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BOUNDEDNESS OF SOME MULTI-PARAMETER FIBER-WISE MULTIPLIER OPERATORS

FRÉDÉRIC BERNICOT AND POLONA DURCIK

ABSTRACT. We prove L^p estimates for various multi-parameter bi- and trilinear operators with symbols acting on fibers of the two-dimensional functions. In particular, this yields estimates for the general bi-parameter form of the twisted paraproduct studied in [14].

1. Introduction

The classical Coifman-Meyer theorem [4, 5] is concerned with bilinear operators of the form

$$T_m(F_1, F_2)(x) = \int_{\mathbb{R}^{2n}} \widehat{F_1}(\xi)\widehat{F_2}(\eta) m(\xi,\eta) e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta,$$

defined for test functions $F_1, F_2 : \mathbb{R}^n \to \mathbb{C}$ and m a bounded function on $\mathbb{R}^{2n}$. The Coifman-Meyer theorem states that if m, in addition, satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \leq C|(\xi,\eta)|^{-|\alpha|-|\beta|} \tag{1.1}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ up to a sufficently large finite order and all $(\xi,\eta) \neq 0$, with $0 \leq C < \infty$, then the operator T_m maps $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2}$ to L^{p_3} whenever $1 < p_1, p_2 \leq \infty$, $1/2 < p_3 < \infty$, and $1/p_1 + 1/p_2 = 1/p_3$. A notable application of this result is to the fractional Leibniz rule by Kato and Ponce [12], which has further applications to nonlinear PDE; see for instance the work by Christ and Weinstein [3].

Multi-parameter variants of the Coifman-Meyer theory arise by considering multilinear operators with symbols behaving as tensor products of symbols (1.1). A simple bi-parameter example can be obtained by considering the operator T_m with

$$m(\xi,\eta) = m_1(\xi) m_2(\eta), \tag{1.2}$$

where m_1 and m_2 are smooth away from the origin and satisfy the analogous estimates as in (1.1). This case immediately splits into a pointwise product of two linear Calderón-Zygmund operators. It can be observed that the symbol m in (1.2) satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \leq C|\xi|^{-|\alpha|}|\eta|^{-|\beta|}. \tag{1.3}$$

In contrast to this example, a major early contribution to the theory of bi-parameter operators by Grafakos and Kalton [10] states that the condition (1.3) is in general not sufficient for the L^p boundedness of T_m.

Further developments of the multi-parameter theory were driven by interest in obtaining various fractional Leibniz-type rules such as in the works by Muscalu, Pipher, Tao, Thiele [22, 23]. In particular, these papers show bounds for the operators T_m with symbols satisfying

$$|\partial_{(\xi_1,\eta_1)}^\alpha \partial_{(\xi_2,\eta_2)}^\beta m(\xi,\eta)| \leq C|(\xi_1,\eta_1)|^{-|\alpha|}|(\xi_2,\eta_2)|^{-|\beta|}, \tag{1.4}$$

where $\xi = (\xi_1,\xi_2)$, $\eta = (\eta_1,\eta_2) \in \mathbb{R}^n \times \mathbb{R}^n$. More general operators with so-called flag singularities were studied by Muscalu [20, 21]. Some recent works in the area include the one by Muscalu

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and Zhai [27], who investigate a certain trilinear operator which falls under the class of singular Brascamp-Lieb integrals with non-Hölder scaling, and some related flag paraproducts studied by Lu, Pipher, and Zhang [17].

In the aforementioned papers, the symbols in question generalize products of Coifman-Meyer symbols (1.1), each symbol acting on one or several fibers of the input functions. For instance, the fiber-wise action in (1.4) means that the first factor on the right-hand side of (1.4) concerns only $F_1(\cdot,\xi_2)$ and $F_2(\cdot,\eta_2)$, while the second term concerns only the complementary fibers $F_1(\xi_1,\cdot)$ and $F_2(\eta_1,\cdot)$. Several recent developments in the theory of singular integrals include the study of multilinear operators with symbols acting fiber-wise on the input functions but with an additional "twist" as compared to (1.4), such as a symbol acting on $F_1(\xi_1,\cdot)$ and $F_2(\cdot,\eta_2)$, or $F_1(\cdot,\xi_2)$ and $F_2(\eta_1,\cdot)$. A fiber-wise action of this kind was first studied by Kovač [14] and the first author [2] in the one-parameter setting (1.1) and dimension $n=2$. In this paper we address such a situation in the multi-parameter setting.

We follow customary practice to model symbols by the convolution-type $P-Q$ operators, see for instance [14] or [24]. For $k\in\mathbb{Z}$ and $1\leq i\leq 2$, let $\varphi_{i,k}$ and $\psi_{i,k}$ be smooth functions adapted in the interval $[-2^{k+1},2^{k+1}]$, and let $\psi_{i,k}$ vanish on $[-2^{k-100},2^{k-100}]$. A function ρ adapted to an interval $I\subseteq\mathbb{R}$ means a function supported in I and satisfying

$$\|\partial^\alpha\rho\|_\infty\leq|I|^{-\alpha}$$

for all multi-indices α up to order N for some large N; see [29]. Let $P_{i,k}$ and $Q_{i,k}$ denote the one-dimensional Fourier multipliers with symbols $\varphi_{i,k}$ and $\psi_{i,k}$ respectively, i.e.

$$P_{i,k}f=f*\check{\varphi}_{i,k},\quad Q_{i,k}f=f*\check{\psi}_{i,k}$$

for $f\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$. When we apply such operators to one-dimensional fibers of a two-dimensional function we use a superscript to denote the fiber on which the action takes place. For instance,

$$P^{(1)}_{i,k}F(x_1,x_2)=\big(F(\cdot,x_2)*\check{\varphi}_{i,k}\big)(x_1),\quad P^{(2)}_{i,k}F(x_1,x_2)=\big(F(x_1,\cdot)*\check{\varphi}_{i,k}\big)(x_2),$$

and similarly for $Q^{(1)}_{i,k}F$ and $Q^{(2)}_{i,k}F$. The central objects of this paper are the two-dimensional bi-parameter bilinear operators

$$T_1(F_1,F_2)(x)=\sum_{(k,l)\in\mathbb{Z}^2:k\leq l}(Q^{(1)}_{1,k}P^{(2)}_{2,l}F_1)(x)\,(Q^{(1)}_{2,l}P^{(2)}_{1,k}F_2)(x)\qquad\text{and}$$
$$T_2(F_1,F_2)(x)=\sum_{(k,l)\in\mathbb{Z}^2:k\leq l}(P^{(1)}_{1,k}P^{(2)}_{2,l}F_1)(x)\,(Q^{(1)}_{2,l}Q^{(2)}_{1,k}F_2)(x),$$

defined for test functions $F_1,F_2:\mathbb{R}^2\to\mathbb{C}$. We prove the following bounds in Section 2.

Theorem 1. *The operators T_1 and T_2 are bounded from $\mathrm{L}^{p_1}(\mathbb{R}^2)\times\mathrm{L}^{p_2}(\mathbb{R}^2)$ to $\mathrm{L}^{p_3'}(\mathbb{R}^2)$ whenever $1<p_1,p_2<\infty$, $1<p_3'<2$, and $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3'}$.*

Passing to the Fourier side, the operators T_1 and T_2 can be viewed as multiplier operators which map the tuple (F_1,F_2) to the two-dimensional function defined by

$$x\mapsto\int_{\mathbb{R}^4}\widehat{F_1}(\xi)\widehat{F_2}(\eta)m(\xi,\eta)e^{2\pi ix\cdot(\xi+\eta)}d\xi d\eta \tag{1.5}$$

for a suitable bounded function m on $\mathbb{R}^4$. In the case of T_1 and T_2, the function m satisfies

$$|\partial^\alpha_{(\xi_1,\eta_2)}\partial^\beta_{(\xi_2,\eta_1)}m(\xi,\eta)|\leq C|(\xi_1,\eta_2)|^{-|\alpha|}|(\xi_2,\eta_1)|^{-|\beta|} \tag{1.6}$$

for all $\alpha,\beta\in\mathbb{N}_0^2$ up to order N and all $(\xi,\eta)\neq0$. In general, the estimates (1.6) alone are not sufficient for boundedness of the multiplier operator (1.5). We elaborate on this in Section 2.4 by reducing a special case of (1.5) to the aforementioned counterexample from [10].

Here and in the sequel, the notion of bi- and multi-parameter is related to the number of parameters of frequency scales. For instance, each factor on the right-hand side of (1.6) gives rise to one parameter. Given the constraint $k \leq l$, the symbols of T_1, T_2 do not split into the tensor products of two symbols $m_1(\xi_1, \eta_2)$ and $m_2(\xi_2, \eta_1)$. However, bounds for T_1, T_2 imply bounds on the multiplier operator (1.5) in the case when m is indeed of tensor type, i.e.

$$m(\xi, \eta) = m_1(\xi_1, \eta_2) m_2(\xi_2, \eta_1), \tag{1.7}$$

where m_1, m_2 satisfy the estimates

$$|\partial^{\alpha}_{(\zeta_1,\zeta_2)} m_i(\zeta_1, \zeta_2)| \leq C |(\zeta_1, \zeta_2)|^{-\alpha} \tag{1.8}$$

for all $\alpha \in \mathbb{N}_0^2$ up to order N and all $0 \neq (\zeta_1, \zeta_2) \in \mathbb{R}^2$. This can be seen by the classical cone decomposition.

