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Schur analysis over the unit spectral ball

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A B S T R A C T

We begin a study of Schur analysis when the variable is now a matrix rather than a complex number. We define the corresponding Hardy space, Schur multipliers and their realizations, and interpolation. Possible applications of the present work include matrices of quaternions, matrices of split quaternions, and other algebras of hypercomplex numbers.

Contents

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1. Introduction

The classical theory of Hilbert spaces of functions analytic in a neighborhood of the origin, i.e., with elements of the form

$$f(z)=\sum_{n=0}^{\infty} z^n f_n \tag{1.1}$$

where the $f_n \in \mathbb{C}$ satisfy

$$\sum_{n=0}^{\infty} \gamma_n |f_n|^2 < \infty \tag{1.2}$$

encompasses spaces such as the Hardy space, the Fock space, the Bergman space, and many others. In (1.2) the numbers γ_n (the weights) are strictly positive numbers such that

$$R=\sqrt{\liminf_{n\to\infty} \gamma_n^{1/n}} > 0.$$

From

$$|\sum_{n=0}^{\infty} z^n f_n|^2 \le \left(\sum_{n=0}^{\infty} \frac{|z|^{2n}}{\gamma_n}\right)\left(\sum_{n=0}^{\infty} \gamma_n |f_n|^2\right)$$

the functions are analytic for z such that

$$|z|^2 < \frac{1}{\limsup_{n\to\infty} \frac{1}{\gamma_n^{1/n}}} = \liminf_{n\to\infty} \gamma_n^{1/n} = R^2$$

($R=\infty$ is allowed; one then has a space of entire functions). Furthermore the corresponding reproducing kernel is

$$K(z,w)=\sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{\gamma_n}. \tag{1.3}$$

One can also allow some of the weights γ_n to vanish, as for instance for the Dirichlet kernel

$$-\ln(1-z\overline{w}) = \sum_{n=1}^{\infty} \frac{z^n\overline{w}^n}{n},$$

where z, w belong to the open unit disk $\mathbb{D}$. Then there is no constant term in the expansions (1.2) and (1.3).

The theory of such spaces can be extended to the case where the coefficients f_n, and possibly the weights γ_n, are matrices; see e.g., [4]. In the present paper we consider the case where, in (1.1), not only f_n but also z is replaced by a matrix. Thus we consider expressions of the form

$$F = F(Z) = \sum_{n=0}^{\infty} Z^n F_n \tag{1.4}$$

where[1] $Z \in \mathbb{C}^{p\times p}$ and the $F_n \in \mathbb{C}^{p\times p}$, or more generally belong to a Hilbert space which is also a right and left $\mathbb{C}^{p\times p}$-module. When $p = 1$ it may be that (1.4) converges only for $Z = z = 0$. When $p > 1$, there are always non-zero nilpotent matrices and the set of convergence of (1.4) is not reduced to a point, but it does not contain necessarily an open set (i.e. may have an empty interior), as follows from Proposition 2.3 below. See Corollary 2.4.

Expressions of the form (1.4) are not closed under pointwise product, and following [42] we define on monomials Z^nA and Z^mB the convolution (or Cauchy; see [42]) product as follows: Let $n, m \in \mathbb{N}_0$ and $A, B \in \mathbb{C}^{p\times p}$. One sets

$$(Z^nA) \star (Z^mB) = Z^{n+m}AB, \tag{1.5}$$

extended by linearity to the linear span of the monomials. This is called the Cauchy product and appears in various places in non-commutative settings; see [42] and, for the quaternionic setting, see [35] for slice functions and [56] for hypercomplex regular functions.

Among other related non-commutating settings let us mention for example the calculus on diagonals developed in [14,15] (see [53] for a predecessor to this calculus).

By specializing to subspaces of $\mathbb{C}^{p\times p}$ we obtain known cases such as the quaternions and the split quaternions, for $p = 2$ and Z, W of the form

$$\begin{pmatrix} z_1 & -z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_1 & z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix} \tag{1.6}$$

respectively, and matrices of these for p even, say $p = 2h$ ($h \in \mathbb{N}$) and z_1 and z_2 being elements of $\mathbb{C}^{h\times h}$ in (1.6). Bicomplex numbers will correspond to matrices of the form

$$\begin{pmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{pmatrix},$$

[1] We will freely use both notations F and $F(Z)$.

and hyperbolic numbers correspond to the case where z_1 and z_2 are real in this latter expression. Here too, our approach allows to study matrices of bicomplex and hyperbolic numbers.

The paper consists of seven sections, of which this introduction is the first. In the second section we give first properties of the power series of the form (1.4), and examples such as the counterparts of the Fock space, the Hardy space, the Wiener algebra and rational functions in the present setting. In the third section we study in greater details the Hardy space. We discuss a version of the Beurling-Lax theorem in the present setting and discuss the notion of Blaschke factor here. In Section 4 we solve a Nevanlinna-Pick interpolation problem in the Hardy space. Schur multipliers and their coisometric realizations are studied in Section 5 using two approaches. The first uses the theory of linear isometry relations and the second is based on operator ranges. In particular we prove a version of Leech's factorization theorem, needed in the arguments. The notion of Carathéodory multipliers is studied in Section 6 with different methods; we reduce the problem to the case of a complex variable. In the last section we discuss various future directions of research and links with hypercomplex analysis.

Throughout the paper $B(0, R)$ denotes the open unit disk in the complex plane centered at the origin, and with radius R. The spectral radius of the matrix $A \in \mathbb{C}^{p\times p}$ will be denoted by $\rho(A)$.

2. Generalities and first examples

2.1. Preliminary results

We first introduce:

Definition 2.1. Given $R > 0$, the ring $\mathcal{H}_R$ consists of the power series of the form

$$F(Z) = \sum_{n=0}^{\infty} Z^n F_n, \quad \text{where } Z, F_0, F_1, \ldots \in \mathbb{C}^{p\times p} \tag{2.1}$$

and the latter are such that

$$\limsup_{n\to\infty} \|F_n\|^{\frac{1}{n}} \leq \frac{1}{R}. \tag{2.2}$$

Note that (2.1) converges for Z such that $\rho(Z) < R$.

Proposition 2.2. *For any $A \in \mathbb{C}^{p\times p}$ with $\rho(A) < R$, the series*

$$F(A) = \sum_{n=0}^{\infty} A^n F_n \tag{2.3}$$

converges absolutely.

Proof. We have

$$\limsup_{n\to\infty} \|A^n F_n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} \cdot \limsup_{n\to\infty} \|F_n\|^{\frac{1}{n}} \leq \frac{\rho(A)}{R} < 1, \tag{2.4}$$

and so the series with running term $\|A^n F_n\|$ converges. □

Proposition 2.3. *In the above notation, assume*

$$\limsup_{n\to\infty} \|F_n\|^{\frac{1}{n}} = \infty. \tag{2.5}$$

Then, the set of convergence of (2.1) does not contain invertible matrices.

Proof. Let $A \in \mathbb{C}^{p\times p}$ be invertible. Then,

$$F_n = A^{-n} A^n F_n$$

and so

$$\|F_n\|^{1/n} \leq \|A^{-n}\|^{1/n} \|A^n F_n\|^{1/n}$$

Since A is invertible, it holds that $\lim \|A^{-n}\|^{1/n} = \rho(A^{-1}) > 0$ and we have

$$\limsup_{n\to\infty} \|F_n\|^{1/n} = \infty \quad \Longrightarrow \quad \limsup_{n\to\infty} \|A^n F_n\|^{1/n} = \infty$$

and (2.3) cannot converge. □

Corollary 2.4. *If (2.5) holds, the set where the power series converges has an empty interior.*

Proof. This follows from the fact that the set of invertible matrices is dense and open in $\mathbb{C}^{p\times p}$. □

Remark 2.5. When $p > 1$ besides the case where A is nilpotent, one has examples where $AF_n = 0$ for all $n \geq 1$; then $F(A) = F_0$ exists even if (2.5) is in force.

Proposition 2.6. *For any $F, G \in \mathcal{H}_R$ and A with $\rho(A) < R$,*

$$(F+G)(A) = F(A) + G(A) \quad \textit{and} \quad (F \star G)(A) = \sum_{n=0}^{\infty} A^n F(A) G_n, \tag{2.6}$$

where $G = \sum_{n=0}^{\infty} Z^n G_n$ with $G_0, G_1, \ldots \in \mathbb{C}^{p\times p}$ and where $\star$ denotes the Cauchy product (1.5). In particular, $(F \star G)(A) = 0$ whenever $F(A) = 0$.

Proof. We only prove the second claim. We have

$$\begin{aligned} F \star G &= F \star \left(\sum_{n=0}^{\infty} Z^n G_n \right) \\ &= \sum_{n=0}^{\infty} (F \star Z^n) G_n \\ &= \sum_{n=0}^{\infty} Z^n F G_n, \end{aligned}$$

where the exchange of sum and $\star$-product is justified since

$$\| \sum_{n=0}^{\infty} Z^n F G_n \| \leq \sum_{n=0}^{\infty} \|F\| \cdot \|Z^n\| \cdot \|G_n\|,$$

and using $\rho(Z) < R$ and (2.2) for G. This ends the proof since

$$(Z^n F G_n)(A) = A^n F(A) G_n. \quad \square$$

Definition 2.7. Let $R > 0$ and let $F(Z) = \sum_{n=0}^{\infty} Z^n F_n \in \mathcal{H}_R$. We define

$$F(zI_p) = \sum_{n=0}^{\infty} z^n F_n, \quad |z| < R. \tag{2.7}$$

Lemma 2.8. *Let $F, G \in \mathcal{H}_R$. It holds that*

$$(F \star G)(zI_p) = F(zI_p) G(zI_p), \quad |z| < R. \tag{2.8}$$

Proof. We have

$$\begin{aligned} \left(\left(\sum_{n=0}^{\infty} Z^n F_n \right) \star \left(\sum_{n=0}^{\infty} Z^n G_n \right) \right) (zI_p) &= \left(\sum_{n=0}^{\infty} Z^n \left(\sum_{k=0}^{n} F_k G_{n-k} \right) \right) (zI_p) \\ &= \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} F_k G_{n-k} \right) \\ &= \left(\sum_{n=0}^{\infty} z^n F_n \right) \left(\sum_{n=0}^{\infty} z^n G_n \right) \\ &= F(zI_p) G(zI_p), \end{aligned}$$

where the various exchanges of sums hold since $|z| < R$. $\square$

Theorem 2.9. *Let* $R > 0$ *and let* $F = \sum_{n=0}^{\infty} Z^n F_n \in \mathcal{H}_R$. *Let*

$$F(zI_p) = (f_{ij}(z))_{i,j=1}^{p}$$

and let $A \in \mathbb{C}^{p\times p}$ *with spectrum inside* $B(0,R)$. *Then,*

$$F(A) = \frac{1}{2\pi i}\int\limits_{\gamma} (zI_p - A)^{-1} F(zI_p)dz \tag{2.9}$$

where γ *is a closed Jordan curve inside* $B(0,R)$, *and which encloses the spectrum of* A.

Proof. With $F_n = (f_{ij}^{(n)})_{i,j=1}^{p}$, and with

$$f_{ij}(z) = \sum_{n=0}^{\infty} z^n f_{ij}^{(n)},$$

we have

$$F(zI_p) = \sum_{n=0}^{\infty} z^n F_n = (f_{ij}(z))_{i,j=1}^{p}\,.$$

With E_{ij}, $i,j = 1,\dots,p$ denoting the standard basis in $\mathbb{C}^{p\times p}$ we further have

$$\begin{aligned}
\sum_{n=0}^{\infty} A^n F_n &= \sum_{n=0}^{\infty} A^n \sum_{i,j=1}^{p} f_{ij}^{(n)} E_{ij} \\
&= \sum_{i,j=1}^{p} \left(\underbrace{\sum_{n=0}^{\infty} A^n f_{ij}^{(n)}}_{f_{ij}(A)} \right) E_{ij} \\
&= \sum_{i,j=1}^{p} \left(\frac{1}{2\pi i}\int\limits_{\gamma} (zI_p - A)^{-1} f_{ij}(z)dz \right) E_{ij}dz \\
&= \frac{1}{2\pi i}\int\limits_{\gamma} (zI_p - A)^{-1} \left(\sum_{i,j=1}^{p} f_{ij}(z)E_{ij} \right) dz \\
&= \frac{1}{2\pi i}\int\limits_{\gamma} (zI_p - A)^{-1} F(zI_p)dz. \quad \square
\end{aligned}$$

Theorem 2.10. *Assume that there is a summable positive measure* μ *of the form*

$$d\mu(z) = m(|z|)dxdy, \quad |r| < R$$

such that

$$2\pi \int_0^R r^{2n+1} m(r)dr = \gamma_n, \quad n = 0, 1, \dots \tag{2.10}$$

Then for $F = \sum_{n=0}^{\infty} Z^n F_n$ *such that* (2.2) *is in force, it holds that*

$$\int_{|z|<R} (F(zI_p))^* F(zI_p) d\mu(z) = \sum_{n=0}^{\infty} \gamma_n F_n^* F_n. \tag{2.11}$$

Proof. The power series $F(Z)$, and in particular $F(zI_p)$, converges in $|z| < R$ in view of (2.2). To check (2.11), let

$$f_N(z) = \sum_{n,m=0}^{N} F_n^* F_m \overline{z}^n z^m.$$

By Cauchy inequality, we have that, for $|z| \le r < R$,

$$\begin{aligned} |f_N(z)| &\le \sum_{n,m=0}^{N} |z|^n |z|^m \|F_n\| \|F_m\| \\ &= \left(\sum_{n=0}^{N} \frac{|z|^n}{\sqrt{\gamma_n}} \sqrt{\gamma_n} \|F_n\| \right)^2 \\ &\le \left(\sum_{n=0}^{N} \frac{|z|^{2n}}{\gamma_n} \right) \left(\sum_{n=0}^{N} \gamma_n \|F_n\|^2 \right) \\ &\le \left(\sum_{n=0}^{\infty} \frac{r^{2n}}{\gamma_n} \right) \left(\sum_{n=0}^{\infty} \gamma_n \|F_n\|^2 \right). \end{aligned}$$

Since $r < R$ we have that $\sum_{n=0}^{\infty} \frac{r^{2n}}{\gamma_n} < \infty$, and we can apply the dominated convergence theorem since $d\mu$ is assumed to be summable. We then obtain

$$\begin{aligned} \int_{|z|<r} (F(zI_p))^* F(zI_p) d\mu(z) &= \sum_{n,m=0}^{\infty} F_n^* F_m \iint_{|z|<r} \overline{z}^n z^m d\mu(z) \\ &= \sum_{n,m=0}^{\infty} F_n^* F_m \int_{\rho=0}^{r} \int_{\theta=0}^{2\pi} \rho^{n+m} e^{i\theta(n-m)} m(\rho) \rho d\rho d\theta \\ &= \sum_{n=0}^{\infty} F_n^* F_n \left(2\pi \int_0^r \rho^{2n+1} m(\rho) d\rho \right) \end{aligned}$$

where the integrals are finite thanks to (2.10). The monotone convergence theorem allows to let r tend to R and obtain the result using here too (2.10). □

Remark 2.11. The previous result can be adapted to other cases, such as Dirichlet, where derivatives appear, or to the case of the Hardy space. Another case where the previous result can be adapted is the time-varying setting as developed in [15,37]. Let

$$U = \sum_{n=0}^{\infty} \mathcal{Z}^n D_n$$

where $\mathcal{Z}$ is now the unitary shift from $\ell_2(\mathbb{Z}, \mathbb{C})$ into itself and $D_0, D_1, \ldots$ are diagonal operators. The Zadeh transform $U(z)$ of U is

$$\sum_{n=0}^{\infty} z^n \mathcal{Z}^n D_n, \quad z \in \mathbb{C} \tag{2.12}$$

(see [5,6], and the unpublished manuscript [2]). Since for the unitary shift, $\mathrm{Z}^{*m}\mathcal{Z}^n = \mathcal{Z}^{n-m}$, we have

$$\frac{1}{2\pi i} \int\limits_{|z|=r} D_m^* \mathcal{Z}^{*m} \overline{z^m} z^n \mathcal{Z}^n D_n \frac{dz}{z} = \begin{cases} 0 & \text{if } n \neq m, \\ r^{2n} D_n^* D_n & \text{if } n \neq m \end{cases}$$

we have, using the same methods as above, and with the same notations,

$$\int\limits_{|z|<R} (U(z))^* U(z) d\mu(z) = \sum_{n=0}^{\infty} D_n^* D_n. \tag{2.13}$$

2.2. The Fock space

The classical Fock space corresponds to $\gamma_n = n!$ in (1.2). See [28]. Besides Bargmann's celebrated characterization ($M_z^* = \partial$, where M_z is multiplication by z and ∂ denotes differentiation), the Fock space can be seen as the only Hilbert space of functions analytic in a (say convex open) neighborhood of the origin in which the backward shift operator

$$(R_0 f)(z) = \begin{cases} \dfrac{f(z) - f(0)}{z}, & z \in V \setminus \{0\} \\ f'(0), & z = 0, \end{cases} \tag{2.14}$$

is bounded and has for adjoint the integration operator

$$(\mathbf{I} f)(z) = \int\limits_{[0,z]} f(s) ds. \tag{2.15}$$

See [7]. The above motivates the following definitions, which are the counterpart of (2.14) and (2.15) in our present setting.

Definition 2.12. Let $F(Z) = \sum_{n=0}^{\infty} Z^n F_n$ be a matrix power series converging in a neighborhood of $0_{p\times p}$. We define

$$(R_0 F)(Z) = \sum_{n=1}^{\infty} Z^{n-1} F_n. \tag{2.16}$$

$$\mathbf{I}\left(\sum_{n=0}^{\infty} Z^n F_n\right) = \sum_{n=0}^{\infty} \frac{1}{n+1} Z^n F_n. \tag{2.17}$$

It is not difficult to check that

$$\mathbf{I}(F) = \int_{[0,z]} F(sZ)ds. \tag{2.18}$$

Definition 2.13. The Fock space $\mathfrak{F}$ consists of the matrix power series $F(Z) = \sum_{n=0}^{\infty} Z^n F_n$ for which

$$\mathrm{Tr}\,[F,F]_{\mathfrak{F}} < \infty, \tag{2.19}$$

where

$$[F,F]_{\mathfrak{F}} = \sum_{n=0}^{\infty} n! F_n^* F_n. \tag{2.20}$$

We can apply Theorem 2.10 with $m(r) = e^{-r^2}$ and obtain:

$$\sum_{n=0}^{\infty} n! F_n^* F_n = \frac{1}{\pi} \iint_{\mathbb{C}} F(zI_p)^* F(zI_p) e^{-|z|^2} dxdy \tag{2.21}$$

For the case $p = 1$ the following result has been proved in [7, Lemma 2.1].

