathbb{R}$. The space of scaled hypercomplex numbers is:

$$\mathcal{H}_2^t = \left\{ \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C} \right\}. \quad (1.4)$$

This space is an algebra over the field of real numbers, with the usual addition and multiplication of matrices.

In another characterization we will see that \mathcal{H}_2^t becomes a real vector space.

Remark 1.2. We note that \mathcal{H}_2^t is not closed under the usual matrix conjugation when $t \neq \pm 1$, since

$$q_t^* = \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} \notin \mathcal{H}_2^t \quad (1.5)$$

if $t \neq \pm 1$ or if $b \neq 0$.

One can define two natural adjoints (or conjugations) associated to (1.4), and which leave \mathcal{H}_2^t invariant, namely

$$q_t^{\otimes} = \begin{pmatrix} \bar{a} & -tb \\ -\bar{b} & a \end{pmatrix} \quad (1.6)$$

(that is, the adjugate matrix of q_t) and, see (2.21),

$$q_t^{[*]} = \begin{pmatrix} a & t\bar{b} \\ b & \bar{a} \end{pmatrix}. \quad (1.7)$$

See [4, 5, 7] and [8]. We note that these two adjoints are contravariant (see Lemma 2.1 and Lemma 2.17)

$$(q_t p_t)^{\otimes} = p_t^{\otimes} q_t^{\otimes} \quad \text{and} \quad (q_t p_t)^{[*]} = p_t^{[*]} q_t^{[*]}, \quad p_t, q_t \in \mathcal{H}_2^t,$$

are involutive, and commute:

$$(q_t^{\oplus})^{[*]} = \left(q_t^{[*]}\right)^{\oplus} = \begin{pmatrix} \bar{a} & -t\bar{b} \\ -b & a \end{pmatrix}.$$

It is therefore not possible to obtain another involutive contravariant from iterated compositions of these adjoints.

It is easy to see that \mathcal{H}_2^t is ring-isomorphic to the ring \mathbb{H}_t defined below.

Definition 1.3. Let \mathbb{H}_t be the space of elements of the form:

$$q_t = a + bj_t, \quad a, b \in \mathbb{C}[i], \quad (1.8)$$

where i, j_t and $k_t = ij_t$ satisfy the generalized Cayley laws, induced by the matrix multiplication in Definition 1.1:

| | | | | |
|------------|-------|--------|-------|--------|
| \nearrow | 1 | i | j_t | k_t |
| 1 | 1 | i | j_t | k_t |
| i | i | -1 | k_t | $-j_t$ |
| j_t | j_t | $-k_t$ | t | $-ti$ |
| k_t | k_t | j_t | ti | t |

Writing $a = x_0 + x_1i$ and $b = x_2 + x_3i$ we have that the ring \mathbb{H}_t is the vector space over \mathbb{R} of elements of the form:

$$q_t = x_0 + x_1i + x_2j_t + x_3k_t, \quad x_0, x_1, x_2, x_3 \in \mathbb{R}. \quad (1.9)$$

We will often in this paper pass from one representation to the other and denote both of them by q_t .

We now introduce a third equivalent definition, where scaled hypercomplex numbers are seen as pairs of complex numbers. For an arbitrarily fixed non-zero scale $t \in \mathbb{R}$, and $q_t = a + bj_t$ as in (1.8), we write the corresponding pair of complex numbers $(a, b)_t$ and we define the scaled multiplicative operation \cdot_t on \mathbb{C}^2 induced by the multiplication of matrices as:

$$(a_1, b_1)_t \cdot_t (a_2, b_2)_t \stackrel{\text{def}}{=} (a_1a_2 + tb_1\bar{b}_2, a_1b_2 + b_1\bar{a}_2)_t. \quad (1.10)$$

As proven in [4, 5] the space of pairs $(a, b)_t$ forms a unital ring with unity $(1, 0)$, where $+$ is the usual vector addition, and \cdot_t is the operation (1.10) above.

We will use these three definitions interchangeably, whenever each becomes more convenient to describe the associated differential theory, spaces of functions, domains of holomorphy and so forth.

Observe that, if

$$a = x_0 + x_1i \quad \text{and} \quad b = x_2 + x_3i$$

are complex numbers with $i = \sqrt{-1}$ and $x_0, x_1, x_2, x_3 \in \mathbb{R}$, and $(a, b)_t \in \mathbb{H}_t$, with some abuse of notation, we have that the realization of $(a, b)_t$ is identified with:

$$q_t = a + bj_t = (a, b)_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} x_0 + x_1i & tx_2 + tx_3i \\ x_2 - x_3i & x_0 - x_1i \end{pmatrix}. \quad (1.11)$$

We then identify the basis elements using:

$$q_t = x_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + x_2 \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix}, \quad (1.12)$$

and we obtain:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (1, 0)_t, \quad i_t = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = (i, 0)_t,$$

and

$$j_t = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} = (0, 1)_t, \quad k_t = \begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix} = (0, i)_t,$$

in \mathcal{H}_2^t , which means that every element q_t of \mathcal{H}_2^t is expressed by

$$q_t = x_0 (1, 0)_t + x_1 (i, 0)_t + x_2 (0, 1)_t + x_3 (0, i)_t,$$

for some $x_l \in \mathbb{R}$, $0 \leq l \leq 3$, (as seen in [4, 5, 7]).

The \mathbb{R} -basis $\mathbf{B}_t = \{(1, 0)_t, (i, 0)_t, (0, 1)_t, (0, i)_t\}$ satisfies the following properties in the Cayley table of \mathbb{H}_t , i.e.:

$$\begin{aligned} i_t^2 &= (-1, 0)_t = -1, & j_t^2 &= t(1, 0)_t = t = k_t^2, \\ i_t \cdot_t j_t &= k_t, & j_t \cdot_t k_t &= -ti_t, \\ k_t \cdot_t i_t &= j_t, & k_t \cdot_t j_t &= ti \\ j_t \cdot_t i_t &= -k_t, & i_t \cdot_t k_t &= -j_t. \end{aligned}$$

Remark 1.4. From now on we write i rather than i_t , meaning that we view the ring \mathbb{H}_t containing a fixed copy of the complex numbers.

When $t = 1$, that is, the split quaternion case, we have

$$q_1^{\otimes} = \begin{pmatrix} \bar{a} & -b \\ -\bar{b} & a \end{pmatrix} \quad \text{while} \quad q_1^{[*]} = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}.$$

With i_1, j_1, k_1 corresponding to $t = 1$, and using the notation (1.8), one has $q_1 = a - bj_1$ and

$$q_1^{\otimes} = \bar{a} - bj_1 \quad \text{and} \quad q_1^{[*]} = a + \bar{b}j_1. \quad (1.13)$$

The latter case was the one considered in [17].

When $t = -1$, that is, for the quaternions, we have

$$q_{-1}^{\otimes} = \begin{pmatrix} \bar{a} & b \\ -\bar{b} & a \end{pmatrix} \quad \text{and} \quad q_{-1}^{[*]} = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}.$$

Here q_1^{\otimes} corresponds to the usual quaternion conjugate, and $q_1^{[*]}$ leads to new directions in analytic quaternionic theory. With i, j, k corresponding to $t = -1$, one has $q_{-1} = a + bj$ and

$$q_{-1}^{\otimes} = \bar{a} - bj = \overline{a + bj} = \overline{q_{-1}} \quad (1.14)$$

(since bj has a zero real part) corresponding to classical quaternionic analysis, while

$$q_{-1}^{[*]} = a - \bar{b}j = a + \bar{b}\bar{j}. \quad (1.15)$$

As just mentioned a few lines above, the corresponding theory is new in quaternionic analysis.

The paper consists of seven sections and two appendices, of which this introduction (Sect. 1) is the first. In Sect. 2 we introduce the algebra of hypercomplex numbers, their conjugates, and properties, as well as an introduction to a theory of functions on these spaces.

In Sect. 3 we introduce the scaled global operator V_t in this case and introduced a new set of Fueter-type variables associated to this operator. In the same section (Sect. 3) we also build parallel theories of regular functions using these new variables (see Definition 3.1) and the previously introduced Fueter-variables associated to the scaled Fueter operator (see Definition 3.6).

In Sect. 4 we introduce and describes Hardy spaces with respect to the conjugate \circledast , define the Blaschke product in this case and discuss the interpolation problem in this context, while in Sect. 5 we do the same for the $[*]$ conjugate.

In Sect. 6 we build the associated Averson space related to the new scaled Fueter variables associated to the scaled global operator, while in Sect. 7 we establish the premises of a theory of rational functions in this case. Both of these sections are further developed in works that the authors are currently in a preliminary stage of completion.

For clarity, we also write Appendix A and Appendix B, to remind the reader of the concepts and properties of Krein and Pontryagin spaces, as they apply to our work.

2. Conjugates, Their Properties, and a Theory of Functions

In this section, as mentioned in the introduction, one can see that \mathcal{H}_2^t is not closed under the usual matrix conjugation as in (1.5) when $t \neq \pm 1$, unless $t \neq \pm 1$ or q_t is a real scalar matrix. However, as in (1.6) and (1.7), one can define two natural adjoints (or conjugations, or conjugate) associated to (1.4) which leave \mathcal{H}_2^t invariant, namely

$$q_t^{\circledast} = \begin{pmatrix} \bar{a} & -tb \\ -\bar{b} & a \end{pmatrix} \quad q_t^{[*]} = \begin{pmatrix} a & t\bar{b} \\ b & \bar{a} \end{pmatrix}.$$

These conjugates lead to different theories, see [4, 7] and [6] respectively. On the other hand, just as for the usual $*$ -matrix conjugation, we have:

Lemma 2.1. *The two new conjugates are contravariant with respect to the matrix product (or, equivalently the product \cdot_t in the representation of the ring in terms of pairs).*

Proof. For the first conjugate we have with $q_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix}$ and $p_t = \begin{pmatrix} c & td \\ \bar{d} & \bar{c} \end{pmatrix}$,

$$(q_t p_t)^{\circledast} = \left(\begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} c & td \\ \bar{d} & \bar{c} \end{pmatrix} \right)^{\circledast}$$

$$\begin{aligned}
& \begin{pmatrix} ac + t\bar{b}\bar{d} & t(ad + b\bar{c}) \\ \bar{c}\bar{b} + \bar{a}\bar{d} & t\bar{b} + \bar{a}\bar{c} \end{pmatrix}^{\otimes} \\
&= \begin{pmatrix} \bar{a}\bar{c} + t\bar{b}\bar{d} & -(c\bar{b} + \bar{a}\bar{d}) \\ -(ad + b\bar{c}) & ac + t\bar{b}\bar{d} \end{pmatrix} \\
&= \begin{pmatrix} \bar{c} & -t\bar{d} \\ -\bar{d} & c \end{pmatrix} \begin{pmatrix} \bar{a} & -t\bar{b} \\ -\bar{b} & a \end{pmatrix} \\
&= p_t^{\otimes} q_t^{\otimes}.
\end{aligned}$$

The second conjugate is considered in Lemma 2.17. \square

2.1. The Regular Matrix Adjoint *

\mathcal{H}_2^t is not closed under the usual matrix conjugation when $t \neq \pm 1$,

$$q_t^* = \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} \notin \mathcal{H}_2^t$$

if $t \neq \pm 1$ or if q_t is not a real scalar matrix, however this conjugate is useful in defining the matrix Hilbert Schmidt norm and the operator norm. For q_t as above, the Hilbert–Schmidt norm is equal to

$$\|q_t\|_{HS} = \sqrt{\text{Tr } q_t q_t^*} = \sqrt{2|a|^2 + (1+t^2)|b|^2}. \quad (2.1)$$

Let us define the operator norm $\|q_t\|_{op}$ as follows:

Definition 2.2. The *operator norm* of $q_t \in \mathcal{H}_t$ is denoted by $\|q_t\|_{op}$ and it is defined to be the square root of the largest eigenvalue of $q_t q_t^*$, using the regular adjoint q_t^* as in (1.5).

For ease of calculations, we now compute the operator norm of q_t :

Proposition 2.3. For any $q_t \in \mathcal{H}_t$ it holds that

$$\|q_t\|_{op} = \sqrt{\frac{|b|^2(1+t^2) + 2|a|^2 + \sqrt{|b|^4(1-t^2)^2 + 4|a|^2|b|^2(1+t)^2}}{2}} \quad (2.2)$$

Proof. We have

$$\begin{aligned}
\det(q_t q_t^* - \lambda I_2) &= \det \begin{pmatrix} |a|^2 + t^2|b|^2 - \lambda & ab(1+t) \\ \bar{a}\bar{b}(1+t) & |b|^2 + |a|^2 - \lambda \end{pmatrix} \\
&= \lambda^2 - \lambda(|b|^2 + |a|^2 + |a|^2 + t^2|b|^2) + (|b|^2 + |a|^2)(|a|^2 + t^2|b|^2) \\
&\quad - |a|^2|b|^2(1+t)^2
\end{aligned}$$

and the discriminant, say Δ , of the corresponding eigenvalue equation is:

$$\begin{aligned}
\Delta &= (|b|^2 + |a|^2 + |a|^2 + t^2|b|^2)^2 - 4((|b|^2 + |a|^2)(|a|^2 + t^2|b|^2) - |a|^2|b|^2(1+t)^2) \\
&= (|b|^2 + |a|^2 - |a|^2 - t^2|b|^2)^2 + 4|a|^2|b|^2(1+t)^2 \\
&= |b|^4(1-t^2)^2 + 4|a|^2|b|^2(1+t)^2.
\end{aligned}$$

Thus

$$\lambda = \frac{|b|^2(1+t^2) + 2|a|^2 \pm \sqrt{|b|^4(1-t^2)^2 + 4|a|^2|b|^2(1+t)^2}}{2}$$

with largest eigenvalue corresponding to the + sign, and hence the result. \square

Corollary 2.4. *We have the following:*

$$\|q_t\|_{op} = \|q_t^{\otimes}\|_{op} = \|q_t^{[*]}\|_{op} \quad (2.3)$$

and

$$\|q_t\|_{HS} = \|q_t^{\otimes}\|_{HS} = \|q_t^{[*]}\|_{HS}. \quad (2.4)$$

Proof. This is because $\|q_t\|_{op}$ depends only on the absolute values of a and b . \square

Remark 2.5. For $t = -1$ we get $\|q_{-1}\| = \sqrt{|a|^2 + |b|^2}$ (as it should be, usually written as $|q| = \sqrt{|a|^2 + |b|^2}$) while we get

$$\|q_1\| = |a| + |b|$$

for $t = 1$. Note that the latter is not a Hilbert space norm.

We recall that \mathcal{H}_2^t endowed with the real-valued bilinear form

$$\langle p_t, q_t \rangle = \text{Tr } q_t^* p_t$$

is a real four dimensional Hilbert space.

This norm is cumbersome to deal with, so we define a similar norm that induces the same topology, a scaled Euclidean-type norm.

Definition 2.6. We will now define the *scaled Euclidean norm* of \mathbb{H}_t to be

$$\|q_t\|_E = \sqrt{|a|^2 + |t||b|^2} = \sqrt{x_0^2 + x_1^2 + |t|(x_2^2 + x_3^2)}, \quad (2.5)$$

and the scaled inner product associated $\|\cdot\|_E$ by $[\cdot, \cdot]_E$.

For $q_t = x_0 + x_1i + x_2j_t + x_3k_t$ and $p_t = y_0 + y_1i + y_2j_t + y_3k_t$ the associated inner product is:

$$[q_t, p_t]_E = x_0y_0 + x_1y_1 + |t|(x_2y_2 + x_3y_3). \quad (2.6)$$

Remark 2.7. This norm gives the same topology as the operator norm given in (2.2) or the Hilbert–Schmidt norm given in (2.1).

We conclude this subsection with a remark on matrices with entries in \mathcal{H}_2^t . Consider Let $G = (g_{uv})_{u,v=1}^N \in (\mathcal{H}_2^t)^{N \times N}$. Identifying \mathcal{H}_2^t with a subset of matrices of $\mathbb{C}^{2 \times 2}$ one can identify G with a matrix in $\mathbb{C}^{2N \times 2N}$. Even if this new matrix is invertible in $\mathbb{C}^{2N \times 2N}$, it is not clear that it will have the structure corresponding to an element of $(\mathcal{H}_2^t)^{N \times N}$. First a definition:

Definition 2.8. Let $G = (g_{uv})_{u,v=1}^N \in (\mathcal{H}_2^t)^{N \times N}$. We say that G is invertible in \mathcal{H}_2^t if there exist a matrix $H = (h_{uv})_{u,v=1}^N \in (\mathcal{H}_2^t)^{N \times N}$ such that $GH = HG = I_{(\mathcal{H}_2^t)^{N \times N}}$. The matrices $G_m = (G_{uv})_{u,v=1}^m$ are called the main minors of G .

Before we consider this above invertibility question in greater details we mention a number of formulas, often presented for complex matrices, but in fact valid for matrices with entries in any algebra, and can be traced at least with the work of Schur [43, 44]. More recent references include [31, 37], [38, p. 18], [35, p. 113].

Lemma 2.9. *Let $(A, B, C, D) \in (\mathcal{H}_2^t)^{n \times n} \times (\mathcal{H}_2^t)^{n \times m} \times (\mathcal{H}_2^t)^{m \times n} \times (\mathcal{H}_2^t)^{m \times m}$, and assume that A (resp. D) is invertible. Then:*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ CA^{-1} & I_m \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I_n & A^{-1}B \\ 0 & I_m \end{pmatrix}, \quad (2.7)$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_n & BD^{-1} \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_n & 0 \\ D^{-1}C & I_m \end{pmatrix} \quad (2.8)$$

respectively.

As a consequence, with

$$A^\times = A - BD^{-1}C \quad \text{and} \quad D^\square = D - CA^{-1}B, \quad (2.9)$$

we have:

Lemma 2.10. *Assume A invertible. Then, M is invertible if and only if D^\square is invertible, and it holds that:*

$$\begin{aligned} M^{-1} &= \begin{pmatrix} I_n & -A^{-1}B \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & (D^\square)^{-1} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -CA^{-1} & I_m \end{pmatrix} \\ &= \begin{pmatrix} A^{-1} + A^{-1}B(D^\square)^{-1}CA^{-1} & -A^{-1}B(D^\square)^{-1} \\ -(D^\square)^{-1}CA^{-1} & (D^\square)^{-1} \end{pmatrix}. \end{aligned} \quad (2.10)$$

Similarly, assuming D invertible, the matrix M is invertible if and only if A^\times is invertible, and we then have:

$$\begin{aligned} M^{-1} &= \begin{pmatrix} I_n & 0 \\ -D^{-1}C & I_m \end{pmatrix} \begin{pmatrix} (A^\times)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I_n & -BD^{-1} \\ 0 & I_m \end{pmatrix} \\ &= \begin{pmatrix} (A^\times)^{-1} & -(A^\times)^{-1}BD^{-1} \\ -D^{-1}C(A^\times)^{-1} & D^{-1} + D^{-1}C(A^\times)^{-1}BD^{-1} \end{pmatrix}. \end{aligned} \quad (2.11)$$

As a preliminary example, consider the case $N = 2$, and $G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$. Assume that g_{11} is invertible, and define

$$g_{22}^\square = g_{11} - g_{12}g_{22}^{-1}g_{21}.$$

Then it follows from the formula (2.10) that G is invertible if g_{22}^\square is invertible. Using the above formulas one can give *sufficient conditions* in terms of N element of \mathcal{H}_2^t for the matrix G to be invertible in $(\mathcal{H}_2^t)^{N \times N}$.

2.2. The Adjoint \circledast

As written in the introduction we define the new adjoint \circledast as follows:

Definition 2.11. For $q_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix}$, $q_t \in \mathcal{H}_t$ we define

$$q_t^{\circledast} = \begin{pmatrix} \bar{a} & -tb \\ -\bar{b} & a \end{pmatrix}.$$

The bilinear real form associated to \circledast is

$$[q_t, p_t]_{\circledast} = a\bar{c} + \bar{a}c - t(\bar{b}d + b\bar{d}) \quad (2.12)$$

with $p_t = \begin{pmatrix} c & td \\ \bar{d} & \bar{c} \end{pmatrix}$.

Lemma 2.12. *The \circledast -adjoint satisfies*

$$q_t q_t^{\circledast} = q_t^{\circledast} q_t = (\det q_t) I_2 = (|a|^2 - t|b|^2) I_2. \quad (2.13)$$

As we saw above, the associated bilinear real form is

$$[q_t, p_t]_{\circledast} = \text{Tr } q_t^{\circledast} p_t = a\bar{c} + \bar{a}c - t(\bar{b}d + b\bar{d}). \quad (2.14)$$

This \circledast adjoint leads to a theory which is completely parallel to the study of slice hyperholomorphic functions.

We set

$$\frac{q_t + q_t^{\circledast}}{2} = \text{Re } q_t$$

We first note that

$$\text{Re } q_t = (\text{Re } a) I_2 = x_0 I_2, \quad (2.15)$$

where we use the notation of Eq. (1.12)

Lemma 2.13. *It holds that $q_t^{\circledast} = q_t$ if and only if q is a scalar matrix, $q_t = x I_2$ for some $x \in \mathbb{R}$.*

Proof. This is a direct computation. \square

Formula (2.13) shows that $q_t q_t^{\circledast}$ need not be positive for $t > 0$. More precisely:

Theorem 2.14. *Let $t \in \mathbb{R}^*$. The space \mathcal{H}_2^t endowed with the form*

$$[p_t, q_t]_{\circledast} = \text{Tr } (q_t^{\circledast} p_t) = \text{Tr } (p_t^{\circledast} q_t) \quad (2.16)$$

is a real Hilbert space of dimension four for $t < 0$ and a real Pontryagin space of dimension four, with (real) index of negativity equal to two, for $t > 0$.