Corollary 2. *Let m be given as in* (1.7)*. Then the associated operator* (1.5) *is bounded from* $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2}$ *to* $\mathrm{L}^{p_3'}$ *whenever* $1 < p_1, p_2 < \infty$, $1 < p_3' < 2$, *and* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3'}$.

The multiplier operator (1.5) with the symbol (1.7) was suggested by Camil Muscalu. The case when m_1 and m_2 are localized to cones in the frequency plane is sometimes called twisted paraproduct. A special case when one of the symbols m_i is constantly equal to one, i.e.

$$x \mapsto \int_{\mathbb{R}^4} \widehat{F_1}(\xi) \widehat{F_2}(\eta) m(\xi_1, \eta_2) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta, \tag{1.9}$$

has been previously studied by Kovač [14] and the first author [2]. It is a degenerate case of the two-dimensional bilinear Hilbert transform investigated by Demeter and Thiele [6]. The one-parameter operator (1.9) is known to map $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2} \to \mathrm{L}^{p_3'}$ in a larger range than stated in Corollary 2, namely whenever the exponents satisfy

$$1 < p_1, p_2 < \infty, \ \frac{1}{2} < p_3' < 2, \quad \text{and} \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3'}.$$

Indeed, the techniques developed in [14] yield bounds whenever $2 < p_1, p_2 < \infty$, $1 < p_3' < 2$. Fiber-wise Calderón-Zygmund decomposition from [2] can be then used to extend the range of exponents.

Let us also mention that there are further connections of (1.9) with several objects in ergodic theory. Indeed, the techniques developed in [14] have been subsequently refined and used to study larger classes of multilinear forms generalizing (1.9), which are motivated by problems on quantifying norm convergence of ergodic averages: the papers by Kovač [15] and Kovač, Škreb, Thiele and the second author [8] study ergodic averages with respect to two commuting transformations, while Škreb [25] studies certain cubic averages. Other applications of operators related to the twisted paraproduct are in stochastic integration obtained by Kovač and Škreb [16] and also in Euclidean Ramsey theory when investigating patterns in large subsets of the Euclidean space; see for instance the work by Kovač and the second author [7]. We also refer to the paper by Stipčić [26] and the references therein.

More generally, one can consider the class of all bilinear operators (1.5) where m is of the tensor type

$$m(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = m_1((\zeta_a)_{a \in S_1}) m_2((\zeta_a)_{a \in S_2}) m_3((\zeta_a)_{a \in S_3}) m_4((\zeta_a)_{a \in S_4}), \tag{1.10}$$

where $S_i \subseteq \{1, 2, 3, 4\}$ and m_i are symbols on $\mathbb{R}^{|S_i|}$, each of them satisfying the estimates analogous to (1.8). Then (1.7) is a special case with $S_3 = S_4 = \emptyset$. For the purpose of this paper, let us restrict ourselves to the case when the sets S_i are pairwise disjoint and consider (1.5) with such a symbol. If $S_1 = \{1\}$, then due to boundedness of the one-dimensional Calderón-Zygmund operators we may replace the function $\widehat{F_1}$ in (1.5) by

$$m_1(\xi_1)\widehat{F_1}(\xi). \tag{1.11}$$

Performing the analogous step for any singleton S_i, we may reduce (1.5) to the case where each non-empty set S_i satisfies $|S_i| \geq 2$. Up to symmetries, mapping properties of the cases that arise are discussed in Table 1 below.

Table 1. Structure of m and the range of boundedness

	Structure of $m(\xi, \eta)$	Known range of boundedness
1	$m_1(\xi_1, \eta_1)$	$1 < p_1, p_2 \leq \infty,\ \frac{1}{2} < p_3' < \infty$
2	$m_1(\xi_1, \xi_2)$	$1 < p_1, p_2 < \infty,\ \frac{1}{2} < p_3' < \infty$
3	$m_1(\xi_1, \eta_2)$	$1 < p_1, p_2 < \infty,\ \frac{1}{2} < p_3' < 2$
4	$m_1(\xi_1, \xi_2)m_2(\eta_1, \eta_2)$	$1 < p_1, p_2 < \infty,\ \frac{1}{2} < p_3' < \infty$
5	$m_1(\xi_1, \eta_1)m_2(\xi_2, \eta_2)$	$1 < p_1, p_2 \leq \infty,\ \frac{1}{2} < p_3' < \infty$
6	$m_1(\xi_1, \eta_2)m_2(\xi_1, \eta_1)$	$1 < p_1, p_2 < \infty,\ 1 < p_3' < 2$
7	$m_1(\xi_1, \xi_2, \eta_1)$	$1 < p_1, p_2 < \infty,\ \frac{1}{2} < p_3' < \infty$
8	$m_1(\xi_1, \xi_2, \eta_1, \eta_2)$	$1 < p_1, p_2 \leq \infty,\ \frac{1}{2} < p_3' < \infty$

The operator corresponding to Case 1 is a classical Coifman-Meyer multiplier [4, 5] acting on the first fibers of the functions F_1, F_2. Case 2 is a pointwise product of a linear Fourier multiplier with the identity operator and can be treated analogously as (1.11). The estimates for Case 2 stated in Table 1 are obtained by Hölder's inequality. Case 3 is the operator (1.9), which has been studied in [14]. Case 4 is a pointwise product of two linear operators and the analogous reduction as in (1.11) applies. Case 5 is a bi-parameter version of a Coifman-Meyer multiplier and its boundedness follows by [22]. As remarked earlier, in [22] the authors are able to handle multipliers m which are not necessarily of tensor type. This is in contrast with Case 6, which is the content of Corollary 2. It remains to consider Case 8, which is the classical Coifman-Meyer multiplier, and Case 7. We prove the following bounds for Case 7 in Section 3.

Theorem 3. *If* $m(\xi, \eta) = m_1(\xi_1, \xi_2, \eta_1)$*, then the associated operator* (1.5) *is bounded from* $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2}$ *to* $\mathrm{L}^{p_3'}$ *whenever* $1 < p_1, p_2 < \infty$, $\frac{1}{2} < p_3' < \infty$, *and* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3'}$.

We emphasize that extending the range of exponents for p_3' in Cases 3 and 6 remains open.

Dualizing the operators in (1.5) with symbols of the form (1.10), the corresponding trilinear forms are particular examples of singular Brascamp-Lieb integrals with several singular kernels, see also the survey [9]. Indeed, they can be represented by the trilinear forms

$$\int_{\mathbb{R}^6} F_1(x + A_1 s + B_1 t)F_2(x + A_2 s + B_2 t)F_3(x)K(s, t)dxdsdt \tag{1.12}$$

for suitable matrices $A_1, B_1, A_2, B_2 \in M_2(\mathbb{R})$ and a kernel K, whose Fourier transform $\widehat{K}$ is of the form (1.10). In particular, the operator with (1.7) is associated with a tensor product of two

Calderón-Zygmund kernels

$$K(s,t) = K_1(s)K_2(t),$$

where $K_1 = \check{m}_1$, $K_2 = \check{m}_2$. In this case, if one of the kernels K_1, K_2 in (1.12) specializes to the Dirac delta, then the object reduces to a one-parameter family of the two-dimensional bilinear Hilbert transforms [6]. Studying (1.12) for an arbitrary choice of matrices A_i, B_i and any Calderón-Zygmund kernels K_1, K_2 remains an open problem.

Furthermore, the operators T_1 and T_2 can be viewed as particular fiber-wise versions of the paraproducts studied by Muscalu, Tao, and Thiele in [24]. Following the setup from [24], let $n \geq 1$ and let $\Omega_n \in \mathbb{Z}^n$ be a convex polytope of the form

$$\Omega_n = \{(k_1, \dots, k_n) \in \mathbb{Z}^n : k_{i_\alpha} \geq k_{i'_\alpha} \text{ for all } 1 \leq \alpha \leq K\},$$

where $i_\alpha, i'_\alpha \in \{1, \dots, n\}$ and $K \geq 0$ an integer. Consider the operators which maps an n-tuple $(F_i)_{i=1}^n$ of test functions on $\mathbb{R}^2$ to the two-dimensional function

$$x \mapsto \sum_{(k_1,\dots,k_{2n}) \in \Omega_{2n}} \prod_{i=1}^{n} (Q^{(1)}_{i,k_i} Q^{(2)}_{i+n,k_{i+n}} F_i)(x), \tag{1.13}$$

where each $Q_{i,k}$ is a Fourier multiplier with symbol $\psi_{i,k}$, which is a bump function adapted in $[-2^{k+1}, 2^{k+1}]$ and vanishes on $[-2^{k-100}, 2^{k-100}]$. When Ω_{2n} is of the form $\Omega_n \times \mathbb{Z}^n$, bounds for (1.13) follow from [24]. When $n = 1$, (1.13) reduces to a classical linear Calderón-Zygmund operator. When $n = 2$, one can classify the cases similarly as in Table 1. Indeed, this can be seen by summing over Ω_4 in at least two parameters k_i in (1.13). Using the fact that $\sum_{k\in\mathbb{Z}} Q_{i,k}$ are linear Calderón-Zygmund operators and hence satisfy the desired bounds, the problem is reduced to objects with the summation over Ω_2, such as T_1, T_2, and other bi-parameter analogues in [22]. This yields $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2}$ to $\mathrm{L}^{p'_3}$ bounds whenever $1 < p_1, p_2 < \infty$ and $1 < p'_3 < 2$, but the known range may, in particular cases, be larger. However, as it is the case for T_1 and T_2, the symbols of (1.13) are in general not of tensor type because of the constraint on Ω_n.