Proposition 2.14. *The Fock space functions are defined for all $Z \in \mathbb{C}^{p\times p}$ and the Fock space can be characterized, up to a positive multiplicative factor for the inner product, as the only space of matrix power series a priori defined near the origin and for which*

$$R_0^* = \mathbf{I}. \tag{2.22}$$

Proof. The first claim follows from $\|Z^n\| \le \|Z\|^n$. For the second claim, and with the understanding that $R_0 C = 0$ we have

$$
\begin{aligned}
[R_0^*(Z^nC), Z^mD]_{\mathfrak{F}} &= \begin{cases} [Z^nC, Z^{m-1}D]_{\mathfrak{F}}, & m = 1, 2, \ldots \\ 0, & m = 0, \end{cases} \\
&= \delta_{n,m-1}(m-1)!D^*C \\
&= \delta_{n+1,m}[\frac{1}{n+1}Z^{n+1}C, Z^mD]_{\mathfrak{F}} \\
&= [\mathbf{I}(Z^nC), Z^mD]_{\mathfrak{F}}. \quad \square
\end{aligned}
$$

2.3. Wiener algebra

The Wiener algebra of the unit circle, denoted here $\mathcal{W}$, consists of the trigonometric series of the form

$$f(e^{it}) = \sum_{n\in\mathbb{Z}} f_n e^{int} \tag{2.23}$$

where the complex numbers f_n satisfy $\sum_{n\in\mathbb{Z}} |f_n| < \infty$. The celebrated Wiener-Lévy theorem states that f is invertible in $\mathcal{W}$ if and only if it is pointwise invertible (that is, pointwise different from 0). The Wiener algebra is an example of Banach algebra. Two important subalgebras of the Wiener algebra consists of $\mathcal{W}_+$ (resp. $\mathcal{W}_-$) for which $f_n = 0$, $n < 0$ (resp. $f_n = 0$, $n > 0$). The Wiener algebra still makes sense when the $f_n \in \mathbb{C}^{p\times p}$. The counterpart of the Wiener-Lévy theorem involves then the determinant of the function.

We now define the counterpart of $\mathcal{W}_+$ in the present framework.

Definition 2.15. We denote by $\mathfrak{W}_+$ the space of functions

$$F(Z) = \sum_{n\in\mathbb{N}_0} Z^n F_n$$

with $\rho(Z) \le 1$ and $\sum_{n=0}^\infty \|F_n\| < \infty$.

We leave to the reader to check that $\mathfrak{W}_+$ endowed with the $\star$ product is an algebra.

Theorem 2.16. *$F \in \mathfrak{W}_+$ is invertible in $\mathfrak{W}_+$ if and only if*

$$\det(\sum_{n=0}^\infty z^n F_n) \neq 0, \quad |z| \le 1.$$

Proof. We first note that $F(zI_p) \in \mathcal{W}_+^{p\times p}$. So it will be invertible in $\mathcal{W}_+^{p\times p}$ if and only if $\det F(zI_p) \neq 0$ in the closed unit disk. Assume this condition in force, and let $g(z) = \sum_{n=0}^\infty z^n G_n \in \mathcal{W}_+^{p\times p}$ (the lower case g is not a misprint) be such that

$$\sum_{n=0}^\infty \|G_n\| < \infty$$

and $F(zI_p)g(z) = I_p$, for $|z| \le 1$. Then, we have

$$\begin{aligned} F_0G_0 &= I_p \\ F_0G_1 + F_1G_0 &= 0 \\ &\vdots \\ F_0G_n + \cdots + F_nG_0 &= 0 \\ &\vdots \end{aligned}$$

These equalities express that $g(z) = G(zI_p)$ where $G = \sum_{n=0}^{\infty} Z^n G_n \in \mathfrak{W}_+$. The converse statement is proved by reading backwards these arguments. □

The definition of the counterpart of $\mathcal{W}$ will involve a unitary variable Z. The case of the counterpart of $\mathcal{W}_-$ will be more problematic since one then requires invertible Z. These aspects will be considered in a different publication, where a counterpart of the Wiener-Lévy inversion theorem is also considered.

2.4. Rational functions

We consider $(\mathbb{C}^{p\times p})^{u\times v}$ as a right module over $\mathbb{C}^{p\times p}$ and a left module over $\mathbb{C}^{p\times p}$ in the following ways:

Definition 2.17. Right module structure: For every $u, v \in \mathbb{N}$, every $A_{jk} \in \mathbb{C}^{p\times p}$, $j = 1, \ldots, u$, $k = 1, \ldots, v$, and every $C \in \mathbb{C}^{p\times p}$

$$\left((A_{jk})_{\substack{j=1,\ldots u \\ k=1,\ldots v}} \right) C = (A_{jk}C)_{\substack{j=1,\ldots u \\ k=1,\ldots v}} \tag{2.24}$$

Left module structure: in the above notation

$$C \left((A_{jk})_{\substack{j=1,\ldots u \\ k=1,\ldots v}} \right) = (CA_{jk})_{\substack{j=1,\ldots u \\ k=1,\ldots v}}. \tag{2.25}$$

We consider power series of the form

$$\sum_{n=0}^{\infty} Z^n A_n, \quad A_n \in (\mathbb{C}^{p\times p})^{u\times v}. \tag{2.26}$$

We extend R_0 on these power series by

$$R_0(\sum_{n=0}^{\infty} Z^n A_n) = \sum_{n=1}^{\infty} Z^{n-1} A_n.$$

Proposition 2.18. *Let $\mathfrak{M}$ be a finitely generated right module of power series of the form (2.26). Assume that $\mathfrak{M}$ is R_0-invariant. Then $\mathfrak{M}$ is generated by the block columns of a matrix function of the form*

$$E(Z) = C \star (I_{pN} - ZA)^{-\star}$$

where $A \in (\mathbb{C}^{p\times p})^{N\times N}$ and $C \in (\mathbb{C}^{p\times p})^{u\times N}$.

Proof. Let $F_1, \ldots, F_M$ a generating family for $\mathfrak{M}$. Every F_j is $(\mathbb{C}^{p\times p})^{u\times v}$-valued and any $F \in \mathfrak{M}$ can be written (possibly in a non-unique way) as

$$F = \sum_{m=1}^{M} F_m C_m, \quad C_1, \ldots, C_M \in \mathbb{C}^{p\times p}.$$

Let

$$R_0 F_j = \sum_{r=1}^{M} F_r A_{rj}.$$

Let $E = \begin{pmatrix} F_1 & F_2 & \cdots & F_M \end{pmatrix}$. Then,

$$\begin{aligned} R_0 F &= \begin{pmatrix} R_0 F_1 & R_0 F_2 & \cdots & R_0 F_M \end{pmatrix} \\ &= \begin{pmatrix} F_1 A_{11} + F_2 A_{21} + \cdots & F_1 A_{12} + F_2 A_{22} + \cdots & \cdots & F_1 A_{1M} + F_2 A_{2M} + \cdots \end{pmatrix} \\ &= \begin{pmatrix} F_1 & F_2 & \cdots & F_M \end{pmatrix} \underbrace{\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1M} \\ A_{21} & A_{22} & \cdots & A_{2M} \\ & & & \\ A_{M1} & A_{M2} & & A_{MM} \end{pmatrix}}_{A \in (\mathbb{C}^{p\times p})^{M\times m}}. \end{aligned}$$

Let $E = \sum_{n=0}^{\infty} Z^n E_n, \quad E_n \in (\mathbb{C}^{p\times p})^{M\times M}$. The above equation can be rewritten as

$$\sum_{n=1}^{\infty} Z^{n-1} E_n = \sum_{n=0}^{\infty} Z^n E_n A.$$

Hence

$$E_{n+1} = E_n A, \quad n = 0, 1, \ldots$$

and so $E_n = E_0 A^n$, $n = 0, 1, \ldots$. Thus

$$E(Z) = \sum_{n=0}^{\infty} Z^n E_0 A^n = E_0 \star (I_{Np} - ZA)^{-\star}. \quad \square$$

Definition 2.19. E of the form (2.26) is rational if the right linear module over $\mathbb{C}^{p\times p}$ generated by R_0^jE, $j=1,\ldots$ is finitely generated.

Theorem 2.20. *E is rational if and only if*

$$E(Z) = D + ZC\star(I_{pN}-ZA)^{-\star}B. \tag{2.27}$$

Proof. Let $\mathfrak{M}$ be the module generated by R_0^jE. Then $\mathfrak{M}$ is generated by $C\star(I_{pN}-ZA)^{-\star}$. Write

$$R_0E = C\star(I_{PN}-ZA)^{-\star}B.$$

So

$$ZR_0E = E - E_0 = ZC\star(I_{PN}-ZA)^{-\star}B,$$

and hence the result holds. □

We note that $E(zI_p)$ is rational in the classical sense and the restriction of (2.27) to $Z = zI_p$ gives a realization in the classical sense; see [29].

3. The Hardy space

3.1. Definition

The classical Hardy space $\mathbf{H}_2(\mathbb{D})$ of the open unit disk $\mathbb{D}$ is the space of power series $f(z)=\sum_{n=0}^\infty z^nf_n$ for which

$$\sum_{n=0}^\infty |f_n|^2<\infty,$$

i.e., corresponding to $\gamma_n\equiv 1$ in (1.2). It is the reproducing kernel Hilbert space with reproducing kernel

$$k(z,w)=\frac{1}{1-z\overline{w}}=\sum_{n=0}^\infty z^n\overline{w}^n, \tag{3.1}$$

where z,w run through $\mathbb{D}$, and plays a key role in operator theory. It has numerous extensions and generalizations. In the present paper we consider its counterpart when the complex numbers are replaced by elements in $\mathbb{C}^{p\times p}$, and the Szegö kernel (3.1) replaced by

$$K(Z,W)=\sum_{n=0}^\infty Z^nW^{*n} \tag{3.2}$$

where Z, W runs through the set of elements of $\mathbb{C}^{p\times p}$ with spectral radius less than 1.

Definition 3.1. We denote by

$$\mathbb{K} = \left\{ Z \in \mathbb{C}^{p\times p}\, ;\, \rho(Z) < 1 \right\} \tag{3.3}$$

where $\rho(Z)$ denotes the spectral radius of Z.

Definition 3.2. The Hardy space $\mathbf{H}_2(\mathbb{K})$ consists of functions of the form

$$F(Z) = \sum_{n=0}^{\infty} Z^n F_n, \quad Z \in \mathbb{K}, \tag{3.4}$$

where $(F_0, F_1, \ldots)$ satisfy

$$\mathrm{Tr}\left(\sum_{n=0}^{\infty} F_n^* F_n \right) < \infty. \tag{3.5}$$

Theorem 3.3. *When endowed with the $\mathbb{C}^{p\times p}$-valued Hermitian form*

$$[F, G]_2 = \sum_{n=0}^{\infty} G_n^* F_n, \tag{3.6}$$

(where $G(Z) = \sum_{n=0}^{\infty} Z^n G_n$) and associated norm (3.5), $\mathbf{H}_2(\mathbb{K})$ the reproducing kernel Hilbert module with reproducing kernel (3.2), meaning that

$$[F(\cdot), K(\cdot, W)C] = C^* F(W), \quad W \in \mathbb{K} \text{ and } C \in \mathbb{C}^{p\times p}. \tag{3.7}$$

Proof. We consider the Hilbert space $\ell_2(\mathbb{N}_0, \mathbb{C}^{p\times p})$ of sequences $\mathbf{F} = (F_0, F_1, \ldots)$ of sequences of elements of $\mathbb{C}^{p\times p}$ with finite norm (3.5). Note that $\ell_2(\mathbb{N}_0, \mathbb{C}^{p\times p})$ is a right $\mathbb{C}^{p\times p}$ module: it holds that

$$\mathbf{F} = (F_0, F_1, \ldots) \in \ell_2(\mathbb{N}_0, \mathbb{C}^{p\times p}) \quad \Longrightarrow \quad \mathbf{F}A = (F_0 A, F_1 A, \ldots) \in \ell_2(\mathbb{N}_0, \mathbb{C}^{p\times p}),$$
$$\forall A \in \mathbb{C}^{p\times p},$$

as well as the linearity conditions

$$(\mathbf{F} + \mathbf{G})A = \mathbf{F}A + \mathbf{G}A$$
$$\mathbf{F}(A + B) = \mathbf{F}A + \mathbf{F}B,$$

for all $\mathbf{F}, \mathbf{G} \in \ell_2(\mathbb{N}_0, \mathbb{C}^{p\times p})$ and $A, B \in \mathbb{C}^{p\times p}$; see e.g. [48, p. 117].

We define

$$[\mathbf{C}, \mathbf{D}] = \sum_{n=0}^{\infty} G_n^* F_n, \quad \mathbf{C} \text{ and } \mathbf{D} \in \ell_2(\mathbb{N}_0, \mathbb{C}^{p\times p}). \tag{3.8}$$

It holds that

$$\operatorname{Tr}[\mathbf{C}, \mathbf{D}] = \sum_{n=0}^{\infty} \operatorname{Tr} G_n^* F_n, \tag{3.9}$$

which is the inner product associated to (3.5).

$\mathbf{H}_2(\mathbb{K})$ is a Hilbert space since $\ell_2(\mathbb{N}_0, \mathbb{C}^{p\times p})$ is a Hilbert space, and since the coefficients F_n in (3.4) are uniquely determined by $F(Z)$ (and in fact by $F(zI_p)$ with $z \in \mathbb{C}$). Furthermore we have with F as in (3.4),

$$\begin{aligned}
[F(\cdot), K(\cdot, W)C]_{\mathbf{H}_2(\mathbb{K})} &= [\sum_{n=0}^{\infty} Z^n F_n, \sum_{n=0}^{\infty} Z^n W^{*n} C]_{\mathbf{H}_2(\mathbb{K})} \\
&= \sum_{n=0}^{\infty} C^* W^n F_n \\
&= [F(W), C]_{\mathbb{C}^{p\times p}}. \quad \square
\end{aligned}$$

Theorem 3.4. *Let $F = \sum_{n=0}^{\infty} Z^n F_n \in \mathcal{H}_1$. It holds that*

$$\lim_{\substack{r\uparrow 1\\ r\in(0,1)}} \frac{1}{2\pi} \int_0^{2\pi} F(re^{i\theta} I_p)^* F(re^{i\theta} I_p) d\theta = \sum_{n=0}^{\infty} F_n^* F_n \tag{3.10}$$

where both sides simultaneously converge or diverge.

Proof. The arguments as in the proof of Theorem 2.10 show that

$$\frac{1}{2\pi} \int_0^{2\pi} F(re^{i\theta} I_p)^* F(re^{i\theta} I_p) d\theta = \sum_{n=0}^{\infty} r^{2n} F_n^* F_n.$$

Thus, for every $w \in \mathbb{C}^p$,

$$\frac{1}{2\pi} \int_0^{2\pi} w^* F(re^{i\theta} I_p)^* F(re^{i\theta} I_p) w d\theta = \sum_{n=0}^{\infty} r^{2n} w^* F_n^* F_n w.$$

Then use the monotone convergence theorem and the polarization identity

$$w^* X z = \frac{1}{4} \sum_{k=0}^{3} i^{-k} (w + i^k z)^* X (w + i^k z), \quad w, z \in \mathbb{C}^p, \quad X \in \mathbb{C}^{p\times p}. \quad \square \tag{3.11}$$

In what follows we will make frequent use of the operators

$$(M_Z F)(Z) = \sum_{n=0}^{\infty} Z^{n+1} F_n, \tag{3.12}$$

$$(M_A F)(Z) = \sum_{n=0}^{\infty} Z^n A F_n. \tag{3.13}$$

Note that M_A satisfies

$$(M_A)^* = M_{A^*}. \tag{3.14}$$

Note also that M_Z is an isometry with adjoint R_0:

$$(M_Z^*)\left(\sum_{n=0}^{\infty} Z^n F_n\right) = \sum_{n=1}^{\infty} Z^{n-1} F_n. \tag{3.15}$$

Note that the right hand side of (3.15) makes sense for converging matrix power series even if the series does not belong to the Hardy space. Furthermore, for $p = 1$ the formula reduces to the classical backward-shift operator defined by (2.14) for a function f analytic in a neighborhood V of the origin.

3.2. Resolvent operators and resolvent equations

We follow [23, §2.3] suitably adapted to the present setting. Recall that M_A denotes the operator of multiplication of the coefficients on the left by A; see (3.13). We define an operator R_A on power series in Z via:

$$(R_A F)(zI_p) = (zI_p - A)^{-1}(F(zI_p) - F(A)). \tag{3.16}$$

When $A = 0_{p\times p}$ this is the operator R_0 defined in (2.16).