Proof. With $p_t = \begin{pmatrix} c & td \\ \bar{d} & \bar{c} \end{pmatrix}$ we have

$$\text{Tr } (q_t^{\circledast} p_t) = \text{Tr } (p_t^{\circledast} q_t) = a\bar{c} + \bar{a}c - t(\bar{b}d + b\bar{d}). \quad (2.17)$$

Furthermore,

$$\begin{aligned} [p_t, q_t]_{\circledast} &= \text{Tr } (q_t^{\circledast} p_t) \\ &= a\bar{c} + \bar{a}c - t(\bar{b}d + b\bar{d}) \\ &= \frac{1}{2} \{ (a + \bar{a})(c + \bar{c}) - (a - \bar{a})(c - \bar{c}) - t((b + \bar{b})(d + \bar{d}) - (b - \bar{b})(d - \bar{d})) \} \\ &= \frac{1}{2} \left\{ (a + \bar{a})(c + \bar{c}) + (a - \bar{a})\overline{(c - \bar{c})} - t \left((b + \bar{b})(d + \bar{d}) + (b - \bar{b})\overline{(d - \bar{d})} \right) \right\}, \end{aligned}$$

which shows that in the case of $t > 0$ the form has two positive squares and two negative squares and \mathcal{H}_2^t becomes a Pontryagin space with $[p_t, q_t]_{\circledast}$ and it is a Hilbert space for $t < 0$. which ends the proof (see also [7] and [8]). \square

As a corollary we have the following result.

Corollary 2.15. *Let \mathcal{H}_2^t be the scaled hypercomplex space for a fixed t , where we assume that $t \neq 0$. If, moreover, for a fixed $p_t \in \mathcal{H}_2^t$ we have $\text{Tr } q_t^{\otimes} p_t = 0$ for all $q_t \in \mathcal{H}_2^t$, then $p_t = 0$.*

Proof. Let

$$p_t = \begin{pmatrix} c & td \\ \bar{d} & \bar{c} \end{pmatrix} \quad \text{and} \quad q_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix}.$$

Then, as in (2.14),

$$\text{Tr } q_t^{\otimes} p_t = a\bar{c} + \bar{a}c - t(\bar{b}d + b\bar{d}).$$

Assume $\text{Tr } q_t^{\otimes} p_t = 0$ and first take $b = 0$. The choice $a = 1$ and $a = i$ lead respectively to $\text{Re } c = 0$ and $\text{Im } c = 0$ so that $c = 0$. The case of d is treated similarly. \square

The first adjoint \otimes leads to a theory which is completely parallel to the study of slice hyperholomorphic functions and the authors are preparing another work in this direction as well.

In order to properly define spaces of functions, we will write the connection between the scaled norm and the \otimes inner product as follows. Let J_{\otimes} be the matrix:

$$J_{\otimes} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{t}{|t|} & 0 \\ 0 & 0 & 0 & -\frac{t}{|t|} \end{pmatrix}_{\mathbb{R}} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{t}{|t|} \end{pmatrix}_{\mathbb{C}} \quad (2.18)$$

J_{\otimes} is a signature matrix (both self-adjoint and equal to its inverse) and induces a signature operator (which we still denote by J_{\otimes} on \mathbb{H}_t such that

$$[q_t, q_t]_{\otimes} = [q_t, J_{\otimes} q_t]_E \quad (2.19)$$

where $[\cdot, \cdot]_E$ is the inner product associated to the norm $\|\cdot\|_E$, as in (2.6).

Note that, using the representation (1.8) we have

$$J_{\otimes} q_t = \bar{a} - bj_t. \quad (2.20)$$

2.3. The Adjoint $[\ast]$

We now turn to the definition of adjoint $q_t^{[\ast]}$, as in the introduction.

Definition 2.16. For a fixed scalar $t \in \mathbb{R}$, and $q_t = (a, b)_t \in \mathbb{H}_t$ we define $q_t^{[\ast]}$ by

$$(a, b)_t^{[\ast]} \stackrel{\text{def}}{=} (a, \bar{b})_t \quad \text{in } \mathbb{H}_t. \quad (2.21)$$

Equivalently, if we take q_t to be an element of \mathcal{H}_2^t rather than \mathbb{H}_t then $q_t^{[\ast]}$ becomes:

$$\begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix}^{[\ast]} = \begin{pmatrix} a & t\bar{b} \\ b & \bar{a} \end{pmatrix} \quad (2.22)$$

on \mathcal{H}_2^t .

The second adjoint $[*]$ satisfies the following identity in matrix form:

$$q_t^{[*]} = J q_t^* J \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.23)$$

Note that, even if, q_t^* and J are not in \mathcal{H}_t for general t , we have that $q_t^{[*]} \in \mathcal{H}_t$ and hence it becomes a Pontryagin space adjoint on $\mathbb{C}^{2 \times 2}$ (see Sect. [Appendix A](#) and Example [A.3](#); see also [\[7\]](#)).

We also note that, in general, q_t is not normal for this adjoint, (i.e. it does not commute with $q_t^{[*]}$):

$$q_t q_t^{[*]} \neq q_t^{[*]} q_t. \quad (2.24)$$

The adjoint $[*]$ is associated to the bilinear real form

$$[q_t, p_t]_{[*]} = ac + \overline{ac} - t(bd + \overline{bd}) \quad (2.25)$$

(as compared with [\(2.14\)](#)), and leads to a new theory, more “non-commutative”. A way to see this is to compare the definitions of the corresponding Blaschke factors later on in Sect. [6](#) (see formulas [\(4.18\)](#) and [\(5.2\)](#) respectively).

The second adjoint $[*]$ satisfies

$$q_t^{[*]} = J q_t^* J \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.26)$$

Lemma 2.17. *It holds that, for every $p_t, q_t \in \mathbb{H}_t$*

$$(q_t^{[*]})^{[*]} = q_t \quad (2.27)$$

$$(p_t \cdot_t q_t)^{[*]} = q_t^{[*]} \cdot_t p_t^{[*]} \quad (2.28)$$

Proof. The proof follows from [\(2.26\)](#) and of the corresponding properties for the regular adjoint. We leave the details to the reader. \square

We now make more precise the fact (see [\(2.24\)](#) above) that q_t is not normal in general with respect to this adjoint.

Proposition 2.18. *Let $q_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathcal{H}_2^t$. Then,*

$$\begin{aligned} q_t q_t^{[*]} &= \begin{pmatrix} a^2 + tb^2 & 2\operatorname{Re} a\bar{b} \\ 2\operatorname{Re} a\bar{b} & \bar{a}^2 + \bar{t}\bar{b}^2 \end{pmatrix} \\ q_t^{[*]} q_t &= \begin{pmatrix} a^2 + \bar{t}\bar{b}^2 & 2t\operatorname{Re} ab \\ 2\operatorname{Re} ab & \bar{a}^2 + tb^2 \end{pmatrix}. \end{aligned} \quad (2.29)$$

Proof. The formulas [\(2.29\)](#) are shown by straightforward computations. See [\[6\]](#) for details. \square

Lemma 2.19. *The form*

$$[q_t, p_t]_{[*]} = \operatorname{Tr} \left(p_t^{[*]} q_t \right), \quad p, q \in \mathcal{H}_2^t$$

is bilinear on \mathcal{H}_2^t with respect to the real numbers, satisfies

$$[q_t, p_t]_{[*]} = [p_t, q_t]_{[*]}$$

and is not degenerate on \mathcal{H}_2^t .

Proof. Set

$$q_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix} \quad \text{and} \quad p_t = \begin{pmatrix} c & td \\ \bar{d} & \bar{c} \end{pmatrix},$$

and assume that $q \in \mathbb{H}_t$ is such that $[q_t, p_t]_{[*]} = 0$ for all $p \in \mathbb{H}_t$. Since

$$q_t^{[*]} p_t = \begin{pmatrix} ac + t\bar{b}\bar{d} & t(ad + \bar{b}\bar{c}) \\ bc + \bar{a}\bar{d} & tbd + \bar{a}\bar{c} \end{pmatrix}$$

we have

$$[q_t, p_t]_{[*]} = \text{Re}(ac + t\bar{b}\bar{d}).$$

The choice $d = 0$ and $c \in \mathbb{R}$ gives $\text{Re } a = 0$ and the choice $c \in \mathbb{R}$ gives $\text{Im } a = 0$. So $a = 0$. To see that $b = 0$ take then d real and then d purely imaginary. \square

By the above analysis, one can get the following result (see [6]).

Theorem 2.20. *If $t \neq 0$, the pair $(\mathcal{H}_2^t, [\cdot, \cdot]_{[*]})$ is a Pontryagin space over \mathbb{R} .*

Proof. We have that:

$$\begin{aligned} [q_t, p_t]_{[*]} &= 2 \text{Re}(ac + t\bar{b}\bar{d}) \\ &= ac + \bar{a}\bar{c} + t(b\bar{d} + \bar{b}d) \\ &= \frac{1}{2} \left\{ (a + \bar{a})(c + \bar{c}) - (a - \bar{a})(c - \bar{c}) + t \left((b + \bar{b})(d + \bar{d}) - (b - \bar{b})(d - \bar{d}) \right) \right\}, \end{aligned}$$

which shows that the form has two positive squares and two negative squares and the result follows. \square

We will now write the connection between the scaled norm and the $[\cdot, \cdot]_{[*]}$ inner product as follows. Let $J_{[*]}$ be the matrix:

$$J_{[*]} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{t}{|t|} & 0 \\ 0 & 0 & 0 & -\frac{t}{|t|} \end{pmatrix}_{\mathbb{R}} \quad (2.30)$$

$J_{[*]}$ is a signature matrix (both self-adjoint and equal to its inverse) and induces a signature operator (which we still denote by $J_{[*]}$ on \mathbb{H}_t such that such that

$$[q_t, q_t]_{[*]} = [q_t, J_{[*]} q_t]_E. \quad (2.31)$$

Note that, using the representation $q_t = a + bj_t$ we have

$$J_{[*]} q_t = a + \bar{b} j_t. \quad (2.32)$$

Remark 2.21. As a reminder, we now have $(\mathcal{H}_2^t, [\cdot, \cdot]_{[*]})$ a Pontryagin space for all $t \neq 0$, while $(\mathcal{H}_2^t, [\cdot, \cdot]_{\otimes})$ is a Pontryagin space for $t > 0$ and a Hilbert space for $t < 0$.

Remark 2.22. Let $t = \pm 1$. It holds that $q_1^* = q_1^{[*]}$ if and only if $b = \bar{b}$ and $a = \bar{a}$, i.e. if and only if q_1 is an hyperbolic number. Similarly, let $t = -1$. Then $q_1^* = q_1^{[*]}$ if and only if $b = -\bar{b}$ and $a = \bar{a}$.

In a table form, the adjoints of i , j_t and k_t with respect to \circledast and $[*]$ are:

| | \circledast | $[*]$ |
|-------|---------------|--------|
| 1 | 1 | 1 |
| i | $-i$ | i |
| j_t | $-j_t$ | j_t |
| k_t | $-k_t$ | $-k_t$ |

or, equivalently, in terms of matrices:

| | \circledast | $[*]$ |
|--|---|---|
| $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ |
| $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ | $-\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ | $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ |
| $\begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}$ | $-\begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}$ |
| $\begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix}$ | $-\begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix}$ | $-\begin{pmatrix} 0 & ti \\ -i & 0 \end{pmatrix}$ |

Throughout the paper we will make use of the three equivalent definitions of scaled hypercomplex numbers and their conjugates and use either the matrix, vector, or complex pair notation as needed. As a reminder, we include them here:

$$q_t^* = \begin{pmatrix} \bar{a} & b \\ t\bar{b} & a \end{pmatrix} \quad q_t^{\circledast} = \begin{pmatrix} \bar{a} & -tb \\ -\bar{b} & a \end{pmatrix} \quad q_t^{[*]} = \begin{pmatrix} a & t\bar{b} \\ b & \bar{a} \end{pmatrix}.$$

These conjugates will lead interesting function theories and it is worth mentioning here the example of $f(q_t) = \sqrt{1 - q_t}$ which will have a familiar corresponding power series decomposition depending on the two adjoints \circledast and $[*]$.

In this paper, we do not develop the degenerate case $t = 0$, as it is not relevant here. This case is currently being worked on in another project.

2.4. Spaces of Functions on \mathcal{H}_2^t

We start with considering the spectrum of a matrix given by the \circledast adjoint. This is the only case that can be considered, since, for the $[*]$ adjoint we have that q_t is not normal for this adjoint (see (2.24)) and that $q_t q_t^{[*]}$ is not scalar.

Proposition 2.23. *For $q_t \in \mathbb{H}_t$ and $A \in \mathbb{H}_t^{m \times m}$ we have that:*

$$\sum_{n=0}^{\infty} q_t^n A^n = (1 - q_t^{\circledast} A)(q_t q_t^{\circledast} A^2 - 2(\operatorname{Re}(q_t))A + I_2)^{-1}, \quad (2.33)$$

whenever the sum on the left is convergent.

Proof. We prove (2.33); We have

$$\begin{aligned}
& \left(\sum_{n=0}^{\infty} q_t^n A^n \right) (q_t q_t^{\otimes} A^2 - 2(\operatorname{Re}(q_t^{\otimes}))A + I_2) \\
&= \sum_{n=0}^{\infty} q_t^n (q_t q_t^{\otimes}) A^{n+2} - 2 \sum_{n=0}^{\infty} q_t^n (\operatorname{Re}(q_t)) A^{n+1} + \sum_{n=0}^{\infty} q_t^n A^n \\
&= \sum_{n=0}^{\infty} q_t^{n+1} q_t^{\otimes} A^{n+2} - \sum_{n=0}^{\infty} q_t^n (q_t + q_t^{\otimes}) A^{n+1} + \sum_{n=0}^{\infty} q_t^n A^n \\
&= \sum_{n=0}^{\infty} q_t^{n+1} q_t^{\otimes} A^{n+2} - \sum_{n=0}^{\infty} q_t^{n+1} A^{n+1} - \sum_{n=0}^{\infty} q_t^n q_t^{\otimes} A^{n+1} + \sum_{n=0}^{\infty} q_t^n A^n \\
&= \underbrace{\sum_{n=0}^{\infty} q_t^{n+1} q_t^{\otimes} A^{n+2} - \sum_{n=0}^{\infty} q_t^n q_t^{\otimes} A^{n+1}}_{-q_t^{\otimes} A} + \underbrace{\sum_{n=0}^{\infty} q_t^n A^n - \sum_{n=0}^{\infty} q_t^{n+1} A^{n+1}}_{I_2} \\
&= I_2 - q_t^{\otimes} A.
\end{aligned}$$

□

Corollary 2.24. *For any $p_t, q_t \in \mathbb{H}_2^t$ we have that:*

$$\sum_{n=0}^{\infty} q_t^n p_t^n = (1 - q_t^{\otimes} p_t)(q_t q_t^{\otimes} p_t^2 - 2(\operatorname{Re}(q_t))p_t + 1)^{-1}, \quad (2.34)$$

whenever the sum on the left is convergent.

Proof. Setting then $A = p_t$ in (2.33) gives (2.34). □

Remark 2.25. In the case where $t = 1$, the quaternionic case, this spectrum first appeared in [25–27].

We now introduce the Cauchy product of power series in \mathcal{H}_2^t . In view of the non-commutativity there are various possible ways to define power series and their product; here we follow [32], and consider power series of the form

$$f(q_t) = \sum_{n=0}^{\infty} q_t^n \alpha_n, \quad (2.35)$$

where both the variable q_t and the coefficients $\alpha_0, \alpha_1, \dots$ are in \mathcal{H}_2^t .

Definition 2.26. Let $f(q_t) = \sum_{n=0}^{\infty} q_t^n \alpha_n$ and $g(q_t) = \sum_{n=0}^{\infty} q_t^n \beta_n$, then the Cauchy-star product of f and g is:

$$(f \star g)(q_t) = \sum_{n=0}^{\infty} \sum_{k=0}^n q_t^{n-k} \alpha_{n-k} \star q_t^k \beta_k \quad (2.36)$$

where:

$$q_t^l \alpha \star_t q_t^m \beta = q_t^{l+m} \alpha \cdot_t \beta, \quad l, m \in \mathbb{N}_0, \quad \alpha, \beta \in \mathcal{H}_2^t,$$

is called the star-product.

Throughout the paper we will call both of the above \star products. The \star -product does not respect point-evaluation, but as in the quaternionic setting we have:

Lemma 2.27. *In the above notation assume that $f(q_t) \neq 0$. We then have:*

$$(f \star g)(q_t) = f(q_t)g \left(f(q_t)^{-1}q_t f(q_t) \right). \quad (2.37)$$

Proof. In the above notation we have:

$$\begin{aligned} (f \star g)(q_t) &= \sum_{n=0}^{\infty} q_t^n f(q_t) \beta_n \\ &= \sum_{n=0}^{\infty} f(q_t) f(q_t)^{-1} q_t^n f(q_t) \beta_n \\ &= \sum_{n=0}^{\infty} f(q_t) \left(f(q_t)^{-1} q_t f(q_t) \right)^n \beta_n \\ &= f(q_t) \left(\sum_{n=0}^{\infty} f(q_t)^{-1} q_t f(q_t)^n \beta_n \right) \\ &= f(q_t)g \left(f(q_t)^{-1} q_t f(q_t) \right). \end{aligned}$$

□

The proof of the following important lemma is easy and will be omitted.

Lemma 2.28. *The power series (2.35) is uniquely determined by the values $f(x)$, where x is real and in an open interval of the form $(-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$. Furthermore the restriction to the real line of the \star -product reduces to the regular (non-commutative) product of power series of a real variable with matrix coefficients.*

Thus these are series of matrices (with a certain structure), and the results from series of matrices will hold, using the operator norm, and the claims can be proved by first considering the case $q_t \in \mathbb{R}$. In particular:

Proposition 2.29. *Let $f(q_t) = \sum_{n=0}^{\infty} q_t^n \alpha_n$ and $g(q_t) = \sum_{n=0}^{\infty} q_t^n \beta_n$ be power series of the form (2.35), converging in a neighborhood $\|q_t\|_{op} < \epsilon$ of the zero matrix in \mathcal{H}_2^t . The star product $f \star g$ converges in a (possibly smaller) neighborhood of the origin.*

Proof. Let (γ_n) be the corresponding convolution sequence:

$$\gamma_n = \sum_{k=0}^n \alpha_k \beta_{n-k}, \quad n = 0, 1, \dots$$

Restricting $q_t = x_0 I_2$ where $x \in \mathbb{R}$, and resorting to the classical theory, we see that the series $\sum_{n=0}^{\infty} x_0^n \gamma_n$ converges for $|x_0| < \eta$ for some $\eta > 0$. Thus

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \|\gamma_n\|_{op}^{\frac{1}{n}} > 0.$$

For $\|q_t\| < R$ the series $\sum_{n=0}^{\infty} \|q_t\|_{op}^n \cdot \|\gamma_n\|_{op}$ converges and so does $f \star g$. □

Proposition 2.30. Assume $\alpha_0 = 1$, and write $f(q_t) = 1 - g(q_t)$ with $g(q_t) = \sum_{k=1}^{\infty} q_t^k \alpha_k$. Then, the inverse $f^{-\star}(q_t)$ is equal to a convergent power series of the form (2.35) in a neighborhood of the origin and moreover

$$f^{-\star}(q_t) = \sum_{n=0}^{\infty} g^{\star n}(q_t)$$

where $g^{\star n} = g \star g \cdots \star g$, n -times, and the convergence is in the topology of $\mathbb{C}^{2 \times 2}$.

Proof. The result is true for $q_t = x \in \mathbb{R}$ and then follows since the restriction to any subinterval of the real line uniquely determines a power series of the form (2.35) (see Lemma 2.28). \square

The following lemma will play a key role in the study of rational functions.

Lemma 2.31. Let $\alpha = (\alpha_n)_{n \in \mathbb{N}_0}$ be a sequence of elements in \mathcal{H}_2^t and let $\alpha^{\circledast} = (\alpha_n^{\circledast})_{n \in \mathbb{N}_0}$ denote the corresponding sequence of \circledast -adjoints. Then the sequence of self-convolution $\alpha \otimes \alpha^{\circledast}$ is real.

Proof. We have

$$(\alpha \otimes \alpha^{\circledast})_n = \sum_{\substack{\ell, k \in \{0, \dots, n\} \\ k + \ell = n}} \alpha_k \alpha_{\ell}^{\circledast}$$

which is equal to its own \circledast -adjoint since

$$\sum_{\substack{\ell, k \in \{0, \dots, n\} \\ k + \ell = n}} \alpha_k \alpha_{\ell}^{\circledast} = \sum_{\substack{\ell, k \in \{0, \dots, n\} \\ k + \ell = n}} \alpha_{\ell} \alpha_k^{\circledast} \quad (2.38)$$

\square

Definition 2.32. Let $f(q_t) = \sum_{n=0}^{\infty} q_t^n \alpha_n$ be converging in a neighborhood $\|q_t\| < \epsilon$ of the zero element of \mathcal{H}_2^t . We define

$$f^{\circledast}(q_t) = \sum_{n=0}^{\infty} q_t^n \alpha_n^{\circledast}. \quad (2.39)$$

We note that, for $q_t = x \in \mathbb{R}$ for which the power series converges,

$$f^{\circledast}(x) = (f(x))^{\circledast}.$$

Proposition 2.33. The power series $f(q_t) \star f^{\circledast}(q_t)$ has real coefficients and commute with every other converging power series.