The main idea used in the proof of Theorem 1 is to reduce the problem to the vector-valued estimates for the operator (1.9) with one constant symbol. The key step is in observing that the operator is localized in frequency due to the frequency supports of the fibers of the input functions. This can be seen in sharp contrast with (1.9) itself, where such localization does not occur. Localization of the operator allows replacements of some low-frequency projections, acting on the input functions, with the identity operators. This, in turn, allows for applications of Hölder's inequality. We prove Theorem 1 and Corollary 2 in Section 2. In the case of (1.9), quasi-Banach estimates can be proven using the fiber-wise Calderón-Zygmund decomposition from [2]. This decomposition does not seem applicable in the context of Theorem 1, as the symbols act on both fibers of the input functions. Extending the range of exponents in Theorem 1 remains an open problem.

Theorem 3 is proven in Section 4 and in the Banach case it relies on the bounds for the operator (1.9). In Theorem 4, the operator acts on only one fiber of the function F_2; in this case we are able to use the fiber-wise Calderón-Zygmund decomposition to prove quasi-Banach estimates as stated in Theorem 3.

Multilinear and multi-parameter generalizations. At present there is only partial understanding of the multilinear generalizations of Theorem 1. In the case of (1.7), multilinear operators with only one non-constant symbol can be described in the language of bipartite graphs and were studied by Kovač in [13] in a dyadic model.

In this paper, we discuss a particular tri-parameter trilinear example, which can be seen as a natural generalization of (1.7). Let m_1, m_2, m_3 be symbols satisfying (1.8) for $1 \leq i \leq 3$. Define the trilinear operator which maps a triple (F_1, F_2, F_3) of functions on $\mathbb{R}^2$ to the two-dimensional function given by

$$x \mapsto \int_{\mathbb{R}^6} \widehat{F}_1(\xi)\widehat{F}_2(\eta)\widehat{F}_2(\tau) m_1(\xi_1,\eta_2) m_2(\eta_1,\tau_2) m_3(\tau_1,\xi_2) e^{2\pi i x\cdot(\xi+\eta+\tau)} d\xi d\eta d\tau, \tag{1.14}$$

where $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2)$, $\tau = (\tau_1, \tau_2)$. We prove the following bounds.

Theorem 4. *The operator* (1.14) *is bounded from* $\mathrm{L}^{p_1}\times\mathrm{L}^{p_2}\times\mathrm{L}^{p_3}$ *to* $\mathrm{L}^{p_4'}$ *when* $1 < p_1, p_2, p_3 < \infty$, $2 < p_4 < \infty$, $\sum_{i=1}^4 \frac{1}{p_i} = 1$, *and* $\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{2}$, $\frac{1}{p_2} + \frac{1}{p_3} > \frac{1}{2}$, $\frac{1}{p_1} + \frac{1}{p_3} > \frac{1}{2}$.

Note that the range in Theorem 4 is non-empty. For example, it contains exponents in vicinity of $3 < p_1 = p_2 = p_3 < 4$. The proof of Theorem 4 is detailed in Section 4 and can be seen as an iteration of the steps used in the proof of Theorem 1 and its corollary, by gradually reducing to the vector-valued estimates for the operators with one or more constant symbols. Iterating this procedure and applying estimates for one- and two-parameter operators, which hold in restricted ranges of exponents, is the reason for further restriction of the range in Theorem 4.

The proof of Theorem 4 does not immediately generalize to all higher degrees of multilinearities. Beside facing the issues with the exponent range, objects with constant symbols which arise in the proof may not be localized in frequency, which prevents further iterations of the approach. Similar issues also arise when trying to generalize Theorems 1 and 4 to higher dimensions. Obtaining bounds for a larger class of multi-parameter objects, such as multilinear operators in (1.13), is closely related to studying a large class of maximally truncated singular integrals, one- and multi-parameter. One such instance is the maximally truncated one-parameter twisted paraproduct which maps a tuple (F_1, F_2) of functions on $\mathbb{R}^2$ to the two-dimensional function

$$x \mapsto \sup_{N>0} \Big| \sum_{|k|<N} (P^{(1)}_{1,k} F_1)(x)\, (Q^{(2)}_{1,k} F_2)(x) \Big|. \tag{1.15}$$

Its boundedness seems to be out of reach of the currently available techniques. Further motivation for proving such maximal estimates and also stronger variational estimates is provided by questions on pointwise convergence of certain ergodic averages. Establishing L^p estimates for the operator which maps a tuple (F_1, F_2) of functions on $\mathbb{R}^2$ to the one-dimensional function

$$x_2 \mapsto \sup_{N>0} \Big\| \sum_{|k|<N} (P^{(1)}_{1,k} F_1)(x_1,x_2)\, (Q^{(2)}_{1,k} F_2)(x_1,x_2) \Big\|_{\mathrm{L}^p_{x_1}(\mathbb{R})}$$

could be considered as an intermediate case between (1.9) and (1.15). Such bounds also remain open.

Notation. For two non-negative quantities A, B we write $A \lesssim B$ if there exists an absolute constant C such that $A \le CB$. If C depends on the parameters $P_1, \ldots, P_n$, we write $A \lesssim_{P_1,\ldots,P_n} B$.

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2. The bi-parameter bilinear operators T_1 and T_2

This section is devoted to the proof of Theorem 1. Throughout this section we will use the shorthand notation $k \ll l$ to denote $k < l - 200$. Similarly, $k \gg l$ will denote $k > l + 200$ and $k \sim l$ will mean $l - 200 \le k \le l + 200$.

It will be evident from the proof that the argument will not rely on the particular choice of the bump functions $\varphi_{i,k}, \psi_{i,k}$ as long as they satisfy the assumptions in the definition of T_1 and T_2. For simplicity of notation, we shall therefore only discuss the case $\varphi_{1,k} = \varphi_{2,k} = \varphi_k$ and $\psi_{1,k} = \psi_{2,k} = \psi_k$ for each $k \in \mathbb{Z}$. We shall also write $P_{1,k} = P_{2,k} = P_k$ and $Q_{1,k} = Q_{2,k} = Q_k$.

2.1. **Boundedness of T_1.** First we split the summation over $k \leq l$ into the regimes where $k \sim l$ and $k \ll l$ respectively, i.e.

$$T_1(F,G) = \sum_{\substack{(k,l)\in\mathbb{Z}^2 \\ l-200\leq k\leq l}} (Q_k^{(1)} P_l^{(2)} F_1)\,(Q_l^{(1)} P_k^{(2)} F_2) + \sum_{\substack{(k,l)\in\mathbb{Z}^2 \\ k\ll l}} (Q_k^{(1)} P_l^{(2)} F_1)\,(Q_l^{(1)} P_k^{(2)} F_2). \tag{2.1}$$

Bounds for the sum over $l - 200 \leq k \leq l$ follow by Cauchy-Schwarz. Indeed, using Cauchy-Schwarz in $l \in \mathbb{Z}$ we pointwise bound this term as

$$\Big| \sum_{-200\leq s\leq 0} \sum_{l\in\mathbb{Z}} (Q_{l+s}^{(1)} P_l^{(2)} F_1)\,(Q_l^{(1)} P_{l+s}^{(2)} F_2)\Big| \leq \sum_{-200\leq s\leq 0} \|Q_{l+s}^{(1)} P_l^{(2)} F_1\|_{\ell_l^2(\mathbb{Z})} \|Q_l^{(1)} P_{l+s}^{(2)} F_2\|_{\ell_l^2(\mathbb{Z})}.$$

We consider the product of terms on the right-hand side for each fixed $-200 \leq s \leq 0$. To estimate the $\mathrm{L}^{p_3'}$ norm of the product we apply Hölder's inequality and use bounds for the classical square function. We obtain

$$\begin{aligned} &\big\| \|Q_{l+s}^{(1)} P_l^{(2)} F_1\|_{\ell_l^2(\mathbb{Z})} \|Q_l^{(1)} P_{l+s}^{(2)} F_2\|_{\ell_l^2(\mathbb{Z})} \big\|_{\mathrm{L}^{p_3'}(\mathbb{R}^2)} \\ &\leq \|Q_{l+s}^{(1)} P_l^{(2)} F_1\|_{\mathrm{L}^{p_1}(\ell_l^2)} \, \|Q_l^{(1)} P_{l+s}^{(2)} F_2\|_{\mathrm{L}^{p_2}(\ell_l^2)} \lesssim_{p_1,p_2} \|F_1\|_{\mathrm{L}^{p_1}(\mathbb{R}^2)} \|F_2\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)} \end{aligned}$$

whenever $1/p_3' = 1/p_1 + 1/p_2$ and $1 < p_1, p_2 < \infty$. In the end it remains to sum the individual contributions of these finitely many terms.