Lemma 3.5. *Let* $F = \sum_{n=0}^{\infty} Z^n F_n$. *Then,*

$$R_A F = \sum_{k=0}^{\infty} Z^k \left(\sum_{n=k+1}^{\infty} A^{n-1-k} F_n\right) \tag{3.17}$$

Proof. We have

$$\begin{aligned}R_AF(zI_p) &= (zI_n - A)^{-1}\left(\sum_{n=1}^{\infty}(z^nI_p - A^n)F_n\right)\\&= \sum_{n=0}^{\infty}(zI_p - A)^{-1}(z^nI_p - A^n)F_n\\&= \sum_{n=0}^{\infty}\sum_{k=0}^{n-1-k}(z^kA^{n-1-k})F_n\\&= \sum_{k=0}^{\infty}z^k\left(\sum_{n=k+1}^{\infty}A^{n-1-k}F_n\right)\end{aligned}$$

and hence the result holds. □

Lemma 3.6. *(see also [23, (2.32) p. 265] for the time-varying counterpart)*

$$R_A = R_0(I - M_AR_0)^{-1}. \tag{3.18}$$

Proof. We follow [23] and set $G = (I - M_AR_0)F$. We have:

$$\begin{aligned}G &= F - M_AR_0F\\&= \sum_{n=0}^{\infty}Z^nF_n - M_A\sum_{n=1}^{\infty}Z^{n-1}F_n\\&= F_0 + \sum_{n=1}^{\infty}(Z^n - Z^{n-1}A)F_n\\&= F_0 + \sum_{n=0}^{\infty}Z^{n-1}(Z - A)F_n\end{aligned}$$

and so $G(A) = F_0$ and

$$G(zI_p) = F_0 + (zI_p - A)(R_0F)(zI_p). \tag{3.19}$$

Thus

$$\begin{aligned}(R_AG)(zI_p) &= (zI_p - A)^{-1}(G(zI_p) - G(A))\\&= (zI_n - A)^{-1}((G - F_0)(zI_p))\\&= (R_0F)(zI_p)\end{aligned}$$

so that

$$R_A(I - M_A R_0) = R_0$$

and hence the result holds. □

We now have the resolvent equation:

Theorem 3.7. *Let $A, B \in \mathbb{C}^{p\times p}$. It holds that:*

$$R_A - R_B = R_A(M_A - M_B)R_B. \tag{3.20}$$

Proof. Using

$$R_0(I - M_A R_0)^{-1} = (I - R_0 M_A)^{-1} R_0$$

we can write

$$\begin{aligned}
R_A - R_B &= R_0(I - M_A R_0)^{-1} - R_0(I - M_B R_0)^{-1} \\
&= (I - R_0 M_A)^{-1} R_0 - R_0(I - M_B R_0)^{-1} \\
&= (I - R_0 M_A)^{-1} \left(R_0 - R_0 M_B R_0 - R_0 + R_0 M_A R_0\right) R_0(I - M_B R_0)^{-1} \\
&= (I - R_0 M_A)^{-1} \left(R_0 M_A R_0 - R_0 M_B R_0\right) R_0(I - M_B R_0)^{-1} \\
&= (I - R_0 M_A)^{-1} R_0(M_A - M_B) R_0(I - M_B R_0)^{-1} \\
&= R_0(I - M_A R_0)^{-1}(M_A - M_B) R_0(I - M_B R_0)^{-1}
\end{aligned}$$

and hence the result holds. □

3.3. Blaschke factor

Given a matrix $A \in \mathbb{C}^{p\times p}$ with $\rho(A) < 1$, the unique solution to the Stein equation

$$\Gamma_A - A\Gamma_A A^* = I_p \tag{3.21}$$

is given by the converging series

$$\Gamma_A = \sum_{n=0}^{\infty} A^n A^{*n}. \tag{3.22}$$

Observe that $\Gamma_A \geq I_p$ and hence is invertible. If we let

$$L_A = \Gamma_A - \Gamma_A A^* \Gamma_A^{-1} A \Gamma_A,$$

then it follows by the Sherman-Morrison formula that

$$L_A^{-1} = A^*A + \Gamma_A^{-1} \geq 0. \tag{3.23}$$

Let us now introduce the power series

$$\begin{aligned} U_A(Z) &:= (Z - A) \star (I - Z\Gamma_A A^* \Gamma_A^{-1})^{-\star} L_A^{\frac{1}{2}} \\ &= -AL_A^{\frac{1}{2}} + \sum_{n=1}^{\infty} Z^n (I - A\Gamma_A A^* \Gamma_A^{-1})(\Gamma_A A^* \Gamma_A^{-1})^{n-1} L_A^{\frac{1}{2}} \\ &= -AL_A^{\frac{1}{2}} + \sum_{n=1}^{\infty} Z^n A^{*(n-1)} \Gamma_A^{-1} L_A^{\frac{1}{2}}. \end{aligned} \tag{3.24}$$

The first power series representation of U_A above follows since

$$(I - Z\Gamma_A A^* \Gamma_A^{-1})^{-\star} = \sum_{k=0}^{\infty} Z^k (\Gamma_A A^* \Gamma_A^{-1})^k,$$

while the next representation follows since

$$\begin{aligned} (I - A\Gamma_A A^* \Gamma_A^{-1})(\Gamma_A A^* \Gamma_A^{-1})^{n-1} &= (I - A\Gamma_A A^* \Gamma_A^{-1})\Gamma_A A^{*(n-1)} \Gamma_A^{-1} \\ &= (\Gamma_A - A\Gamma_A A^*) A^{*(n-1)} \Gamma_A^{-1} = A^{*(n-1)} \Gamma_A^{-1} \end{aligned}$$

for all $n \geq 1$, due to (3.21).

Proposition 3.8. *Let $U_A(Z)$ be defined as in (3.24). Then*

$$\left[M_Z^n U_A,\, M_Z^k U_A\right]_2 = \delta_{n,k} I_p \quad \textit{for all} \quad n, k \geq 0. \tag{3.25}$$

Proof. By (3.15), it suffices to verify (3.25) for $k = 0$. For $n = 0$, we have by (3.24) and definitions (3.6) and (3.22),

$$\begin{aligned} [U_A, U_A]_2 &= L_A^{\frac{1}{2}} A^* A L_A^{\frac{1}{2}} + \sum_{j=0}^{\infty} L_A^{\frac{1}{2}} \Gamma_A^{-1} A^j A^{*j} \Gamma_A^{-1} L_A^{\frac{1}{2}} \\ &= L_A^{\frac{1}{2}} \left(A^*A + \Gamma_A^{-1} \left(\sum_{j=0}^{\infty} A^j A^{*j}\right) \Gamma_A^{-1}\right) L_A^{\frac{1}{2}} \\ &= L_A^{\frac{1}{2}} \left(A^*A + \Gamma_A^{-1}\right) L_A^{\frac{1}{2}} = L_A^{\frac{1}{2}} L_A L_A^{\frac{1}{2}} = I_p. \end{aligned}$$

For $n > 0$, we have

$$Z^n U_A(Z) = -Z^n A L_A^{\frac{1}{2}} + \sum_{j=1}^{\infty} Z^{j+n} A^{*(j-1)} \Gamma_A^{-1} L_A^{\frac{1}{2}},$$

and subsequently,

$$\begin{aligned}[M_Z^n U_A,\, U_A]_2 &= -L_A^{\frac{1}{2}}\Gamma_A^{-1}A^{n-1}AL_A^{\frac{1}{2}} + \sum_{j=0}^{\infty} L_A^{\frac{1}{2}}\Gamma_A^{-1}A^{n+j}A^{*j}\Gamma_A^{-1}L_A^{\frac{1}{2}} \\ &= L_A^{\frac{1}{2}}\Gamma_A^{-1}A^n\big(-I_p + \Big(\sum_{j=0}^{\infty} A^jA^{*j}\Big)\Gamma_A^{-1}\big)L_A^{\frac{1}{2}} \\ &= L_A^{\frac{1}{2}}\Gamma_A^{-1}A^n\big(-I_p + \Gamma_A\Gamma_A^{-1}\big)L_A^{\frac{1}{2}} = 0,\end{aligned}$$

which completes the proof. □

Corollary 3.9. *The operator $M_{U_A} : F(z) \mapsto U_A(z) \star F(z)$ is an isometry on $\mathbf{H}_2(\mathbb{K})$ in the following sense:*

$$[U_A \star F,\, U_A \star F]_2 = [F,\, F]_2 \quad \textit{for any} \quad F \in \mathbf{H}_2(\mathbb{K}). \tag{3.26}$$

Indeed, if we take $F \in \mathbf{H}_2(\mathbb{K})$ in the form (3.4), then we have, by (3.25),

$$\begin{aligned}[U_A \star F,\, U_A \star F]_2 &= \Bigg[\sum_{j=0}^{\infty} M_Z^j U_A C_j,\, \sum_{j=0}^{\infty} M_Z^j U_A C_j\Bigg]_2 \\ &= \sum_{j=0}^{\infty} [U_A C_j,\, U_A C_j]_2 = \sum_{j=0}^{\infty} [C_j,\, C_j]_2 = \sum_{j=0}^{\infty} C_j^* C_j = [F,\, F]_2 .\end{aligned}$$

Proposition 3.10. $\mathbf{H}_2(\mathbb{K}) \ominus M_{U_A}\mathbf{H}_2(\mathbb{K}) = \{K(Z,A)C :\ C \in \mathbb{C}^{p\times p}\}$.

Proof. Since $U_A(A) = 0$, it follows by Proposition 2.6 that $(U_A \star F)(A) = 0$ for any $F \in \mathbf{H}_2(\mathbb{K})$. Therefore, by (3.7) we have

$$[U_A \star F,\, K(Z,A)C]_2 = C^*(U_A \star F)(A) = 0$$

for all $F \in \mathbf{H}_2(\mathbb{K})$ and $C \in \mathbb{C}^{p\times p}$. Therefore, the submodules $\{K(Z,A)C :\ C \in \mathbb{C}^{p\times p}\}$ and $M_{U_A}\mathbf{H}_2(\mathbb{K})$ of $\mathbf{H}_2(\mathbb{K})$ are orthogonal with respect to the form (3.6).

We next take an arbitrary $G(Z) = \sum_{j=0}^{\infty} Z^j G_j \in \mathbf{H}_2(\mathbb{K})$, which is orthogonal to $M_{U_A}\mathbf{H}_2(\mathbb{K})$. In particular, it is orthogonal to $Z^n U_A$ for all $n \geq 0$. In other words,

$$[G,\, M_Z^n U_A]_2 = 0 \quad \text{for all} \quad n \geq 0,$$

which can be written in terms of coefficients of G and U_A (see (3.24)) as

$$-L_A^{\frac{1}{2}}A^*G_n + \sum_{j=1}^{\infty} L_A^{\frac{1}{2}}\Gamma_A^{-1}A^{j-1}G_{n+j} = 0,$$

or equivalently,

$$-\Gamma_A A^* G_n + \sum_{j=1}^{\infty} A^{j-1} G_{n+j} = 0 \tag{3.27}$$

for all $n \geq 0$. Replacing n by $n+1$ in (3.27) and multiplying both parts by A on the left we get

$$-A\Gamma_A A^* G_{n+1} + \sum_{j=1}^{\infty} A^j G_{n+j+1} = 0.$$

Upon subtracting the latter equality from (3.27) we get

$$\begin{aligned} 0 &= \Gamma_A A^* G_n - A\Gamma_A A^* G_{n+1} - G_{n+1} \\ &= \Gamma_A A^* G_n - (I + A\Gamma_A A^*) G_{n+1} = \Gamma_A A^* G_n - \Gamma_A G_{n+1}. \end{aligned}$$

Therefore, $G_{n+1} = A^{*n} G_n$ for all $n \geq 0$ and hence $G_n = A^{*n} G_0$, so that

$$G(z) = \sum_{j=0}^{\infty} Z^j A^{*n} G_0 = K(Z, A) G(0),$$

which completes the proof. □

Corollary 3.11. *An element F of $\mathbf{H}_2(\mathbb{K})$ satisfies $F(A) = 0$ if and only if it is of the form $F = U_A \star G$ for some $G \in \mathbf{H}_2(\mathbb{K})$.*

Proof. The "if" part follows from Remark 2.6 since $U_A(A) = 0$. On the other hand, if $F \in \mathbf{H}_2(\mathbb{K})$ is subject to $F(A) = 0$, then

$$[F, K(\cdot, A)C]_2 = C^* F(A) = 0, \quad C \in \mathbb{C}^{p \times p},$$

and hence, $F \in M_{U_A} \mathbf{H}_2(\mathbb{K})$, by Proposition 3.10. □

The preceding results are of special interest when $Z = zI_p$. The function $b_A(z) \overset{\text{def.}}{=} U_A(zI_p)$ is then a $\mathbb{C}^{p \times p}$-valued rational function contractive in the open unit disk and unitary on the unit circle. As such one can apply the results of [19,20] which characterize such functions, as we now explain. We rewrite first

$$\begin{aligned} b_A(z) &= -AL_A^{1/2} + z(I_p - zA^*)^{-1} \Gamma_A^{-1} L_A^{1/2} &(3.28)\\ &= (zI_p - A)(I_p - z\Gamma_A A^* \Gamma_A^{-1})^{-1} L_A^{1/2} &(3.29)\\ &= (I_p - zA^*)^{-1}(zI_P - AL_A^{-1}) L_A^{1/2}. &(3.30) \end{aligned}$$

These equations are a direct consequence of (3.24) when $Z = zI_p$. When $A = aI_p$ with $a \in \mathbb{D}$, we have

$$b_A(z) = \frac{z-a}{1-z\overline{a}} I_p.$$

Furthermore, we set

$$\begin{aligned} \mathcal{A} &= A^* \\ \mathcal{B} &= \Gamma_A^{-1} L_A^{1/2} \\ \mathcal{C} &= I_p \\ \mathcal{D} &= -A L_A^{1/2} \end{aligned} \tag{3.31}$$

so that

$$b_A(z) = \mathcal{D} + z\mathcal{C}(I_p - z\mathcal{A})^{-1}\mathcal{B}.$$

Theorem 3.12. (3.31) *is a minimal realization of* $b_A(z)$ *and it holds that*

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}^* \begin{pmatrix} \Gamma_A & 0 \\ 0 & I_p \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} \Gamma_A & 0 \\ 0 & I_p \end{pmatrix}. \tag{3.32}$$

Proof. We need to verify that

$$A\Gamma_A A^* + I_p = \Gamma_A \tag{3.33}$$

$$L_A^{1/2}\Gamma_A^{-1}\Gamma_A A^* - L_A^{1/2}A^* = 0 \tag{3.34}$$

$$L_A^{1/2}\Gamma_A^{-1}\Gamma_A\Gamma_A^{-1}L_A^{1/2} + L_A^{1/2}A^*AL_A^{1/2} = I_p. \tag{3.35}$$

The first identity is just (3.21), the second one is trivial and the last identity follows from (3.23). □

Remark 3.13. In the language of [20], Γ_A is the associated Hermitian matrix associated to the minimal realization (3.31).

As a corollary of the previous arguments we see that an element $f(z) = \sum_{n=0}^\infty z^n F_n \in \mathbf{H}_2(\mathbb{D})^{p\times p}$, with $F_0, F_1, \ldots \in \mathbb{C}^{p\times p}$ satisfies $f(A) = 0$ if and only if it can be written as

$$f(z) = b_A(z)g(z)$$

where $\mathbf{H}_2(\mathbb{D})^{p\times p}$, and $\|f\| = \|g\|$. One could also obtain this result directly from [3] or [27] for instance.

Particular cases:
In the case of quaternions, we take

$$Z = \begin{pmatrix} z_1 & -z_2 \\ \overline{z}_2 & \overline{z}_1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a_1 & -a_2 \\ \overline{a}_2 & \overline{a}_1 \end{pmatrix}, \quad |a_1|^2 + |a_2|^2 < 1. \tag{3.36}$$

Since $AA^* = A^*A = (|a_1|^2 + |a_2|^2)I_2$, we derive from (3.22) $\Gamma_a = (1 - |a_1|^2 - |a_2|^2)^{-1}$, $L_A = I_2$, and then (3.24) amounts to

$$U_A = (Z - A) \star (I_2 - ZA^*)^{-\star}.$$

See [10], where the quaternionic notation rather than matrix notation is used.

The split quaternions correspond to

$$A = \begin{pmatrix} a_1 & a_2 \\ \overline{a_2} & \overline{a_1} \end{pmatrix}, \quad a_1, a_2 \in \mathbb{C}.$$

See [21,43,52]. We now have, with

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$AJA^* = (|a_1|^2 - |a_2|^2)J.$$

Thus, an indefinite metric appears and the current theory has to be appropriately extended.

3.4. A Beurling-Lax type theorem

Let $\mathfrak{M}$ be a closed subspace of $\mathbf{H}_2(\mathbb{K})$ invariant under M_Z. Then, the set of functions of a complex variable $F(zI_p)$ with $F \in \mathfrak{M}$ is a z-invariant subspace of the classical Hardy space $\mathbf{H}_2(\mathbb{D}) \otimes \mathbb{C}^{p\times p}$ (i.e. $\mathbb{C}^{p\times p}$-valued functions with entries in $\mathbf{H}_2(\mathbb{D})$). By the classical Beurling-Lax theorem there is a Hilbert space $\mathcal{C}$ and a $L(\mathcal{C}, \mathbb{C}^{p\times p})$-valued function Θ which takes coisometric values on the unit circle and such that

$$\mathfrak{M} = \Theta \mathbf{H}_2(\mathbb{D}, \mathcal{C}).$$

Let $\Theta(z) = \sum_{n=0}^{\infty} z^n \Theta_n$ and for $H \in \mathbf{H}_2(\mathbb{D}, \mathcal{C})$, set $H(z) = \sum_{n=0}^{\infty} z^n H_n$. Let $F = \Theta H$, with $F(z) = \sum_{n=0}^{\infty} z^n F_n$. *A priori* $\mathcal{C}$ is not a left $\mathbb{C}^{p\times p}$-module and we cannot lift directly to the setting of the paper, i.e., cannot write from

$$F(z) = \sum_{n=0}^{\infty} z^n F_n = \Theta(z)H(z) \tag{3.37}$$

$$\sum_{n=0}^{\infty} Z^n F_n = \left(\sum_{n=0}^{\infty} Z^n \Theta_n\right) \star \left(\sum_{n=0}^{\infty} Z^n H_n\right).$$

But (3.37) can be rewritten as

$$F(z) = \sum_{n=0}^{\infty} z^n F_n = \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} \Theta_k H_{n-k}\right) \tag{3.38}$$

The operator products $\Theta_k H_{n-k}$ make sense by construction, and (3.38) is equivalent to

$$\sum_{n=0}^{\infty} Z^n F_n = \sum_{n=0}^{\infty} Z^n \left(\sum_{k=0}^{n} \Theta_k H_{n-k}\right). \tag{3.39}$$

Hence:

Theorem 3.14. *Let $\mathfrak{M}$ be a closed subspace of $\mathbf{H}_2(\mathbb{K})$ invariant under M_Z. There exist a Hilbert space $\mathcal{C}$ and an operator-valued function $\Theta = \sum_{n=0}^{\infty} Z^n \Theta_n$ such that $F = \sum_{n=0}^{\infty} Z^n F_n \in \mathfrak{M}$ if and only if (3.38) holds for some $H \in \mathbf{H}_2(\mathbb{D}, \mathcal{C})$.*

4. Interpolation

We want to solve:

Problem 4.1. *Given $A_1, B_1, \ldots, A_N, B_N \in \mathbb{C}^{p\times p}$ (the interpolation data), describe the set of all functions $F \in \mathbf{H}_2(\mathbb{K})$ such that*

$$F(A_j) = B_j, \quad j = 1, \ldots, N \tag{4.1}$$

We follow the classical approach to interpolation in reproducing kernel spaces (and, more generally, modules; see [4]). We look for a solution of the form

$$F = \sum_{j=1}^{N} (I - ZA_j^*)^{-\star} C_j$$

where $C_1, \ldots, C_N \in \mathbb{C}^{p\times p}$ are to be found. Since

$$F = \sum_{j=1}^{N} \sum_{n=0}^{\infty} Z^n A_j^{*n} C_j$$

the interpolation conditions lead to

$$B_k = F(A_k) = \sum_{j=1}^{N} \sum_{n=0}^{\infty} A_k^n A_j^{*n} C_j, \quad k = 1, \ldots, N,$$

and so

$$\underbrace{\begin{pmatrix} \sum_{n=0}^{\infty} A_1^n A_1^{*n} & \sum_{n=0}^{\infty} A_1^n A_2^{*n} & \cdots & \sum_{n=0}^{\infty} A_1^n A_N^{*n} \\ \sum_{n=0}^{\infty} A_2^n A_1^{*n} & \sum_{n=0}^{\infty} A_2^n A_2^{*n} & \cdots & \sum_{n=0}^{\infty} A_2^n A_N^{*n} \\ \\ \sum_{n=0}^{\infty} A_N^n A_1^{*n} & \sum_{n=0}^{\infty} A_N^n A_2^{*n} & \cdots & \sum_{n=0}^{\infty} A_N^n A_N^{*n} \end{pmatrix}}_{\mathbf{G}} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{pmatrix} \tag{4.2}$$

The Gram matrix $\mathbf{G}$ is positive semi-definite. When it is positive definite, one can solve and get the C_j. The fact that $\mathbf{G} > 0$ means that the interpolation points are "far away" enough one from the other. Similar phenomenon occurs in the time-varying setting. See [38].