Proof. This follows from Lemma 2.31. \square

Proposition 2.34. Assume that α_0 in (2.35) is invertible. Then,

$$(f(q_t))^{-\star} = f^{\circledast}(q_t)(f(q_t) \star f^{\circledast}(q_t))^{-1} = (f(q_t) \star f^{\circledast}(q_t))^{-1} f^{\circledast}(q_t). \quad (2.40)$$

Proof. We observe that, since $f(q_t) \star f^{\circledast}(q_t)$ is real valued and \star is associative:

$$\begin{aligned} (f(q_t))^{-\star} (f(q_t) \star f^{\circledast}(q_t)) &= (f(q_t))^{-\star} \star (f(q_t) \star f^{\circledast}(q_t)) \\ &= (f(q_t))^{-\star} \star f((q_t)) \star f^{\circledast}(q_t) = f^{\circledast}(q_t). \end{aligned}$$

\square

In preparation of the study of rational functions we specialize the previous result to polynomial

Corollary 2.35. *Let $P(x) = \sum_{n=0}^N x^n \alpha_n$ be a polynomial of the real variable x , with coefficients $\alpha_0, \dots, \alpha_N \in \mathcal{H}_2^t$. Then the polynomial*

$$P(x)(P(x))^{\otimes} \quad (2.41)$$

has real coefficients and

$$(P(q_t))^{-\star} = P^{\otimes}(q_t)(P(q_t) \star P^{\otimes}(q_t))^{-1} = (P(q_t) \star P^{\otimes}(q_t))^{-1} P^{\otimes}(q_t). \quad (2.42)$$

We note that (2.42) can be rewritten as

$$(P(q_t))^{-\star} = \frac{P^{\otimes}(q_t)}{(P(q_t) \star P^{\otimes}(q_t))}.$$

We present an illustration of the previous corollary, which pertains to Blaschke factors; see Sect. 4.4 for the latter.

Example 2.36. Let $\alpha \in \mathcal{H}_2^t$. Then,

$$\begin{aligned} (1 - q_t \alpha)^{-\star} &= (1 - q_t \alpha^{\otimes})((1 - q_t \alpha) \star (1 - q_t \alpha^{\otimes}))^{-1} \\ &= \frac{1 - q_t \alpha^{\otimes}}{q_t^2(\alpha \alpha^{\otimes}) - 2q_t \operatorname{Re} \alpha + 1} \end{aligned}$$

We conclude this section with:

Proposition 2.37. *We have that $q_t - \alpha$ and $(1 - q_t \alpha) \star$ -commute. It then follows that $q_t - \alpha$ and $(1 - q_t \alpha)^{-\star}$ also \star -commute.*

Proof. The claim amounts to show that

$$(1 - q_t \alpha) \star (q_t - \alpha) = (q_t - \alpha) \star (1 - q_t \alpha),$$

but both sides are equal to $-q_t^2 \alpha + q_t(\alpha^2 + 1) - \alpha$.

The second part follows. \square

In fact the above proposition is a special case of the following result, whose proof is left to the reader.

Proposition 2.38. *Let $f(q_t) = \sum_{n=0}^{\infty} q_t^n \alpha_n$ and $g(q_t) = \sum_{n=0}^{\infty} q_t^n \beta_n$ be power series of the form (2.35), converging in a neighborhood $\|q_t\|_{op} < \epsilon$ of zero. If their coefficients pairwise commute, i.e. $\alpha_n \beta_m = \beta_m \alpha_n$, $\forall n, m \in \mathbb{N}_0$, then f and g will also \star -commute.*

We conclude with a definition generalizing the quaternionic setting:

Definition 2.39. Let $\alpha \in \mathbb{H}_t$. We denote by $[\alpha]$, and call sphere associated to α , the set of points $\beta \in \mathbb{H}_t$ such that

$$\operatorname{Re} \alpha = \operatorname{Re} \beta \quad (2.43)$$

$$\alpha \alpha^{\otimes} = \beta \beta^{\otimes} \quad (2.44)$$

3. The Scaled Global Operator for Scaled Hypercomplex Numbers

For the scaled variable q_t we have its vector part:

$$\vec{q}_t = ix_1 + j_tx_2 + k_tx_3,$$

and, similar to the global operator G_q on the space of quaternions, we can define the global operator associated to the scaled quaternions \mathbb{H}_t to be:

$$G_t = \vec{q}_t \frac{\partial}{\partial x_0} - \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) \quad (3.1)$$

and its associated, simplified version:

$$V_t = \frac{\partial}{\partial x_0} - \frac{1}{ix_1 + j_tx_2 + k_tx_3} \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) \quad (3.2)$$

3.1. The Gleason Problem Associated to the V_t -Operator and the V_t -Fueter Variables

As in [15] we solve the Gleason problem associated to the new V_t -operator and define $\mu_{t,l}$, the Fueter-type variables which are in its kernel and which generate the series solutions to $V_t f = 0$, where f is real-analytic.

Assume f smooth and in the kernel of V_t , which yields:

$$\frac{\partial f}{\partial x_0} = \frac{1}{ix_1 + j_tx_2 + k_tx_3} \left(x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} \right).$$

Varying x by a real parameter u , we obtain:

$$\begin{aligned} \frac{df(ux)}{du} &= x_0 \frac{\partial f}{\partial x_0}(ux) + x_1 \frac{\partial f}{\partial x_1}(ux) + x_2 \frac{\partial f}{\partial x_2}(ux) + x_3 \frac{\partial f}{\partial x_3}(ux) \\ &= \frac{x_0}{q_t} \left(x_1 \frac{\partial f}{\partial x_1}(ux) + x_2 \frac{\partial f}{\partial x_2}(ux) + x_3 \frac{\partial f}{\partial x_3}(ux) \right) \\ &\quad + x_1 \frac{\partial f}{\partial x_1}(ux) + x_2 \frac{\partial f}{\partial x_2}(ux) + x_3 \frac{\partial f}{\partial x_3}(ux) \\ &= \mu_{1,t}(x) \frac{\partial f}{\partial x_1}(ux) + \mu_{2,t}(x) \frac{\partial f}{\partial x_2}(ux) + \mu_{3,t}(x) \frac{\partial f}{\partial x_3}(ux) \end{aligned} \quad (3.3)$$

where:

$$\mu_{l,t}(x) = x_l \left(1 + \frac{x_0}{q_t} \right), \quad l = 1, 2, 3. \quad (3.4)$$

Definition 3.1. The functions $\mu_{l,t}$, $l = 1, 2, 3$, are called the V_{q_t} -Fueter variables.

For $t = -1$, corresponding to the field of quaternions \mathbb{H}_{-1} , these variables were introduced in [15].

With $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ we recall that

$$|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$$

and we use the multi-variable notation for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, and more generally for commuting variables:

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}.$$

We can now define:

Definition 3.2. For the V_t -Fueter variables μ_u and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$, we define the product:

$$\mu_t^\alpha(x) = \mu_{1,t}^{\alpha_1}(x) \mu_{2,t}^{\alpha_2}(x) \mu_{3,t}^{\alpha_3}(x), \quad (3.5)$$

for every $x \in \mathbb{H}_t^*$.

Proposition 3.3. *The V_t -Fueter variables pairwise commute and for $\alpha \in \mathbb{N}_0^3$ it holds that*

$$\begin{aligned} \mu_t^\alpha &= \mu_{1,t}^{\alpha_1} \mu_{2,t}^{\alpha_2} \mu_{3,t}^{\alpha_3} \\ &= x^\alpha \left(1 + \frac{x_0}{\vec{q}_t} \right)^{|\alpha|} \end{aligned} \quad (3.6)$$

and

$$\mu_t^\alpha|_{x_0=0} = x^\alpha \quad (3.7)$$

Proof. Since here too the V_t -Fueter variables μ_l commute, in the above definition we do not need to use the symmetric product and the proof goes along the lines of the quaternionic case which was presented in [15]. \square

We can now prove that the V_t -Fueter products μ_t^α are in the kernel of V_t on \mathbb{H}_t^* :

Theorem 3.4. *It holds that μ_t^α are in the kernel of the operator V_t on any open domain $\Omega \subset \mathbb{H}_t^*$. Moreover, we have:*

$$V_t \mu_t^\alpha(x) = 0, \quad (3.8)$$

for every $x \in \Omega$.

Proof. Following the argument in [15], we divide the verification into a number of steps.

STEP 1: *It holds that*

$$\begin{aligned} \frac{\partial}{\partial x_1} \frac{1}{\vec{q}_t} &= \frac{i}{\vec{q}_t^2} + \frac{2x_1}{\vec{q}_t^3}, \\ \frac{\partial}{\partial x_2} \frac{1}{\vec{q}_t} &= \frac{j_t}{\vec{q}_t^2} + \frac{2x_2 t}{\vec{q}_t^3}, \\ \frac{\partial}{\partial x_3} \frac{1}{\vec{q}_t} &= \frac{k_t}{\vec{q}_t^2} + \frac{2x_3 t}{\vec{q}_t^3}. \end{aligned} \quad (3.9)$$

Indeed, for the first case we have

$$\frac{1}{\vec{q}_t} = \frac{\vec{q}_t}{\vec{q}_t^2} = \frac{-\vec{q}_t}{x_1^2 + (x_2^2 + x_3^2)t}.$$

Hence

$$\begin{aligned} \frac{\partial}{\partial x_1} \frac{1}{\vec{q}_t} &= \frac{-i(x_1^2 + (x_2^2 + x_3^2)t) + 2x_1 \vec{q}_t}{(x_1^2 + (x_2^2 + x_3^2)t)^2} \\ &= \frac{i\vec{q}_t^2}{\vec{q}_t^4} + \frac{2x_1 \vec{q}_t}{\vec{q}_t^3} \end{aligned}$$

and the result follows. The other cases are similar and Eq. (3.9) are proven.

STEP 2: *It now holds that*

$$\begin{aligned}
 \frac{\partial}{\partial x_1} \left(1 + \frac{x_0}{\vec{q}t} \right)^{|\alpha|} &= \sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|}} \left(1 + \frac{x_0}{\vec{q}t} \right)^t \left(\frac{x_0 i}{\vec{q}t^2} + \frac{2x_1 x_0}{\vec{q}t^3} \right) \left(1 + \frac{x_0}{\vec{q}t} \right)^s, \\
 \frac{\partial}{\partial x_2} \left(1 + \frac{x_0}{\vec{q}t} \right)^{|\alpha|} &= \sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|}} \left(1 + \frac{x_0}{\vec{q}t} \right)^t \left(\frac{x_0 j_t}{\vec{q}t^2} + \frac{2tx_2 x_0}{\vec{q}t^3} \right) \left(1 + \frac{x_0}{\vec{q}t} \right)^s, \\
 \frac{\partial}{\partial x_3} \left(1 + \frac{x_0}{\vec{q}t} \right)^{|\alpha|} &= \sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|}} \left(1 + \frac{x_0}{\vec{q}t} \right)^t \left(\frac{x_0 k_t}{\vec{q}t^2} + \frac{2tx_3 x_0}{\vec{q}t^3} \right) \left(1 + \frac{x_0}{\vec{q}t} \right)^s.
 \end{aligned} \tag{3.10}$$

(3.10) is a direct consequence of (3.9) and of the formula for the derivative of f^n when f is a matrix-valued function (and in particular a global quaternionic valued) of (say) a real variable w :

$$\frac{df^n}{dw} = \sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|}} f^t f' f^s. \tag{3.11}$$

STEP 3: *We have*

$$\frac{\partial}{\partial x_0} \left(1 + \frac{x_0}{\vec{q}t} \right)^{|\alpha|} = \frac{|\alpha|}{\vec{q}t} \left(1 + \frac{x_0}{\vec{q}t} \right)^{|\alpha|-1}. \tag{3.12}$$

This is because $1 + \frac{x_0}{\vec{q}t}$ commutes with its derivative with respect to x_0 , and formula (3.11) reduces then to the classical formula.

STEP 4: *We now calculate*

$$\frac{1}{\vec{q}t} \sum_{u=1}^3 x_u \frac{\partial}{\partial x_u} \mu^\alpha.$$

We have:

$$\begin{aligned}
 &\frac{1}{\vec{q}t} \sum_{u=1}^3 x_u \frac{\partial}{\partial x_u} x^\alpha \left(1 + \frac{x_0}{\vec{q}t} \right)^{|\alpha|} = \frac{1}{\vec{q}t} \\
 &\quad \cdot \left(x_1 \frac{\partial}{\partial x_1} x^\alpha \left(1 + \frac{x_0}{\vec{q}t} \right)^{|\alpha|} + x_2 \frac{\partial}{\partial x_2} x^\alpha \left(1 + \frac{x_0}{\vec{q}t} \right)^{|\alpha|} + x_3 \frac{\partial}{\partial x_3} x^\alpha \left(1 + \frac{x_0}{\vec{q}t} \right)^{|\alpha|} \right) \\
 &= \frac{1}{\vec{q}t} \left[x_1 \alpha_1 x^{\alpha-(1,0,0)} \left(1 + \frac{x_0}{\vec{q}t} \right)^{|\alpha|} \right. \\
 &\quad \left. + x_1 x^\alpha \sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|}} \left(1 + \frac{x_0}{\vec{q}t} \right)^t \left(\frac{x_0 i}{\vec{q}t^2} + \frac{2x_1 x_0}{\vec{q}t^3} \right) \left(1 + \frac{x_0}{\vec{q}t} \right)^s \right] \\
 &\quad + \frac{1}{\vec{q}t} \left[x_2 \alpha_2 x^{\alpha-(0,1,0)} \left(1 + \frac{x_0}{\vec{q}t} \right)^{|\alpha|} \right. \\
 &\quad \left. + x_2 x^\alpha \sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|}} \left(1 + \frac{x_0}{\vec{q}t} \right)^t \left(\frac{x_0 j_t}{\vec{q}t^2} + \frac{2tx_2 x_0}{\vec{q}t^3} \right) \left(1 + \frac{x_0}{\vec{q}t} \right)^s \right]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\bar{q}_t} \left[x_3 \alpha_3 x^{\alpha-(0,0,1)} \left(1 + \frac{x_0}{\bar{q}_t} \right)^{|\alpha|} \right. \\
& \quad \left. + x_3 x^\alpha \sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|}} \left(1 + \frac{x_0}{\bar{q}_t} \right)^t \left(\frac{x_0 k_t}{\bar{q}_t^2} + \frac{2tx_3 x_0}{\bar{q}_t^3} \right) \left(1 + \frac{x_0}{\bar{q}_t} \right)^s \right], \\
& = \frac{1}{\bar{q}_t} \left[|\alpha| x^\alpha \left(1 + \frac{x_0}{\bar{q}_t} \right)^{|\alpha|} \right] \\
& \quad + \frac{1}{\bar{q}_t} \left[x^\alpha \sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|-1}} \left(1 + \frac{x_0}{\bar{q}_t} \right)^t \left(\frac{x_1 x_0 i + x_2 x_0 j + x_3 x_0 k}{\bar{q}_t^2} \right. \right. \\
& \quad \left. \left. + \frac{2x_1^2 x_0 + 2tx_2^2 x_0 + 2tx_3^2 x_0}{\bar{q}_t^3} \right) \left(1 + \frac{x_0}{\bar{q}_t} \right)^s \right] \\
& = \frac{x^\alpha}{\bar{q}_t} \left[|\alpha| \left(1 + \frac{x_0}{\bar{q}_t} \right)^{|\alpha|} + \sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|-1}} \left(1 + \frac{x_0}{\bar{q}_t} \right)^t \left(\frac{x_0 \bar{q}_t}{\bar{q}_t^2} - \frac{2\bar{q}_t^2 x_0}{\bar{q}_t^3} \right) \left(1 + \frac{x_0}{\bar{q}_t} \right)^s \right] \\
& = \frac{x^\alpha}{\bar{q}_t} \left[|\alpha| \left(1 + \frac{x_0}{\bar{q}_t} \right)^{|\alpha|} - \sum_{\substack{t,s \in \mathbb{N}_0 \\ t+s=|\alpha|-1}} \left(1 + \frac{x_0}{\bar{q}_t} \right)^t \frac{x_0}{\bar{q}_t} \left(1 + \frac{x_0}{\bar{q}_t} \right)^s \right] \\
& = \frac{x^\alpha}{\bar{q}_t} \left[|\alpha| \left(1 + \frac{x_0}{\bar{q}_t} \right)^{|\alpha|} - |\alpha| \left(1 + \frac{x_0}{\bar{q}_t} \right)^{|\alpha|-1} \frac{x_0}{\bar{q}_t} \right].
\end{aligned}$$

STEP 5: We can now compute $V_t \mu_t^\alpha$.

Using (3.12) and the previous step we have:

$$\begin{aligned}
V_t \mu_t^\alpha & = x^\alpha \frac{|\alpha|}{\bar{q}_t} \left(1 + \frac{x_0}{\bar{q}_t} \right)^{|\alpha|-1} - \frac{x^\alpha}{\bar{q}_t} \left[|\alpha| \left(1 + \frac{x_0}{\bar{q}_t} \right)^{|\alpha|} - |\alpha| \left(1 + \frac{x_0}{\bar{q}_t} \right)^{|\alpha|-1} \frac{x_0}{\bar{q}_t} \right] \\
& = \frac{|\alpha| x^\alpha}{\bar{q}_t} \left(1 + \frac{x_0}{\bar{q}_t} \right)^{|\alpha|-1} \left(1 - \left(1 + \frac{x_0}{\bar{q}_t} \right) + \frac{x_0}{\bar{q}_t} \right) \\
& = 0.
\end{aligned}$$

This ends the proof. \square

Remark 3.5. We observe that both functions μ_t^α and ζ_t^α coincide with x^α when $x_0 = 0$. It is important to note that these are two different extensions of the same real function x^α leading to two different regular function theories. In fact, μ_t^α is the V_t -regular extension of x^α while ζ_t^α gives the classical global Fueter extension. However, the classical global Fueter variables ζ_t^α extend x^α to the whole space of quaternions while μ_t^α extend x^α to domains of \mathbb{H}_t^* .

In their work paper [8] two of the co-authors have introduced the extension of the Fueter variables corresponding to the Fueter operator to the scaled case, namely to scaled Fueter variables for the scaled Fueter operator:

Definition 3.6. The scaled Fueter ζ_t variables are:

$$\zeta_{1,t} = x_1 - x_0 i, \quad \zeta_{2,t} = x_2 + \frac{\text{sign}(t)x_0}{\sqrt{|t|}} j_t, \quad \zeta_{3,t} = x_3 + \frac{\text{sign}(t)x_0}{\sqrt{|t|}} k_t. \quad (3.13)$$

In the case of $t = -1$, namely, in the case of quaternions, we have the following (see [15] for details)

Proposition 3.7. *Let Ω be an open domain in \mathbb{H}^* . For every $x \in \Omega \subset \mathbb{H}^*$ it holds that*

$$|\mu^\alpha|^2 = |x|^{2\alpha} \left(1 + \frac{x_0^2}{x_1^2 + x_2^2 + x_3^2} \right)^{|\alpha|}, \quad (3.14)$$

and, in particular, we have:

$$|\mu_u(x)|^2 \leq |\zeta_u(x)|^2, \quad u = 1, 2, 3. \quad (3.15)$$

Proof. This follows from the fact that $\frac{x_0}{q}$ has no real part on Ω and:

$$|\mu_u(x)|^2 = x_u^2 + \frac{x_u^2 x_0^2}{x_1^2 + x_2^2 + x_3^2} \leq x_u^2 + x_0^2 = |\zeta_u(x)|^2. \quad (3.16)$$

□

In the case of the scaled hypercomplex numbers, using the scaled Euclidean norm we have a similar result:

Proposition 3.8. *Let Ω_t be an open domain in \mathbb{H}_t^* . For every $x \in \Omega \subset \mathbb{H}_t^*$ it holds that*

$$|\mu_t^\alpha|_E^2 = |x|_E^{2\alpha} \left(1 + \frac{x_0^2}{x_1^2 + |t|(x_2^2 + x_3^2)} \right)^{|\alpha|}, \quad (3.17)$$

and, in particular, we have:

$$|\mu_{u,t}(x)|_E^2 \leq |\zeta_{u,t}(x)|^2, \quad u = 1, 2, 3. \quad (3.18)$$

Proof. This follows from the fact that $\frac{x_0}{q}$ has no real part on Ω_t and it is easy to check that:

$$|\mu_{u,t}(x)|_E^2 = x_u^2 + \frac{x_u^2 x_0^2}{x_1^2 + |t|(x_2^2 + x_3^2)} \leq x_u^2 + \frac{x_0^2}{|t|} = |\zeta_{u,t}(x)|_E^2,$$

for $u = 2, 3$ and that:

$$|\mu_{1,t}(x)|_E^2 = x_1^2 + \frac{x_1^2 x_0^2}{x_1^2 + |t|(x_2^2 + x_3^2)} \leq x_1^2 + x_0^2 = |\zeta_{1,t}(x)|_E^2,$$

□

Proposition 3.9. *For any $n \in \mathbb{N}$, the function q_t^n is in $\ker V_t$ and that, moreover,*

$$q_t^n = \sum_{|\alpha|=n} \mu_t^\alpha c_{\alpha,n}$$

where, with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$$c_{\alpha,n} = \frac{n!}{\alpha!} i^{\times \alpha_1} \times j_t^{\times \alpha_2} \times k_t^{\times \alpha_3}, \quad (3.19)$$

where the symmetric product is taken among all the products of the units \mathbf{e}_u .

Proof. In \mathbb{H}_t^* we have

$$\begin{aligned} q_t^n &= (x_0 + \vec{q}_t)^n \\ &= \left(1 + \frac{x_0}{\vec{q}_t}\right)^n (\vec{q}_t)^n \\ &= \left(1 + \frac{x_0}{\vec{q}_t}\right)^n \left(\sum_{|\alpha|=n} x^\alpha c_{\alpha,n}\right) \\ &= \sum_{|\alpha|=n} x^\alpha \left(1 + \frac{x_0}{\vec{q}_t}\right)^n c_{\alpha,n}, \end{aligned}$$

for some $c_{\alpha,n} \in \mathbb{H}_t$ which can be expressed in term of symmetrized products as in (3.19) by known formulas. \square

We note that (3.19) does not take into account the Cayley table of multiplication for the quaternions.