It remains to consider the case $k \ll l$ in (2.1). By duality it suffices to study the corresponding trilinear form and show

$$\Big| \sum_{(k,l)\in\mathbb{Z}^2 : k\ll l} \int_{\mathbb{R}^2} (Q_k^{(1)} P_l^{(2)} F_1)\,(Q_l^{(1)} P_k^{(2)} F_2)\, F_3 \Big| \lesssim_{p_1,p_2,p_3} \|F_1\|_{\mathrm{L}^{p_1}(\mathbb{R}^2)} \|F_2\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)} \|F_3\|_{\mathrm{L}^{p_3}(\mathbb{R}^2)}$$

for any choice of exponents $1 < p_1, p_2 < \infty$, $2 < p_3 < \infty$ with $1/p_1 + 1/p_2 + 1/p_3 = 1$. By the frequency supports of F_1, F_2, the form on the left-hand side can be written up to a constant as

$$\sum_{(k,l)\in\mathbb{Z}^2 : k\ll l} \int_{\mathbb{R}^2} (Q_k^{(1)} P_l^{(2)} F_1)\,(Q_l^{(1)} P_k^{(2)} F_2)\,(\mathcal{Q}_l^{(1)} \mathcal{P}_l^{(2)} F_3), \tag{2.2}$$

where $\mathcal{P}_l$ and $\mathcal{Q}_l$ are Fourier multipliers with symbols adapted to $[-2^{l+3}, 2^{l+3}]$, the symbol of $\mathcal{P}_l$ is constant on $[-2^{l+2}, 2^{l+2}]$, and the symbol of $\mathcal{Q}_l$ vanishes on $[-2^{l-90}, 2^{l-90}]$.

Then we write $P_l = \varphi_l(0)I + (P_l - \varphi_l(0)I)$ where I is the identity operator, and plug this decomposition into the form (2.2). This yields

$$(2.2) = \sum_{(k,l)\in\mathbb{Z}^2 : k\ll l} \mathcal{M}_{k,l} \quad + \sum_{(k,l)\in\mathbb{Z}^2 : k\ll l} \mathcal{E}_{k,l}, \tag{2.3}$$

where we have defined

$$\begin{aligned} \mathcal{M}_{k,l} &= \int_{\mathbb{R}^2} (Q_k^{(1)} F_1)\,(Q_l^{(1)} P_k^{(2)} F_2)\,(\varphi_l(0) \mathcal{Q}_l^{(1)} \mathcal{P}_l^{(2)} F_3), \\ \mathcal{E}_{k,l} &= \int_{\mathbb{R}^2} (Q_k^{(1)} (P_l^{(2)} - \varphi_l(0) I^{(2)}) F_1)\,(Q_l^{(1)} P_k^{(2)} F_2)\,(\mathcal{Q}_l^{(1)} \mathcal{P}_l^{(2)} F_3). \end{aligned}$$

First we consider the term involving $\mathcal{M}_{k,l}$. Since $|\varphi_l(0)| \leq 1$, up to redefining $\mathcal{P}_l$ we may assume that $\varphi_l(0) = 1$ for each l. We split the summation as

$$\sum_{(k,l)\in\mathbb{Z}^2:k\ll l} \mathcal{M}_{k,l} = \sum_{(k,l)\in\mathbb{Z}^2} \mathcal{M}_{k,l} - \sum_{(k,l)\in\mathbb{Z}^2:k\sim l} \mathcal{M}_{k,l} - \sum_{(k,l)\in\mathbb{Z}^2:k\gg l} \mathcal{M}_{k,l}. \tag{2.4}$$

By the frequency support of the first fibers of the functions F_i it follows that the term over $k \gg l$ vanishes. To estimate the term with summation over $k \sim l$ we use Hölder's inequality in l and in the integration, yielding

$$\Big|\sum_{l\in\mathbb{Z}}\sum_{s\sim 0} \mathcal{M}_{l+s,s}\Big| \leq \sum_{s\sim 0} \|Q^{(1)}_{l+s}F_1\|_{\mathrm{L}^{p_1}(\ell^\infty_l)} \|Q^{(1)}_l P^{(2)}_{l+s} F_2\|_{\mathrm{L}^{p_2}(\ell^2_l)} \|\mathcal{Q}^{(1)}_l \mathcal{P}^{(2)}_l F_3\|_{\mathrm{L}^{p_3}(\ell^2_l)}.$$

It remains to use bound on the one-dimensional maximal function and the two square functions, which hold uniformly in $s \sim 0$, and finally sum in s.

Thus, it suffices to estimate the case when the sum is unconstrained, i.e. over $(k,l) \in \mathbb{Z}^2$. By Cauchy-Schwarz in l and Hölder's inequality in the integration we estimate

$$\Big|\sum_{(k,l)\in\mathbb{Z}^2} \mathcal{M}_{k,l}\Big| \leq \Big\|\sum_{k\in\mathbb{Z}} (Q^{(1)}_k F_1)\,(Q^{(1)}_l P^{(2)}_k F_2)\Big\|_{\mathrm{L}^{p_3'}(\ell^2_l)} \|\mathcal{Q}^{(1)}_l \mathcal{P}^{(2)}_l F_3\|_{\mathrm{L}^{p_3}(\ell^2_l)}, \tag{2.5}$$

where $1/p_3 + 1/p_3' = 1$. The second term on the right-hand side is a classical square function. To bound the first term we use $\mathrm{L}^{p_1}(\mathbb{R}^2)\times\mathrm{L}^{p_2}(\ell^2) \to \mathrm{L}^{p_3'}(\ell^2)$ vector-valued estimates for the operator (1.9), which hold whenever $1 < p_3' < 2$ and $1 < p_1, p_2 < \infty$. These vector-valued estimates are obtained by freezing the function $F_1 \in \mathrm{L}^{p_1}$ and using the linear inequalities of Marcinkiewicz and Zygmund [18], together with scalar-valued boundedness of (1.9). We obtain

$$\Big\|\sum_{k\in\mathbb{Z}} (Q^{(1)}_k F_1)\,(Q^{(1)}_l P^{(2)}_k F_2)\Big\|_{\mathrm{L}^{p_3'}(\ell^2_l)} \lesssim_{p_1} \|F_1\|_{\mathrm{L}^{p_1}(\mathbb{R}^2)} \|Q^{(1)}_l F_2\|_{\mathrm{L}^{p_2}(\ell^2_l)} \lesssim_{p_1,p_2} \|F_1\|_{\mathrm{L}^{p_1}(\mathbb{R}^2)} \|F_2\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)},$$

where we have used the Littlewood-Paley inequality for the last bound. This yields the desired estimate for the first term in (2.3).

It remains to estimate the second term in (2.3). By frequency consideration in the second fiber, one has

$$\sum_{(k,l)\in\mathbb{Z}^2:k\ll l} \mathcal{E}_{k,l} = c \sum_{(k,l)\in\mathbb{Z}^2:k\ll l} \int_{\mathbb{R}^2} (Q^{(1)}_k \widetilde{\mathcal{Q}}^{(2)}_l F_1)\,(Q^{(1)}_l P^{(2)}_k F_2)\,(\mathcal{Q}^{(1)}_l \mathcal{P}^{(2)}_l F_3), \tag{2.6}$$

where c is an absolute constant and $\widetilde{\mathcal{Q}}_l$ is a Fourier multiplier with symbol adapted to $[-2^{l+3}, 2^{l+3}]$ which vanishes at the origin. Indeed, note that this is the case both when $\varphi_l(0) = 0$ and when $\varphi_l(0) \neq 0$. We split the summation in the regions where $(k,l) \in \mathbb{Z}^2$, $k \ll l$ and $k \gg l$. By the analogous considerations as in the paragraphs from (2.4) to (2.5) we note that it suffices to instead consider the case when the sum is unconstrained, i.e.

$$\sum_{(k,l)\in\mathbb{Z}^2} \int_{\mathbb{R}^2} (Q^{(1)}_k \widetilde{\mathcal{Q}}^{(2)}_l F_1)\,(Q^{(1)}_l P^{(2)}_k F_2)\,(\mathcal{Q}^{(1)}_l \mathcal{P}^{(2)}_l F_3).$$

By the Cauchy-Schwarz inequality in l and Hölder's inequality in the integration we bound the last display by

$$\Big\|\sum_{k\in\mathbb{Z}} (Q^{(1)}_k \widetilde{\mathcal{Q}}^2_l F_1)\,(Q^{(1)}_l P^{(2)}_k F_2)\Big\|_{\mathrm{L}^{p_3'}(\ell^2_l)} \|\mathcal{Q}^{(1)}_l \mathcal{P}^{(2)}_l F_3\|_{\mathrm{L}^{p_3}(\ell^2_l)}.$$

The second term is a square function. Bounds for the first term follow from $\mathrm{L}^p(\ell^2)\times\mathrm{L}^q(\ell^2) \to \mathrm{L}^r(\ell^2)$ vector-valued estimates for the twisted paraproduct (1.9) and two applications of the Littlewood-Paley inequality. The vector valued estimates which we need in this case are a consequence of scalar boundedness of the operator (1.9) and a result by Grafakos and Martell [11, Theorem 9.1]. This yields the desired bound for the second term in (2.3) and in turn establishes the claim for T_1.