By the Beurling-Lax theorem considered in the previous section one can then consider the functions for which interpolation is met with $B_1 = \cdots = B_N = 0$ and get a description of all solutions, which we now present.

Let

$$\mathcal{C} = \underbrace{\begin{pmatrix} I_p & I_p & \cdots & I_p \end{pmatrix}}_{N \text{ times}} \quad \text{and} \quad \mathcal{A} = \operatorname{diag}(A_1^*, A_2^*, \ldots, A_N^*). \tag{4.3}$$

The matrix $\mathbf{G}$ satisfies the Stein equation

$$\mathbf{G} - \mathcal{A}^* \mathbf{G} \mathcal{A} = \mathcal{C}^* \mathcal{C} \tag{4.4}$$

Proposition 4.2. *Define (with $\star$ product performed block-wise)*

$$\Theta(Z) = I_p - (I_p - Z) \star \begin{pmatrix} (I_p - ZA_1^*)^{-\star} & \cdots & (I_p - ZA_N^*)^{-\star} \end{pmatrix} \mathbf{G}^{-1} \begin{pmatrix} (I_p - A_1)^{-1} \\ \vdots \\ (I_p - A_N)^{-1} \end{pmatrix} \tag{4.5}$$

It holds that

$$\Theta(A_j) = 0, \quad j = 1, \ldots, N. \tag{4.6}$$

Proof. We have

$$
\begin{aligned}
&(I_p - Z)\star C \star \begin{pmatrix} (I_p - ZA_1^*)^{-\star} & 0 & 0 & \cdots & 0 \\ 0 & (I_p - ZA_2^*)^{-\star} & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & & & 0 \\ 0 & 0 & \cdots & & (I_p - ZA_N^*)^{-\star} \end{pmatrix}(A_1) = \\
&= \underbrace{\begin{pmatrix} I_p - Z & I_p - Z & \cdots \end{pmatrix}}_{N \text{ times}} \star \begin{pmatrix} \sum_{n=0}^\infty Z^n A_1^{n*} & 0 & 0 & \cdots & 0 \\ 0 & \sum_{n=0}^\infty Z^n A_2^{n*} & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & & & 0 \\ 0 & 0 & \cdots & & \sum_{n=0}^\infty Z^n A_N^{n*} \end{pmatrix}(A_1) \\
&= \begin{pmatrix} \sum_{n=0}^\infty (Z^n - Z^{n+1})A_1^{n*} & \sum_{n=0}^\infty (Z^n - Z^{n+1})A_2^{n*} & \cdots & \sum_{n=0}^\infty (Z^n - Z^{n+1})A_N^{n*} \end{pmatrix}(A_1) \\
&= \begin{pmatrix} (I_p - A_1)G_{11} & (I_p - A_1)G_{12} & \cdots & (I_p - A_1)G_{1N} \end{pmatrix} \\
&= (I_p - A_1)\begin{pmatrix} G_{11} & G_{12} & \cdots & G_{1N} \end{pmatrix}.
\end{aligned}
$$

Hence (here too with $\times$ denoting regular matrix multiplication)

$$
\begin{aligned}
\Theta(A_1) &= I_p - (I_p - A_1)\begin{pmatrix} G_{11} & G_{12} & \cdots & G_{1N} \end{pmatrix}\mathbf{G}^{-1} \times \\
&\quad \times \begin{pmatrix} (I_p - A_1)^{-1} & 0 & 0 & \cdots & 0 \\ 0 & (I_p - A_2)^{-1} & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & & & 0 \\ 0 & 0 & \cdots & & (I_p - A_N)^{-1} \end{pmatrix} C^* \\
&= I_p - (I_p - A_1)\begin{pmatrix} I_p & 0 & \cdots & 0 \end{pmatrix} \times \\
&\quad \times \begin{pmatrix} (I_p - A_1)^{-1} & 0 & 0 & \cdots & 0 \\ 0 & (I_p - A_2)^{-1} & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & & & 0 \\ 0 & 0 & \cdots & & (I_p - A_N)^{-1} \end{pmatrix} C^* \\
&= I_p - \begin{pmatrix} I_p & 0 & \cdots & 0 \end{pmatrix} C^* \\
&= 0.
\end{aligned}
$$

The same argument works of course for $A_2, \ldots, A_N$. □

We now set

$$
\psi(z) = \Theta(zI_p) = I_p - (1 - z)C(I_{Np} - zA)^{-1}\mathbf{G}^{-1}(I_{Np} - A^*)^{-1}C^*
$$

Proposition 4.3. *It holds that*

$$
\frac{I_p - \psi(z)\psi(w)^*}{1 - z\overline{w}} = C(I_{Np} - zA)^{-1}\mathbf{G}^{-1}(I_{Np} - \overline{w}A^*)^{-1}C^* \tag{4.7}
$$

and in particular ψ is a rational inner function and so the operation of multiplication by ψ is an isometry from $\mathbf{H}_2(\mathbb{D}, \mathbb{C}^{p\times p})$ into itself.

(4.7) is a classical computation based on the identity (4.4), which originates with L. de Branges' work; see [36] for the latter (see also [40]). We refer to [1, Exercise 7.1.17 p. 375 and p. 402] for a recent presentation of the computation. Rather than proving (4.7) we present a minimal realization of ψ and compute its associated Hermitian matrix, as in Theorem 3.12.

Theorem 4.4. *With $\mathcal{A}$ and $\mathcal{C}$ as in (4.3) and*

$$\mathcal{B} = \mathbf{G}^{-1}(I_{Np} - \mathcal{A}^*)^{-1}\mathcal{C}^* \tag{4.8}$$

$$\mathcal{D} = I_p - \mathcal{C}\mathbf{G}^{-1}(I_{Np} - \mathcal{A}^*)^{-1}\mathcal{C}^* \tag{4.9}$$

we have

$$\psi(z) = \mathcal{D} + z\mathcal{C}(I_{Np} - z\mathcal{A})^{-1}\mathcal{B}$$

and

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}^* \begin{pmatrix} \mathbf{G} & 0 \\ 0 & I_p \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} \mathbf{G} & 0 \\ 0 & I_p \end{pmatrix}. \tag{4.10}$$

Proof. We set $A = \mathrm{diag}(A_1, A_2, \ldots, A_N)$. The $(2,1)$ block equality in (4.10) is

$$\mathcal{A}^*\mathbf{G}\mathcal{A} + \mathcal{C}^*\mathcal{C} = \mathbf{G}$$

which is (4.4). The $(1,2)$ block amounts to

$$\mathcal{A}^*\mathbf{G}\mathcal{B} + \mathcal{C}^*\mathcal{D} = 0,$$

which is equivalent to

$$A\mathbf{G}(I_{Np} - A)^*\mathbf{G}^{-1}(I_{Np} - A)^{-1}\mathcal{C} + \mathcal{C}^*\left(I_p - \mathcal{C}\mathbf{G}^{-1}(I_{Np} - A)^{-1}\mathcal{C}^*\right) = 0.$$

Equivalently

$$\mathcal{C}^* + \left(A\mathbf{G}(I_{Np} - A^*) + \mathcal{C}^*\mathcal{C}\right)\mathbf{G}^{-1}(I_{Np} - A)^{-1}\mathcal{C}^* = 0,$$

i.e., after using (4.4)

$$\mathcal{C}^* + (A - I_{Np})\mathbf{G}\mathbf{G}^{-1}(I_{Np} - A)^{-1}\mathcal{C}^* = 0,$$

which clearly holds. To conclude we verify that identity holds in the $(2,2)$-block, i.e. that we have

$$\begin{aligned}I_p &= \mathcal{C}(I_{Np}-A^*)^{-1}\mathbf{G}^{-1}(I_{Np}-A)\mathbf{G}(I_{Np}-A^*)\mathbf{G}^{-1}(I_{Np}-A)^{-1}\mathcal{C}^*+\\ &\quad + I_p - \mathcal{C}(I_{Np}-A^*)^{-1}\mathbf{G}^{-1}\mathcal{C}^* - \mathcal{C}\mathbf{G}^{-1}(I_{Np}-A)^{-1}\mathcal{C}^*+\\ &\quad + \mathcal{C}(I_{Np}-A^*)^{-1}\mathbf{G}^{-1}\mathcal{C}^*\mathcal{C}\mathbf{G}^{-1}(I_{Np}-A)^{-1}\mathcal{C}^*.\end{aligned}$$

This amounts to check that

$$\mathcal{C}(I_{Np}-A^*)^{-1}\mathbf{G}^{-1}\Delta\mathbf{G}^{-1}(I_{Np}-A)^{-1}\mathcal{C}^*=0$$

where

$$\Delta=(I_{Np}-A)\mathbf{G}(I_{Np}-A^*)+\mathcal{C}^*\mathcal{C}-(I_{Np}-A)\mathbf{G}-\mathbf{G}(I_{Np}-A^*)=0.$$

But it is readily seen that $\Delta=0$ using (4.4). □

We now come back to the original interpolation problem.

Proposition 4.5. *Let $A_1, A_2, \ldots, A_N \in \mathbb{K}$ be such that the matrix $\mathbf{G}$ (defined by (4.2)) is strictly positive. Then, $F \in \mathbf{H}_2(\mathbb{K})$ vanishes at $A_1, A_2, \ldots, A_N$ if and only if it is in the range of M_Θ.*

Proof. It follows by the characterization (3.4) of the inner product in $\mathbf{H}_2(\mathbb{K})$ that the operator M_Θ of star-multiplication on the left by Θ is an isometry from $\mathbf{H}_2(\mathbb{K})$ into itself. Thus (and taking into account that M_Θ is an isometry)

$$\begin{aligned}\mathbf{H}_2(\mathbb{K}) &= \operatorname{Ran}(I-M_\Theta M_\Theta^*)\oplus\operatorname{Ran}(M_\Theta M_\Theta^*)\\ &= \operatorname{Ran}(I-M_\Theta M_\Theta^*)\oplus\operatorname{Ran}M_\Theta.\end{aligned}$$

To characterize $\operatorname{Ran}(I-M_\Theta M_\Theta^*)$ we first remark that $\frac{I_p-\psi(z)\psi(w)^*}{1-z\overline{w}}$ is the reproducing kernel of $\mathbf{H}_2(\mathbb{D},\mathbb{C}^{p\times p})\ominus\psi\mathbf{H}_2(\mathbb{D},\mathbb{C}^{p\times p})$ and that

$$\mathbf{H}_2(\mathbb{D},\mathbb{C}^{p\times p})\ominus\psi\mathbf{H}_2(\mathbb{D},\mathbb{C}^{p\times p})=\operatorname{Ran}(I-M_\psi M_\psi^*). \tag{4.11}$$

From (4.7) follows that $\operatorname{Ran}(I-M_\psi M_\psi^*)$ is spanned by the functions $(I_p-zA_j^*)^{-1}$ with coefficients on the right belonging to $\mathbb{C}^{p\times p}$, and this ends the proof since

$$\left((I_p-ZA_j^*)^{-\star}D\right)(zI_p)=(I_p-zA_j^*)^{-1}D,\quad j=1,\ldots,N\quad\text{and}\quad D\in\mathbb{C}^{p\times p}. \tag{4.12}$$

This concludes the proof since a function $F\in\mathbf{H}_2(\mathbb{K})$ vanishes at $A_1,\ldots,A_N$ if and only if it is orthogonal to the functions (4.12). □

Combining with the beginning of the section we have:

Theorem 4.6. *Let $A_1, A_2, \ldots, A_N \in \mathbb{K}$ be such that the matrix G (defined by (4.2)) is strictly positive, and let $B_1, \ldots, B_N \in \mathbb{C}^{p\times p}$. Then F satisfies the interpolation conditions (4.1)*

$$F(A_j) = B_j, \quad j = 1, \ldots, N \tag{4.13}$$

if and only if F is of the form

$$F = F_{\min} + \Theta \star G \tag{4.14}$$

where

$$F_{\min} = \begin{pmatrix} (I_p - ZA_1^*)^{-\star} & (I_p - ZA_2^*)^{-\star} & \cdots & (I_p - ZA_N^*)^{-\star} \end{pmatrix} \mathbf{G}^{-1} \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix}$$

and G runs through $\mathbf{H}_2(\mathbb{K})$. The decomposition (4.14) is orthogonal.

5. Schur multipliers

5.1. Definition

We denote by $\mathcal{S}_p$ the Schur class of $\mathbb{C}^{p\times p}$-valued Schur multipliers, meaning that the operator M_S of multiplication by $S \in \mathcal{S}_p$ on the left is a contraction from the Hardy space $(\mathbf{H}_2(\mathbb{D}))^p$. In this section we consider the case of $\mathbb{C}^{p\times p}$-valued Schur multipliers for the space $\mathbf{H}_2(\mathbb{K})$. The operator-valued case, where now S takes values in $\mathcal{L}(\mathfrak{H}, \mathbb{C}^{p\times p})$ for some Hilbert space $\mathfrak{H}$ will be considered in Section 5.6.

Theorem 5.1. *Let $S(Z) = \sum_{j=0}^{\infty} Z^j S_j$. Then the kernel*

$$K_S(Z, W) = \sum_{k=0}^{\infty} Z^k (I_p - S(Z)S(W)^*) W^{*k} \tag{5.1}$$

is positive definite on $\mathbb{K}$ if and only if $s(z) := S(zI_p)$ belongs to the Schur class $\mathcal{S}_p$.

Proof. The "only if" part: the positivity of the kernel (5.1) means that for any choice of matrices $P_1, \ldots, P_n \in \mathbb{K}$, the following matrix is positive semidefinite:

$$\left(\sum_{k=0}^{\infty} P_i^k (I_p - S(P_i)S(P_j)^*) P_j^{*k} \right)_{i,j=1,\ldots n} \geq 0. \tag{5.2}$$

Letting $P_i = z_i I_p$ ($|z_i| < 1$) gives

$$\left(\sum_{k=0}^{\infty} z_i^k (I_p - s(z_i)s(s_j)^*)\overline{z_j}^k\right)_{i,j=1,\ldots,n} = \left(\frac{I_p - s(z_i)s(s_j)^*}{1 - z_i\overline{z_j}}\right)_{i,j=1,\ldots,n} \geq 0.$$

Therefore, the kernel

$$K_s(z,w) = \frac{I_p - s(z)s(w)^*}{1 - z\overline{w}}$$

is positive on $\mathbb{D}$ and hence s is a $\mathbb{C}^{p\times p}$-valued Schur-class function.

Conversely, let us assume that the function $s(z) = \sum_{j=0}^{\infty} S_j z^j$ belongs to $\mathcal{S}_p$. Then it admits a coisometric (observable) realization

$$s(z) = S_0 + zC(I_{\mathcal{X}} - zA)^{-1}B = S_0 + \sum_{j=1}^{\infty} z^j CA^{j-1}B$$

with the state space $\mathcal{X}$ (the de Branges-Rovnyak space $\mathcal{H}(K_s)$, for example). Therefore,

$$S_j = CA^{j-1}B \quad \text{for all} \quad j \geq 1.$$

Then

$$S(Z) = S_0 + \sum_{j=1}^{\infty} Z^j CA^{j-1}B = S_0 + ZC \star (I_{\mathcal{X}} - ZA)^{-\star}B.$$

Then the kernel (5.1) is positive definite on $\mathbb{K}$, by Theorem 5.9, below. □

Having in view the quaternionic setting we recall the following result, which complements Theorem 5.1. For a proof, see [39].

Theorem 5.2. *Let s be a matrix-valued function defined on a subset of the open unit disk, having an accumulation point in the open unit disk (as opposed to on the unit circle). Assume that the kernel $K_s(z,w)$ is positive-definite on Ω. Then s is the restriction to Ω of a uniquely defined function analytic and contractive in the open unit disk.*

Theorem 5.2 is used in the theory of slice-holomorphic function taking Ω to be some open subinterval of $(-1,1)$. See [13].