3.2. Left Regular Functions on \mathbb{H}_t Where $t \neq 0$ in \mathbb{R}

In this section, we review some main results of [8], providing motivations of our works. Let $t \in \mathbb{R}$ and $\mathbb{H}_t = \text{span}_{\mathbb{R}}(\{1, i, j_t, k_t\})$, the t -scaled hypercomplexes. We consider functions,

$$f : \mathbb{H}_t \rightarrow \mathbb{H}_t,$$

in the t -scaled hypercomplex variable,

$$w = x_0 + x_1 i + x_2 j_t + x_3 k_t, \text{ with } x_l \in \mathbb{R},$$

for all $l = 0, 1, 2, 3$. Especially, we are interested in \mathbb{R} -differentiable functions on an open connected set Ω of the t -hypercomplex \mathbb{R} -space \mathbb{H}_t . For motivations, see e.g., [17, 18, 29, 36] and [40].

Definition 3.10. Let $t \in \mathbb{R}$ be fixed, and $U \subset \mathbb{H}_t$, an open subset. Define a set,

$$\mathcal{F}_{t,U} \stackrel{\text{def}}{=} \{f : \mathbb{H}_t \rightarrow \mathbb{H}_t \mid f \text{ is } \mathbb{R}\text{-differentiable on } U\}.$$

Remark that the openness on \mathbb{H}_t , here, is determined by the semi-norm topology for \mathbb{H}_t in terms of the semi-norm $\|\cdot\|_t$ of (2.1.10), for a fixed $t \in \mathbb{R}$. We first, let $t \neq 0$ in \mathbb{R} , and $\mathcal{F}_{t,U}$, the family of \mathbb{R} -differentiable functions on an open subset U of the t -scaled hypercomplexes \mathbb{H}_t . Define the differential operators ∇_t and ∇_t^{\otimes} on $\mathcal{F}_{t,U}$ by

$$\nabla_t = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} - j_t \frac{\text{sign}(t)}{\sqrt{|t|}} \frac{\partial}{\partial x_2} - k_t \frac{\text{sign}(t)}{\sqrt{|t|}} \frac{\partial}{\partial x_3}, \quad (3.20)$$

and

$$\nabla_t^{\otimes} = \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1} i + \frac{\text{sign}(t)}{\sqrt{|t|}} \frac{\partial}{\partial x_2} j_t + \frac{\text{sign}(t)}{\sqrt{|t|}} \frac{\partial}{\partial x_3} k_t, \quad (3.21)$$

where

$$\text{sign}(t) = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0, \end{cases}$$

for all $t \in \mathbb{R} \setminus \{0\}$.

The definitions (3.20) and (3.21) of the differential operators ∇_t and ∇_t^{\otimes} on $\mathcal{F}_{t,U}$ generalizes those on \mathcal{F}_{-1} (the quaternionic case) and those on \mathcal{F}_1 (the split-quaternionic case), studied in [17–19, 29, 36]

Theorem 3.11. *If ∇_t and ∇_t^{\otimes} are the differential operators of (3.20) and (3.21), then*

$$\nabla_t^{\otimes} \nabla_t = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} - \text{sign}(t) \frac{\partial^2}{\partial x_2^2} - \text{sign}(t) \frac{\partial^2}{\partial x_3^2} \quad (3.22)$$

Proof. See [8] for details. \square

By (3.22), one obtains the following corollary.

Corollary 3.12. *Let ∇_t and ∇_t^{\otimes} be the differential operators on \mathcal{F}_t , and Δ_t , the t -Laplacian. Then*

$$\Delta_t = \begin{cases} \Delta_{-1} = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} & \text{if } t < 0 \\ \Delta_1 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} & \text{if } t > 0. \end{cases}$$

Proof. It is proven by (3.22), whenever $t \neq 0$ in \mathbb{R} . \square

Definition 3.13. Let ∇_t be the operator 3.20 on $\mathcal{F}_{t,U}$, and $f \in \mathcal{F}_{t,U}$. If

$$\nabla_t f = \frac{\partial f}{\partial x_0} + i \frac{\partial f}{\partial x_1} - j_t \frac{\text{sign}(t)}{\sqrt{|t|}} \frac{\partial f}{\partial x_2} - k_t \frac{\text{sign}(t)}{\sqrt{|t|}} \frac{\partial f}{\partial x_3} = 0,$$

then f is said to be left t -(scaled)-regular on U . If

$$f \nabla_t = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1} i - \frac{\text{sign}(t)}{\sqrt{|t|}} \frac{\partial f}{\partial x_2} j_t - \frac{\text{sign}(t)}{\sqrt{|t|}} \frac{\partial f}{\partial x_3} k_t = 0,$$

then f is said to be right t -(scaled)-regular on U . If $f \in \mathcal{F}_{t,U}$ is both left and right t -regular, then it is called t -(scaled)-regular.

While, a function $f \in \mathcal{F}_{t,U}$ is t -(scaled)-harmonic, if

$$\Delta_t f = \frac{\partial^2 f}{\partial x_0^2} + \frac{\partial^2 f}{\partial x_1^2} - \text{sign}(t) \frac{\partial^2 f}{\partial x_2^2} - \text{sign}(t) \frac{\partial^2 f}{\partial x_3^2} = 0,$$

where Δ_t is the t -Laplacian (3.22).

The following theorem illustrates the relation between t -regularity and t -harmonicity.

Theorem 3.14. *Let $f \in \mathcal{F}_{t,U}$, for $U \in \mathcal{T}_t$ in \mathbb{H}_t . Then if f is left t -regular on U , it is t -harmonic on U .*

Proof. If f is left t -regular in $\mathcal{F}_{t,U}$, then $\nabla_t f = 0$ on U , and hence,

$$\Delta_t f = \nabla_t^{\otimes} (\nabla_t f) = \nabla_t^{\otimes} (0) = 0,$$

implying the t -harmonicity of f on U . \square

In Definition 3.6 we have introduced the entire \mathbb{R} -differentiable functions $\{\zeta_l\}_{l=1}^3 \subset \mathcal{F}_{t,\mathbb{H}_t}$, associated to a \mathbb{H}_t -variable $q_t = x_0 + x_1 i + x_2 j_t + x_3 k_t$, with $x_l \in \mathbb{R}$, $0 \leq l \leq 3$. In [8] the following result was proven:

Theorem 3.15. If $\zeta_{l,t}$ are the functions (3.13) in $\mathcal{F}_{t,\mathbb{H}_t}$, then $\zeta_{l,t}$ are t -harmonic and t -regular on \mathbb{H}_t , $\forall l = 1, 2, 3$.

Now, define the so-called symmetrized product on \mathbb{H}_t . One can do that since the \mathbb{R} -vector space \mathbb{H}_t has a well-defined vector-multiplication $(\cdot) = (\cdot)_t$.

Definition 3.16. Let $h_1, \dots, h_N \in \mathbb{H}_t$, for $N \in \mathbb{N}$. Then the symmetrized product of h_1, \dots, h_N is defined by a new hypercomplex number,

$$\bigtimes_{n=1}^N h_n \stackrel{\text{denote}}{=} h_1 \times \cdots \times h_N \stackrel{\text{def}}{=} \frac{1}{N!} \sum_{\sigma \in S_N} h_{\sigma(1)} h_{\sigma(2)} \cdots h_{\sigma(N)}, \quad (3.23)$$

where S_N is the symmetric group over $\{1, \dots, N\}$.

Remark 3.17. Recall that we are considering the cases where $t \neq 0$ in \mathbb{R} , however, the symmetrized product (\times) of (3.23) is well-defined for “all” $t \in \mathbb{R}$, including the case where $t = 0$.

Let $f_1, \dots, f_N : \mathbb{H}_t \rightarrow \mathbb{H}_t$ be functions for $N \in \mathbb{N}$. By applying (3.23), define a symmetrized-product function of f_1, \dots, f_N by

$$\bigtimes_{n=1}^N f_n = \frac{1}{N!} \sum_{\sigma \in S_N} f_{\sigma(1)} f_{\sigma(2)} \cdots f_{\sigma(N)}, \quad (3.24)$$

where

$$(f_{\sigma(1)} f_{\sigma(2)} \cdots f_{\sigma(N)})(h) = f_{\sigma(1)}(h) \cdots f_{\sigma(N)}(h), \quad \forall h \in \mathbb{H}_t,$$

for all $\sigma \in S_N$. So, one obtains

$$f_j^{(n)} = \underbrace{f_j \times f_j \times \cdots \times f_j}_{n\text{-times}} = f_j^n, \quad \forall n \in \mathbb{N},$$

where we define f^0 to be:

$$f^{(0)} = 1, \text{ the constant 1-function on } \mathbb{H}_t,$$

for all functions $f : \mathbb{H}_t \rightarrow \mathbb{H}_t$, where $1 = 1 + 0i + 0j + 0k \in \mathbb{H}_t$.

Definition 3.18. Let $\mathbf{n} \stackrel{\text{denote}}{=} (n_1, n_2, n_3) \in \mathbb{N}_0^3$ be a triple of numbers in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $\{\zeta_l\}_{l=1}^3$ be the t -harmonic t -regular functions of (3.13). Define a new function $\zeta^{\mathbf{n}} \in \mathcal{F}_{1,\mathbb{H}_t}$ by

$$\zeta^{\mathbf{n}} \stackrel{\text{def}}{=} \frac{1}{\mathbf{n}!} \left(\zeta_1^{(n_1)} \times \zeta_2^{(n_2)} \times \zeta_3^{(n_3)} \right), \quad (3.25)$$

where

$$\mathbf{n}! = (n_1!)(n_2!)(n_3!) \in \mathbb{N},$$

and

$$\zeta_{l+1}^{(n_l)} = \underbrace{\zeta_{l+1} \times \zeta_{l+1} \times \cdots \times \zeta_{l+1}}_{n_l\text{-times}} = \zeta_{l+1}^{n_l}, \quad \forall l = 1, 2, 3.$$

Theorem 3.19. Let $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{N}_0^3$, and let $\zeta^{\mathbf{n}} \in \mathcal{F}_{t,\mathbb{H}_t}$ be the function (3.25). Then it is a t -harmonic t -regular function, i.e.,

$$\zeta^{\mathbf{n}} \nabla_t = \nabla_t \zeta^{\mathbf{n}} = 0, \text{ and } \Delta_t \zeta^{\mathbf{n}} = 0, \text{ on } \mathbb{H}_t.$$

Proof. Theorem 3.19 follows from Theorem 3.14. See [8] for details. \square

By Theorem 3.19, all functions $\{\zeta^{\mathbf{n}} : \mathbf{n} \in \mathbb{N}_0^3\}$ of (3.25) are t -harmonic t -regular functions on \mathbb{H}_t . We will now see that they also generate the space of t -regular functions in this case.

In [8] the authors have proven the following:

Theorem 3.20. *Let $t \neq 0$ in \mathbb{R} , and $f \in \mathcal{F}_{t,U}$, where $0 = 0 + 0i + 0j_t + 0k_t \in U$ in \mathbb{H}_t . If f is \mathbb{R} -analytic on U , then*

f is left t -regular on U ,

if and only if

$$f = f(0) + \sum_{\mathbf{n} \in \mathbb{N}^3} \zeta^{\mathbf{n}} f_{\mathbf{n}}, \quad (3.26)$$

with

$$f_{\mathbf{n}} = \frac{1}{\mathbf{n}!} \frac{\partial^{n_1+n_2+n_3} f}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}}(0), \quad \forall \mathbf{n} \in \mathbb{N}^3.$$

Proof. See [8] for details. \square

Note that if $t < 0$, then the \mathbb{R} -analyticity of a t -regular function $f \in \mathcal{F}_{t,U}$ is automatically guaranteed by (3.26). Thus, if $t < 0$ in \mathbb{R} then, without the \mathbb{R} -analyticity assumption of f , the above characterization (3.26) holds. However, if $t \geq 0$, then the \mathbb{R} -analyticity assumption is needed in (3.26). See e.g., [8].

Remark 3.21. Please note that for the Fueter operator in the case of split quaternions other notations exist in literature and we write the equivalence of notation in this table:

| | This paper | Notation in paper [17] |
|--|---|---|
| Variables | x_3 | x_1 |
| | x_1 | x_3 |
| Hypercomplex basis | i | i |
| | j_1 | k_2 |
| | k_1 | k_1 |
| Differential operator | ∇_1 | $\nabla_{\mathbb{R}}^+$ |
| | $\frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} - j_1 \frac{\partial}{\partial x_2} -$ | $\frac{\partial}{\partial x_0} - k_1 \frac{\partial}{\partial x_1} - k_2 \frac{\partial}{\partial x_2} +$ |
| | $k_1 \frac{\partial f}{\partial x_3}$ | $i \frac{\partial f}{\partial x_3}$ |
| Fueter variables (3.5)–(3.7), p. 332 of [17] | $\zeta_{1,1}(q_1) = x_1 - x_0 i$ | $\zeta_1(x) = x_1 + x_0 k_1$ |
| | $\zeta_{2,1}(q_1) = x_2 + x_0 j_1$ | $\zeta_2(x) = x_2 + x_0 k_2$ |
| | $\zeta_{3,1}(q_1) = x_3 + x_0 k_1$ | $\zeta_3(x) = x_3 - x_0 i$ |

In general, other linear changes of variables can be considered for the setting of Fueter variables, for example in [12].

4. Hardy Space and Interpolation for the First Conjugate \otimes

4.1. Prelude

The Hardy space $\mathbf{H}_2(\mathbb{D})$ is the reproducing kernel Hilbert space of functions analytic in the open unit disk and with reproducing $\frac{1}{1-z\bar{w}}$. It is also the unique (up to a positive multiplicative factor in the inner product) Hilbert space of functions analytic in \mathbb{D} such that $M_z^* = R_0$, where M_z denotes the operator of multiplication by z and R_0 is the backward-shift operator. Important issues pertaining to the classical theory include the Beurling-Lax theorem, Schur multipliers and related interpolation problems. In the present section we first study the counterparts of this space in the setting of the rings \mathcal{H}_2^t . For $t = -1$ this was done in [10]. We mention that a difference occurred there between the scalar and matrix-valued case in view of the lack of commutativity; see [11, p. 1767 (and in particular (62.38))]. The counterpart of the Hardy space is now the following real Hilbert space:

Definition 4.1. We denote by \mathbf{H}_2^t the space of power series of the form

$$f(q_t) = \sum_{n=0}^{\infty} q_t^n \alpha_n \quad (4.1)$$

where $\alpha_0, \alpha_1, \dots \in \mathbb{H}_t$ and such that

$$\sum_{n=0}^{\infty} \|\alpha_n\|_E^2 < \infty \quad (4.2)$$

where the scaled norm (2.5) is used in (4.2).

Note that (4.2) is induced by the inner-product

$$[f, g]_E = 2 \sum_{n=0}^{\infty} \operatorname{Re} (a_n \bar{c}_n + |t| b_n \bar{d}_n)$$

where

$$\alpha_n = a_n + b_n j_t, \quad n = 0, 1, \dots$$

and with $g(q_t) = \sum_{n=0}^{\infty} q_t^n \gamma_n$,

$$\gamma_n = c_n + d_n j_t, \quad n = 0, 1, \dots$$

We also note that, under (4.2), the power series converges for $\|q_t\| < 1$, where $\|\cdot\|$ is the operator norm. The case $t = -1$, corresponding to the quaternionic setting, was first studied in [10]. In that work and related ones, one views the Hardy space as a quaternionic Hilbert space. This is not possible for general t since \mathbb{H}_t will not be a skew field in general. We will endow \mathbf{H}_2^t with two real symmetric forms, namely

$$[f, g]_{\otimes} = \sum_{n=0}^{\infty} [g_n, f_n]_{\otimes}, \quad (4.3)$$

and

$$[f, g]_{[*]} = \sum_{n=0}^{\infty} [g_n, f_n]_{[*]} \quad (4.4)$$

where $f(q_t) = \sum_{n=0}^{\infty} q_t^n f_n$ and $g(q_t) = \sum_{n=0}^{\infty} q_t^n g_n$ are elements of \mathbf{H}_2^t .

Theorem 4.2. *The space \mathbf{H}_2^t endowed with either of the form (4.3) or (4.4) is a Krein space. In case of (4.3) it is a Hilbert space for $t < 0$. When using the form (4.3) we denote this space by $\mathbf{H}_{2,\otimes}^t$, and by $\mathbf{H}_{2,[*]}^t$ when using (4.4).*

Proof. The claim follows from the corresponding structure of the coefficient space \mathbb{H}_t from Theorems 2.14 and 2.20 endowed with the forms

$$[p_t, q_t]_{\otimes} = \text{Tr} (q_t^{\otimes} p_t) = a\bar{c} + \bar{a}c - t(b\bar{d} + \bar{b}d)$$

and

$$[p_t, q_t]_{[*]} = \text{Tr} (q_t^{[*]} p_t) = ac + \bar{a}\bar{c} - t(bd + \bar{b}\bar{d}),$$

where

$$q_t = \begin{pmatrix} a & tb \\ \bar{b} & \bar{a} \end{pmatrix}, \quad p_t = \begin{pmatrix} c & td \\ \bar{d} & \bar{c} \end{pmatrix}.$$

□

Definition 4.3. We denote by $\mathbf{H}_{2,\otimes}$ the Hardy space \mathbf{H}_2^t endowed with the form (4.3) and by $\mathbf{H}_{2,[*]}^t$ the Hardy space \mathbf{H}_2^t endowed with the form (4.4), which, by Theorem 4.2, become Hilbert and Krein spaces, respectively, depending on sign t .

These Hilbert and Krein spaces are studied in Sects. 4.2 and 5 respectively.

In several complex variables, one considers in Schur analysis the Arveson space rather than the Hardy space of the ball. In the last part of the present section we present the analog of the Arveson space for the μ_t variables. In the cases $t = 1$ this was first done in [19].

4.2. The Hardy Space in One Variable: Case 1

In this subsection, we consider the Hardy space $\mathbf{H}_{2,\otimes}^t$ when the adjoint \otimes is considered,

$$q_t^{\otimes} = \begin{pmatrix} \bar{a} & -tb \\ -\bar{b} & a \end{pmatrix}. \quad (4.5)$$

We recall that \mathcal{H}_2^t endowed with the inner product (2.16)

$$[p_t, q_t]_{\otimes} = \text{Tr} (q_t^{\otimes} p_t)$$

is a real Pontryagin space of dimension 4 and index of negativity 2, in the case $t > 0$ and it is a Hilbert space when $t < 0$; see Lemma 2.14.

In what follows we will use the spaces \mathcal{H}_2^t and \mathbb{H}_t interchangeably, for example the matrix I_2 in the first becomes the number 1 in the second.

Theorem 4.4. $\mathbf{H}_{2,\otimes}^t$ is a reproducing kernel Krein right \mathcal{H}_2^t -module (or an \mathbb{H}_t -module) with reproducing kernel

$$K(q_t, p_t) = \sum_{n=0}^{\infty} q_t^n p_t^{\otimes n} \quad (4.6)$$

when endowed with the form (we use the same notation as for \mathcal{H}_2^t ; see (4.5))

$$[f, g]_{\otimes} = \sum_{n=0}^{\infty} \delta_n^{\otimes} \gamma_n \quad (4.7)$$

with $g(q_t) = \sum_{n=0}^{\infty} q_t^n \delta_n$, and associated inner product (indefinite when $t > 0$ and definite when $t < 0$)

$$\mathrm{Tr} [f, g]_{\otimes} = \sum_{n=0}^{\infty} \mathrm{Tr} \delta_n^{\otimes} \gamma_n \quad (4.8)$$

Proof. The space of sequences $(\gamma_n)_{n \in \mathbb{N}_0}$ of elements of \mathcal{H}_2^t such that (4.2) is in force is a real Hilbert space, and so is $\mathbf{H}_{2, \otimes}^t$ by transfer of structure. The norm (4.2) corresponds to the inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \mathrm{Tr} \delta_n^* \gamma_n$$

where $g(q_t) = \sum_{n=0}^{\infty} q_t^n \delta_n$. To check the reproducing kernel property, we write, with f as in (4.1) and the kernel defined by (4.6):

$$\mathrm{Tr} [f(\cdot), K(\cdot, p_t) \alpha]_{\otimes} = \mathrm{Tr} \left(\sum_{n=0}^{\infty} \alpha^{\otimes} p_t^n \gamma_n \right) = \mathrm{Tr} \alpha^{\otimes} f(p_t).$$

□

Theorem 4.5. *The operator M_{q_t} of star-multiplication on the left by q_t is a continuous Krein space isometry, with adjoint the backward-shift operator*

$$(M_{q_t}^* f)(q_t) = \sum_{n=1}^{\infty} q_t^{n-1} \alpha_n, \quad \text{with} \quad f(q_t) = \sum_{n=0}^{\infty} q_t^n \alpha_n. \quad (4.9)$$

We also have that $M_{q_t}^* M_{q_t} f = f$ for $f \in \mathbf{H}_{2, \otimes}^t$.

Proof. We have with f and g as above,

$$\begin{aligned} [M_{q_t} f, g]_{\otimes} &= \mathrm{Tr} \left(\sum_{n=0}^{\infty} \delta_{n+1}^{\otimes} \gamma_n \right) \\ &= [f, M_{q_t}^* g]_{\otimes}. \end{aligned}$$

□

4.3. Interpolation in the Complex Case

In the following subsections we begin a study of interpolation in the spaces $\mathbf{H}_{2, \otimes}^t$ and $\mathbf{H}_{2, [*]}^t$. In the classical complex scalar setting, the most simple problem would be:

Given N different points a_1, \dots, a_N in the open unit disk, describe the set of functions f in the Hardy space such that

$$f(a_j) = 0, \quad j = 1, \dots, N. \quad (4.10)$$

It is a simple exercise, and a particular instance of Beurling's theorem, that the set of solutions of this problem consists of the functions of the form $f(z) = b(z)g(z)$ where $b(z) = \prod_{n=1}^N \frac{z - a_n}{1 - \bar{z} \bar{a}_n}$ and g runs through the Hardy

space. To sharpen the difference between the following subsection and the following two ones we present the following well-known result from J -theory (see [41, Theorem 1, p. 145, p. 146 line 5 and Theorem 2, p. 146], [31, 46] and [2, Lemma 4.1.7, p. 80] for more information).