2.2. **Boundedness of T_2.** The proof for T_2 proceeds in the analogous way as the proof for T_1 and we only sketch the necessary ingredients. By duality it suffices to bound

$$\Big|\sum_{(k,l)\in\mathbb{Z}^2:k\leq l}\int_{\mathbb{R}^2}(P_k^{(1)}P_l^{(2)}F_1)\,(Q_l^{(1)}Q_k^{(2)}F_2)\,F_3\Big|\lesssim_{p_1,p_2,p_3}\|F_1\|_{\mathrm{L}^{p_1}(\mathbb{R}^2)}\|F_2\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)}\|F_3\|_{\mathrm{L}^{p_3}(\mathbb{R}^2)}$$

for any choice of exponents $1<p_1,p_2<\infty$, $2<p_3<\infty$ with $1/p_1+1/p_2+1/p_3=1$. The case when $l-200\leq k\leq l$ is bounded by Hölder's inequality, so it suffices to consider $k\ll l$.

By frequency considerations, the form on the left-hand side is a constant multiple of

$$\sum_{(k,l)\in\mathbb{Z}^2:k\ll l}\int_{\mathbb{R}^2}(P_k^{(1)}P_l^{(2)}F_1)\,(Q_l^{(1)}Q_k^{(2)}F_2)\,(\mathcal{Q}_l^{(1)}\mathcal{P}_l^{(2)}F_3)$$

for frequency projections $\mathcal{Q}_l$ and $\mathcal{P}_l$ as in (2.2). As above we split $P_l=\varphi(0)I+(P_l-\varphi(0)I)$. By considerations as in the discussion after (2.3) it remains to estimate the analogue of the term associated with $\mathcal{M}_{k,l}$, i.e.

$$\sum_{(k,l)\in\mathbb{Z}^2}\int_{\mathbb{R}^2}(P_k^{(1)}F_1)\,(Q_l^{(1)}Q_k^{(2)}F_2)\,(\varphi(0)\mathcal{Q}_l^{(1)}\mathcal{P}_l^{(2)}F_3),$$

and the analogue of the term associated with $\mathcal{E}_{k,l}$, i.e.

$$\sum_{(k,l)\in\mathbb{Z}^2}\int_{\mathbb{R}^2}(P_k^{(1)}\widetilde{\mathcal{Q}}_l^{(2)}F_1)\,(Q_l^{(1)}Q_k^{(2)}F_2)\,(\mathcal{Q}_l^{(1)}\mathcal{P}_l^{(2)}F_3),$$

where $\widetilde{\mathcal{Q}}_l$ is as in (2.6). The proofs for each of these terms proceed analogously as for (2.5) and (2.6) respectively, reducing to vector-valued estimates for the operator (1.9).

Remark 5. An alternative way to prove bounds for T_2 is to deduce them from the bounds for T_1 via a telescoping identity, which "swaps" the P- and Q-type operators. Indeed, this can be achieved in a special case when T_1 and T_2 are related by a condition on the bump functions as in Proposition 8 in Section 4 below. Then, one can deduce a general case of T_2 from the special case by an averaging argument. We perform such arguments in a trilinear tri-parameter setting in Section 4 below.

2.3. **Proof of Corollary 2.** This corollary can be deduced from Theorem 1 by a classical cone decomposition, as performed for instance in [19]. For completeness we outline the relevant steps.

Let m be a function on $\mathbb{R}^2\setminus\{0\}$ satisfying

$$|\partial^\alpha m(\zeta)|\lesssim|\zeta|^{-|\alpha|}$$

for all α up to a large finite order and all $\zeta\neq 0$ in $\mathbb{R}^2$. Let φ be a smooth function supported in $[-2,2]$ and constantly equal to 1 on $[-1,1]$. Let $\psi=\varphi-\varphi(2\cdot)$. Then $\sum_{k\in\mathbb{Z}}\psi(2^{-k}\tau)=0$ for each $\tau\neq 0$. We can write

$$m(\zeta)=\sum_{(k,l)\in\mathbb{Z}^2}m(\zeta)\psi(2^{-k}\zeta_1)\psi(2^{-l}\zeta_2).$$

Splitting the sum into regions when $k\leq l$ and $k>l$, respectively, and summing in the smaller parameter we obtain

$$m(\zeta)=\sum_{k\in\mathbb{Z}}m(\zeta)\varphi(2^{-k}\zeta_1)\psi(2^{-k}\zeta_2)+\sum_{k\in\mathbb{Z}}m(\zeta)\psi(2^{-k}\zeta_1)\varphi(2^{-(k-1)}\zeta_2).$$

We consider the first sum; the second is treated analogously.

Note that for each $k \in \mathbb{Z}$, the summand is supported in

$$\{\zeta \in \mathbb{R}^2 : |\zeta_1| \le 2^{k+1},\, 2^{-k-1} \le |\zeta_2| \le 2^{k+1}\}.$$

The smooth restriction of m to that region can be decomposed into a double Fourier series, which yields

$$\sum_{k\in\mathbb{Z}} m(\zeta)\varphi(2^{-k}\zeta_1)\psi(2^{-k}\zeta_2) = \sum_{(n_1,n_2)\in\mathbb{Z}^2}\sum_{k\in\mathbb{Z}} C^k_{n_1,n_2}\varphi(2^{-k}\zeta_1)e^{c\pi i 2^{-k}\zeta_1 n_1}\psi(2^{-k}\zeta_2)e^{c\pi i 2^{-k}\zeta_2 n_2}$$

where c is a fixed constant and the Fourier coefficients $C^k_{n_1,n_2}$ satisfy

$$|C^k_{n_1,n_2}| \lesssim (1+|n_1|)^{-N}(1+|n_2|)^{-N}$$

for any $N>0$ up to a large order, uniformly in $k\in\mathbb{Z}$. For details we refer to [19], Chapter 2.13.

Normalizing the bump functions and the coefficients, the last display can be written as an absolute constant times

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2}(1+|n_1|)^{-2}(1+|n_2|)^{-2}\sum_{k\in\mathbb{Z}}\widetilde{C}^k_{n_1,n_2}\varphi_{k,n_1}(\zeta_1)\psi_{k,n_2}(\zeta_2),$$

where

$$\varphi_{k,n}(\zeta_1) = c_1\varphi(\zeta_1)e^{c\pi i\zeta_1 n_1},\ \psi_{k,n_2}(\zeta_2) = c_2\psi(\zeta_2)e^{c\pi i\zeta_2 n_2},$$

and the constants c_1, c_2 are such that both functions are adapted to $[-2^{k+1}, 2^{k+1}]$. Moreover, $\varphi_{k,n_1}(0)$ is the same constant for each $k\in\mathbb{Z}$, ψ_{k,n_2} vanishes in $[-2^{k-1}, 2^{k-1}]$, and $|\widetilde{C}^k_{n_1,n_2}|\le 1$. This reduces the matters to considering symbols for each fixed n_1, n_2 with bounds uniform in n_1, n_2. Note that the coefficients $\widetilde{C}^k_{n_1,n_2}$ can assumed to be equal to 1 by subsuming them into the definitions of ψ_{k,n_2}.

We perform this decomposition for the symbols m_1 and m_2 in (1.7). Then the bounds for the associated operator follow from bounds on T_1 and T_2.

2.4. **A counterexample.** In this section we show that the estimates (1.6) alone are not sufficient for boundedness of the operator (1.5). We restrict ourselves to the Banach regime. More precisely, we show that there exists a bounded function m on $\mathbb{R}^4$ satisfying

$$|\partial^\alpha_{(\xi_1,\eta_2)}\partial^\beta_{(\xi_2,\eta_1)}m(\xi,\eta)| \lesssim_{\alpha,\beta} |(\xi_1,\eta_2)|^{-|\alpha|}|(\xi_2,\eta_1)|^{-|\beta|}$$

for all multi-indices $\alpha,\beta\in\mathbb{N}_0$ such that the associated operator (1.5) does not satisfy $\mathrm{L}^{p_1}\times\mathrm{L}^{p_2}$ to $\mathrm{L}^{p_3'}$ estimates for any exponents $1<p_1,p_2,p_3<\infty$.

To see this we first recall a result from [10]; see the remark at the end of Section 3 in [10]. Existence is shown of a symbol m_0 on $\mathbb{R}^2\setminus\{0\}$ satisfying

$$|\partial^\alpha_{\xi_1}\partial^\beta_{\eta_1}m_0(\xi_1,\eta_1)| \lesssim_{\alpha,\beta} |\xi_1|^{-\alpha}|\eta_1|^{-\beta}$$

for all multi-indices $\alpha,\beta\in\mathbb{N}_0$, such that the one-dimensional operator mapping a tuple (f_1,f_2) of functions on $\mathbb{R}$ to a one-dimensional function

$$x\mapsto\int_{\mathbb{R}^2}\widehat{f}_1(\xi_1)\widehat{f}_2(\eta_1)m_0(\xi_1,\eta_1)e^{2\pi i x(\xi_1+\eta_1)}d\xi_1 d\eta_1, \tag{2.7}$$

does not satisfy any $\mathrm{L}^{p_1}\times\mathrm{L}^{p_2}$ to $\mathrm{L}^{p_3'}$ bounds.