Theorem 5.3. *Let S be a function from $\mathbb{K}$ into $\mathbb{C}^{p\times p}$. The following are equivalent:*
(1) *The function*

$$K_S(A,B) = \sum_{n=0}^{\infty} A^n (I_p - S(A)S(B)^*)B^{*n} \tag{5.3}$$

is positive definite on $\mathbb{K}$.
(2) *The function* S *is a power series:* $S(Z) = \sum_{n=0}^{\infty} Z^n S_n$ *and the operator of* $\star$ *multiplication by* S *on the left is a contraction from* $\mathbf{H}_2(\mathbb{K})$ *into itself.*

Proof. Assume first the kernel (5.3) positive definite in $\mathbb{K}$. Setting $A = B$, and since $K_S(A, A) \geq 0$, we get that

$$\sum_{n=0}^{\infty} A^n S(A) S(A)^* A^{*n} \leq \sum_{n=0}^{\infty} A^n A^{*n} < \infty, \quad A \in \mathbb{K},$$

and so the function $\sum_{n=0}^{\infty} Z^n S(A)^* A^{*n} C$ belongs to $\mathbf{H}_2(\mathbb{K})$ for every $C \in \mathbb{C}^{p\times p}$. The positivity of the kernel then implies that the linear relation spanned by the pairs

$$(\sum_{n=0}^{\infty} Z^n A^{*n} C, \sum_{n=0}^{\infty} Z^n S(A)^* A^{*n} C) \subset \mathbf{H}_2(\mathbb{K}) \times \mathbf{H}_2(\mathbb{K}) \tag{5.4}$$

extends to the graph of a contraction, say T. One then computes

$$T^*(Z^m D) = S(Z) \star (Z^m D).$$

Conversely, assume that the operator M_S of $\star$ multiplication by S is a contraction from $\mathbf{H}_2(\mathbb{K})$ into itself. We compute $M_S^*(K(\cdot, A)C)$ for $A \in \mathbb{K}$ and $C \in \mathbb{C}^{p\times p}$:

$$\begin{aligned}
[M_S^*(K(\cdot, A)C), Z^m D]_{\mathbf{H}_2(\mathbb{K})} &= [K(\cdot, A,)C, Z^m S(Z) D]_{\mathbf{H}_2(\mathbb{K})} \\
&= [Z^m S(Z) D, K(\cdot, A,)C]^*_{\mathbf{H}_2(\mathbb{K})} \\
&= (C^* A^m S(A)^* D)^* \\
&= D^* S(A)^* A^{*m} C
\end{aligned}$$

and so, by continuity,

$$\begin{aligned}
[M_S^*(K(\cdot, A)C), K(\cdot, B)D]_{\mathbf{H}_2(\mathbb{K})} &= \sum_{n=0}^{\infty} [K(\cdot, A,)C, Z^n S(Z) B^{*n} D]_{\mathbf{H}_2(\mathbb{K})} \\
&= \sum_{n=0}^{\infty} D^* B^n S(A)^* A^{*n} C
\end{aligned}$$

and so

$$M_S^*(K(\cdot, A)C) = \sum_{n=0}^{\infty} Z^n S(A)^* A^{*n} C \tag{5.5}$$

and similarly

$$M_S^*(K(\cdot,B)D)=\sum_{n=0}^\infty Z^nS(B)^*B^{*n}D.$$

Thus

$$[M_S^*(K(\cdot,B)D),M_S^*(K(\cdot,A)C)]=C^*\sum_{n=0}^\infty A^nS(A)S(B)^*B^{*n}D.$$

The result follows by expressing that M_S is a contraction. $\square$

Remark 5.4. We note that the block matrix representation of M_S is

$$\begin{pmatrix} S_0 & 0 & 0 & \cdots & \cdots \\ S_1 & S_0 & 0 & \ddots & \cdots \\ S_2 & S_1 & S_0 & 0 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix} \tag{5.6}$$

This follows from the convolution formula

$$G_n=\sum_{u=0}^n S_uF_{n-u},\quad u=0,1,\dots \tag{5.7}$$

for the coefficients $G_0,G_1,\dots\in\mathbb{C}^{p\times p}$ of $S\star F=\sum_{n=0}^\infty Z^nG_n$, where $F=\sum_{n=0}^\infty Z^nF_n\in\mathbf{H}_2(\mathbb{K})$.

By the definition of the inner product, the contractivity is expressed as

$$\sum_{n=0}^\infty\left(\sum_{u=0}^n S_u^*F_{n-u}\right)^*\left(\sum_{u=0}^n S_u^*F_{n-u}\right)\le\sum_{n=0}^\infty F_n^*F_n. \tag{5.8}$$

Remark 5.5. (analytic extension) The positivity of the kernel (5.3) implies that M_S is a contraction from $\mathbf{H}_2(\mathbb{K})$ into itself. In particular $S=M_S(I_p)\in\mathbf{H}_2(\mathbb{K})$, and is of the form $S=\sum_{n=0}^\infty Z^nS_n$. When $p=1$ a much stronger result holds (see [39]): if (for $p=1$), S is supposed **defined** on an open set, say Ω, of $\mathbb{D}$ (or more generally a subset of $\mathbb{D}$ having an accumulation point in $\mathbb{D}$) and if the corresponding kernel (5.3) is positive in Ω, then S is the restriction to Ω of a uniquely defined function S analytic and contractive in the open unit disk. We do not know if there is a counterpart of the result for $p>1$.

Remark 5.6. We note that $K_S(A,B)$ is the unique solution of the matrix equation

$$K_S(A,B)-AK_S(A,B)B^*=I_p-S(A)S(B)^*,\quad A,B\in\mathbb{K}. \tag{5.9}$$

This "structural identity" is the tool needed to extend the theory of $\mathcal{H}(S)$ spaces (see e.g., [44,45]) from the complex scalar case to the present setting. In the case of quaternions a similar equation and analysis hold; see [10]. But an important difference is that $q\overline{q} \in [0, \infty)$ for a quaternion q with conjugate $\overline{q}$, corresponding here to (3.36).

Note that, with M_A^r being the operator of multiplication *on the right* by A, i.e.

$$M_A^r(Z^m B) = Z^m BA, \quad A, B \in \mathbb{C}^{p\times p},\ n \in \mathbb{N},$$

we have:

$$M_S M_Z = M_Z M_S \tag{5.10}$$

$$M_S M_A^r = M_A^r M_S \tag{5.11}$$

We now present a version of the Bochner-Chandrasekharan theorem (see [30, Theorem 72, p. 144]) in our present setting.

Theorem 5.7. *Let T be linear and contractive from $\mathbf{H}_2(\mathbb{K})$ into itself, and satisfying*

$$TM_Z = M_Z T \tag{5.12}$$

$$TM_A^r = M_A^r T, \quad \forall A \in \mathbb{C}^{p\times p}. \tag{5.13}$$

Then T is a Schur multiplier.

Proof. Let $A \in \mathbb{C}^{p\times p}$ and $n \in \mathbb{N}$. We have from (5.12)

$$TM_{Z^n}A = Z^n T(A), \tag{5.14}$$

and

$$T(A) = T(I_p A) = T(I_p)A. \tag{5.15}$$

Thus

$$\begin{aligned} T\left(\sum_{n=0}^{\infty} Z^n F_n\right) &= \sum_{n=0}^{\infty} Z^n T(F_n) \qquad \text{(using} \quad \text{(5.12))} \\ &= \sum_{n=0}^{\infty} Z^n T(I_p) F_n \qquad \text{(using} \quad \text{(5.13))} \\ &= T(I_n) \star \left(\sum_{n=0}^{\infty} Z^n F_n\right) \quad \square \end{aligned}$$

Theorem 5.8. *Let $S = \sum_{n=0}^{\infty} Z^n S_n$ be a Schur multiplier, with $S_n \in \mathbb{C}^{p\times p}$. Then, $\widetilde{S} = \sum_{n=0}^{\infty} Z^n S_n^*$ is also a Schur multiplier*

Proof. We proceed in a number of steps.

STEP 1: *Assume S be a Schur multiplier. Then the function $S(zI_p)$ is analytic and contractive in the open unit disk.*

Setting $A = zI_p$ and $B = wI_p$ in (5.9) with $z, w \in \mathbb{C}$ we get that the kernel

$$\sum_{n=0}^{\infty} z^n (I_p - S(zI_p)S(I_pw)^*)\overline{w}^n = \frac{I_p - S(zI_p)S(I_pw)^*}{1 - z\overline{w}}$$

is positive definite in the open unit disk (since elements of the form zI_p with z in the open unit disk $\mathbb{D}$ are inside $\mathbb{K}$) and so series $S(zI_p) = \sum_{n=0}^{\infty} z^n S_n$ converges in the open unit disk $\mathbb{D}$ and is contractive there (and so is a $\mathbb{C}^{p\times p}$-valued Schur function of the open unit disk).

STEP 2: *The operator of multiplication by the function*

$$S(\overline{z})^* = \sum_{n=0}^{\infty} z^n S_n^* \tag{5.16}$$

on the right is a contraction from $\mathbf{H}_2(\mathbb{D})^{p\times p}$ into itself (i.e. the function $S(\overline{z})^*$ is still a $\mathbb{C}^{p\times p}$-valued Schur function of the open unit disk).

Using the integral representation of the inner product in $\mathbf{H}_2(\mathbb{D})^{p\times p}$ and since

$$(S(e^{-it})^*)^* S(e^{-it})^* \leq I_p, \quad a.e. \text{ on the unit circle } \mathbb{T},$$

we get

$$\frac{1}{2\pi}\int_0^{2\pi} (f(e^{it}))^* (S(e^{-it})^*)^* S(e^{-it})^* f(e^{it}) dt \leq \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})^* f(e^{it}) dt$$

where $f \in \mathbf{H}_2(\mathbb{D})^{p\times p}$ and where the inequality is between matrices, and so

$$\operatorname{Tr}\left(\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})^* (S(e^{-it})^*)^* S(e^{-it})^* f(e^{it} dt\right) \leq \operatorname{Tr}\left(\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})^* f(e^{it}) dt\right),$$

which expresses the asserted contractivity.

STEP 3: *The matrix representation of the operator multiplication by the function $S(\overline{z})^*$ on the right has block matrix representation the block-Toeplitz operator with block matrix representation*

$$\begin{pmatrix} S_0^* & 0 & 0 & \cdots & \cdots \\ S_1^* & S_0^* & 0 & \ddots & \cdots \\ S_2^* & S_1^* & S_0^* & 0 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \vdots \end{pmatrix} \tag{5.17}$$

As in Remark 5.4 this follows from the convolution formula

$$G_n = \sum_{u=0}^{n} S_u^* F_{n-u}, \quad u = 0, 1, \ldots$$

for the coefficients $G_0, G_1, \ldots \in \mathbb{C}^{p\times p}$ of $S(\overline{z})^* f(z) = \sum_{n=0}^{\infty} z^n G_n$, where $f(z) = \sum_{n=0}^{\infty} z^n F_n \in \mathbf{H}_2(\mathbb{C}^{p\times p})$).

STEP 4: $\widetilde{S}$ *is Schur multiplier in* $\mathbf{H}_2(\mathbb{K})$

This just follows from (5.8) with the S_n^* in lieu of the S_n since the contractivity is expressed in the same way on the level of the coefficients in both cases. □

5.2. Leech theorem

The main results in this section are based on discussions of the authors with Professor Bolotnikov. We thank Professor Bolotnikov for allowing us to include the material presented in this subsection.

Theorem 5.9. *Let $\mathcal{X}$ be a Hilbert space and let us assume that the operator*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \mathbb{C}^p \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \mathbb{C}^p \end{pmatrix} \tag{5.18}$$

is a contraction. Then the power series

$$S(Z) = D + ZC \star (I_{\mathcal{X}} - ZA)^{-\star} B = D + \sum_{k=0}^{\infty} Z^{k+1} C A^k B \tag{5.19}$$

is a Schur multiplier. Moreover, the kernel K_S (5.1) can be expressed as

$$K_S(Z, W) = \Gamma(Z)\Gamma(W)^* + \sum_{k=0}^{\infty} \Lambda_k(Z)(I - UU^*)\Lambda_k(W)^* \tag{5.20}$$

where

$$\Gamma(Z) = \sum_{k=0}^{\infty} Z^k C A^k \quad \textit{and} \quad \Lambda_k(Z) = Z^k \begin{pmatrix} Z\Gamma(Z) & I_p \end{pmatrix}. \tag{5.21}$$

Proof. Since U is a contraction, A is bounded by 1 in norm, and the $\mathcal{L}(\mathcal{X}, \mathbb{C}^p)$–valued power series $\sum_{k=0}^{\infty} Z^n CA^n$ converges in norm for each $Z \in \mathbb{K}$. It follows from (5.19) and (5.21) that

$$\Gamma(Z) = C + Z\Gamma(Z)A, \quad S(Z) = D + ZC\Gamma(Z)B$$

and subsequently,

$$\Lambda_k(Z)U = Z^k \begin{pmatrix} \Gamma(Z) & S(Z) \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
&\Gamma(Z)\Gamma(W)^* + \sum_{k=0}^{\infty} \Lambda_k(Z)(I - UU^*)\Lambda_k(W)^* \\
&= \Gamma(Z)\Gamma(W)^* + \sum_{k=0}^{\infty} Z^k \left(I_p + Z\Gamma(Z)\Gamma(W)^* W^*\right) W^{*k} \\
&\quad - \sum_{k=0}^{\infty} Z^k \left(\Gamma(Z)\Gamma(W)^* + S(Z)S(W)^*\right) W^{*k} \\
&= \sum_{k=0}^{\infty} Z^k \left(I_p - S(Z)S(W)^*\right) W^{*k} = K_S(Z, W)
\end{aligned}$$

which confirms (5.20). Since the kernel on the right side of (5.20) is positive on $\mathbb{K}$, the proof is complete. □

Our next result is Leech's factorization theorem in the present setting.

Theorem 5.10. *Given two power series $P(Z)$ and $Q(Z)$, the following are equivalent:*

(1) There is a Schur multiplier $S(Z)$ such that

$$Q(Z) = (P \star S)(Z) \quad \textit{for all} \quad Z \in \mathbb{K}. \tag{5.22}$$

(2) The kernel

$$K_{P,Q}(Z, W) = \sum_{k=0}^{\infty} Z^k (P(Z)P(W)^* - Q(Z)Q(W)^*) W^{*k} \tag{5.23}$$

is positive on $\mathbb{K}$.

Proof. One direction is easy: if (5.22) holds for some

$$S(Z)=\sum_{k=0}^{\infty}Z^kS_k,$$

then for every $Z,\, W\in\mathbb{K}$ and any $n\geq 0$, we have $Z^nP(z)=P(Z)\star Z^n$ and

$$Z^nQ(Z)=Z^n(P\star S)(Z)=\sum_{k=0}^{\infty}Z^{n+k}P(Z)S_k=P(Z)\star(Z^nS(Z)),$$

and subsequently,

$$\begin{aligned}K_{P,Q}(Z,W)=&\sum_{n=0}^{\infty}Z^n(P(Z)P(W)^*-Q(Z)Q(W)^*)W^{*n}\\=&\sum_{n=0}^{\infty}P(Z)\star(Z^nW^{*n}-Z^nS(Z)S(W)^*W^{*n})\star_r P(W)^*\\=&P(Z)\star K_S(Z,W)\star_r P(W)^*,\end{aligned}$$

and the latter kernel is positive, as is seen on translating the above equality on the level of the coefficients; see [11, Proposition 5.3 p. 855] for a similar argument for quaternionic kernels.

Conversely, let us assume that the kernel (5.23) is positive on $\mathbb{K}$. Then (see e.g. [50]) it admits a Kolmogorov factorization, i.e., there exists a Hilbert space $\mathcal{X}$ and an $\mathcal{L}(\mathcal{X},\mathbb{C}^p)$-valued power series $H(Z)$ converging weakly for all $Z\in\mathbb{K}$ such that

$$K_{P,Q}(Z,W)=H(Z)H(W)^*\quad\text{for all}\quad Z,W\in\mathbb{K}.\tag{5.24}$$

Combining the latter identity with (5.23) we conclude that

$$H(Z)H(Z)^*-ZH(Z)H(W)^*W^*=P(Z)P(W)^*-Q(Z)Q(W)^*\tag{5.25}$$

for all $Z,W\in\mathbb{K}$. The latter identity tells us that the linear map

$$V:\begin{pmatrix}H(W)^*W^*x\\P(W)^*x\end{pmatrix}\mapsto\begin{pmatrix}H(W)^*x\\Q(W)^*x\end{pmatrix}\tag{5.26}$$

extends by linearity and continuity to an isometry (still denoted by V) from

$$\mathcal{D}_V=\bigvee_{W\in\mathbb{K},x\in\mathbb{C}^p}\begin{pmatrix}H(W)^*W^*x\\P(W)^*x\end{pmatrix}\subset\mathcal{X}\oplus\mathbb{C}^p\quad\text{onto}$$

$$\mathcal{R}_V=\bigvee_{W\in\mathbb{K},x\in\mathbb{C}^p}\begin{pmatrix}H(W)^*x\\Q(W)^*x\end{pmatrix}\subset\mathcal{X}\oplus\mathbb{C}^p.$$

Let us extend V to a contraction

$$\widehat{V} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \mathbb{C}^p \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \mathbb{C}^p \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} H(W)^* W^* x \\ P(W)^* x \end{pmatrix} = \begin{pmatrix} H(W)^* x \\ Q(W)^* x \end{pmatrix},$$

from which it follows that

$$A^* H(W)^* W^* x + C^* P(W)^* x = H(W)^* x, \tag{5.27}$$

$$B^* H(W)^* W^* x + D^* P(W)^* x = Q(W)^* x. \tag{5.28}$$

Since A is a contraction, we recover $H(W)^* x$ from (5.27) as

$$H(W)^* x = \sum_{n=0}^{\infty} A^{*n} C^* P(W)^* W^* x.$$

Substituting the latter representation into (5.28) gives

$$Q(W)x = D^* P(W)^* x + B^* \sum_{n=0}^{\infty} A^{*n} C^* P(W)^* W^{*(n+1)} x.$$

Taking adjoints and using the arbitrariness of $x \in \mathbb{C}^p$ we get

$$\begin{aligned} Q(W) &= P(W) D + \sum_{n=0}^{\infty} W^{n+1} P(W) C A^n B \\ &= P(W) \star \left(D + \sum_{n=0}^{\infty} W^{n+1} C A^n B \right) \end{aligned} \tag{5.29}$$

holding for all $W \in \mathbb{K}$. The formula

$$S(W) = D + \sum_{n=0}^{\infty} W^{n+1} C A^n B$$

defines a Schur multiplier by Theorem 5.9. On the other hand, equality (5.29) means that (5.22) is in force. □

5.3. The coisometric realization

We associate to every Schur multiplier a co-isometric realization with state space the reproducing kernel Hilbert space module $\mathcal{H}(S)$ with reproducing kernel $K_S(A,B)$. The complex-variable case was first developed, using complementation theory by de Branges and Rovnyak in [33]. Here, we use the theory of linear relations as applied in [18] for the complex-variable setting and in [22, §5] for the time-varying case.