Lemma 4.6. *Let $E_1, E_2 \in \mathbb{C}^{u \times v}$ be strict contractions. Then, the matrices*

$$(I_u - E_1 E_1^*)^{-1/2} (E_2 + E_1) (I_v + E_1^* E_2)^{-1} (I_v - E_1^* E_1)^{1/2} \quad (4.11)$$

and

$$(I_u - E_1 E_1^*)^{1/2} (I_v + E_2 E_1^*) (E_2 + E_1) (I_v - E_1^* E_1)^{-1/2} \quad (4.12)$$

are also strictly contractive.

Assume that $u = v$ and $E_1 E_1^* = c I_u$ for some real number $c > 0$. Then, $E_1^* E_1 = c I_u$ and

$$(E_2 + E_1) (I_v + E_1^* E_2)^{-1} \quad (4.13)$$

and

$$(I_v + E_1^* E_2)^{-1} (E_2 + E_1) \quad (4.14)$$

are strictly contractive.

When $u = 1$ and replacing $E_1 = -a \in \mathbb{C}^{1 \times v}$ and $E_2 = z \in \mathbb{C}^{1 \times v}$ such that $zz^* < 1$ we get the counterpart of the Blaschke product in the unit ball of \mathbb{C}^v ,

$$b_a(z) = \sqrt{1 - aa^*} (z - a) (I_v - a^* z)^{-1} (I_v - a^* a)^{-1/2}$$

which satisfies the formula (with $w \in \mathbb{C}^{1 \times v}$ such that $ww^* < 1$)

$$\frac{1 - aa^*}{(1 - za^*)(1 - aw^*)} = \frac{1 - b_a(z)b_a(w)^*}{1 - zw^*} \quad (4.15)$$

see [42].

We remark that Blaschke factors, and more generally Blaschke products, are examples of rational functions taking unitary (or co-isometric in the case of \mathbb{C}^d) values on the boundary.

4.4. Blaschke Factor and Interpolation: Case 1

Since $\alpha\alpha^{\otimes}$ is a scalar we can use (4.13) to define the \otimes -Blaschke factor.

Definition 4.7. Assume $\|\alpha\| < 1$. We define the \otimes -Blaschke factor with a zero at α by:

$$b_\alpha(q_t) = (q_t - \alpha) \star (1 - q_t \alpha^{\otimes})^{-\star}, \quad (4.16)$$

where $S^{-\star}$ represents the \star inverse of S .

Lemma 4.8. Let $\alpha \in \mathcal{H}_2^t$ of norm strictly less than 1. Then,

$$b_\alpha(q_t) = -\alpha + \sum_{n=1}^{\infty} q_t^n (I_2 - \alpha\alpha^{\otimes}) \alpha^{\otimes(n-1)} \quad (4.17)$$

where the convergence is in $\mathbb{C}^{2 \times 2}$.

Proof. For $q_t = xI_2$ where x is real and of absolute value less than 1 we have

$$\begin{aligned}
 b_\alpha(q_t) &= (x - \alpha) \left(\sum_{n=0}^{\infty} x^n \alpha^{\otimes n} \right) \\
 &= \sum_{n=0}^{\infty} x^{n+1} \alpha^{\otimes n} - \alpha - \sum_{n=1}^{\infty} x^n \alpha \alpha^{\otimes n} \\
 &= -\alpha + \sum_{n=1}^{\infty} x^n (\alpha^{\otimes(n-1)} - \alpha \alpha^{\otimes n}) \\
 &= -\alpha + \sum_{n=1}^{\infty} x^n (I_2 - \alpha \alpha^{\otimes}) \alpha^{\otimes(n-1)} \\
 &= -\alpha + (\det \alpha) \sum_{n=1}^{\infty} x^n \alpha^{\otimes(n-1)}
 \end{aligned}$$

and hence the result by Proposition 2.29. \square

As in the slice-hyperholomorphic setting we have the following new phenomenon:

Lemma 4.9. *Let α of norm strictly less than 1. It holds that:*

$$(b_\alpha \star b_{\alpha^{\otimes}})(q_t) = \frac{q_t^2 - 2q_t \operatorname{Re} \alpha + \det \alpha}{q_t^2 \det \alpha - 2q_t \operatorname{Re} \alpha + 1} \quad (4.18)$$

Proof. We can write

$$\begin{aligned}
 (b_\alpha \star b_{\alpha^{\otimes}})(q_t) &= (q_t - \alpha) \star \underbrace{(1 - q_t \alpha^{\otimes})^{-\star} \star (q_t - \alpha^{\otimes})}_{\star\text{-commute; see Proposition 2.37}} \star (1 - q_t \alpha^{\otimes})^{-\star} \\
 &= (q_t - \alpha) \star (q_t - \alpha^{\otimes}) \star (1 - q_t \alpha^{\otimes})^{-\star} \star (1 - q_t \alpha)^{-\star} \\
 &= (q_t^2 - 2q_t \operatorname{Re} \alpha) + \alpha \alpha^{\otimes} \star (1 - q_t \alpha^{\otimes})^{-\star} \star (1 - q_t \alpha)^{-\star} \\
 &= (q_t^2 - 2q_t \operatorname{Re} \alpha) + \alpha \alpha^{\otimes} \star ((1 - q_t \alpha) \star (1 - q_t \alpha^{\otimes}))^{-\star} \\
 &= (q_t^2 - 2q_t \operatorname{Re} \alpha) + \alpha \alpha^{\otimes} \star (q_t^2 \det \alpha - 2q_t \operatorname{Re} \alpha + 1)^{-\star},
 \end{aligned}$$

and hence the result since the elements of the quotients have real coefficients, and hence the *star*-product can be replaced by the usual product. \square

In view of the following theorem note that there is a technical difficulty: a densely defined isometry in a Krein (as opposed to Hilbert) space need not be continuous, and so it is not enough to check isometry on a dense set.

Theorem 4.10. *The operator M_{b_α} of \star -multiplication by b_α on the left is a continuous isometry from the Krein space $\mathbf{H}_{2,\otimes}$ into itself.*

Proof. We divide the proof into steps.

STEP 1: M_{b_α} is an isometry on a dense set.

Indeed, since

$$(M_{b_\alpha} q_t^u \gamma)(M_{b_\alpha} q_t^v \delta)^{\otimes} = \delta^{\otimes} [M_{b_\alpha} q_t^u \gamma, M_{b_\alpha} q_t^v]_{\otimes} \gamma$$

we can assume $\gamma = \delta = I_2$. We first assume $u = v$. We have

$$\begin{aligned}
& [M_{b_\alpha} q_t^u, M_{b_\alpha} q_t^u]_{\otimes} \\
&= \left[-q_t^u \alpha + \sum_{n=1}^{\infty} q_t^{n+u} (I_2 - \alpha \alpha^{\otimes}) \alpha^{\otimes(n-1)}, -q_t^u \alpha + \sum_{m=1}^{\infty} q_t^{m+u} (I_2 - \alpha \alpha^{\otimes}) \alpha^{\otimes(m-1)} \right]_{\otimes} \\
&= \alpha^{\otimes} \alpha + \sum_{n=1}^{\infty} \alpha^{n-1} (I_2 - \alpha \alpha^{\otimes})^2 \alpha^{\otimes(n-1)} \\
&= \alpha^{\otimes} \alpha + (I_2 - \alpha \alpha^{\otimes})^2 (I_2 - \alpha \alpha^{\otimes})^{-1} \\
&= I_2,
\end{aligned}$$

where we have used that $\alpha \alpha^{\otimes} = \alpha^{\otimes} \alpha$ to compute

$$\sum_{n=1}^{\infty} \alpha^{n-1} \alpha^{\otimes(n-1)} = (I_2 - \alpha \alpha^{\otimes})^{-1}$$

for $\|\alpha\| < 1$ and the fact that $[q_t^{n+u}, q_t^{m+u}]_{\otimes} = 0$ if $m \neq n$.

We now assume $u \neq v$. We now have $[q_t^{n+u}, q_t^{m+v}]_{\otimes} = 0$ unless $n + u = m + v$. Without loss of generality we will assume $u > v$, so that $m = n + (u - v) > n$ and write:

$$\begin{aligned}
& [M_{b_\alpha} q_t^u, M_{b_\alpha} q_t^v]_{\otimes} \\
&= \left[-q_t^u \alpha + \sum_{n=1}^{\infty} q_t^{n+u} (I_2 - \alpha \alpha^{\otimes}) \alpha^{\otimes(n-1)}, -q_t^v \alpha + \sum_{m=1}^{\infty} q_t^{m+v} (I_2 - \alpha \alpha^{\otimes}) \alpha^{\otimes(m-1)} \right]_{\otimes} \\
&= \underbrace{\left[-q_t^u \alpha, \sum_{m=1}^{\infty} q_t^{m+v} (I_2 - \alpha \alpha^{\otimes}) \alpha^{\otimes(m-1)} \right]_{\otimes}}_{\text{only } m=u-v \text{ contributes}} + \underbrace{\left[\sum_{n=1}^{\infty} q_t^{n+u} (I_2 - \alpha \alpha^{\otimes}) \alpha^{\otimes(n-1)}, -q_t^v \alpha \right]_{\otimes}}_{=0 \text{ since } n+u > v} \\
&\quad + \left[\sum_{n=1}^{\infty} q_t^{n+u} (I_2 - \alpha \alpha^{\otimes}) \alpha^{\otimes(n-1)}, \sum_{n=1}^{\infty} q_t^{n+u} (I_2 - \alpha \alpha^{\otimes}) \alpha^{\otimes(n+u-v-1)} \right]_{\otimes} \\
&= -(I_2 - \alpha \alpha^{\otimes}) \alpha^{u-v} + \alpha^{u-v} (I_2 - \alpha \alpha^{\otimes})^2 \left[\sum_{n=1}^{\infty} q_t^{n+u} \alpha^{\otimes(n-1)}, \sum_{n=1}^{\infty} q_t^{n+u} \alpha^{\otimes(n-1)} \right]_{\otimes} \\
&= -(I_2 - \alpha \alpha^{\otimes}) \alpha^{u-v} + (I_2 - \alpha \alpha^{\otimes})^2 \alpha^{u-v} (I_2 - \alpha \alpha^{\otimes})^{-1} \\
&= 0.
\end{aligned}$$

STEP 2: M_{b_α} is continuous. We note that $M_{b_\alpha} = M_{q_t - \alpha} \circ M_{(1 - q_t \alpha^{\otimes})^{-1}}$. The result then follows from the fact that M_{q_t} is continuous of norm equal to 1 and so

$$\|M_{(1 - q_t \alpha^{\otimes})^{-1}}\| \leq \frac{1}{1 - \|\alpha\|} < \infty \quad (4.19)$$

It follows that M_{b_α} has a continuous extension to \mathbf{H}_2^t , which is an isometry since the form (4.3) is continuous with respect to the inner product of \mathbf{H}_2^t . \square

Theorem 4.11. *A function $f \in \mathbf{H}_{2,\otimes}$ vanishes at α if and only if it can be written as $f = b_\alpha \star g$ where $g \in \mathbf{H}_{2,\otimes}$ and, moreover*

$$[f, f]_{\otimes} = [g, g]_{\otimes}$$

Proof. We proceed in a number of steps; the arguments are an adaptation of the classical arguments to the present setting. We denote by \mathfrak{M} the real linear space spanned by the functions $(1 - q_t\alpha)^{-\star}b$, where b varies in \mathcal{H}_2^t .

STEP 1: For any $g \in \mathbf{H}_2^t$ and α such that $\|\alpha\| < 1$ (operator norm), we have

$$(b_\alpha \star g)(\alpha) = 0. \quad (4.20)$$

Indeed, with $h(q_t) = (I_2 - q_t\alpha)^{-\star} \star g(q_t) \stackrel{\text{ref}}{=} \sum_{n=0}^{\infty} q_t t^n h_n$ we can write

$$\begin{aligned} b_\alpha(q_t) \star g(q_t) &= (q_t - \alpha) \star h(q_t) \\ &= \sum_{n=0}^{\infty} ((q_t - \alpha) \star q_t^n) h_n \\ &= \sum_{n=0}^{\infty} (q_t^{n+1} - q_t^n \alpha) h_n \end{aligned}$$

which vanishes for $q_t = \alpha$.

STEP 2: We have $\mathfrak{M} \subset \mathbf{H}_2^t \ominus b_a \star \mathbf{H}_2^t$.

It suffices to verify that \mathfrak{M} and $b_a \star \mathbf{H}_2^t$ are orthogonal. For $f \in \mathbf{H}_2^t$ we have

$$[f(q_t), (1 - q_t\alpha)^{-\star}b]_{\otimes} = \text{Tr } b^{\otimes} f(\alpha) = [f(\alpha), b]_{\otimes}$$

and so the claim follows from Step 1.

STEP 3: We have $\mathfrak{M} \supset \mathbf{H}_2^t \ominus b_a \star \mathbf{H}_2^t$.

We follow the proof of Proposition 3.10 in [9]. Let $g(q_t) = \sum_{n=0}^{\infty} q_t^n g_n \in \mathbf{H}_2^t \ominus b_a \star \mathbf{H}_2^t$. Then for every $k \in \mathbb{N}_0$,

$$[g, b_a \star q_t^k]_{\otimes} = 0.$$

Using (4.17) we can write

$$[g, b_a \star q_t^k]_{\otimes} = -\alpha^{\otimes} g_k + \sum_{j=1}^{\infty} (I_2 - \alpha\alpha^{\otimes}) \alpha^{j-1} g_{k+j}.$$

Hence,

$$\begin{aligned} \alpha^{\otimes} g_k &= \sum_{j=1}^{\infty} (I_2 - \alpha\alpha^{\otimes}) \alpha^{j-1} g_{k+j} \\ &= (I_2 - \alpha\alpha^{\otimes}) g_{k+1} + (I_2 - \alpha\alpha^{\otimes}) \alpha g_{k+2} \\ &\quad + (I_2 - \alpha\alpha^{\otimes}) \alpha^2 g_{k+3} + \cdots, \quad k = 0, 1, \dots \end{aligned} \quad (4.21)$$

We multiply on the left these equalities by α to obtain

$$\begin{aligned} \alpha\alpha^{\otimes} g_k &= \alpha(I_2 - \alpha\alpha^{\otimes}) g_{k+1} + \alpha(I_2 - \alpha\alpha^{\otimes}) \alpha g_{k+2} \\ &\quad + \alpha(I_2 - \alpha\alpha^{\otimes}) \alpha^2 g_{k+3} + \cdots, \quad k = 0, 1, \dots \end{aligned}$$

For $k+1$ instead of k this identity becomes

$$\begin{aligned} \alpha\alpha^{\otimes} g_{k+1} &= \alpha(I_2 - \alpha\alpha^{\otimes}) g_{k+2} + \alpha(I_2 - \alpha\alpha^{\otimes}) \alpha g_{k+3} \\ &\quad + \alpha(I_2 - \alpha\alpha^{\otimes}) \alpha g_{k+4} + \cdots, \quad k = 0, 1, \dots \end{aligned}$$

Subtracting from (4.21)

$$\begin{aligned}\alpha^{\otimes} g_k &= (I_2 - \alpha\alpha^{\otimes})g_{k+1} + (I_2 - \alpha\alpha^{\otimes})\alpha g_{k+2} \\ &\quad + (I_2 - \alpha\alpha^{\otimes})^2 \alpha^2 g_{k+3} + c \dots, \quad k = 0, 1, \dots\end{aligned}$$

we obtain

$$\alpha\alpha^{\otimes} g_{k+1} - \alpha^{\otimes} g_k = -(I_2 - \alpha\alpha^{\otimes})g_{k+1}, \quad k = 0, 1, \dots$$

and so

$$g_{k+1} = \alpha^{\otimes} g_k, \quad k = 0, 1, \dots$$

Thus $g_k = \alpha^{\otimes k} g_0$, and

$$g(q_t) = \sum_{k=0}^{\infty} q_t \alpha^{\otimes k} g_0 = (I_2 - q_t \alpha^{\otimes})^{-\star} g_0 \in \mathfrak{M}.$$

To conclude the proof, we write for $f \in \mathbf{H}_2^t$,

$$f(q_t) = \underbrace{f(q_t) - (1 - q_t \alpha^{\otimes})^{-\star} (1 - \alpha\alpha^{\otimes}) f(\alpha)}_{\text{vanishes at } \alpha} + \underbrace{(1 - q_t \alpha^{\otimes})^{-\star} (1 - \alpha\alpha^{\otimes}) f(\alpha)}_{\in \mathfrak{M}}$$

We have thus $f(q_t) - (1 - q_t \alpha^{\otimes})^{-\star} (1 - \alpha\alpha^{\otimes}) f(\alpha) \in b_a \star \mathbf{H}_2^t$, and the result follows. \square

Remark 4.12. Unfortunately the previous result cannot be used iteratively because of the non-commutativity. If α_1 and $\alpha_2 \in \mathbb{H}_t$ are of small enough norm, then we indeed have $f(q_t) = b_{\alpha_1}(q_1) \star g(q_t)$ when $f(\alpha_1) = 0$. On the other hand the condition $f(\alpha_2) = 0$, i.e.

$$(b_{\alpha_1}(q_t) \star g(q_t))(\alpha_2) = 0$$

does not translate in a way conducive to computations, even when using Lemma 2.27. Indeed, let α_1 and α_2 be in \mathbb{H}_t and small enough. Assume that $f \in \mathbf{H}_{2,\otimes}^t$ is such that $f(\alpha_1) = 0$. Then

$$f(q_t) = b_{\alpha_1}(q_t) \star g(q_t)$$

where $g \in \mathbf{H}_{2,\otimes}^t$. We now require $f(\alpha_2) = 0$. Assuming $b_{\alpha_1}(\alpha_2)$ invertible, we can write

$$f(\alpha_2) = (b_{\alpha_1}(\alpha_2)) g(\alpha_3)$$

where

$$\alpha_3 = (b_{\alpha_1}(\alpha_2))^{-1} \alpha_2 (b_{\alpha_1}(\alpha_2)) g(\alpha_3) \neq \alpha_2$$

in general. A special case occurs when $\alpha_2 = \alpha_1^{\otimes}$. Then (see Proposition 2.37),

$$\begin{aligned}b_{\alpha_1}(q_t) b_{\alpha_1^{\otimes}}(q_t) &= (q_t - \alpha) \star (1 - q_t \alpha^{\otimes})^{-\text{star}} \star (q - \alpha^{ct}) \star (1 - q_t \alpha)^{-\star} \\ &= (q_t - \alpha) \star (q - \alpha^{ct}) \star (1 - q_t \alpha^{\otimes})^{-\text{star}} \star (1 - q_t \alpha)^{-\star} \\ &= (q_t^2 - 2q_t(\text{Re } \alpha) + \alpha\alpha^{\otimes})(q_t(\alpha\alpha^{\otimes}) - 2q_t(\text{Re } \alpha) + 1)^{-1}\end{aligned}$$

and so we have interpolation on the sphere associated to α (see Definition 2.39 for the latter).

In general one needs to proceed differently, in a global way. The following extension of Theorem 4.11 can be seen as a finite dimensional version of the Beurling–Lax theorem in the present setting. First some notations. We define a matrix $\mathbf{G}_\otimes \in \mathbf{H}_t^{N \times N}$ via

$$\mathbf{G}_\otimes = \begin{pmatrix} \sum_{n=0}^{\infty} \alpha_1^n \alpha_1^{\otimes n} & \sum_{n=0}^{\infty} \alpha_1^n \alpha_2^{\otimes n} & \cdots & \sum_{n=0}^{\infty} \alpha_1^n \alpha_N^{\otimes n} \\ \sum_{n=0}^{\infty} \alpha_2^n \alpha_1^{\otimes n} & \sum_{n=0}^{\infty} \alpha_2^n \alpha_2^{\otimes n} & \cdots & \sum_{n=0}^{\infty} \alpha_2^n \alpha_N^{\otimes n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{n=0}^{\infty} \alpha_N^n \alpha_1^{\otimes n} & \sum_{n=0}^{\infty} \alpha_N^n \alpha_2^{\otimes n} & \cdots & \sum_{n=0}^{\infty} \alpha_N^n \alpha_N^{\otimes n} \end{pmatrix}. \quad (4.22)$$

where $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{H}_t$ are the interpolation points, assumed to be in the operator unit ball of \mathbf{H}_t , i.e. such that

$$\|\alpha_j\|_{op} < 1, \quad j = 1, \dots, N. \quad (4.23)$$

Then, see Corollary 2.4, the \otimes -adjoints also are in the operator unit ball, and thus the power series in (5.9) converge.

Theorem 4.13. Assume \mathbf{G}_\otimes invertible in $\mathbb{H}_t^{N \times N}$ and define

$$\Theta(q_t) = 1 - (1 - q_t) \star ((1 - q_t \alpha_1^{\otimes})^{-\star} \cdots (1 - q_t \alpha_N^{\otimes})^{-\star}) \mathbf{G}_\otimes^{-1} \begin{pmatrix} (1 - \alpha_1)^{-1} \\ \vdots \\ (1 - \alpha_N)^{-1} \end{pmatrix}. \quad (4.24)$$

A function $f \in \mathbf{H}_{2,\otimes}$ vanishes at $\alpha_1, \alpha_2, \dots, \alpha_N$ if it can be written as $f = b_\alpha \star g$ where $g \in \mathbf{H}_{2,\otimes}$ and, moreover

$$[f, f]_\otimes = [g, g]_\otimes$$

Before presenting the proof we need some preliminary results. We first note that (4.20) has the following generalization:

Lemma 4.14. Assume that $f \star g$ makes sense and that $f(\alpha) = 0$. Then, $(f \star g)(\alpha) = 0$.