To show that the estimates (1.6) are in general not sufficient we reduce a special case of (1.5) to this counterexample. Let us define

$$m(\xi_1,\xi_2,\eta_1,\eta_2) = m_0(\xi_1,\eta_1)\widetilde{m}(\xi_1,\eta_2)\widetilde{m}(\eta_1,\xi_2),$$

where $\widetilde{m}$ is a smooth symbol on $\mathbb{R}^2 \setminus \{0\}$ supported in the cone $\{(\zeta_1, \zeta_2) : |\zeta_2| \lesssim |\zeta_1|\}$, satisfying $\widetilde{m}(\zeta_1, 0) = 1$ whenever $|\zeta_1| \neq 0$, and

$$|\partial^\alpha \widetilde{m}(\zeta_1, \zeta_2)| \lesssim_\alpha |(\zeta_1, \zeta_2)|^{-|\alpha|}$$

for all $\alpha \in \mathbb{N}_0^2$ and all $(\zeta_1, \zeta_2) \neq 0$. Then m satisfies (1.6) and we have $m(\xi_1, 0, \eta_1, 0) = m_0(\xi_1, \eta_1)$. We dualize the operator (1.5) associated with this multiplier and consider the corresponding trilinear form, which reads

$$\int_{\mathbb{R}^4} \widehat{F_1}(\xi)\widehat{F_2}(\eta)\widehat{F_3}(\xi+\eta)m(\xi,\eta)d\xi d\eta. \tag{2.8}$$

Let $\lambda > 0$ and let $1 < p_1, p_2, p_3 < \infty$ be such that $1/p_1 + 1/p_2 + 1/p_3 = 1$. For $1 \leq i \leq 3$ we set

$$F_i(x_1, x_2) = f_i(x_1)\lambda^{-1/p_i}\varphi(\lambda^{-1}x_2),$$

where f_i and φ are one-dimensional smooth compactly supported functions and $\widehat{\varphi} \geq 0$. Plugging these particular functions F_i into the trilinear form (2.8) we obtain

$$\lambda^{-1}\int_{\mathbb{R}^4} \widehat{f_1}(\xi_1)\lambda\widehat{\varphi}(\lambda\xi_2)\widehat{f_2}(\eta_1)\lambda\widehat{\varphi}(\lambda\eta_2)\widehat{f_3}(\xi_1+\eta_1)\lambda\widehat{\varphi}(\lambda(\xi_2+\eta_2)) \\ m(\xi_1,\xi_2,\eta_1,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2. \tag{2.9}$$

By rescaling in ξ_2 and η_2 it suffices to consider

$$\int_{\mathbb{R}^4} \widehat{f_1}(\xi_1)\widehat{\varphi}(\xi_2)\widehat{f_2}(\eta_1)\widehat{\varphi}(\eta_2)\widehat{f_3}(\xi_1+\eta_1)\widehat{\varphi}(\xi_2+\eta_2)m(\xi_1,\lambda^{-1}\xi_2,\eta_1,\lambda^{-1}\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2.$$

Letting $\lambda \to \infty$ and then integrating in ξ_2, η_2 we obtain a non-zero constant multiple of

$$\int_{\mathbb{R}^2} \widehat{f_1}(\xi_1)\widehat{f_2}(\eta_1)\widehat{f_3}(\xi_1+\eta_1)m_0(\xi_1,\eta_1)d\xi_1 d\eta_1,$$

which can be recognized as (2.7) paired with a function f_3. Therefore, the form in the last display does not satisfy any L^p estimates. Since $\|F_i\|_{\mathrm{L}^{p_i}(\mathbb{R}^2)}$ equals $\|f_i\|_{\mathrm{L}^{p_i}(\mathbb{R})}\|\varphi\|_{\mathrm{L}^{p_i}(\mathbb{R})}$, it follows that (2.9) does not satisfy any L^p estimates as well.

3. A one-parameter bilinear operator

This section is devoted to the proof of Theorem 3. By a three-dimensional analogue of the cone decomposition outlined in Section 2.3 it suffices to estimate the operators mapping (F_1, F_2) to two-dimensional functions given by

$$x \mapsto \sum_{k\in\mathbb{Z}} (Q_k^{(1)} P_{1,k}^{(2)} F_1)(x)\,(P_{2,k}^{(1)} F_2)(x), \tag{3.1}$$

$$x \mapsto \sum_{k\in\mathbb{Z}} (P_{1,k}^{(1)} P_{2,k}^{(2)} F_1)(x)\,(Q_k^{(1)} F_2)(x), \tag{3.2}$$

and

$$x \mapsto \sum_{k\in\mathbb{Z}} (P_{1,k}^{(1)} Q_k^{(2)} F_1)(x)\,(P_{2,k}^{(1)} F_2)(x), \tag{3.3}$$

where $P_{i,k}$, Q_k are Fourier multipliers with symbols $\varphi_{i,k}$ and ψ_k, respectively, which are adapted to $[-2^{k+1}, 2^{k+1}]$, and in addition ψ_k vanishes on $[-2^{k-1}, 2^{k-1}]$.

We will first prove bounds for each of them in the open Banach range. Then we will extend the range with a fiber-wise Calderón-Zygmund decomposition from [2].

3.1. **Boundedness in the open Banach range.** In this section we show that the operators (3.1), (3.2) and (3.3) are bounded from $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2}$ to $\mathrm{L}^{p_3'}$ whenever $1 < p_1, p_2, p_3 < \infty$ and $1/p_1 + 1/p_2 + 1/p_3 = 1$.

To bound the first operator (3.1), we first reduce to the case when the symbol of $P_{2,k}$ is supported in $[-2^{k-99}, 2^{k-99}]$. Indeed, if this is not the case we split

$$P_{2,k} = \widetilde{P}_{2,k} + Q_{2,k},$$

where the symbols of $\widetilde{P}_{2,k}$ and $Q_{2,k}$ are supported in $[-2^{k-99}, 2^{k-99}]$ and $[-2^{k+1}, -2^{k-100}] \cup [2^{k-100}, 2^{k+1}]$, respectively. We split the operator accordingly and note that the term with $Q_{2,k}$ immediately reduces to two square functions by Cauchy-Schwarz.

Now we dualize and consider the corresponding trilinear form. By the frequency localization in the first fibers of the functions, it suffices to show

$$\Big|\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2} (Q_k^{(1)} P_{1,k}^{(2)} F_1)\, (\widetilde{P}_{2,k}^{(1)} F_2)\, (\mathcal{Q}_k^{(1)} F_3)\Big| \lesssim_{p_1,p_2,p_3} \|F_1\|_{\mathrm{L}^{p_1}(\mathbb{R}^2)} \|F_2\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)} \|F_3\|_{\mathrm{L}^{p_3}(\mathbb{R}^2)}$$

for any choice of exponents $1 < p_1, p_2, p_3 < \infty$ and $1/p_1 + 1/p_2 + 1/p_3 = 1$. Here $\mathcal{Q}_k$ is adapted to $[-2^{k+3}, 2^{k+3}]$ and the corresponding symbol vanishes at the origin. This estimate follows immediately by Hölder's inequality in k and in the integration, together with bounds on the square and maximal function.

For the second operator (3.2) we proceed in the same way; this time reducing to the case when the symbol of $P_{1,k}$ is supported in $[-2^{k-99}, 2^{k-99}]$.

It remains to consider (3.3). By considerations as above, we may assume that $P_{1,k}$ is supported in $[-2^{k-99}, 2^{k-99}]$. By duality it suffices to study the corresponding trilinear form

$$\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2} (P_{1,k}^{(1)} Q_k^{(2)} F_1)\, (P_{2,k}^{(1)} F_2)\, F_3 = c \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2} (P_{1,k}^{(1)} Q_k^{(2)} F_1)\, \mathcal{Q}_k^{(2)} \Big((P_{2,k}^{(1)} F_2)\, F_3 \Big), \tag{3.4}$$

where c is a constant and $\mathcal{Q}_k$ satisfies the properties as in the previous display. By Cauchy-Schwarz in k and Hölder's inequality in the integration, we bound the last display by

$$\|P_{1,k}^{(1)} Q_k^{(2)} F_1\|_{\mathrm{L}^{p_1}(\ell_k^2)} \Big\| \mathcal{Q}_k^{(2)} \Big((P_{2,k}^{(1)} F_2)\, F_3 \Big) \Big\|_{\mathrm{L}^{p_1'}(\ell_k^2)}.$$

The first term is a square function bounded in the full range. For the second term, we note that by Fatou's lemma, it suffices to restrict the ℓ_k^2 norm to $|k| \le K$ for some $K > 0$ and prove bounds uniform in K. By Kintchine's inequality, this term reduces to having to estimate

$$\Big\| \sum_{|k|\le K} a_k \mathcal{Q}_k^{(2)} \Big((P_{2,k}^{(1)} F_2) F_3 \Big) \Big\|_{\mathrm{L}^{p_1'}(\mathbb{R}^2)}$$

for uniformly bounded coefficients a_k. Dualizing with $G \in \mathrm{L}^{p_1}(\mathbb{R}^2)$ it suffices to show

$$\Big| \sum_{|k|\le K} \int_{\mathbb{R}^2} a_k (\mathcal{Q}_k^{(2)} G)\, (P_{2,k}^{(1)} F_2)\, F_3 \Big| \lesssim_{p_1,p_2,p_3} \|G\|_{\mathrm{L}^{p_1}(\mathbb{R}^2)} \|F_2\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)} \|F_3\|_{\mathrm{L}^{p_3}(\mathbb{R}^2)}.$$

This estimate holds in the range $1 < p_1, p_2 < \infty$, $2 < p_3 < \infty$, and Hölder scaling, since the left-hand side is an operator of the form (1.9) paired with the function F_3.