We consider the linear relation $\mathcal{R} \subset (\mathcal{H}(S) \oplus \mathbb{C}^{p\times p}) \times (\mathcal{H}(S)) \oplus \mathbb{C}^{p\times p})$ spanned by the pairs

$$\left(\begin{pmatrix} K_S(\cdot,A)A^*G \\ H\end{pmatrix}, \begin{pmatrix}(K_S(\cdot,A)-K_S(\cdot,0))G+K_S(\cdot,0)H \\ (S(A)^*-S(0)^*)A^*G+S(0)^*H\end{pmatrix}\right)$$

when A runs through $\mathbb{K}$ and G,H run through $\mathbb{C}^{p\times p}$.

Proposition 5.11. *$\mathcal{R}$ is densely defined and isometric, and thus extends to the graph of an everywhere defined isometry from $\mathbf{H}_2(\mathbb{K}) \oplus \mathbb{C}^{p\times p}$ into itself.*

Proof. Let

$$\left(\begin{pmatrix} K_S(\cdot,B)B^*E \\ F\end{pmatrix}, \begin{pmatrix}(K_S(\cdot,B)-K_S(\cdot,0))E+K_S(\cdot,0)H \\ (S(B)^*-S(0)^*)E+S(0)^*F\end{pmatrix}\right)$$

be another element of $\mathcal{R}$, with $B \in \mathbb{K}$ and $E,F \in \mathbb{C}^{p\times p}$. We want to show that

$$\begin{aligned}&\langle K_S(\cdot,B)B^*E, K_S(\cdot,A)A^*G\rangle_S + \operatorname{Tr} H^*F = \\ &\quad = \langle (K_S(\cdot,B)-K_S(\cdot,0))E+K_S(\cdot,0)F, (K_S(\cdot,A)-K_S(\cdot,0))G+K_S(\cdot,0)H\rangle_S + \\ &\qquad + \operatorname{Tr}((S(B)^*-S(0)^*)E+S(0)^*F)^*((S(A)^*-S(0)^*)G+S(0)^*H).\end{aligned}$$

Considering the $\mathbb{C}^{p\times p}$-valued forms associated to the inner products, the above equality can be rewritten in the form

$$G^*\square_1 E + G^*\square_2 F + H^*\square_3 E + H^*\square_4 F = 0,$$

for appropriate expressions $\square_j$, $j = 1,2,3,4$ which we now show to be equal to 0. We have

$$\begin{aligned}\square_1 &= -AK_S(A,B)B^* + K(A,B) - K_S(A,0) - K_S(0,B) + K_S(0,0) \\ &\qquad + (S(A)-S(0))(S(B)^*-S(0)^*) \\ &= I_p - S(A)S(B)^* - (I_p - S(A)S(0)^*) - (I_p - S(0)S(B)^*) + (I_p - S(0)S(0)^*) + \\ &\qquad + S(A)S(B)^* - S(A)S(0)^* - S(0)S(B)^* + S(0)S(0)^* \\ &= 0,\end{aligned}$$

where we have used (5.9) to go from the first equality to the second one.

Similarly

$$\begin{aligned}\square_2 &= K_S(A,0) - K_S(0,0) + (S(A) - S(0))S(0)^* \\ &= I_p - S(A)S(0)^* - I_p + S(0)S(0)^* + S(A)S(0)^* - S(0)S(0)^* \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\square_3 &= K_S(0,B) - K_S(0,0) + S(0)(S(B)^* - S(0)^*) \\ &= I_p - S(0)S(B)^* - I_p + S(0)S(0)^* + S(0)S(B)^* - S(0)S(0)^* \\ &= 0.\end{aligned}$$

As for $\square_4$ one has:

$$\square_4 = -I_p + K_S(0,0) + S(0)S(0)^* = 0.$$

To conclude we prove that the domain of $\mathcal{R}$ is dense in $\mathcal{H}(S) \oplus \mathbb{C}^{p\times p}$. Any element $\begin{pmatrix} F \\ C \end{pmatrix} \in \mathcal{H}(S) \oplus \mathbb{C}^{p\times p}$ orthogonal to the domain of $\mathcal{R}$ will satisfy

$$\mathrm{Tr}(G^*AF(A) + H^*C) = 0.$$

Letting $G = 0$ and H run through $\mathbb{C}^{p\times p}$ we get $C = 0$. We thus have $\mathrm{Tr}(G^*AF(A) = 0$ for all $G \in \mathbb{C}^{p\times p}$, and so $AF(A) = 0$ and so $aF(aI_p) = 0$ for all a near the origin and so that $F = 0$. $\square$

We write the isometry whose graph extends $\mathcal{R}$ in the form

$$\begin{pmatrix} \mathscr{T} & \mathscr{F} \\ \mathscr{G} & \mathscr{H} \end{pmatrix}^*,$$

that is

$$\mathscr{T}^*\,(K_S(\cdot,A)A^*G) = (K_S(\cdot,A) - K_S(\cdot,0))G \tag{5.30}$$

$$\mathscr{F}^*\,(K_S(\cdot,A)A^*G) = (S(A)^* - S(0)^*)G \tag{5.31}$$

$$\mathscr{G}^*H = K_S(\cdot,0)H \tag{5.32}$$

$$\mathscr{H}^*H = S(0)^*H. \tag{5.33}$$

Proposition 5.12. *Let $F \in \mathcal{H}(S)$, $A \in \mathbb{K}$ and $G \in \mathbb{C}^{p\times p}$. It holds that*

$$A(\mathscr{T}F)(A) = F(A) - F(0) \tag{5.34}$$

$$A(\mathscr{F}G)(A) = (S(A) - S(0))G. \tag{5.35}$$

$$\mathscr{G}F = F(0), \tag{5.36}$$

$$\mathscr{H}H = S(0)H. \tag{5.37}$$

Proof. Apply both sides of (5.30) to F to obtain

$$\langle F, \mathscr{T}^* (K_S(\cdot, A)A^*G)\rangle_S = \langle F, K_S(\cdot, A) - K_S(\cdot, 0))G\rangle_S, \tag{5.38}$$

that is

$$G^*A(\mathscr{T}F)(A) = G^*(F(A) - F(0)). \quad \square$$

Theorem 5.13. *Let S be a Schur multiplier. In the above notation we have*

$$S(A)G = \mathscr{H}G + \sum_{n=0}^{\infty} A^{n+1}\mathscr{G}\mathscr{T}^n\mathscr{F}(G), \quad G \in \mathbb{C}^{p\times p}. \tag{5.39}$$

Proof. Let C_A denote evaluation at $A \in \mathbb{K}$. Equation (5.34) gives

$$C_A = C_0 + AC_A\mathscr{T}$$

from which we get

$$C_A = \sum_{n=0}^{\infty} A^n C_0 \mathscr{T}^n$$

which converges in $\mathcal{H}(S)$ in the operator norm. Applying to $F = \mathscr{F}G$ we obtain

$$(\mathscr{F}G(A)) = \sum_{n=0}^{\infty} A^n C_0 \mathscr{T}^{n+1}\mathscr{F}G,$$

and so

$$A(\mathscr{F}G(A)) = \sum_{n=0}^{\infty} A^{n+1} C_0 \mathscr{T}^{n+1}\mathscr{F}G,$$

i.e.

$$S(A)G - S(0)G = \sum_{n=0}^{\infty} A^{n+1} C_0 \mathscr{T}^{n+1}\mathscr{F}G.$$

This concludes the proof since $C_0 = \mathscr{G}$ and $S(0) = \mathscr{H}$. $\square$

5.4. Operator ranges and complementation

In this section we give another proof of the realization theorem for Schur multipliers using complementation theory and the theory of operator ranges. We refer to [41, §4] and [55] for the latter. Similar arguments for the time-varying setting can be found in [22], and in [9] for the quaternionic setting. We first recall a well-known result on operator ranges. We provide a proof for completeness.

Theorem 5.14. *Let $\mathfrak{H}$ be a Hilbert space and let A be a bounded positive operator from $\mathfrak{H}$ into itself. Let π denote the orthogonal projection onto* $\ker A$. *Then the space* $\operatorname{ran}\sqrt{A}$ *endowed with the norm (called the range norm)*

$$\|\sqrt{A}h\|_{\operatorname{ran}\sqrt{A}} = \|(I_{\mathfrak{H}} - \pi)h\|_{\mathfrak{H}}, \quad h \in \mathfrak{H}, \tag{5.40}$$

is a Hilbert space. Furthermore,

$$\|Ah\|_{\operatorname{ran}\sqrt{A}} = \|\sqrt{A}h\|_{\mathfrak{H}}, \quad h \in \mathfrak{H}, \tag{5.41}$$

and the range of A is dense in the range of $\sqrt{A}$ in this norm. Finally in the associated inner product it holds that

$$\langle\sqrt{A}h, \sqrt{A}g\rangle_{\operatorname{ran}\sqrt{A}} = \langle(I_{\mathfrak{H}} - \pi)h, g\rangle_{\mathfrak{H}} \tag{5.42}$$

$$\langle Ah, Ag\rangle_{\operatorname{ran}\sqrt{A}} = \langle Ah, g\rangle_{\mathfrak{H}} \tag{5.43}$$

and

$$\langle Ah, \sqrt{A}g\rangle_{\operatorname{ran}\sqrt{A}} = \langle\sqrt{A}h, g\rangle_{\mathfrak{H}} \tag{5.44}$$

for $h, g \in \mathfrak{H}$.

Proof. That (5.40) is indeed a norm follows from

$$\sqrt{A}h = 0 \quad \iff \pi h = h.$$

Let now a Cauchy sequence $(\sqrt{A}h_n)$ in $\operatorname{ran}\sqrt{A}$. Then $(I_{\mathfrak{H}} - \pi)h_n$ is a Cauchy sequence in $\mathfrak{H}$ converging to an element, say k. We have by continuity of π that $k = (I_{\mathfrak{H}} - \pi)k$ and so $\sqrt{A}h_n)$ has limit $\sqrt{A}k$ and $\operatorname{ran}\sqrt{A}$ is closed in the range norm. Formula (5.42) follows from (5.40) by polarization. The last two formulas follow from

$$A(I_{\mathfrak{H}} - \pi) = A. \quad \square$$

We now recall the operator range characterization from [41] which we will use. Note that [41] consider one Hilbert space and we consider two possibly different Hilbert spaces, but the proof is the same for the latter case. Theorem 5.15 below is [41, Theorem 4.1 p. 275] with $\sqrt{I_{\mathfrak{H}} - TT^*}$ rather than T. We present a proof since the passage from one to the other involves *a priori* polar representations. We give a direct proof, explicitly adapted from [41]. See also [9] where it is presented in the quaternionic setting.

Theorem 5.15. *Let T be a contraction from the Hilbert space $\mathfrak{G}$ into the Hilbert space $\mathfrak{H}$. Then, $f \in \operatorname{ran} \sqrt{I_{\mathfrak{H}} - TT^*}$ if and only if*

$$\sup_{g\in\mathfrak{G}} \left(\|f + Tg\|^2_{\mathfrak{G}} - \|g\|^2_{\mathfrak{G}}\right) < \infty. \tag{5.45}$$

Proof. As mentioned above, the proof is directly adapted from the proof of [41, Theorem 4.1 p. 275] with $\sqrt{I_{\mathfrak{H}} - TT^*}$ rather than T. We have for $g \in \mathfrak{G}$ and $h \in \mathfrak{H}$

$$\begin{aligned}
\|\sqrt{I_{\mathfrak{H}} - TT^*}h + Tg\|^2_{\mathfrak{H}} - \|g\|^2_{\mathfrak{G}} &= \\
&= \|\sqrt{I_{\mathfrak{H}} - TT^*}h\|^2_{\mathfrak{H}} + \|Tg\|^2_{\mathfrak{H}} - \|g\|^2 + 2\mathrm{Re}\,\langle\sqrt{I_{\mathfrak{H}} - TT^*}h, Tg\rangle_{\mathfrak{H}} - \|g\|^2_{\mathfrak{G}} \\
&= \|h\|^2_{\mathfrak{H}} - \|T^*h\|^2_{\mathfrak{G}} + 2\mathrm{Re}\,\langle T^*h, \sqrt{I_{\mathfrak{G}} - T^*T}g\rangle_{\mathfrak{G}} - \|\sqrt{I_{\mathfrak{G}} - T^*T}g\|^2_{\mathfrak{G}} \\
&= \|h\|^2_{\mathfrak{H}} - \|T^*h - \sqrt{I_{\mathfrak{G}} - T^*T}g\|^2_{\mathfrak{G}} \\
&\le \|h\|^2_{\mathfrak{H}}
\end{aligned}$$

Conversely, let $f \in \mathfrak{H}$ such that the supremum in (5.45) is finite, and denote it by K. The choice $g = 0$ in (5.45) gives $\|u\|^2_{\mathfrak{H}} \le K$. We can write

$$\|f + Tg\|^2_{\mathfrak{H}} \le K + \|g\|^2_{\mathfrak{G}}, \quad \forall g \in \mathfrak{G}$$

and so

$$2\mathrm{Re}\,\langle f, Tg\rangle_{\mathfrak{H}} \le K - \|u\|^2_{\mathfrak{H}} + \|g\|^2_{\mathfrak{G}} - \|Tg\|^2_{\mathfrak{H}} \tag{5.46}$$

If $\langle f, Tg\rangle_{\mathfrak{H}} = 0$, this inequality is trivial. If $\langle f, Tg\rangle_{\mathfrak{H}} \neq 0$ and replace g by $tge^{i\theta}$ where $t \in \mathbb{R}$ and $\theta \in \mathbb{R}$ is chosen such that

$$e^{-i\theta}\langle f, Tg\rangle_{\mathfrak{H}} = |\langle f, Tg\rangle_{\mathfrak{H}}|.$$

Using the previous equality for every g we rewrite (5.46) as

$$2t|\langle f, Tg_1\rangle_{\mathfrak{H}}| \le K - \|f\|^2_{\mathfrak{H}} + t^2\|\sqrt{I_{\mathfrak{G}} - T^*T}g_1\|^2_{\mathfrak{G}}, \quad \forall t \in \mathbb{R},\ \forall g_1 \in \mathfrak{H}.$$

It follows that

$$0 \le t^2\|\sqrt{I_{\mathfrak{G}} - T^*T}g_1\|^2_{\mathfrak{G}} - 2t|\langle f, Tg_1\rangle_{\mathfrak{H}}| + K - \|f\|^2_{\mathfrak{H}}, \quad \forall t \in \mathbb{R},\ \forall g_1 \in \mathfrak{H}.$$

Thus

$$|\langle f, Tg_1\rangle_{\mathfrak{H}}| \leq \sqrt{K - \|f\|^2_{\mathcal{H}}}\left(\|\sqrt{I_{\mathfrak{G}} - T^*T}g_1\|_{\mathfrak{G}}\right).$$

By Theorem 5.14 we have

$$\|\sqrt{I_{\mathfrak{G}} - T^*T}g_1\|_{\mathfrak{G}} = \|(I_{\mathfrak{G}} - T^*T)g_1\|_{\mathrm{ran}\,(I_{\mathfrak{G}} - T^*T)}$$

and since the range of $I_{\mathfrak{G}} - T^*T$ is dense in the range of $\sqrt{I_{\mathfrak{G}} - T^*T}$ in the range norm, the map $g_1 \mapsto \langle Tg_1, u\rangle_{\mathfrak{H}}$ is continuous on the range of $\sqrt{I_{\mathfrak{G}} - T^*T}$. By Riesz representation theorem for continuous linear functionals on a Hilbert space there exists $r \in \mathfrak{H}$ such that

$$\begin{aligned}\langle Tg_1, u\rangle_{\mathfrak{H}} &= \langle (I_{\mathfrak{G}} - T^*T)g_1, \sqrt{I_{\mathfrak{G}} - T^*T}r\rangle_{\mathrm{ran}\,\sqrt{I_{\mathfrak{G}} - T^*T}} \\ &= \langle \sqrt{I_{\mathfrak{G}} - T^*T}g_1, r\rangle_{\mathfrak{G}}\end{aligned}$$

as follows from (5.44). It follows that $T^*f = \sqrt{I_{\mathfrak{G}} - T^*T}r$. So

$$\begin{aligned}f &= f - TT^*f + TT^*f \\ &= (I_{\mathfrak{H}} - TT^*)f + T\sqrt{I_{\mathfrak{G}} - T^*T}r \\ &= (I_{\mathfrak{H}} - TT^*)f + \sqrt{I_{\mathfrak{G}} - TT^*}r\end{aligned}$$

belongs to ran $\sqrt{I_{\mathfrak{H}} - TT^*}$. □

We now apply the previous results to our present setting.

Proposition 5.16. *It holds that*

$$\mathcal{H}(S) = \left\{ G \in \mathbf{H}_2(\mathbb{K})\ ;\ \sup_{H \in \mathbf{H}_2(\mathbb{K})} \|G + M_S H\|^2_{\mathbf{H}_2(\mathbb{K})} - \|H\|^2_{\mathbf{H}_2(\mathbb{K})} < \infty \right\}$$

and the above supremum is then the norm of G in $\mathcal{H}(S)$.