Proof. Let $f(q_t) = \sum_{n=0}^{\infty} q_t^n f_n$ and $g(q_t) = \sum_{m=0}^{\infty} q_t^m g_m$ where the coefficients f_n and g_m belong to \mathbb{H}_t . The claim follows from the formula

$$(f \star g)q_t = \sum_{m=0}^{\infty} q_t^m f(q_t)g_m.$$

□

Proposition 4.15. Assume that the points $\alpha_1, \dots, \alpha_N$ are in the open unit operator ball of \mathbb{H}_t , and let

$$A = \text{diag}(\alpha_1^{\otimes}, \dots, \alpha_N^{\otimes}) \\ C = (1 \ 1 \ \cdots \ 1).$$

The matrix \mathbf{G}_\otimes is the unique solution of the Stein equation

$$\mathbf{G}_\otimes - A^{\otimes} \mathbf{G}_\otimes A = C^{\otimes} C. \quad (4.25)$$

and is given by the formula

$$\mathbf{G}_{\otimes} = \sum_{n=0}^{\infty} A^{\otimes n} C^{\otimes} C A^n. \quad (4.26)$$

Proof. From the hypothesis on the interpolation points we have $\|A\|_{op} < 1$ and the series on the right hand side of (4.26) converges. On the other hand, iterating (4.25) $(M-1)$ -times we get

$$\mathbf{G}_{\otimes} = \sum_{n=1}^M A^{\otimes n} C^{\otimes} C A^n + A^{\otimes(M+1)} \mathbf{G}_{\otimes} A^{(M+1)}$$

The result follows by letting $M \rightarrow \infty$. \square

Proof. We will proceed in steps and follow arguments from [9].

STEP 1: It holds that $\Theta(\alpha_j) = 0$, $j = 1, \dots, N$

Without loss of generality we consider the case $j = 1$. We have that

$$((1 - q_t \alpha_1^{\otimes})^{-\star} \cdots (1 - q_t \alpha_N^{\otimes})^{-\star}) (\alpha_1)$$

is the first row of the matrix G and that

$$(1 - q_t) \star ((1 - q_t \alpha_1^{\otimes})^{-\star} \cdots (1 - q_t \alpha_N^{\otimes})^{-\star}) = (1 - \alpha_1) (g_{11} \ g_{12} \ \dots \ g_{1N}).$$

Thus

$$\Theta(\alpha_1) = 1 - (1 - \alpha_1) (g_{11} \ g_{12} \ \dots \ g_{1N}) \mathbf{G}_{\otimes}^{-1} \begin{pmatrix} (1 - \alpha_1)^{-1} \\ \vdots \\ (1 - \alpha_N)^{-1} \end{pmatrix} = 0$$

STEP 2: It holds that

$$\Theta(q_t) = D + q_t C \star (I_{\mathbb{H}_t^N} - q_t A)^{-\star} B \quad (4.27)$$

where (note that A and C were defined earlier in Proposition 4.15)

$$A = \text{diag}(\alpha_1^{\otimes}, \dots, \alpha_N^{\otimes}) \quad (4.28)$$

$$B = (I_{\mathbb{H}_t^N} - A) \mathbf{G}_{\otimes}^{-1} (I_{\mathbb{H}_t^N} - A^{\otimes})^{-1} C^{\otimes}. \quad (4.29)$$

$$C = (1 \ 1 \ \dots \ 1) \quad (4.30)$$

$$D = 1 - C \mathbf{G}_{\otimes}^{-1} (I_{\mathbb{H}_t^N} - A^{\otimes})^{-1} C^{\otimes}. \quad (4.31)$$

The computations presented in the next step are used later, see Example 7.8, but we need at this stage equality (4.34) in Step 4.

STEP 3: The following three equalities hold:

$$A^{\otimes} \mathbf{G}_{\otimes} A + C^{\otimes} C = \mathbf{G}_{\otimes}, \quad (4.32)$$

$$B^{\otimes} \mathbf{G}_{\otimes} A + D^{\otimes} C = 0, \quad (4.33)$$

$$B^{\otimes} \mathbf{G}_{\otimes} B + D^{\otimes} D = I. \quad (4.34)$$

Indeed, the first equality is the Stein equation (4.25). We now check (4.33) and write

$$\begin{aligned}
 B^{\otimes} \mathbf{G}_{\otimes} A + D^{\otimes} C &= C(I_N - A)^{-1} \mathbf{G}_{\otimes}^{-1} (I_N - A^{\otimes}) \mathbf{G}_{\otimes} A \\
 &\quad + (1 - C(I - A)^{-1} \mathbf{G}_{\otimes}^{-1} C^{\otimes}) C \\
 &= C(I_N - A)^{-1} \mathbf{G}_{\otimes}^{-1} \left\{ (I_N - A^{\otimes}) \mathbf{G}_{\otimes} A + \mathbf{G}_{\otimes} (I_N - A) - C^{\otimes} C \right\} \\
 &= C(I_N - A)^{-1} \mathbf{G}_{\otimes}^{-1} \left\{ -A^{\otimes} \mathbf{G}_{\otimes} A + \mathbf{G}_{\otimes} - C^{\otimes} C \right\} \\
 &= 0
 \end{aligned}$$

thanks to the Stein equation (4.25). Finally to check (4.34) we write

$$\begin{aligned}
 B^{\otimes} \mathbf{G}_{\otimes} B + D^{\otimes} D &= C(I_N - A)^{-1} \mathbf{G}_{\otimes}^{-1} (I_N - A^{\otimes}) \mathbf{G}_{\otimes} \\
 &\quad (I_{(\mathbf{H}_2^t)^N} - A) \mathbf{G}_{\otimes}^{-1} (I_{(\mathbf{H}_2^t)^N} - A^{\otimes})^{-1} C^{\otimes} \\
 &\quad + (1 - C(I - A)^{-1} \mathbf{G}_{\otimes}^{-1} C^{\otimes}) (1 - C \mathbf{G}_{\otimes}^{-1} (I_{(\mathbf{H}_2^t)^N} - A^{\otimes})^{-1} C^{\otimes}) \\
 &= 1 + C(I_N - A)^{-1} \mathbf{G}_{\otimes}^{-1} \left\{ (I_N - A^{\otimes}) \mathbf{G}_{\otimes} (I_{(\mathbf{H}_2^t)^N} - A) + \right. \\
 &\quad \left. - (I_N - A^{\otimes}) \mathbf{G}_{\otimes} - \mathbf{G}_{\otimes} (I_N - A) + C^{\otimes} C \right\} \mathbf{G}_{\otimes}^{-1} \\
 &\quad (I_{(\mathbf{H}_2^t)^N} - A^{\otimes})^{-1} C^{\otimes} \\
 &= 1 + C(I_N - A)^{-1} \mathbf{G}_{\otimes}^{-1} \left\{ -\mathbf{G}_{\otimes} + A^{\otimes} \mathbf{G}_{\otimes} A + C^{\otimes} C \right\} \\
 &\quad \times \mathbf{G}_{\otimes}^{-1} (I_{(\mathbf{H}_2^t)^N} - A^{\otimes})^{-1} C^{\otimes} \\
 &= 0,
 \end{aligned}$$

here too thanks to the Stein equation (4.25).

This follows from (4.19) in the second step of Theorem 4.10 applied to each of the α_j .

STEP 4: The operator M_{Θ} of \star -multiplication on the left by Θ is an isometry from $\mathbf{H}_{2,\otimes}$ into itself.

The fact that M_{Θ} is bounded follows from (4.19) in the second step of Theorem 4.10 applied to each of the α_j . We now prove that M_{Θ} is isometric on polynomials. The continuity of M_{Θ} implies then the isometry property on the whole Hardy space. We first remark that $\Theta = \sum_{n=0}^{\infty} q_t^n \theta_n$ with

$$\theta_n = \begin{cases} D, & n = 0, \\ CA^{n-1}B, & n = 1, 2, \dots \end{cases} \quad (4.35)$$

So

$$\begin{aligned}
 \theta_0^{\otimes} \theta_0 + \sum_{n=1}^{\infty} \theta_n^{\otimes} \theta_n &= D^{\otimes} D + B^{\otimes} \left(\sum_{n=0}^{\infty} A^{\otimes n} C^{\otimes} C A^n \right) B \\
 &= D^{\otimes} B + B^{\otimes} \mathbf{G}_{\otimes} B \\
 &= 1,
 \end{aligned} \quad (4.36)$$

where we have used (4.34) and the formula (4.26) for \mathbf{G}_{\otimes} , and for $k > 0$,

$$\begin{aligned}
 \theta_0^{\otimes} \theta_k + \sum_{n=1}^{\infty} \theta_n^{\otimes} \theta_{n+k} &= D^{\otimes} C A^{k-1} B + B^{\otimes} \left(\sum_{n=0}^{\infty} A^{\otimes n} C^{\otimes} C A^n \right) A^{k-1} B \\
 &= (D^{\otimes} C + B^{\otimes} \mathbf{G}_{\otimes}) A^{k-1} B
 \end{aligned}$$

$$= 0, \quad (4.37)$$

where we have used (4.33) and here too the formula (4.26) for \mathbf{G}_{\otimes} .

We follow STEP 1 in the proof of Theorem 4.10 and compute $[M_{\Theta} q_t^u \gamma, M_{\Theta} q_t v \delta]_{\otimes}$ for $\gamma = \delta = 1$. We may, and shall, assume that $u \geq v$ and set $k = u - v$. We have

$$\begin{aligned} [M_{\Theta} q_t^u, M_{\Theta} q_t^v] &= [M_{\Theta} 1, M_{\Theta} q_t^v]_{\otimes} \\ &= \left[\sum_{n=0}^{\infty} q_t^n \theta_n, \sum_{n=0}^{\infty} q_t^{n+k} \theta_n \right]_{\otimes} \\ &= \theta_0^{\otimes} \theta_k + \sum_{n=1}^{\infty} \theta_n^{\otimes} \theta_{n+k} \\ &= \delta_{k,0} \end{aligned}$$

by the previous step, and hence the result by linearity and continuity.

STEP 5: Let \mathfrak{M} be the real linear span of the functions $(1 - q_t \alpha_j)^{-*} b$, where $j = 1, \dots, N$ and $b \in \mathbb{H}_t$. Then

$$M_{\Theta} \mathbf{H}_{2,\otimes}^t \subset \mathfrak{M}^{\perp_{\otimes}}, \quad (4.38)$$

where \perp_{\otimes} is the Krein orthogonal with respect to $[\cdot, \cdot]_{\otimes}$

This follows from Step 1. \square

The above result is a finite dimensional Beurling–Lax type theorem in the present setting.

5. Hardy Space and Interpolation for the Second Conjugate $[*]$

5.1. The Hardy Space in One Variable: Case 2

We recall from Sect. 4.1 that the Hardy space $\mathbf{H}_2(\mathbb{D})$ is the reproducing kernel Hilbert space of functions analytic in the open unit disk and with reproducing $\frac{1}{1-z\bar{w}}$. We also recall that the counterpart of the Hardy space is now the following real Hilbert space \mathbf{H}_2^t the space of power series of the form

$$f(q_t) = \sum_{n=0}^{\infty} q_t^n \alpha_n$$

as in (4.2).

We now focus on \mathbf{H}_2^t with the real $[*]$ symmetric form (4.4), namely

$$[f, g]_{[*]} = \sum_{n=0}^{\infty} [g_n, f_n]_{[*]}$$

where $g(q_t) = \sum_{n=0}^{\infty} q_t^n g_n$ is another element of \mathbf{H}_2^t and we recall that Theorem 4.2 for this case states:

Theorem 5.1. *The space \mathbf{H}_2^t endowed with the form (4.4) is a Krein space. This space was denote by $\mathbf{H}_{2,[*]}^t$ in Definition 4.3.*

Proposition 5.2. *The space $\mathbf{H}_{2,[*]}^t$ endowed with the real symmetric form (4.4) is a reproducing kernel Krein space right \mathcal{H}_2^t -module with reproducing kernel with reproducing kernel*

$$k(q_t, p_t) = \sum_{n=0}^{\infty} q_t^n (p_t^n)^{[*]}. \quad (5.1)$$

Proof. We write

$$\begin{aligned} [*] &= \left[\sum_{n=0}^{\infty} q_t^n f_n, \sum_{n=0}^{\infty} q_t^n (p_t^n)^{[*]} b \right]_{[*]} \\ &= \sum_{n=0}^{\infty} [f_n, (p_t^n)^{[*]} b]_{[*]} \\ &= \sum_{n=0}^{\infty} \text{Tr} (b^{[*]} p_t^n f_n) \\ &= \text{Tr} b^{[*]} f(p_t) \\ &= [f(p_t), b]_{[*]}. \end{aligned}$$

□

We see that $\mathbf{H}_{2,[*]}^t$ is an \mathbb{H}_t -module. For $t = -1$ the second adjoint gives a new theory for quaternions.

5.2. Star Product and Blaschke Factor

We remind the reader of the precursor to this theory, described in Sect. 4.3. Since $\alpha\alpha^{[*]}$ is not (in general) a scalar matrix, one cannot define Blaschke factors using the formula $(q_t - \alpha) \star (I_2 - q_t\alpha^{[*]})^{-\star}$, but one could think of defining a Blaschke factor here by

$$C_\alpha(q_t) = (1 - \alpha\alpha^{[*]})^{-1/2} \star (q_t - \alpha) \star (1 - q_t\alpha^{[*]})^{-\star} (1 - \alpha^{[*]}\alpha)^{1/2}; \quad (5.2)$$

See (4.11). But this definition does not allow for a counterpart of Theorem 4.11. The appropriate definition is

$$B_\alpha(q_t) = (q_t - \alpha) \star (1 - q_t\Gamma_\alpha\alpha^{[*]}\Gamma_\alpha^{-1})^{-\star} L_\alpha^{1/2}$$

which is introduced (for regular adjoint) in [9] in a different context, and takes its inspiration from the formulas for a Blaschke factor in [14, Section 4]. For $t \neq -1$ Γ_α is not self-adjoint, and need not be invertible not have a squareroot. But we have:

Lemma 5.3. Γ_α is $[*]$ self-adjoint. When

$$\sum_{n=1}^{\infty} \|\alpha\|_{op}^{2n} = \frac{\|\alpha\|_{op}^2}{1 - \|\alpha\|_{op}^2} < 1$$

Γ_α is invertible and there exists an element of \mathbb{H}_t , denoted $\Gamma_\alpha^{1/2}$, which is $[*]$ self-adjoint and whose square is Γ_α .

Proof. We write

$$\Gamma_\alpha = 1 + \underbrace{\sum_{n=1}^{\infty} \alpha^n \alpha^{[*]n}}_{\varepsilon}$$

For $\|\alpha\|_{op} < 1$ we have

$$\|\varepsilon\|_{op} < \frac{\|\alpha\|_{op}^2}{1 - \|\alpha\|_{op}^2} \quad (5.3)$$

we can define $\Gamma_\alpha^{1/2}$ via

$$\Gamma_\alpha^{1/2} = 1 + \sum_{n=1}^{\infty} \gamma_n \alpha^n,$$

where the numbers γ_n are defined via

$$\sqrt{1+t} = 1 + \sum_{n=1}^{\infty} \gamma_n t^n, \quad t \in \mathbb{C} \text{ such that } |t| < 1. \quad (5.4)$$

□

Still following [9] and [14] we define

$$L_\alpha = \Gamma_\alpha - \Gamma_\alpha \alpha^{[*]} \Gamma_\alpha^{-1} \alpha \Gamma_\alpha$$

for α such that Γ_α^{-1} exists.

Lemma 5.4. *L_α satisfies $L_\alpha = L_\alpha^{[*]}$ and the set of α for which L_α is invertible and for which there exists K_α satisfying $K_\alpha = K_\alpha^{[*]}$ and $K_\alpha^2 = L_\alpha$ is open and not empty. Furthermore we have*

$$L_\alpha^{-1} = \alpha^{[*]} \alpha + \Gamma_\alpha^{-1} \quad (5.5)$$

Proof. Let as above $\Gamma_\alpha = 1 + \varepsilon$ with α be such that $\|\varepsilon\|_{op} < 1$. Then, the series

$$\sum_{n=0}^{\infty} (-1)^n \varepsilon^n$$

converges in \mathbb{H}_t and (since the operator norm is submultiplicative)

$$\begin{aligned} \|\Gamma_\alpha^{-1}\|_{op} &\leq \frac{1}{1 - \|\varepsilon\|_{op}} \\ &\leq \frac{1}{1 - \frac{\|\alpha\|_{op}^2}{1 - \|\alpha\|_{op}^2}} \\ &= \frac{1 - \|\alpha\|_{op}^2}{1 - 2\|\alpha\|_{op}^2} < 1 \end{aligned}$$

for α small enough. From

$$L_\alpha = 1 + \underbrace{\Gamma_\alpha - 1 - \Gamma_\alpha \alpha^{[*]} \Gamma_\alpha^{-1} \alpha \Gamma_\alpha}_{\varepsilon_1}$$

and using (5.3), we have

$$\begin{aligned}
 \|\varepsilon_1\|_{op} &\leq \frac{\|\alpha\|_{op}^2}{1 - \|\alpha\|_{op}^2} + \|\Gamma_\alpha\|_{op}^2 \|\Gamma_\alpha^{-1}\|_{op} \|\alpha\|_{op}^2 \\
 &\leq \frac{\|\alpha\|_{op}^2}{1 - \|\alpha\|_{op}^2} + \left(1 + \frac{\|\alpha\|_{op}^2}{1 - \|\alpha\|_{op}^2}\right)^2 \frac{1}{1 - \frac{\|\alpha\|_{op}^2}{1 - \|\alpha\|_{op}^2}} \|\alpha\|_{op}^2 \\
 &\leq \frac{\|\alpha\|_{op}^2}{1 - \|\alpha\|_{op}^2} + \left(\frac{1}{1 - \|\alpha\|_{op}^2}\right)^2 \frac{1 - \|\alpha\|_{op}^2}{1 - 2\|\alpha\|_{op}^2} \|\alpha\|_{op}^2 \\
 &< 1
 \end{aligned}$$

for α small enough. For such α , L_α is also invertible and has a squareroot which is $[*]$ symmetric. The argument is the same as in the preceding lemma. We now check (5.5). We have

$$\begin{aligned}
 (\alpha^{[*]}\alpha + \Gamma_\alpha^{-1})L_\alpha &= \alpha^{[*]}\alpha(\Gamma_\alpha - \Gamma_\alpha\alpha^{[*]}\Gamma_\alpha^{-1}\alpha\Gamma_\alpha) + 1 - \alpha^{[*]}\Gamma_\alpha^{-1}\alpha\Gamma_\alpha \\
 &= 1 + \alpha^{[*]}\alpha\Gamma_\alpha + \alpha^{[*]}(1 - \Gamma_\alpha)\Gamma_\alpha^{-1}\alpha\Gamma_\alpha - \alpha^{[*]}\Gamma_\alpha^{-1}\alpha\Gamma_\alpha \\
 &= 1.
 \end{aligned}$$

To conclude the proof we note that M_{B_α} is continuous for α small enough, as is seen from the formula giving B_α , and so the isometry property extends to all of $H_{2,[*]}^t$ (recall that in a Krein space, a densely defined unitary map need not be continuous). \square

Theorem 5.5. *The operator of \star -multiplication by B_α on the left is isometric from the Krein space $\mathbf{H}_{2,[*]}^t$ into itself.*

Proof. It is enough to check on monomials and the proofs of Steps 1, 2, and 3 are similar to the \otimes case. We refer to the proof of Theorem 4.10 for more details.

STEP 1: *It holds that*

$$B_\alpha(q_t) = -\alpha K_\alpha + \sum_{n=1}^{\infty} q_t^n \alpha^{[*](n-1)} \Gamma_\alpha^{-1} K_\alpha \quad (5.6)$$

We set $q_t = x \in \mathbb{R}$ (small enough) in the definition of B_α and write:

$$\begin{aligned}
 B_\alpha(x) &= (x - \alpha) \left(K_\alpha + \sum_{n=0}^{\infty} x^n \Gamma_\alpha \alpha^{[*]n} \Gamma_\alpha^{-1} K_\alpha \right) \\
 &= -\alpha K_\alpha + \sum_{n=0}^{\infty} x^{n+1} \Gamma_\alpha \alpha^{[*]n} \Gamma_\alpha^{-1} K_\alpha - \underbrace{\sum_{n=0}^{\infty} x^{n+1} \alpha \Gamma_\alpha \alpha^{[*]n} \Gamma_\alpha^{-1} K_\alpha}_{n=0 \text{ corresponds to } xK_\alpha} + xK_\alpha \\
 &= -\alpha K_\alpha + \sum_{n=1}^{\infty} x^n \Gamma_\alpha \alpha^{[*](n-1)} \Gamma_\alpha^{-1} K_\alpha - \sum_{n=1}^{\infty} x^n (\alpha \Gamma_\alpha \alpha^{[*]}) \alpha^{[*](n-1)} \Gamma_\alpha^{-1} K_\alpha \\
 &= -\alpha K_\alpha + \sum_{n=1}^{\infty} x^n \alpha^{[*](n-1)} \Gamma_\alpha^{-1} K_\alpha
 \end{aligned}$$

since Γ_α satisfies the Stein equation

$$\Gamma_\alpha - \alpha \Gamma_\alpha \alpha^{[*]} = 1. \quad (5.7)$$

STEP 2: We have

$$[B_\alpha \star q_t^m c, B_\alpha \star q_t^m d]_{[*]} = [c, d]_{[*]}$$

We have

$$[B_\alpha \star q_t^m c, B_\alpha \star q_t^m d]_{[*]} = [\Delta c, d]_{[*]},$$

where

$$\begin{aligned} \Delta &= K_\alpha \alpha^{[*]} \alpha K_\alpha + K_\alpha \Gamma_\alpha^{-1} \left(\underbrace{\sum_{n=1}^{\infty} \alpha^{(n-1)} \alpha^{[*](n-1)}}_{\Gamma_\alpha} \right) \Gamma_\alpha^{-1} K_\alpha \\ &= K_\alpha \alpha^{[*]} \alpha K_\alpha + K_\alpha \Gamma_\alpha^{-1} \Gamma_\alpha \Gamma_\alpha^{-1} K_\alpha \\ &= K_\alpha \left(\alpha^{[*]} \alpha + \Gamma_\alpha^{-1} \right) K_\alpha \\ &= K_\alpha L_\alpha^{-1} K_\alpha \quad (\text{by (5.5)}) \\ &= 1 \end{aligned}$$

where we have used Lemmma 5.4. Note that $K_\alpha L_\alpha^{-1} K_\alpha = 1$ and $K_\alpha^2 = L_\alpha$ are indeed equivalent since both K_α and L_α are invertible.