To obtain bounds for (3.3) in the full range, it suffices to show estimates with $1 < p_1, p_3 < \infty$ and $2 < p_2 < \infty$. The left-hand side of (3.4) can be written as

$$\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2} (P_{1,k}^{(1)} Q_k^{(2)} F_1)\, (P_{2,k}^{(1)} F_2)\, \mathcal{P}_k^{(1)} F_3$$

for a multiplier $\mathcal{P}_k$ with symbol adapted to $[-2^{k+3}, 2^{k+3}]$. Writing $P_{2,k} = \varphi_{2,k}(0)I + (P_{2,k} - \varphi_{2,k}(0)I)$ we reduce the last display to

$$\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2} (P^{(1)}_{1,k} Q^{(2)}_k F_1)\, F_2\, (\varphi_{2,k}(0)\mathcal{P}^{(1)}_k F_3) + \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2} (P^{(1)}_{1,k} Q^{(2)}_k F_1)\, (\mathcal{Q}^{(1)}_k F_2)\, (\mathcal{P}^{(1)}_k F_3)$$

for a suitably defined $\mathcal{Q}_k$ whose symbol vanishes at the origin. The first term is analogous to the left-hand side of (3.4), satisfying the estimates with $1 < p_1, p_3 < \infty$ and $2 < p_2 < \infty$. The second term has two Q-type operators and bounds in the full range follow by Hölder's inequality and bounds on the maximal and square functions.

3.2. **Fiber-wise Calderón-Zygmund decomposition.** In this section we show that the operators (3.1), (3.2) and (3.3) are bounded from $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2}$ to $\mathrm{L}^{p_3'}$ whenever $1 < p_1, p_2 < \infty$, $1/2 < p_3' < \infty$ and $1/p_1 + 1/p_2 = 1/p_3'$.

By real bilinear interpolation, it suffices to prove weak-type bounds with $p_3' < 1$ for the operators (3.1), (3.2) and (3.3). Each of them is of the form

$$T(F_1, F_2)(x) = \sum_{k\in\mathbb{Z}} (P^{(1)}_{1,k} P^{(2)}_{3,k} F_1)(x)\, (P^{(1)}_{2,k} F_2)(x),$$

where $P_{i,k}$ are Fourier multipliers with symbols adapted to $[-2^{k+1}, 2^{k+1}]$. We are going to apply the fiberwise Calderón-Zygmund from [2] to the function F_2.

It suffices to show that for any $1 < p_1, p_2 < \infty$ and $1/2 < p_3' < 1$, the operator T satisfies the weak $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2} \to \mathrm{L}^{p_3',\infty}$ estimates. Fix $F_1 \in \mathrm{L}^{p_1}(\mathbb{R}^2)$ and $F_2 \in \mathrm{L}^{p_2}(\mathbb{R}^2)$. By homogeneity we may assume

$$\|F_1\|_{\mathrm{L}^{p_1}(\mathbb{R}^2)} = \|F_2\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)} = 1.$$

Our goal is to show that for every $\lambda > 0$

$$|\{x \in \mathbb{R}^2 : \ |T(F_1, F_2)(x)| > \lambda\}| \lesssim_{p_1,p_2} \lambda^{-p_3'}.$$

We write $x = (x_1, x_2)$. Fix $\lambda > 0$ and perform a fiber-wise Calderón-Zygmund decomposition of $F_2(\cdot, x_2)$ for fixed $x_2 \in \mathbb{R}$ at level λ^{p_3'/p_2}. This yields functions g_{x_2} and atoms a_{i,x_2} supported on disjoint dyadic intervals intervals I_{i,x_2} such that $F_2(\cdot, x_2) = g_{x_2} + \sum_i a_{i,x_2}$ and for all x_2

$$\|g_{x_2}\|_{\mathrm{L}^{p_2}(\mathbb{R})} \le \|F_2(\cdot, x_2)\|_{\mathrm{L}^{p_2}(\mathbb{R})} \quad \text{and} \quad \|g_{x_2}\|_{\mathrm{L}^\infty(\mathbb{R})} \lesssim \lambda^{p_3'/p_2}.$$

Moreover, for all i, the atom a_{i,x_2} has vanishing mean on I_{i,x_2},

$$\|a_{i,x_2}\|_{\mathrm{L}^{p_2}(I_{i,x_2})} \lesssim \lambda^{p_3'/p_2} |I_{i,x_2}|^{1/p_2},$$

and the intervals satisfy

$$\sum_i |I_{i,x_2}| \lesssim \lambda^{-p_3'} \|F_2(\cdot, x_2)\|^{p_2}_{\mathrm{L}^{p_2}(\mathbb{R})}. \tag{3.5}$$

First we consider the good part $T(F_1, g)$, where $g = (x_1, x_2) \mapsto g_{x_2}(x_1)$. We have

$$\|g\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)} \le \|F_2\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)} = 1 \quad \text{and} \quad \|g\|_{\mathrm{L}^\infty(\mathbb{R}^2)} \lesssim \lambda^{p_3'/p_2}.$$

Therefore, for $q_2 > p_2$ we get

$$\|g\|_{\mathrm{L}^{q_2}(\mathbb{R}^2)} \lesssim_{p_2} \lambda^{p_3'(1/p_2 - 1/q_2)}.$$

By the boundedness in the Banach range obtained in the previous section we obtain

$$\begin{aligned} |\{x \in \mathbb{R}^2 : |T(F_1, g)(x)| > \lambda\}| &\le \lambda^{-s} \|T(F_1, g)\|^s_{\mathrm{L}^s(\mathbb{R}^2)} \\ &\lesssim_{p_1,p_2} \lambda^{-s} \|F_1\|^s_{\mathrm{L}^{p_1}(\mathbb{R}^2)} \|g\|^s_{\mathrm{L}^{q_2}(\mathbb{R}^2)} \lesssim \lambda^{-s + s p_3'(1/p_2 - 1/q_2)} = \lambda^{-p_3'}, \end{aligned}$$

where $1 < q_2, s < \infty$ are any exponents that satisfy $1/p_1 + 1/q_2 = 1/s$, $q_2 > p_2$ and (p_1, q_2, s) belong to the open Banach range. This is the desired estimate on the level set.

It remains to consider the bad part $T(F_1, b)$, where $b = (x_1, x_2) \mapsto \sum_i a_{i,x_2}(x_1)$. Denote

$$E = \bigcup_{x_2 \in \mathbb{R}} \bigcup_i (2I_{i,x_2} \times \{x_2\}),$$

where $2I_{i,x_2}$ is the interval with the same center as I_{i,x_2} but twice the length. Note that

$$|E| \lesssim \int_{\mathbb{R}} |\cup_i 2I_{i,x_2}| dx_2 \lesssim \lambda^{-p_3'} \int_{\mathbb{R}} \|F_2(\cdot, x_2)\|_{\mathrm{L}^{p_2}(\mathbb{R})}^{p_2} dx_2 \lesssim \lambda^{-p_3'} \|F_2\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)}^{p_2} = \lambda^{-p_3'},$$

where we have used (3.5). Therefore, it remains to estimate $|\{x \notin E : |T(F_1, b)(x)| > \lambda\}|$. By the definition we have

$$P_{2,k}^{(1)} b(x) = \sum_i P_{2,k}[a_{i,x_2}](x_1).$$

Fix $x_2 \in \mathbb{R}$ and take $x_1 \in \mathbb{R} \setminus \cup_i 2I_{i,x_2}$. Denote by c_{i,x_2} the center of the interval I_{i,x_2}. Since $P_{2,k}$ is a convolution operator with some smooth function ρ_k (with its Fourier transform adapted to $[-2^{k+1}, 2^{k+1}]$ up to a large order), using the mean zero property of a_{i,x_2}, we obtain

$$\begin{aligned} |P_{2,k}[a_{i,x_2}](x_1)| &\leq \int_{I_{i,x_2}} |a_{i,x_2}(w)| |\rho_k(x_1 - w) - \rho_k(x_1 - c_{i,x_2})| dw \\ &\lesssim \int_{I_{i,x_2}} |a_{i,x_2}(w)|\, 2^{2k} |I_{i,x_2}| \big(1 + 2^k |x_1 - c_{i,x_2}|\big)^{-100} dw \\ &\lesssim \lambda^{p_3'/p_2}\, 2^{2k} |I_{i,x_2}|^2 \big(1 + 2^k |x_1 - c_{i,x_2}|\big)^{-100}. \end{aligned}$$