Proof. This is Theorem 5.15 with $\mathfrak{H} = \mathbf{H}_2(\mathbb{K})$ and $T = M_S$. □

Proposition 5.17. *It holds that*

$$\mathcal{H}(S) = \mathrm{ran}\,\sqrt{I_{\mathbf{H}_2(\mathbb{K})} - M_S M_S^*} \tag{5.47}$$

with the operator range norm

$$\|\sqrt{I_{\mathbf{H}_2(\mathbb{K})} - M_S M_S^*}F\| = \|(I_{\mathbf{H}_2(\mathbb{K})} - \pi)F\|_{\mathbf{H}_2(\mathbb{K})}$$

where π is the orthogonal projection onto ker $\sqrt{I_{\mathbf{H}_2(\mathbb{K})} - M_S M_S^*}$.

Proof. This is Theorem 5.14 with $\mathfrak{H} = \mathbf{H}_2(\mathbb{K})$ and $T = M_S$. □

Theorem 5.18.

$$\mathbf{H}_2(\mathbb{K}) = \operatorname{ran} M_S + \operatorname{ran} \sqrt{I_{\mathbf{H}_2(\mathbb{K})} - M_S M_S^*} \tag{5.48}$$

in the sense of complementation.

5.5. Another approach to the co-isometric realization

We study the co-isometric realization of a Schur multiplier using the results of the previous section, that is, using complementation theory. See [22] for similar computations in the time-varying setting. Recall that R_0 was defined in (2.14).

Proposition 5.19. *Let $F \in \mathcal{H}(S)$. Then, $R_0 F \in \mathcal{H}(S)$ and*

$$\|R_0 F\|^2_{\mathcal{H}(S)} \leq \|F\|^2_{\mathcal{H}(S)} - \|F(0)^* F(0)\| \tag{5.49}$$

Let $C \in \mathbb{C}^{p\times p}$. Then, $R_0 SC \in \mathcal{H}(S)$ and

$$\|R_0(SC)\|^2_{\mathcal{H}(S)} \leq \operatorname{Tr}\left(C(I_p - S(0)S(0)^*)C^*\right). \tag{5.50}$$

Proof. Let $G \in \mathbf{H}_2(\mathbb{K})$.

$$\begin{aligned}
&\|R_0 F + M_S G\|^2_{\mathbf{H}_2(\mathbb{K})} - \|G\|^2_{\mathbf{H}_2(\mathbb{K})}\\
&\qquad = \|M_Z(R_0 F + M_S G)\|^2_{\mathbf{H}_2(\mathbb{K})} - \|G\|^2_{\mathbf{H}_2(\mathbb{K})}\\
&\qquad = \|F - F_0 + M_S ZG)\|^2_{\mathbf{H}_2(\mathbb{K})} - \|G\|^2_{\mathbf{H}_2(\mathbb{K})}\\
&\qquad = \|-F_0\|^2_{\mathbf{H}_2(\mathbb{K})} - 2\mathrm{re}\langle F_0, F + M_S ZG\rangle_{\mathbf{H}_2(\mathbb{K})} +\\
&\qquad\quad + \underbrace{\|F + M_S ZG\|^2_{\mathbf{H}_2(\mathbb{K})} - \|ZG\|^2_{\mathbf{H}_2(\mathbb{K})}}_{\leq \|F\|^2_{\mathcal{H}(S)}}\\
&\qquad \leq \|F\|^2_{\mathcal{H}(S)} - \|F_0 F_0^*\|_{\mathbb{C}^{p\times p}}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\|R_0(SC)+M_SG\|^2_{\mathbf{H}_2(\mathbb{K})}-\|G\|^2_{\mathbf{H}_2(\mathbb{K})}\\
&\qquad=\|M_Z(R_0(SC+M_SG)\|^2_{\mathbf{H}_2(\mathbb{K})}-\|G\|^2_{\mathbf{H}_2(\mathbb{K})}\\
&\qquad=\|SC-S_0C+M_S(ZG)\|^2_{\mathbf{H}_2(\mathbb{K})}-\|G\|^2_{\mathbf{H}_2(\mathbb{K})}\\
&\qquad=\|-S_0C+M_S(C+ZG)\|^2_{\mathbf{H}_2(\mathbb{K})}-\|G\|^2_{\mathbf{H}_2(\mathbb{K})}\\
&\qquad=\|-S_0C\|^2_{\mathbf{H}_2(\mathbb{K})}+2\mathrm{Re}\,\langle -S_0C, M_S(C+ZG)\rangle^2_{\mathbf{H}_2(\mathbb{K})}+\\
&\qquad\quad+\|M_S(C+ZG)\|^2_{\mathbf{H}_2(\mathbb{K})}-\|M_ZG\|^2_{\mathbf{H}_2(\mathbb{K})}\\
&\qquad=\|S_0C\|^2_{\mathbf{H}_2(\mathbb{K})}-2\|S_0C\|^2_{\mathbf{H}_2(\mathbb{K})}+\|M_S(C+ZG)\|^2_{\mathbf{H}_2(\mathbb{K})}-\|G\|^2_{\mathbf{H}_2(\mathbb{K})}\\
&\qquad\le-\|S_0C\|^2_{\mathbf{H}_2(\mathbb{K})}+\|C+ZG\|^2_{\mathbf{H}_2(\mathbb{K})}-\|G\|^2_{\mathbf{H}_2(\mathbb{K})}\\
&\qquad=\|C\|^2_{\mathbf{H}_2(\mathbb{K})}-\|S_0C\|^2_{\mathbf{H}_2(\mathbb{K})}.\quad\square
\end{aligned}$$

5.6. A structure theorem

An important aspect of the theory of de Branges and de Branges-Rovnyak spaces (see [31–33]) is the weakening of isometric inclusion to contractive inclusion, and the associated notion of complementation, which replaces orthogonal sum. The results involve matrix-valued, or more generally operator-valued functions. For example (see [19, p. 24]) the function

$$s(z)=\begin{pmatrix}c_1s_1(z) & c_2s_2(z) & \cdots & c_Ns_N(z)\end{pmatrix}$$

where $s_1,\ldots,s_N$ are inner functions (for instance finite Blaschke products) and $c_1,c_2,\ldots,c_N$ are complex numbers such that

$$\sum_{n=1}^N|c_n|^2=1$$

is such that

$$\frac{1-s(z)s(w)^*}{1-z\overline{w}}=\sum_{n=1}^N|c_n|^2\frac{1-s_n(z)\overline{s_n(w)}}{1-z\overline{w}},$$

and the associated reproducing kernel Hilbert space will, in general, be only contractively included in the Hardy space $\mathbf{H}_2(\mathbb{D})$.

We also remark that in the theory of reproducing kernel Hilbert spaces, complementation and contractive inclusion correspond to the older results on the reproducing kernel Hilbert space associated to a sum of positive definite functions; see [26, p. 353], [47]. In Section 6 this problem is avoided by assuming that one starts with a power series to begin with.

In this section we replace the inequality

$$\|R_0F\|^2_{\mathfrak{M}}=\|F\|^2_{\mathfrak{M}}-\|F(0)\|^2$$

which characterizes isometric inclusion in $\mathbf{H}_2(\mathbb{K})$, by the inequality

$$\|R_0F\|^2_{\mathfrak{M}} \leq \|F\|^2_{\mathfrak{M}} - \|F(0)\|^2. \tag{5.51}$$

So, we wish to study the structure of R_0-invariant subspaces contractively included in $\mathbf{H}_2(\mathbb{K})$. We follow the arguments in the proof of [18, Theorem 3.1.2 p. 85], suitably adapted to the present situation. First note that (5.51) can be rewritten as

$$R_0^*R_0 + C^*C \leq I_{\mathfrak{M}}.$$

So

$$\begin{pmatrix} R_0 \\ C \end{pmatrix}^* \begin{pmatrix} R_0 \\ C \end{pmatrix} \leq I_{\mathfrak{M}}.$$

Since the adjoint of a Hilbert space contraction is a Hilbert space contraction we can write

$$\begin{pmatrix} R_0 \\ C \end{pmatrix} \begin{pmatrix} R_0 \\ C \end{pmatrix}^* \leq I_{\mathfrak{M}\oplus\mathbb{C}^p}.$$

Let $\mathfrak{H} = \mathfrak{M} \oplus \mathbb{C}^{p\times p}$ and let $B \in \mathcal{L}(\mathfrak{H}, \mathfrak{M})$ and $D \in \mathcal{L}(\mathfrak{H}, \mathbb{C}^{p\times p})$ be defined by

$$\begin{pmatrix} B \\ D \end{pmatrix} = \sqrt{I_{\mathfrak{M}\oplus\mathbb{C}^{p\times p}} - \begin{pmatrix} R_0 \\ C \end{pmatrix} \begin{pmatrix} R_0 \\ C \end{pmatrix}^*}$$

Then,

$$M = \begin{pmatrix} R_0 & G \\ C & D \end{pmatrix} \quad \mathfrak{M} \oplus \mathfrak{H} \longrightarrow \mathfrak{H} \oplus \mathbb{C}^{p\times p}$$

is co-isometric. We define

$$S = D + \sum_{n=1}^{\infty} Z^n C R_0^{n-1} G.$$

Note that S is $\mathcal{L}(\mathfrak{H}, \mathbb{C}^{p\times p})$-valued and that $S(A)$ makes sense for all $A \in \mathbb{K}$ as a converging series in the operator norm topology

Remark 5.20. When $Z = \lambda I_p$ the function S coincides with the characteristic function of the colligation M; see e.g. [18, p. 16].

Following [24, p. 38], we associate to S the (possibly unbounded and not everywhere defined) multiplication operator M_S now from $\ell_2(\mathbb{N}_0, \mathfrak{H})$ into $\mathfrak{H}_2(\mathbb{K})$ by

$$M_S(h)=\sum_{n=0}^{\infty} Z^n\left(Dh_n+\sum_{j=0}^{n-1} CR_0^{n-1-j}Gh_j\right), \quad h=(h_j)_{j=0}^{\infty}\in \ell_2(\mathbb{N}_0,\mathfrak{H}). \tag{5.52}$$

Theorem 5.21. *M_S is a contraction from $\ell_2(\mathbb{N}_0,\mathfrak{H})$ into $\mathbf{H}_2(\mathbb{K})$ if and only if the kernel* (5.3) *(now computed for the current operator-valued S) is positive definite in $\mathbb{K}$.*

Proof. We have (using the co-isometry of M)

$$\begin{aligned}
I_p-S(A)S(B)^* &= \underbrace{I_p-DD^*}_{CC^*}-\sum_{n=1}^{\infty} A^nCR_0^{n-1}GD^*-\\
&\quad -\sum_{m=1}^{\infty}\underbrace{DG^*}_{-CR_0^*}R_0^{*(m-1)}C^*B^{*m}-\sum_{n,m=1}^{\infty} A^nCR_0^{n-1}GG^*R_0^{*(m-1)}C^*B^{*m}\\
&= CC^*+\sum_{n=1}^{\infty} A^nCR_0^{n-1}R_0C^*+\sum_{m=1}^{\infty} CR_0^*R_0^{*(m-1)}C^*B^{*m}-\\
&\quad -\sum_{n,m=1}^{\infty} A^nCR_0^{n-1}(I-R_0R_0^*)R_0^{*(m-1)}C^*B^{*m}\\
&= \underbrace{CC^*}_{\overset{\text{def.}}{=}\mathbf{1}}+\underbrace{\sum_{n=1}^{\infty} A^nCR_0^nC^*}_{\overset{\text{def.}}{=}\mathbf{2}}+\underbrace{\sum_{m=1}^{\infty} CR_0^{*m}C^*B^{*m}}_{\overset{\text{def.}}{=}\mathbf{3}}+\\
&\quad +\underbrace{\sum_{n,m=1}^{\infty} A^nCR_0^nR_0^{*m}C^*B^{*m}}_{\overset{\text{def.}}{=}\mathbf{4}}-\underbrace{\sum_{n,m=1}^{\infty} A^nCR_0^{n-1}R_0^{*(m-1)}C^*B^{*m}}_{\overset{\text{def.}}{=}\mathbf{5}}\\
&= \mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{4}-\mathbf{5}
\end{aligned}$$

so that

$$I_p-S(A)S(B)^*=\underbrace{\sum_{n,m=0}^{\infty} A^nCR_0^nR_0^{*m}C^*B^{*m}}_{\mathbf{1+2+3+4}}-\underbrace{\sum_{n.m=0}^{\infty} A^{n+1}CR_0^nR_0^{*m}C^*B^{*(m+1)}}_{\mathbf{5}} \tag{5.53}$$

Define now

$$K(A,B)=\left(\sum_{n=0}^{\infty} A^nCR_0^n\right)\left(\sum_{m=0}^{\infty} B^mCR_0^m\right)^*=\sum_{n,m=0}^{\infty} A^nCR_0^nR_0^{*m}C^*B^{*m}$$

Then $K(A,B)$ is positive definite and

$$K(A,B) - AK(A,B)B^* = \sum_{n,m=0}^{\infty} A^n C R_0^n R_0^{*m} C^* B^{*m} - \sum_{n,m=0}^{\infty} A^{n+1} C R_0^n R_0^{*m} C^* B^{*(m+1)}$$
$$= I - S(A)S(B)^*$$

by (5.53), and so

$$K(A,B) = \sum_{n=0}^{\infty} A^n (I_p - S(A)S(B)^*) B^{*n}, \tag{5.54}$$

which is (5.3), but now with S operator-valued. To prove that M_S is a contraction we adapt the proof of Theorem 5.3 as follows. Recall that $S(A)^*$ is a bounded operator from $\mathbb{C}^{p\times p}$ into $\mathfrak{H}$. The linear relation (5.4) is now a linear subspace of $\mathbf{H}_2(\mathbb{K}) \times \ell_2(\mathbb{N}_0, \mathfrak{H})$ spanned by the pairs

$$(\sum_{n=0}^{\infty} Z^n A^{*n} C, (S(A)^* A^{*n} C)_{n=0}^{\infty}).$$

The positivity of (5.54) implies that this linear relation extends to the graph of a contraction, whose adjoint is M_S.

The converse statement is a direct computation. □

Remark 5.22. For $A = aI_p$ and $B = bI_p$ the function $K(A,B)$ reduces to formulas given in [18, Theorem 2.1.2 p. 44 and p. 97].

Remark 5.23. Besides the scalar case $A = aI_p$ with $a \in \mathbb{C}$, we do not know if and when the positivity of the kernel (5.54) implies that $S(A)$ is a contraction. The counterpart of this question in the quaternionic setting has a negative answer in the matrix-valued case; see [12, (62.38) p. 1767]. We adapt the example from the latter publication to the present setting to find a counterexample for multipliers of $\mathbf{H}_2(\mathbb{K})$. We consider $(\mathbf{H}_2(\mathbb{K}))^2$. Multipliers are elements $S = (S_{uv})_{u,v=1,2}$ of $(\mathbf{H}_2(\mathbb{K}))^{2\times 2}$ such that

$$\sum_{n=0}^{\infty} A^n \left(I_{2p} - \begin{pmatrix} S_{11}(A) & S_{12}(A) \\ S_{21}(A) & S_{22}(A) \end{pmatrix} \begin{pmatrix} (S_{11}(B))^* & (S_{21}(B))^* \\ (S_{12}(B))^* & (S_{22}(B))^* \end{pmatrix} \right) B^{*n}$$

is positive definite in $\mathbb{K}$. Let $p \geq 2$ and $J \in \mathbb{C}^{p\times p}$ be such that $J^* = -J$ and $J^2 = -I_p$ (for instance, if $p = 2$, $J = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$). Let

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} Z & J \\ ZJ & I \end{pmatrix}$$

(compare with [12, (62.38) p. 1767]), and let

$$F=\sum_{n=0}^{\infty} Z^n \begin{pmatrix} A_n \\ B_n \end{pmatrix} \in (\mathbf{H}_2(\mathbb{K}))^2.$$

Then,

$$\begin{aligned}(\sqrt{2}S)\star F &= \sum_{n=0}^{\infty} \begin{pmatrix} Z^{n+1}A_n + Z^n JB_n \\ Z^{n+1}JA_n + Z^n B_n \end{pmatrix} \\ &= \begin{pmatrix} JB_0 \\ B_0 \end{pmatrix} + \sum_{k=0}^{\infty} Z^{2k+1} \begin{pmatrix} A_{2k} + JB_{2k+1} \\ JA_{2k} + B_{2k+1} \end{pmatrix} + \sum_{k=1}^{\infty} Z^{2k} \begin{pmatrix} A_{2k-1} + JB_{2k} \\ JA_{2k-1} + B_{2k} \end{pmatrix}.\end{aligned}$$

Since

$$\begin{aligned}(A_{2k} + JB_{2k+1})^*(A_{2k} + JB_{2k+1}) + (JA_{2k} + B_{2k+1})^*(JA_{2k} + B_{2k+1}) = \\ = A_{2k}^*A_{2k} + B_{2k+1}^*B_{2k+1} - B_{2k+1}^*JA_{2k} + B_{2k+1}^*JA_{2k} \\ = A_{2k}^*A_{2k} + B_{2k+1}^*B_{2k+1}\end{aligned}$$

and similarly

$$\begin{aligned}(A_{2k-1} + JB_{2k})^*(A_{2k-1} + JB_{2k}) + (JA_{2k-1} + B_{2k})^*(JA_{2k-1} + B_{2k}) = \\ = A_{2k-1}^*A_{2k-1} + B_{2k}^*B_{2k}\end{aligned}$$

we have

$$[S\star F, S\star F]_{(\mathbf{H}_2(\mathbb{K}))^2} = [F,F]^2_{(\mathbf{H}_2(\mathbb{K}))^2},$$

and so S is a Schur multiplier. But

$$I_{2p} - S(A)S(A)^* = \frac{1}{2}\begin{pmatrix} I_p - AA^* & AJA^* - J \\ AJA^* - J & I_p - AA^* \end{pmatrix}$$

which is not positive for A unitary such that $AJA^* \neq J$ (we need $p \geq 2$ for ensure this); take for instance $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and J as above.