STEP 3: We have

$$[B_\alpha \star q_t^u c, B_\alpha \star q_t^v d]_{[*]} = 0, \quad u \neq v.$$

Without loss of generality we will assume $u > v$, In a way similar to the proof of Step 2 we have with $h = u - v$:

$$\begin{aligned} &[B_\alpha \star q_t^{(u-v)} c, B_\alpha \star d]_{[*]} \\ &= \left[-q_t^h \alpha K_\alpha c + \sum_{n=1}^{\infty} q_t^{h+n} \alpha^{[*](n-1)} \Gamma_\alpha^{-1} K_\alpha c, \sum_{n=h}^{\infty} q_t^n \alpha^{[*](n-1)} \Gamma_\alpha^{-1} K_\alpha d \right]_{[*]} \\ &= \left[-q_t^h \alpha K_\alpha c + \sum_{n=1}^{\infty} q_t^{h+n} \alpha^{[*](n-1)} \Gamma_\alpha^{-1} K_\alpha c, \alpha^{[*](h-1)} \Gamma_\alpha^{-1} K_\alpha d \right. \\ &\quad \left. + \sum_{n=0}^{\infty} q_t^{h+n} \alpha^{[*](h+n-1)} \Gamma_\alpha^{-1} K_\alpha d \right]_{[*]} \\ &= [\Delta_h c, d]_{[*]}, \end{aligned}$$

where

$$\begin{aligned}\Delta_h &= -K_\alpha \Gamma_\alpha^{-1} \alpha^h K_\alpha + \sum_{n=0}^{\infty} K_\alpha \Gamma_\alpha^{-1} \underbrace{\alpha^{h+n} \alpha^{[*]n}}_{\alpha^h \Gamma_\alpha} \Gamma_\alpha^{-1} K_\alpha \\ &= -K_\alpha \Gamma_\alpha^{-1} \alpha^h K_\alpha + \sum_{n=0}^{\infty} K_\alpha \Gamma_\alpha^{-1} \alpha^h \Gamma_\alpha \Gamma_\alpha^{-1} K_\alpha \\ &= 0.\end{aligned}$$

□

Theorem 5.6. $f \in \mathbf{H}_{2,[*]}^t$ vanishes at α if and only if $f = B_\alpha \star g$ where $g \in \mathbf{H}_{2,[*]}^t$ satisfies $[f, f]_{[*]} = [g, g]_{[*]}$

Proof. One direction is clear, thanks to (4.20) (see Step 1 in the proof of Theorem 4.11. To prove the converse we proceed in a number of steps, and first define the operator

$$M^a q_t^n b = q_t^n a b, \quad a, b \in \mathbb{H}_t. \quad (5.8)$$

STEP 1: M^a is bounded in $\mathbf{H}_{2,[*]}^t$ and it holds that $(M^a)^{[*]} q_t^n b = q_t^n a^{[*]} b$

M^a is defined everywhere and closed (as is seen using the reproducing kernel property), and so is bounded in the Krein space $\mathbf{H}_{2,[*]}^t$. To compute the adjoint we write

$$\begin{aligned}[*] &= [q_t^n b, M^a(q_t^m c)]_{[*]} \\ &= [q_t^n b, q_t^m a c]_{[*]} \\ &= \delta_{m,n} \operatorname{Tr}((ac)^{[*]} b) \\ &= \delta_{m,n} \operatorname{Tr}(c^{[*]} a^{[*]} b) \\ &= \delta_{m,n} [q_t^n a^{[*]} b, q_t^m b]_{[*]}.\end{aligned}$$

STEP 2: The operator $M_{q_t} - M^a$ has closed range and

$$H_{2,[*]}^t = \operatorname{ran} (M_{q_t} - M^a) \overset{+}{[*]} \operatorname{span} \left\{ (1 - q_t a^{[*]})^{-\star} b, b \in \mathbb{H}_1 \right\}$$

The closed range follows from the reproducing kernel property and the previous step. Let now f orthogonal in the $[*]$ inner product to the range. Then

$$(M_{q_t}^{[*]} - (M^a)^{[*]})f = 0$$

With $f(q_t) = \sum_{n=0}^{\infty} q_t^n f_n$ and since, as is readily verified

$$M_{q_t}^{[*]} q_t^n b \begin{cases} 0, & n = 0, \\ q_t^{n-1} b, & n = 1, 2, \dots \end{cases}$$

we get

$$f_n = a^{[*]} f_{n+1}, \quad n = 0, 1, \dots$$

so that

$$f(q_t) = (1 - q_t a^{[*]})^{-\star} f_0$$

STEP 3: We conclude the proof.

Let $f \in \mathbf{H}_{[2,[*]}^t$ be such that $f(a) = 0$. By the previous step, $f \in \text{ran}(M_{q_t} - M^a)$ so that

$$f(q_t) = (q_t - a) \star g(q_t) = B_\alpha(q_t) \star (1 - q_t \Gamma_\alpha \alpha^{[*]} \Gamma_\alpha^{-1})^\star L_\alpha^{-1/2}$$

which concludes the proof. \square

The same conclusions as in Remark 4.12 hold. One can define the counter-part of the matrix \mathbf{G}_\otimes and Problem 4.13, namely

$$\mathbf{G}_{[*]} = \begin{pmatrix} \sum_{n=0}^{\infty} \alpha_1^n \alpha_1^{[*]n} & \sum_{n=0}^{\infty} \alpha_1^n \alpha_2^{[*]n} & \cdots & \sum_{n=0}^{\infty} \alpha_1^n \alpha_N^{[*]n} \\ \sum_{n=0}^{\infty} \alpha_2^n \alpha_1^{[*]n} & \sum_{n=0}^{\infty} \alpha_2^n \alpha_2^{[*]n} & \cdots & \sum_{n=0}^{\infty} \alpha_2^n \alpha_N^{[*]n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{n=0}^{\infty} \alpha_N^n \alpha_1^{[*]n} & \sum_{n=0}^{\infty} \alpha_N^n \alpha_2^{[*]n} & \cdots & \sum_{n=0}^{\infty} \alpha_N^n \alpha_N^{[*]n} \end{pmatrix}. \quad (5.9)$$

where $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{H}_t$ are the interpolation points, assumed to be in the operator unit ball of \mathbf{H}_t , i.e. such that

$$\|\alpha_j\|_{op} < 1, \quad j = 1, \dots, N. \quad (5.10)$$

6. Arveson Space, Blaschke Factors and Related Topics

6.1. Reproducing Kernel Krein Spaces of V_t -Fueter Series

We are interested in studying power series of the form

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^3} \mu_t^\alpha f_\alpha \quad (6.1)$$

where the $f_\alpha \in \mathbb{H}_t$. Since the μ_t are not defined at $x = 0 \in \mathbb{R}^4$ one first needs to make precise where such power series will be considered. For estimates we will need a submultiplicative norm, and take the operator norm (2.2). We define

$$O_{r,\rho} = \{x \in \mathbb{R}^4; r < |x_1^2 + t(x_2^2 + x_3^2)| \quad \text{and} \quad |x_u| < \rho, \quad u = 0, 1, 2, 3\} \quad (6.2)$$

Lemma 6.1. *In $O_{r,R,\rho}$ we have*

$$\|\mu_t^\alpha(x)\|_{op} \leq M_{r,R,\rho}^{|\alpha|} \quad (6.3)$$

with

$$M_{r,R,\rho} = \rho \left(1 + \frac{3\rho^2}{r} \right) \quad (6.4)$$

Proof. In a similar way to our work [15], we have

$$\begin{aligned}
 \|\mu_t^\alpha(x)\|_{op} &= |x_1|^{\alpha_1} |x_2|^{\alpha_2} |x_3|^{\alpha_3} \left\| \left(1 + \frac{x_0}{\vec{q}_t} \right)^{|\alpha|} \right\|_{op} \\
 &\leq \rho^\alpha \left\| 1 + \frac{x_0}{\vec{q}_t} \right\|_{op}^{|\alpha|} \\
 &= \rho^\alpha \left\| 1 - \frac{x_0 \vec{q}_t}{x_1^2 + t(x_2^2 + x_3^2)} \right\|_{op}^{|\alpha|} \\
 &\leq \rho^\alpha \left(1 + \left\| \frac{x_0 \vec{q}_t}{x_1^2 + t(x_2^2 + x_3^2)} \right\|_{op} \right)^{|\alpha|} \\
 &\leq \rho^{|\alpha|} \left(1 + \frac{3\rho^2}{r} \right)^{|\alpha|}
 \end{aligned}$$

□

One could also consider

$$O_{r,\rho,R} = \{x \in \mathbb{R}^4; r < |x_1^2 + t(x_2^2 + x_3^2)| < R \text{ and } |x_u| < \rho, u = 0, 1, 2, 3\} \quad (6.5)$$

The following is the counterpart of [15, Proposition 5.1].

Proposition 6.2. *Let $(c_\alpha)_{\alpha \in \mathbb{N}_0^3}$ be a family of positive fnumbers such that*

$$\sum_{\substack{\alpha \in \mathbb{N}_0^3 \\ c_\alpha > 0}} \frac{M_{r_1, R_1, \rho_1}^{|\alpha|}}{c_\alpha} < \infty$$

for all r_1, ρ_1 such that

$$0 < r < r_1 \quad \text{and} \quad 0 < \rho_1 < \rho.$$

Then the function

$$k_{\mathbf{c}}(x, y) = \sum_{\substack{\alpha \in \mathbb{N}_0^3 \\ c_\alpha \neq 0}} \frac{\mu_t^\alpha(x)(\mu_t^\alpha(y))^{\otimes}}{c_\alpha} \quad (6.6)$$

converges in $O_{r,\rho}$, and is the reproducing kernel of the reproducing kernel Krein space of power series of the form (6.1) endowed with the symmetric form $[\cdot, \cdot]_{\otimes}$, namely

$$\mathfrak{H}(k_{\mathbf{c}}) = \left\{ f = \sum_{\substack{\alpha \in \mathbb{N}_0^3 \\ c_\alpha \neq 0}} \mu_t^\alpha f_\alpha, \quad f_\alpha \in \mathcal{H}_2^t \mid \sum_{\substack{\alpha \in \mathbb{N}_0^3 \\ c_\alpha \neq 0}} c_\alpha \|f_\alpha\|_{op}^2 < \infty \right\} \quad (6.7)$$

Proof. We leave the proof to the reader, as it is a straightforward generalization of our work [15]. □

Proposition 6.3. *Elements of $\mathfrak{H}(k_c)$ are V_t -regular in*

$$O(k_c) = \left\{ x \in \mathbb{R}^4 \setminus \{(0, 0, 0, 0)\} \mid \sum_{\substack{\alpha \in \mathbb{N}_0^3 \\ c_\alpha \neq 0}} \frac{\|\mu_t^\alpha\|_{op}^2}{c_\alpha} < \infty \right\} \quad (6.8)$$

Proof. See our work [15] for the $t = -1$ case, this follows from Theorem 3.4. \square

Similar considerations hold for power series of the form

$$\sum_{\substack{\alpha \in \mathbb{N}_0^3 \\ c_\alpha \neq 0}} \frac{\mu_t^\alpha(x)(\mu_t^\alpha(y))^{[*]}}{c_\alpha} \quad (6.9)$$

The case $c_\alpha = \alpha!$ will correspond to the counterpart of the Arveson space.

6.2. Blaschke Factor (μ_t Variables)

We now denote the vector μ_t of V_t -Fueter variables by:

$$\mu_t = (\mu_{1,t} \ \mu_{2,t} \ \mu_{3,t}), \quad (6.10)$$

where $\mu_{l,t}$ are defined in Eq. (3.4).

We first remark that for $a \in \mathbb{R}^4$ such that

$$\sum_{j=1}^3 \|\mu_{j,t}(a)\| < 1 \quad (6.11)$$

one can define the squareroots

$$\sqrt{1 - \mu_t(a)\mu_t(a)^{\otimes}} \quad \text{and} \quad (I_{\mathbb{H}_t^3} - \mu_t(a)^{\otimes}\mu_t(a))^{-1/2},$$

where

$$\mu_t(a)^{\otimes} = (\mu_{1,t}^{\otimes} \ \mu_{2,t}^{\otimes} \ \mu_{3,t}^{\otimes})^T,$$

via the power series (5.4). Since

$$\mu_t(a)(\mu_t(a)^{\otimes}\mu_t(a))^n = (\mu_t(a)\mu_t(a)^{\otimes})^n\mu_t(a), \quad n = 1, 2, \dots$$

we furthermore have

$$(1 - \mu_t(a)\mu_t(a)^{\otimes})^{1/2}\mu_t(a)(1_{\mathbb{H}_t^3} - \mu_t(a)^{\otimes}\mu_t(a))^{-1/2} = \mu_t(a). \quad (6.12)$$

Definition 6.4. Assume that $a \in \mathbb{R}^4$ satisfies (6.13). We define

$$b_a(q_t) = \sqrt{1 - \mu_t(a)\mu_t(a)^{\otimes}} \star (1 - \mu_t(q_t)\mu_t(a)^{\otimes})^{-\star} \star (\mu_t(q_t) - \mu_t(a)) \cdot (I_{\mathbb{H}_t^3} - \mu_t(a)^{\otimes}\mu_t(a))^{-1/2}$$

Similarly, for $a \in \mathbb{R}^4$ such that

$$\sum_{j=1}^3 \|\mu_{j,t}(a)\| < 1 \quad (6.13)$$

one can define the square roots

$$\sqrt{1 - \mu_t(a)\mu_t(a)^{[*]}} \quad \text{and} \quad (I_{\mathbb{H}_t^3} - \mu_t(a)^{[*]}\mu_t(a))^{-1/2},$$

where

$$\mu_t(a)^{[*]} = \left(\mu_{1,t}^{[*]} \mu_{2,t}^{[*]} \mu_{3,t}^{[*]} \right)^T,$$

via the power series (5.4). Since

$$\mu_t(a)(\mu_t(a)^{[*]}\mu_t(a))^n = (\mu_t(a)\mu_t(a)^{[*]})^n\mu_t(a), \quad n = 1, 2, \dots$$

we furthermore have

$$(1 - \mu_t(a)\mu_t(a)^{[*]})^{1/2}\mu_t(a)(1_{\mathbb{H}_t^3} - \mu(a)^{[*]}\mu_t(a))^{-1/2} = \mu_t(a). \quad (6.14)$$

Definition 6.5. Assume that $a \in \mathbb{R}^4$ satisfies (6.13). We define

$$\begin{aligned} \beta_a(q_t) = & \sqrt{1 - \mu_t(a)\mu_t(a)^{[*]} \star (1 - \mu_t(q_t)\mu_t(a)^{[*]})^{-\star} \star (\mu_t(q_t) - \mu_t(a))} \\ & \cdot (I_{\mathbb{H}_t^3} - \mu_t(a)^{[*]}\mu_t(a))^{-1/2} \end{aligned}$$

Remark 6.6. It is worth noting that this analysis can be written equivalently, for the corresponding Fueter ζ_t variables, and we have the same type of results as in the previous section, but with

$$\zeta_t(q_t) = (\zeta_{1,t} \ \zeta_{2,t} \ \zeta_{3,t})$$

instead of $\mu_{l,t}$.

7. Rational Functions

In this section we discuss rational functions in the case of the $q_t, \mu_{u,t}$ and $\zeta_{u,t}$ variables. We begin with a prologue, where the main features of the classical case are reviewed.

7.1. Prologue

The easiest way to define a matrix-valued rational function in one or several complex variables is certainly to characterize it by having entries being quotient of polynomials (in the corresponding number of variables). Another approach was developped in the theory of linear systems, Schur analysis and operator theory, and is called *the state space method*; see e.g. [22]. There the emphasis is on the notion of transfer function. These various concepts make sense and a corresponding theory of linear system can be developped, when the complex numbers are replaced by a (possibly non-commutative) ring; see for instance [24, 47, 48]. This provides one motivation for the present section, where we consider the counterpart of rational functions in the setting of the \mathbb{H}_t algebras, both for power series and Fueter-like series. We conclude and remark that rational functions with respect to the \star product will be defined independently of the adjoint, but some of their properties will be fundamentally different depending on the chosen adjoint.

We begin with the following definition, taken from [13, Definition 2.1, p. 60], but of course of a much earlier origin, restricted to \mathbb{H}_t .

Definition 7.1. A rational $(\mathbb{H}_t^{n \times m})$ -valued function of d real variables x_1, \dots, x_d , regular at the origin of \mathbb{R}^d is an element of the minimal subring of the $(\mathbb{H}_t^{n \times m})$ -valued function real analytic in a neighborhood of the origin which contains the constant functions the real variables and is closed under inversion.

Theorem 7.2. (see [13, Theorem 2.2 p 61]) *A $\mathbb{H}_t^{n \times m}$ -valued function real analytic in a neighbourhood of the origin of \mathbb{R}^N is rational if and only if it can be written as*

$$\begin{aligned} f(x_1, \dots, x_d) \\ = D + C(I_{\mathbb{H}_t^N} - x_1 A_1 - x_2 A_2 - \dots - x_d A_d)^{-1}(x_1 B_1 + x_2 B_2 + \dots + x_d B_d), \end{aligned} \quad (7.1)$$

where $N \in \mathbb{N}$, $D \in \mathbb{H}_t^{n \times m}$, $C \in \mathbb{H}_t^{n \times N}$, and $A_j \in \mathbb{H}_t^{N \times N}$, $B_j \in \mathbb{H}_t^{N \times m}$, $j = 1, \dots, d$.

When $N = 1$ one rewrites usually (7.1) as

$$f(x) = D + xC(I_{\mathbb{H}_t^N} - xA)^{-1}B. \quad (7.2)$$

Expressions (7.1) and (7.2) are called realizations of the corresponding function f . When $d > 1$, they correspond to transfer functions of a linear system of the Fornasini-Marchesini type; see [33, 34] for the latter.

In terms of power series expansion (7.1) can be rewritten as, in multi-index notation,

$$f(x) = \sum_{k \in \mathbb{N}_0^d} x^k f_k \quad (7.3)$$

where

$$f_k = \frac{(|k| - 1)!}{k!} C (k_1 A^{k_1 - e_1} k_2 A^{k_2 - e_2} \dots k_d A^{k_d - e_d}) B, \quad (7.4)$$

where e_k , $k = 1, \dots, d$, is the multi-index with all entries equal to 0, besides the k -th one, equal to 1. For $d = 1$ we have

$$f_k = CA^{k-1}B, \quad k = 1, 2, \dots \quad (7.5)$$

In this section, we take $N = 1$ and $N = 3$, and wish to replace in (7.1) $x_1 = x$ by q_t (for $N = 1$) and by the Fueter variables μ and ζ for $N = 3$, and use the \star product rather than the pointwise product. We give equivalent definitions in terms of backward-shift operators and related tools.

7.2. Rational Function: Power Series in q_t

Theorem 7.3. *Let $f(q_t) = \sum_{k=0}^{\infty} q_t^k f_k$ be a converging power series with coefficients $f_n \in \mathbb{H}_t^{n \times m}$. Then, the following are equivalent:*

- (1) *There exist $N \in \mathbb{N}_0$ and matrices $(A, B, C) \in \mathbb{H}_t^{N \times N} \times \mathbb{H}_t^{N \times m} \times \mathbb{H}_t^{n \times N}$ such that, with $D = f(0)$,*

$$f(q_t) = D + q_t \star C \star (I_N - q_t \star A)^{-\star} \star B \quad (7.6)$$

- (2) *With A, B, C as above*

$$f_k = CA^{k-1}B, \quad k = 1, 2, \dots \quad (7.7)$$

- (3) *The real linear span $\mathfrak{M}(f)$ of the functions $R_0^k f b$, when $n = 0, 1, \dots$ and b runs through $\mathbb{H}_t^{N \times 1}$ is a finite dimensional real vector space.*

(4) f is the \star quotient of two matrix-valued polynomials

$$f = P_1 \star P_2^{-\star} = P_3^{-\star} \star P_4$$

such that the denominator polynomials P_2 and P_3 do not vanish at the origin.

(5) The restriction of f to the real line is a rational function with no pole at the origin and with coefficients being block matrices with entries in \mathbb{H}_t .

Proof. We proceed into a number of steps.

STEP 1: (1), (2) and (5) are equivalent.

Assume that (1) holds. Letting $q_t = x \in \mathbb{R}$ we get the power series expansion of $f(x)$ and obtain (7.7). Assume (2), then for real x near the origin, and with $f(0) = D$,

$$f(x) = f(0) + \sum_{k=1}^{\infty} f_k x^k = D + xC(I_N - xA)^{-1}B \quad (7.8)$$

and this proves (5). Assume now (5). By the classical theory it admits a realization of the form (7.8), which determines $f(q_t)$ in a unique way via the power expansion in power of x .

STEP 2: (1) and (3) are equivalent.

Assume that (1) holds. Then

$$R_0^k f b = C \star (I_{\mathbb{H}_t^N} - q_t \star A)^{-\star} A^{k-1} B, \quad k = 1, 2, \dots$$

and so the span in question has dimension at most the span (as a real vector space) of the vectors of the form

$$A^{k-1} B b \quad (7.9)$$

which is finite, and so (3) holds. Assume now (3). Then we take v_1, \dots, v_N a basis of $\mathfrak{M}(f)$, and let

$$M(q_t) = (v_1(q_t) \ v_2(q_t) \ \cdots \ v_N(q_t))$$

Since $\mathfrak{M}(f)$ is R_0 -invariant, there is a matrix $A \in \mathbb{R}^{N \times N}$ such that

$$R_0 M = M A$$

from which we get

$$M(q_t) = q_t \star M(q_t) A + M(0)$$

and so¹ $M(q_t) = M(0) \star (I_{\mathbb{H}_t^N} - q_t A)^{-\star}$. To conclude we note that for every $u \in \mathbb{H}_t^N$ there exists a $v \in \mathbb{H}_t^N$ such that

$$R_0 f u = M(q_t) v$$

and so

$$R_0 f = M(q_t) B$$

for some matrix in $\mathbb{H}_t^{N \times m}$, and hence (1) holds.

¹Since this specific A has real entries we could write $M(0)(I_{\mathbb{H}_t^N} - q_t A)^{-1}$

STEP 3: (3) and (5) are equivalent.

Assume (3). Then the restriction of f to the real axis is a rational function of a real variable, with coefficients matrices with entries in \mathbb{H}_t . It admits thus a realization of the form (7.8), and so is the restriction of a uniquely determined function of the form (7.6). The converse is proved by remarking that $(I_{\mathbb{H}_t^N} - xA)^{-1}$ is the quotient of two polynomials. \square

Definition 7.4. The power series $f(q_t) = \sum_{k=0}^{\infty} q_t^k f_k$ is called rational if it satisfies either of the conditions in the previous theorem.

Using the classical properties of realizations one proves:

Theorem 7.5. *The sum and product of two rational functions of compatible sizes are rational. Assuming its value at the origin invertible the inverse of a rational function is invertible.*

Proof. For the inverse, we need to have $n = m$ and D invertible. Then (see e.g. [21]), with $f(x)$ as in (7.2), we have $\det f(x) \neq 0$ and

$$f(x)^{-1} = D^{-1} - xD^{-1}C(I_{\mathbb{H}_t^N} - xA^{\times})^{-1}BD^{-1}, \quad \text{where } A^{\times} = A - BD^{-1}C.$$

It follows that the function

$$D^{-1} - q_t DC^{-1} \star (I_{\mathbb{H}_t^N} - q_t A^{\times})^{-\star} BD^{-1}$$

is the \star -inverse of $f(q_t)$. Similarly, still relying on the formulas from [21], we have (with matrices of compatible sizes)

$$\begin{aligned} & (D_1 + q_t C_1 \star (I_{\mathbb{H}_t^{N_1}} - q_t A_1)^{-\star} B_1) \star (D_2 + q_t C_2 \star (I_{\mathbb{H}_t^{N_2}} - q_t A_2)^{-\star} B_2) \\ &= D + q_t C \star (I_{\mathbb{H}_t^N} - q_t A)^{-\star} B \end{aligned}$$

with $N = N_1 + N_2$ and

$$\begin{aligned} A &= \begin{pmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{pmatrix} \\ B &= \begin{pmatrix} B_1 D_2 \\ B_2 \end{pmatrix} \\ C &= (C_1 \ D_1 C_2) \\ D &= D_1 D_2. \end{aligned}$$

\square

Theorem 7.6. *A function is rational if and only if it can be obtained after a finite number of \star operations involving polynomials*

Proof. The result is true when restricted to the real line and follows by replacing x by q_t and the pointwise product by the \star product. \square

As we noted above, the notion of rational function is the same for the two adjoints \otimes and $[\ast]$, but some of their properties are completely different, in view of the different properties of the two adjoints. In the present discussion, in the case of \otimes one can go one step further:

Theorem 7.7. *The power series $f(q_t) = \sum_{k=0}^{\infty} q_t^k f_k$ is rational if and only if it is the quotient of a matrix-polynomial and of a real scalar polynomial*

Proof. Assume first that f is $\mathbb{H}_t^{a \times b}$ -valued and rational. So each entry f_{ij} , $i = 1, \dots, a$ and $j = 1, \dots, b$ is rational. Using Corollary 2.35 write

$$\begin{aligned} f_{ij}(q_t) &= q_{ij}(q_t) \star (p_{ij}(q_t))^{-\star} \\ &= \frac{(q_{ij} \star p_{ij}^{\circledast})(q_t)}{(p_{ij} \star p_{ij}^{\circledast})(q_t)}, \end{aligned}$$

The result follows by taking as denominator $P(q_t) = \prod_{i,j} (p_{ij} \star p_{ij}^{\circledast})(q_t)$. Each entry f_{ij} can be rewritten as

$$f_{ij}(q_t) = \frac{q_{ij}(q_t) \star \prod_{\substack{k=1, \dots, a \\ \ell=1, \dots, b \\ k \neq i \\ \ell \neq j}} (p_{k\ell} \star p_{k\ell}^{\circledast})(q_t)}{\underbrace{\hspace{10em}}_{\text{real polynomial}} P(q_t)}$$

□

This characterisation is not possible for $[*]$ since $q_t q_t^{[*]}$ is not, in general, a scalar matrix.

Example 7.8. The function Θ_N defined by (4.24) is rational with a realization (4.27) given by (4.28)–(4.31). Furthermore, this realization is \circledast -unitary with respect to \mathbf{G}_{\circledast} in the sense that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\circledast} \begin{pmatrix} \mathbf{G}_{\circledast} & 0 \\ 0 & I_{\mathbb{H}_t^N} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathbf{G}_{\circledast} & 0 \\ 0 & I_{\mathbb{H}_t^N} \end{pmatrix} \quad (7.10)$$

This follows Step 2 in the proof of Theorem 4.13, since one has to verify equalities (4.33)–(4.34).

We note that (7.11) is equivalent to

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{G}_{\circledast}^{-1} & 0 \\ 0 & I_{\mathbb{H}_t^N} \end{pmatrix} \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\circledast} \right)^{-1} = \begin{pmatrix} \mathbf{G}_{\circledast}^{-1} & 0 \\ 0 & I_{\mathbb{H}_t^N} \end{pmatrix}$$

and so is equivalent to

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{G}_{\circledast}^{-1} & 0 \\ 0 & I_{\mathbb{H}_t^N} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\circledast} = \begin{pmatrix} \mathbf{G}_{\circledast}^{-1} & 0 \\ 0 & I_{\mathbb{H}_t^N} \end{pmatrix}, \quad (7.11)$$

In particular the matrix $\mathbf{G}_{\circledast}^{-1}$ satisfies the Stein equation

$$\mathbf{G}_{\circledast}^{-1} - A \mathbf{G}_{\circledast}^{-1} A^{\circledast} = B B^{\circledast}. \quad (7.12)$$

The following result hints to the fact that the theory of structured rational functions can be extended to the present setting; as was shown by a counterexample in [11, (62.38), p. 1767] one cannot directly force conditions on the boundary, but one has to consider conditions in terms of multiplication operators, or symmetric on the real line replacing the metric boundary conditions by operator-type conditions.

Proposition 7.9. *In the notation of (7.11) define Θ by (4.27),*

$$\Theta(q_t) = D + q_t C \star (I_{\mathbb{H}_t^N} - q_t A)^{-\star} B.$$

Then the operator of \star multiplication by Θ on the right is an isometry from the Hardy space $\mathbf{H}_{2,\otimes}$ into itself.

Proof. The proof is the same as the proof of Step 5 in the proof of Theorem 4.13, where only equality (7.11) is used, and not the specific form of A, B, C, D . \square

Similar considerations hold for the second adjoint. Similar claims hold for the Blaschke factor associated to $[*]$, as we now check, and for the Blaschke factors associated to the Fueter variables; see next subsection for the latter.

Proposition 7.10. *The formula*

$$A = a^{[*]} \quad (7.13)$$

$$B = \Gamma_\alpha^{-1} K_\alpha \quad (7.14)$$

$$C = 1 \quad (7.15)$$

$$D = -\alpha K_\alpha \quad (7.16)$$

provide a realization of B_α , which satisfies moreover

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{[*]} \begin{pmatrix} \Gamma_\alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \Gamma_\alpha & 0 \\ 0 & 1 \end{pmatrix} \quad (7.17)$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \Gamma_\alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{[*]} = \begin{pmatrix} \Gamma_\alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} \quad (7.18)$$

Proof. From (5.6) we have

$$\begin{aligned} B_\alpha(q_t) &= -\alpha K_\alpha + \sum_{n=1}^{\infty} q_t^n \alpha^{[*](n-1)} \Gamma_\alpha^{-1} K_\alpha \\ &= -\alpha K_\alpha + q_t \star (1 - q_t \alpha^{[*]})^{-\star} \Gamma_\alpha^{-1} K_\alpha, \end{aligned}$$

and hence we get the realization (7.13)–(7.16). To check (7.17) amounts to check the following three equalities

$$\alpha \Gamma_\alpha \alpha^{[*]} + 1 = \Gamma_\alpha, \quad (7.19)$$

$$\alpha K_\alpha - \alpha K_\alpha = 0, \quad (7.20)$$

$$K_\alpha \Gamma_\alpha^{-1} K_\alpha + K_\alpha a^{[*]} a K_\alpha = 1. \quad (7.21)$$

The first one is the Stein equation (5.7) and the second one is a tautology. The third one can be rewritten as

$$\Gamma_\alpha^{-1} + a^{[*]} a = K_\alpha^{-2} = L_\alpha^{-1}$$

and is just (5.5). To get (7.18) we rewrite (7.17) as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{[*]} \begin{pmatrix} \Gamma_\alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \Gamma_\alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} = I_{\mathbb{H}_t^4},$$

so that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \Gamma_\alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{[*]} \begin{pmatrix} \Gamma_\alpha & 0 \\ 0 & 1 \end{pmatrix} = I_{\mathbb{H}_t^4},$$

and the result follows. \square

For the above computations in the setting of complex matrices see the arXiv preprint [9, Theorem 3.12].

7.3. Rational Functions for Fueter Variables

We will consider this topic in greater details in a future publication but mention the following: For the μ_t variables, the \star -product is defined by

$$\mu_t^\alpha u \star \mu_t^\beta v = \mu_t^{\alpha+\beta} uv, \quad \alpha, \beta \in \mathbb{N}_0^3, \quad u, v \in \mathbb{H}_t. \quad (7.22)$$

The corresponding rational functions regular at the origin are the expressions of the form

$$f(x) = D + C \star (I_{\mathbb{H}_t^N} - \sum_{k=1}^3 \mu_{k,t} A_k)^{-\star} \star \left(\sum_{k=1}^3 \mu_{k,t} B_k \right). \quad (7.23)$$

With

$$\mu_t = (\mu_{1,t} \ \mu_{2,t} \ \mu_{3,t})$$

and

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix},$$

we rewrite (7.1) as

$$f(x) = D + C \star (I_{\mathbb{H}_t^N} - (\mu_t \otimes I_N) A)^{-1} {}^{-\star} (\mu_t \otimes I_N) B \quad (7.24)$$

which we write as

$$f(x) = D + C \star (I_{\mathbb{H}_t^N} - \mu_t A)^{-1} {}^{-\star} \mu_t B \quad (7.25)$$

Note that this function is not defined at the origin of \mathbb{R}^4 .

For the ζ_t variables, the \star -product is defined by

$$\zeta_t^\alpha u \star \zeta_t^\beta v = \zeta_t^{\alpha+\beta} uv, \quad \alpha, \beta \in \mathbb{N}_0^3, \quad u, v \in \mathbb{H}_t. \quad (7.26)$$

$$f(x) = D + C \star \left(I_{\mathbb{H}_t^N} - \sum_{k=1}^3 \zeta_{k,t} A_k \right)^{-\star} \star \left(\sum_{k=1}^3 \zeta_{k,t} B_k \right). \quad (7.27)$$

The formulas are the same as for the previous subsection, and were developed in [19] for the case $t = -1$.

In both cases the corresponding Blaschke factors (6.4) and (6.5) are rational. More precisely, and using (6.14), we can rewrite $b_a(q_t)$ as (see also [16, (4.4), p. 12])

$$\begin{aligned} b_a(x) = & -\mu_t(a) + \\ & + (1 - \mu_t(a) \mu_t(a)^{\otimes})^{1/2} \star (1 - \mu_t(x) \mu_t(a)^{\otimes})^{-\star} \\ & \star \mu_t(a) (I_{\mathbb{H}_t^3} - \mu(a)_t^{\otimes} \mu_t(a))^{1/2}. \end{aligned} \quad (7.28)$$

The realization is equal to

$$\begin{aligned} A &= \mu_t(a)^{\otimes} \\ B &= (I_{\mathbb{H}_t^3} - \mu_t(a)^{\otimes} \mu_t(a))^{1/2} \\ C &= (1 - \mu_t(a) \mu_t(a)^{\otimes})^{1/2} \\ D &= -\mu_t(a), \end{aligned} \tag{7.29}$$

where, as earlier, the squareroots are defined by the power expansion of $\sqrt{1-t}$ for $a \in \mathbb{R}^4$ such that $\|\mu_t(a) \mu_t(a)^{\otimes}\|_{op} < 1$. We repeat the argument: with $\sqrt{1-t} = 1 + \sum_{n=1}^{\infty} c_n t^n$, where $c_1, c_2, \dots \in \mathbb{R}$ (see also (5.4)), the series

$$1 + \sum_{n=1}^{\infty} c_n (\mu_t(a) \mu_t(a)^{\otimes})^n \quad \text{and} \quad I_{\mathbb{H}_t^3} + \sum_{n=1}^{\infty} c_n \mu_t(a)^{\otimes} (\mu_t(a) \mu_t(a)^{\otimes})^{n-1} \mu_t(a)$$

converge since the operator norm is sub-multiplicative.

The realization is \otimes -unitary in the sense that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\otimes} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\otimes} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = I_{\mathbb{H}_t^4 \times 4}.$$

Equality (6.14) is used to prove this equality.

Remark 7.11. Besides the case just encountered, there are three other cases for the Fueter variables, namely the classical Fueter variables, for the two adjoints, and the new Fueter variables, for the $[*]$ adjoint.

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Appendix A. Pontryagin and Krein Spaces

Having in view the needs for the present work we briefly review the main relevant definitions and results on Pontryagin spaces and Krein spaces. For further information we refer in particular to [20, 23, 30, 39]. Let therefore $(\mathcal{V}, [\cdot, \cdot])$ be a vector space on \mathbb{K} where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, endowed with a symmetric form (when $\mathbb{K} = \mathbb{R}$) or an Hermitian form (when $\mathbb{K} = \mathbb{R}$). The discussion below is done for the complex setting and we choose the convention

$$[av, bw]_{\mathcal{V}} = \bar{b}[v, w]_{\mathcal{V}}a, \quad v, w \in \mathcal{V}, \quad \text{and} \quad a, b \in \mathbb{C}, \quad (\text{A.1})$$

for the anti-linear variable. The real case is directly adapted. One can also consider the case of quaternions, but we prefer to include it in the following section, pertaining to modules.

Definition A.1. The space $(\mathcal{V}, [\cdot, \cdot]_{\mathcal{V}})$ is called a Krein space if it can be written in the direct and orthogonal sum

$$\mathcal{V} = \mathcal{V}_+ \dot{+} \mathcal{V}_- \quad (\text{A.2})$$

where

- (1) The spaces $(\mathcal{V}_+, [\cdot, \cdot]_{\mathcal{V}})$ and $(\mathcal{V}_-, -[\cdot, \cdot]_{\mathcal{V}})$ are Hilbert spaces.
- (2) The sum (A.2) is direct, meaning that $\mathcal{V}_+ \cap \mathcal{V}_- = \{0\}$.
- (3) The sum (A.2) is orthogonal, meaning that

$$[v_+, v_-]_{\mathcal{V}} = 0, \quad \forall v_+ \in \mathcal{V}_+ \quad \text{and} \quad v_- \in \mathcal{V}_-. \quad (\text{A.3})$$

(A.2) is called a *fundamental decomposition* and is not unique, but in the case where one of the component reduces to the zero vector space. Every fundamental decomposition generates a Hilbert space structure on \mathcal{V} via

$$\langle v, w \rangle = [v_+, w_+]_{\mathcal{V}} - [v_-, w_-]_{\mathcal{V}}, \quad (\text{A.4})$$

where $v = v_+ + v_-$ and $w = w_+ + w_-$ are the corresponding fundamental decompositions of $v, w \in \mathcal{V}$.

Proposition A.2. (see [23, p. 102 and Corollary IV.6.3, p. 92]) *The space $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ is a Hilbert space, and norms arising from orthogonal decompositions are equivalent, and define therefore the same topology on \mathcal{V} .*

In particular, Riesz theorem for the representation of bounded functions still holds in Krein spaces and the notion of bounded point evaluation in a Krein space of functions leads to the notion of reproducing kernel Krein space.

When \mathcal{V}_- is finite dimensional the space \mathcal{V} is called a Pontryagin space, and $\dim \mathcal{V}_-$ is its index of negativity.

Example A.3. Let $J \in \mathbb{C}^{n \times n}$ satisfy $J = J^* = J^{-1}$ (a *signature matrix*), and define on \mathbb{C}^n

$$[z, w] = w^* J z, \quad z, w \in \mathbb{C}^n. \quad (\text{A.5})$$

Then, $(\mathbb{C}^n, [\cdot, \cdot]_J)$ is a finite dimensional Pontryagin space with negativity index equal to the number of negative eigenvalues of J . The adjoint $A^{[*]}$ of the matrix $A \in \mathbb{C}^{n \times n}$ with respect to this adjoint is

$$A^{[*]} = J A^* J. \quad (\text{A.6})$$

One should be aware that there may be more than one given Krein (as opposed to Hilbert or Pontryagin) space of functions with a given reproducing kernel; this was proved in Laurent Schwartz paper [45, §13] and another counterexample can be found in [1, §4].

In our work Pontryagin, and Hilbert spaces appear in a natural way as described in Sect. 2 and Krein spaces appear in Sect. 4 and beyond, see, for example, Theorem 4.4.

Appendix B. Pontryagin and Krein Right \mathcal{H}_2^t -Modules

We assume now that \mathcal{V} is a right \mathcal{H}_2^t -module, and now replace (A.1) by

$$[vq, wp]_{\mathcal{V}} = p^{\otimes} [v, w]_{\mathcal{V}} q, \quad v, w \in \mathcal{H}, \quad \text{and} \quad p, q \in \mathcal{H}_2^t, \quad (\text{B.1})$$

or

$$[vq, wp]_{\mathcal{V}} = p^{[*]} [v, w]_{\mathcal{V}} q, \quad v, w \in \mathcal{H}, \quad \text{and} \quad p, q \in \mathcal{H}_2^t, \quad (\text{B.2})$$

depending on the chosen adjoint.

Definition B.1. Let \mathcal{K} be a vector space which is also a right \mathcal{H}_2^t -module, and let $[\cdot, \cdot]_{\mathcal{K}}$ be a \mathcal{H}_2^t -valued form which satisfies (B.1) or (B.2). Then, \mathcal{K} is called a \mathcal{H}_2^t module if it is moreover a real Krein space when endowed with the symmetric form

$$\langle v, w \rangle = \text{Tr} \underbrace{[v, w]_{\mathcal{K}}}_{\in \mathcal{H}_2^t}.$$

Definition B.2. In the notation of the previous definition, assume that \mathcal{K} is a space of \mathcal{H}_2^t -valued functions defined on a set Ω . Then, \mathcal{K} is a reproducing kernel Krein space if there exists a function $K(z, w)$ defined for $z, w \in \Omega$ and with the following property

$$\text{Tr} \underbrace{[f(\cdot), K(\cdot, w)p]_{\mathcal{K}}}_{\in \mathcal{H}_2^t} = \text{Tr} p^{\otimes} f(w), \quad w \in \Omega, \quad p \in \mathcal{H}_2^t, \quad (\text{B.3})$$

or

$$\text{Tr} \underbrace{[f(\cdot), K(\cdot, w)p]_{\mathcal{K}}}_{\in \mathcal{H}_2^t} = \text{Tr} p^{[*]} f(w), \quad w \in \Omega, \quad p \in \mathcal{H}_2^t, \quad (\text{B.4})$$

depending on the chosen adjoint.

Proposition B.3. *In the previous notation, it holds that*

$$[f(\cdot), K(\cdot, w)b]_{\mathcal{K}} = b^{\otimes} f(w), \quad w \in \Omega, \quad b \in \mathcal{H}_2^t \quad (\text{B.5})$$

or

$$[f(\cdot), K(\cdot, w)b]_{\mathcal{K}} = b^{[*]} f(w), \quad w \in \Omega, \quad b \in \mathcal{H}_2^t \quad (\text{B.6})$$

depending on the chosen adjoint.

Proof. We consider the case of the first adjoint; the case of $[*]$ is treated similarly. From (B.1) we can rewrite (B.3) as

$$\text{Tr } b^{\otimes} [f(\cdot), K(\cdot, w)]_{\mathcal{K}} = \text{Tr } b^{\otimes} f(w), \quad w \in \Omega, \quad b \in \mathcal{H}_2^t. \quad (\text{B.7})$$

To conclude we use Corollary 2.15. \square

These definitions are readily extended to the case of matrix-valued functions with entries in \mathcal{H}_2^t . As already earlier in Sect. 5, the matrix-valued case already has differences with the scalar case in the quaternionic setting; see [11, p. 1767 (and in particular (62.38))].

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