6. Carathéodory multipliers

Closely related to Schur functions are Carathéodory functions, that is, functions analytic in the open unit disk and with a positive real part there. In [51], see also the collection of papers [46], Herglotz gave an integral representation of such functions. A function φ is a Carathéodory function if and only if it can be written as

$$\varphi(z) = im + \int_0^{2\pi} \frac{e^{is}+z}{e^{is}-z} d\mu(s), \quad z \in \mathbb{D}, \tag{6.1}$$

where $m \in \mathbb{R}$ and where μ is a positive measure on $[0, 2\pi)$. Note that (6.1) can be rewritten as

$$\varphi(z) = im + \int_0^{2\pi} d\mu(s) + 2\sum_{n=1}^{\infty} z^n t_n \tag{6.2}$$

where

$$t_n = \int_0^{2\pi} e^{-ins} d\mu(s), \quad n = 0, 1, \dots \tag{6.3}$$

is the moment sequence associated to $d\mu$.

In terms of kernels, a function φ *defined* on a subset of the open unit disk which possesses an accumulation point in $\mathbb{D}$ is the restriction to Ω of a (uniquely defined) Carathéodory function if and only if the kernel

$$\frac{\varphi(z) + \overline{\varphi(w)}}{1 - z\overline{w}} \tag{6.4}$$

is positive definite in Ω. Note that (6.4) can be rewritten as

$$\frac{\varphi(z) + \overline{\varphi(w)}}{1 - z\overline{w}} = \sum_{n=0}^{\infty} z^n (\varphi(z) + \overline{\varphi(w)}) \overline{w}^n.$$

Remark 6.1. We note that φ need not be bounded in modulus in the open unit disk. Then (and only then), the operator of multiplication by φ will not be a bounded operator from the Hardy space into itself.

The previous discussion motivates the following definition:

Definition 6.2. A $\mathbb{C}^{p\times p}$-valued function Φ defined in $\mathbb{K}$ is called a Carathéodory multiplier if the kernel

$$K_\Phi(A, B) = \sum_{n=0}^{\infty} A^n (\Phi(A) + \Phi(B)^*) B^{*n} \tag{6.5}$$

is positive definite on $\mathbb{K}$.

In this section we prove a realization theorem similar to (6.1) for such functions, in two different ways:

(1) The first approach reduces the study to the complex setting, and does not require that the function $\Phi(A)$, or more generally, that the operator M_Φ of $\star$-multiplication on the left be a bounded operator from $\mathbf{H}_2(\mathbb{K})$ into itself.

(2) The second approach will require this latter hypothesis. Then, Φ is a power series in Z with matrix coefficients on the right, and converging in $\mathbb{K}$, i.e. $\Phi = \sum_{n=0}^{\infty} Z^n \Phi_n$. As just mentioned above, in the classical setting, this hypothesis is not necessary.

Rather than stating the theorem and proving it afterwards, we here prefer to go the other way around, and begin with a discussion and results which lead to the result. The result itself is presented in Theorem 6.3 below. So let us start from a function Φ for which the kernel (6.5) is positive definite in $\mathbb{K}$, and set $\Psi(a) = \Phi(aI_p)$ with $a \in \mathbb{D}$. The kernel

$$\frac{\Psi(a) + \Psi(b)^*}{1 - a\overline{b}} \tag{6.6}$$

is positive definite in the open unit disk. By the matrix version of Herglotz representation theorem (see e.g. [34, Theorem 4.5 p. 23] for the operator-valued version), we can write

$$\Psi(a) = iX + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} dM(t)$$

where $X \in \mathbb{C}^{p\times p}$ is self-adjoint and where M is now a $\mathbb{C}^{p\times p}$-valued positive measure on $[0, 2\pi)$. This formula can be rewritten as

$$\Psi(a) = iX + T_0 + 2\sum_{n=1}^{\infty} a^n T_n \tag{6.7}$$

where $(T_n)_{n\in\mathbb{N}_0}$ is the moment sequence associated to dM. We claim that (recall that Φ is assumed to be a power series in Z with matrix coefficients on the right, converging in $\mathbb{K}$)

$$\Phi(A) = iX + T_0 + 2\sum_{n=1}^{\infty} A^n T_n, \quad A \in \mathbb{K}. \tag{6.8}$$

Indeed, let $\widetilde{\Phi}$ denote the right hand side of (6.8). Both $\widetilde{\Phi}$ and Φ coincide on the matrices aI_p, $a \in \mathbb{D}$, and this restriction completely determines the coefficients T_n. Thus we have proved one direction in the following result. The converse direction is easily proved and will be omitted.

Theorem 6.3. *Let $\Phi = \sum_{n=0}^{\infty} Z^n \Phi_n$ be a power series in Z with matrix coefficients on the right, converging in $\mathbb{K}$ and assume $\Phi_0 = \Phi_0^*$. Then the kernel*

$$\sum_{n=0}^{\infty} A^n (\Phi(A) + \Phi(B)^*) B^{*n}$$

is positive definite in $\mathbb{K}$ *if and only if the sequence* $(\Phi_n)_{n\in\mathbb{N}_0}$ *is the moment sequence of a positive* $\mathbb{C}^{p\times p}$*-valued measure on* $[0, 2\pi)$.

We note that the method used here, different from the one we used to characterize Schur multiplier, is not intrinsic, in the sense that we do not use the reproducing kernel Hilbert space with reproducing kernel (6.5). Denoting this space by $\mathcal{L}(\Phi)$, one can also characterize Φ in terms of the associated backward-shift realization, assuming the operator M_Φ bounded from $\mathbf{H}_2(\mathbb{K})$ into itself.

Theorem 6.4. *Assume that* M_Φ *is bounded from* $\mathbf{H}_2(\mathbb{K})$ *into itself, and let* $\mathcal{L}(\Phi)$ *denote the reproducing kernel space with reproducing kernel* K_Φ*. Then* $\operatorname{Re} M_\Phi \geq 0$ *and*

$$\mathcal{L}(\Phi) = \operatorname{ran} \sqrt{M_\Phi + M_\Phi^*} \tag{6.9}$$

with the operator range norm.

The proof follows the arguments of Section 5.4 and will be omitted. The key in the proof is the formula

$$M_\Phi^*((I - ZB)^{-\star}E) = \sum_{n=0}^{\infty} Z^n \Phi(B)^* B^{*n} E, \quad B \in \mathbb{K}, \quad E \in \mathbb{C}^{p\times p}, \tag{6.10}$$

valid since M_ϕ is bounded, and so for $A, B \in \mathbb{K}$ and $E, H \in \mathbb{C}^{p\times p}$,

$$[(M_\Phi + M_\Phi^*)((I - ZB)^{-\star}E), (I - ZA)^{-\star}H]_{\mathcal{L}(\Phi)} = \sum_{n=0}^{\infty} A^n(\Phi(A) + (\Phi(B))^*)B^{*n}. \tag{6.11}$$

The second realization theorem is now presented and proved.

Theorem 6.5. *Assume that the operator* M_ϕ *is bounded from* $\mathbf{H}_2(\mathbb{K})$ *into itself. Then* R_0 *is a co-isometry from* $\mathcal{L}(\Phi)$ *into itself. Furthermore,* Φ *can be written as* $\Phi = \sum_{n=0}^{\infty} Z^n\Phi_n$

$$\operatorname{Re} \Phi_0 = 2C_0C_0^* \tag{6.12}$$

$$\Phi_n = C_0R_0^nC_0^*, \quad n = 1, \ldots \tag{6.13}$$

where C_0 *is the evaluation at* 0.

Proof. We follow the approach from [25, pp. 708-709], suitably adapted to the present setting. We first note that K_Φ in (6.5) satisfies the equation

$$K_\Phi(A, B) - AK_\Phi(A, B)B^* = \Phi(A) + (\Phi(B))^*, \quad A, B \in \mathbb{K}. \tag{6.14}$$

We then define in $\mathcal{L}(\Phi) \times \mathcal{L}(\Phi)$ the linear relation $\mathscr{R}$ spanned by the pairs (which we write as a row rather than column, in opposition to the notation in Section 5.3)

$$(K_\Phi(\cdot, B)BE, K_\Phi(\cdot, B)E - K_\Phi(\cdot, 0)E), \quad B \in \mathbb{K}, \quad E \in \mathbb{C}^{p\times p}, \tag{6.15}$$

and divide the rest of the proof into steps.

STEP 1: *$\mathscr{R}$ is isometric.*

Indeed, let $A \in \mathbb{K}$ and $H \in \mathbb{C}^{p\times p}$. We have on the one hand

$$[K_\Phi(\cdot, B)BE, K_\phi(\cdot, A)AH]_{\mathcal{L}(\Phi)} = H^*AK_\Phi(A, B)B^*E, \tag{6.16}$$

and on the other hand,

$$\begin{aligned}
&[K_\Phi(\cdot, B)E - K_\Phi(\cdot, 0)E), K_\Phi(\cdot, A)H - K_\Phi(\cdot, 0)H]_{\mathcal{L}(\Phi)} = \\
&\qquad = H^*K_\Phi(A, B)E - H^*K_\Phi(A, 0)E - H^*K_\Phi(0, B)E + H^*K_\Phi(0, 0)E \\
&\qquad = H^*K_\Phi(A, B)E - H^*(\Phi(A) + \Phi(0))E - H^*(\Phi(0) + (\Phi(B))^*)E+ \\
&\qquad\quad + H^*(\Phi(0) + (\Phi(0))^*)E \\
&\qquad = H^*K_\Phi(A, B)E - H^*(\Phi(A)E - H^*(\Phi(B))^*E \\
&\qquad = H^* \{K_\Phi(A, B)E - \Phi(A) - (\Phi(B))^*\} E.
\end{aligned}$$

By (6.14),

$$H^* \{K_\Phi(A, B)E - \Phi(A) - (\Phi(B))^*\} E = H^*A^*K_\Phi(A, B)BE,$$

and hence the isometry property holds.

STEP 2: *$\mathscr{R}$ has a dense domain.*

Indeed, let $F = \sum_{n=0}^{\infty} Z^nF_n$ be orthogonal to the domain of $\mathscr{R}$. Then for every B and E as above, $E^*BF(B) = 0$. Taking $B = bI_p$ with $b \in \mathbb{D}$, this leads to $F_n = 0$ for $n = 0, 1, \ldots$ and so $F = 0$.

It follows from the first two steps that $\mathscr{R}$ extends to the graph of an everywhere defined isometry, say T, which we compute in STEP 3.

STEP 3: *It holds that $T^* = R_0$, and so $\mathcal{L}(\Phi)$ is R_0-invariant.*

Let $F \in \mathcal{L}(\Phi)$, and B, E as above. We have

$$\begin{aligned}
E^*BF(B) &= [TF, K_\Phi(\cdot, B)B^*E]_{\mathcal{L}(\Phi)} \\
&= [F, K_\Phi(\cdot, B)E - K_\Phi(\cdot, 0)E]_{\mathcal{L}(\Phi)} \\
&= E^*(F(B) - F(0)),
\end{aligned}$$

so that

$$B(TF(B)) = F(B) - F(0).$$

Since $\mathcal{L}(\Phi) \subset \mathbf{H}_2(\mathbb{K})$ (see Theorem 6.4) we know that TF is a power series with coefficients on the right it follows that $TF = R_0$.

STEP 4: *We prove* (6.13).

We first note that Φ can be written as $\Phi = \sum_{n=0}^{\infty} Z^n \Phi_n$ since $K_\Phi(\cdot, 0)E \in \mathcal{L}(\Phi)$ for every $E \in \mathbb{C}^{p\times p}$. We note that

$$K_\Phi(\cdot, 0) = \Phi + (\Phi(0))^* \in \mathcal{L}(\Phi)$$

and so

$$C_0 C_0^* = 2\mathrm{Re}\, \Phi_0,$$

which is (6.12). Furthermore,

$$\begin{aligned} C_0(K(\cdot,0)E) &= (\Phi_0 + \Phi_0^*)E \\ C_0(R_0(K(\cdot,0)E) &= C_0(\Phi_1 + Z\Phi_2 + \cdots) = \Phi_1 E \\ &\vdots \\ C_0(R_0^n(K(\cdot,0)E) &= C_0((\Phi_n + Z\Phi_n + \cdots)E) = \Phi_n E \\ &\vdots \quad \square \end{aligned}$$

Remark 6.6. (6.13) implies that, with $Z = zI_p$,

$$\begin{aligned} \Phi E &= \sum_{n=0}^{\infty} z^n \Phi_n E \\ &= (i\mathrm{Im}\, \Phi(0))E + (\mathrm{Re}\, \Phi_0)E + \sum_{n=1}^{\infty} z^n C_0 R_0^{n-1} C_0^* E \\ &= (i\mathrm{Im}\, \Phi(0))E + \frac{C_0 C_0^* E}{2} + \sum_{n=1}^{\infty} z^n C_0 R_0^{n-1} C_0^* E \\ &= (i\mathrm{Im}\, \Phi(0))E + \frac{1}{2} C_0 (I_{\mathcal{L}(\Phi)} - zR_0)^{-1} (I_{\mathcal{L}(\Phi)} + zR_0) C_0^* E. \end{aligned} \tag{6.17}$$

In the above expression, z is a number and zR_0 means the multiplication by this number of the operator R_0 acting in $\mathcal{L}(\Phi)$.

For papers related to this section we mention [6,16,17].

7. Concluding remarks

We conclude with some remarks on possible future work.

7.1. Interpolation

Problem 4.1 is a special case of a much more general bitangential interpolation problem (see e.g. [27]), which can also be set in the framework of Schur multipliers. We will consider these problems in a future publication.

7.2. Matrix polynomials and other applications of the map (2.7)

The map (2.7) which to $F(Z) = \sum_{n=0}^{\infty} Z^n F_n$ associates the matrix-valued function of a complex variable $F(zI_p) = \sum_{n=0}^{\infty} z^n F_n$ allows further applications than the ones presented here. For instance, define $F(Z)$ to be a matrix-polynomial (of the matrix variable Z) if only a finite number of powers of Z arise in the power series expansion of F. Then, $F(Z)$ is a matrix polynomial if and only if $F(zI_p)$ is a classical matrix-polynomials, and factorizations of F in factors of matrix-polynomials coincide with factorization of $F(zI_p)$ into classical matrix-polynomials. These factorizations, and much more, have been considered in [49,54]. We plan to pursue this line of study in a future work.

7.3. Symmetries

We begin with a motivating example. For a general matrix $A = (a_{jk})_{j,k=1}^{h} \in \mathbb{C}^{2h\times 2h}$, define

$$A_\varphi = J_1 \overline{A} J_1^*, \quad \text{where} \quad J = \begin{pmatrix} 0 & I_h \\ -I_h & 0 \end{pmatrix} \quad \text{and} \quad \overline{A} = (\overline{a_{jk}})_{j,k=1}^{h}. \tag{7.1}$$

Clearly

$$(\lambda I_{2h})_\varphi = \overline{\lambda} I_{2h}, \quad \lambda \in \mathbb{C}$$

The proof of the following lemma is easy and will be omitted.

Lemma 7.1. *We have*

$$(A_\varphi)^* = (A^*)_\varphi \quad \text{and} \quad \overline{A_\varphi} = (\overline{A})_\varphi \quad \text{and} \tag{7.2}$$

$$(AB)_\varphi = A_\varphi B_\varphi \tag{7.3}$$

and

$$A \geq 0 \implies A_\varphi \geq 0$$

Furthermore:

Lemma 7.2. *We have*

$$\sqrt{A_\varphi} = (\sqrt{A})_\varphi \tag{7.4}$$

Proof. We have $A = \sqrt{A}\sqrt{A}$ and so from (7.3)

$$A_\varphi = (\sqrt{A})_\varphi(\sqrt{A})_\varphi$$

If $A = A_\varphi$ and since $(\sqrt{A})_\varphi \geq 0$ the uniqueness of the squareroot implies (7.4). □

Lemma 7.3. *If A is invertible, we have*

$$A_\varphi^{-1} = (A_\varphi)^{-1}$$

Proof. This follows from $\overline{A}^{-1} = \overline{A^{-1}}$. □

Finally, a matrix A satisfies $A = A_\varphi$ if and only if it is of the form

$$A = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} \tag{7.5}$$

where A_1 and A_2 belong to $\mathbb{C}^{h\times h}$.

Proposition 7.4. *Restricting A and Z in the formula (3.24) for the Blaschke factor $U_A(Z)$, the latter satisfies*

$$(U_A(Z))_\varphi = U_A(Z).$$

Proof. This follows from (7.4), (7.3) and (7.2). □

When $h = 1$, AA^* is a scalar matrix and we get back the classical Blaschke factor from quaternionic analysis.

Similarly, let now

$$A_\varphi = J_2\overline{A}J_2^*, \quad \text{where now} \quad J_2 = \begin{pmatrix} 0 & I_h \\ I_h & 0 \end{pmatrix}. \tag{7.6}$$

Then, Lemmas 7.1 and 7.2 still hold. Furthermore, a matrix A now satisfies $A = A_\varphi$ if and only if it is of the form

$$A = \begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix} \tag{7.7}$$

where A_1 and A_2 belong to $\mathbb{C}^{h\times h}$. When $h = 1$ we get back the split quaternions, and U_A will be the corresponding Blaschke factor. Even when $h = 1$ we get a new formula since AA^* will not be a scalar matrix.

Remark 7.5. Taking A_1 and A_2 with real values we get the real realization of elements in $\mathbb{C}^{h\times h}$ in the first case and $h \times h$ matrices with components hyperbolic numbers in the second case.

Definition 7.6. We call a map satisfying

$$(AB)_\varphi = A_\varphi B_\varphi \tag{7.8}$$

$$(A + B)_\varphi = A_\varphi + B_\varphi \tag{7.9}$$

$$A \geq 0 \Longrightarrow A_\varphi \geq 0 \tag{7.10}$$

$$(A_\varphi)^* = (A^*)_\varphi \tag{7.11}$$

$$(\lambda I_{2h})_\varphi = \overline{\lambda} I_{2h}, \quad \lambda \in \mathbb{C} \tag{7.12}$$

an admissible symmetry.

For such a symmetry it follows that

$$A_\varphi = (\sqrt{A})_\varphi(\sqrt{A})_\varphi$$

and, when A is invertible,

$$A_\varphi^{-1} = (A_\varphi)^{-1}$$

Definition 7.7. The ring of matrices for which $A = A_\varphi$ is called the associated ring and denoted by $\mathbb{C}_\varphi$.

The various results presented here extend when we replace $\mathbb{C}^{p\times p}$ by $\mathbb{C}_\varphi$. This setting includes quaternions, split-quaternions and corresponding matrix versions. For a related work, see [8].

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: there is no conflict of interests. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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