Here we used

$$\|a_{i,x_2}\|_{\mathrm{L}^1(\mathbb{R})} \leq |I_{i,x_2}|^{1/p_2'} \|a_{i,x_2}\|_{\mathrm{L}^{p_2}(\mathbb{R})} \lesssim \lambda^{p_3'/p_2} |I_{i,x_2}|.$$

Therefore, since $|x - c_{i,x_2}| \geq |I_{i,x_2}|/2$ we have

$$\sum_{k \in \mathbb{Z}} 2^{2k} |I_{i,x_2}|^2 \Big(1 + \frac{|x_2 - c_{i,y}|}{2^{-k}}\Big)^{-100} \lesssim \Big(1 + \frac{|x_2 - c_{i,y}|}{|I_{i,x_2}|}\Big)^{-2}.$$

Therefore, we have showed that for $x_1 \in \mathbb{R} \setminus \cup_i 2I_{i,x_2}$ it holds

$$|T(F_1, b)(x)| \lesssim \lambda^{p_3'/p_2} \mathcal{M}(F_1)(x) H(x),$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function and

$$H(x) = \sum_i \Big(1 + \frac{|x_1 - c_{i,y}|}{|I_{i,y}|}\Big)^{-2}.$$

Next we use boundedness of the Marcinkiewicz functions associated with a disjoint collection of the intervals $(I_{i,x_2})_i$, i.e.

$$\Big\| \sum_i \Big(1 + \frac{|x_1 - c_{i,x_2}|}{|I_{i,x_2}|}\Big)^{-2} \Big\|_{\mathrm{L}_{x_1}^{p_2}(\mathbb{R})} \lesssim_{p_2} \Big(\sum_i |I_{i,x_2}|\Big)^{1/p_2}.$$

(See [28] or Grafakos, Exercise 4.6.6.) This yields

$$\|H\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)} \lesssim_{p_2} \Big\| \Big(\sum_i |I_{i,x_2}|\Big)^{1/p_2} \Big\|_{\mathrm{L}_{x_2}^{p_2}(\mathbb{R})} \lesssim \lambda^{-p_3'/p_2} \| \|F_2(x_1, x_2)\|_{\mathrm{L}_{x_1}^{p_2}(\mathbb{R})} \|_{\mathrm{L}_{x_2}^{p_2}(\mathbb{R})} \lesssim \lambda^{-p_3'/p_2}.$$

Therefore,

$$\begin{aligned}|\{x \notin E : |T(F_1, b)(x)| > \lambda\}| &\lesssim |\{x \in \mathbb{R}^2 : \lambda^{p_3'/p_2} \mathcal{M}(F_1)(x) H(x) \gtrsim \lambda\}| \\ &= |\{x \in \mathbb{R}^2 : \mathcal{M}(F_1)(x) H(x) \gtrsim \lambda^{1-p_3'/p_2}\}| \\ &\lesssim \lambda^{-p_3'+(p_3')^2/p_2} \|\mathcal{M}(F_1)\|_{\mathrm{L}^{p_1}(\mathbb{R}^2)}^{p_3'} \|H\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)}^{p_3'} \\ &\lesssim_{p_1,p_2} \lambda^{-p_3'+(p_3')^2/p_2-(p_3')^2/p_2} \lesssim \lambda^{-p_3}.\end{aligned}$$

Summarizing, we have shown that the operator T satisfies the weak inequality $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2} \to \mathrm{L}^{p_3',\infty}$. By interpolation it is bounded in the range claimed by Theorem 4.

Remark 6. We emphasize that we are able to make use of the fiberwise Calderón-Zygmund in this case (and not in the case of Theorem 1) because the operators (3.1), (3.2) and (3.3) act on only fiber of the function F_2. Alternatively, one could apply the classical two-dimensional Calderón-Zygmund decomposition by decomposing the function F_1.

Remark 7. An alternative approach to prove quasi-Banach estimates for (3.1) and (3.2), which avoids the use of fiberwise Calderón-Zygmund decomposition and in fact proves a stronger claim, namely, that (3.1) and (3.2) are bounded with values in the Hardy space $H^{p_3'} \subseteq L^{p_3'}$, is the following. Note that the operators (3.1) and (3.2) are localized in frequency in the first coordinate at scale k. So we can use the argument from [1] about the inequality $\|f\|_p \lesssim \|S(f)\|_p$ for $0 < p \leq 1$, where S is the Littlewood-Paley square function. More precisely, following the scalar case of [1, Theorem 3.1], one obtains for $0 < p_3' \leq 1$ the bound

$$\|\sum_{k\in\mathbb{Z}} (Q_k^{(1)} P_{1,k}^{(2)} F_1)\,(P_{2,k}^{(1)} F_2)\|_{\mathrm{L}^{p_3'}(\mathbb{R}^2)} \lesssim \|(Q_k^{(1)} P_{1,k}^{(2)} F_1)\,(P_{2,k}^{(1)} F_2)\|_{\mathrm{L}^{p_3'}(\ell_k^2)}.$$

To apply the arguments from [1] it is important is to obtain the localized estimates which are reduced through duality by factorizing a Q_k operator on the dual function, which is possible in this situation. Then the square function on the right-hand side is easily estimated by the product of a maximal and square function, both of them which are bounded. A similar reasoning can be done for the second operator of type (3.2). In that way we recover the $\mathrm{L}^{p_3'}$–boundedness of operators (3.1) and (3.2) and we also prove boundedness in the Hardy space $H^{p_3'}$.

4. A tri-parameter trilinear operator

The goal of this section it to prove Theorem 4. Throughout this section we will use the shorthand notation $k \ll l$ to denote $k < l - 50$. Similarly, $k \gg l$ will denote $k > l + 50$ and $k \sim l$ will mean $l - 50 \leq k \leq l + 50$.

For $k \in \mathbb{Z}$ and $1 \leq i \leq 3$ let $Q_{i,k}$ be Fourier multipliers with symbols $\psi_{i,k}$. For $k \in \mathbb{Z}$ and $4 \leq j \leq 6$ let $P_{j,k}$ be Fourier multipliers with symbols $\varphi_{j,k}$. Here $\psi_{i,k}$ and $\varphi_{j,k}$ are smooth one-dimensional functions. In what follows we will consider various classes of symbols and we will apply more assumptions on them as we proceed. We define the trilinear operators U_1 and U_2 acting on two-dimensional functions $F_1, F_2, F_3 : \mathbb{R}^2 \to \mathbb{C}$ by

$$U_1(F_1, F_2, F_3)(x) = \sum_{(k,l,m)\in\mathbb{Z}^3} (Q_{1,k}^{(1)} P_{4,m}^{(2)} F_1)(x)\,(Q_{2,l}^{(1)} P_{5,k}^{(2)} F_2)(x)\,(Q_{3,m}^{(1)} P_{6,l}^{(2)} F_3)(x),$$

$$U_2(F_1, F_2, F_3)(x) = \sum_{(k,l,m)\in\mathbb{Z}^3} (P_{5,k}^{(1)} P_{4,m}^{(2)} F_1)(x)\,(Q_{2,l}^{(1)} Q_{1,k}^{(2)} F_2)(x)\,(Q_{3,m}^{(1)} P_{6,l}^{(2)} F_3)(x).$$

We keep in mind that they depend on the particular choice of the functions $\psi_{i,k}, \varphi_{j,k}$, but we have suppressed that in the notation.

Let the exponents p_1, p_2, p_3, and p_4' satisfy the assumptions stated in Theorem 4. The first step in the proof of Theorem 4 is to decompose the symbols m_1, m_2, m_3 as in Section 2.3. This gives that it suffices to prove $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2} \times \mathrm{L}^{p_3}$ to $\mathrm{L}^{p_4'}$ estimates for U_1 and U_2 under the assumptions that for $1 \leq i \leq 3$, $\psi_{i,k}$ is adapted to $[-2^{k+1}, 2^{k+1}]$, and for $4 \leq j \leq 6$, $\varphi_{j,k}$ is adapted to $[-2^{k+1}, 2^{k+1}]$. Moreover, each $\psi_{i,k}$ vanishes on $[-2^{k-1}, 2^{k-1}]$ and $\varphi_{j,k}(0) = \varphi_{j,0}(0)$ for each k. Indeed, note that while cone decomposition gives 8 terms, all other terms that are of the form of U_1 and U_2 with P- and Q-type operators interchanged can be deduced from the bounds on U_1 and U_2 by symmetry considerations, that is, up to interchanging the role of the functions F_i and considering their transposes $(x, y) \mapsto F_i(y, x)$.

Bounds for U_1 and U_2 resulting after a cone decomposition will be deduced from Lemmas 8-10. Lemma 8 concerns a particular case of U_1.

Lemma 8. *Let p_1, p_2, p_3 and p_4' be exponents as in Theorem 4. For $1 \leq i \leq 3$ and $k \in \mathbb{Z}$ let $\psi_{i,k}$ be a smooth function supported in $[-2^{k+3}, 2^{k+3}]$, which vanishes on $[-2^{k-3}, 2^{k-3}]$. Assume that for each $1 \leq i \leq 3$, $\psi_{i,k}$ is of the form*

$$\psi_{i,k} = \varphi_{i,k} - \varphi_{i,k-1}, \tag{4.1}$$

where $\varphi_{i,k}$ is a bump function adapted to $[-2^{k+3}, 2^{k+3}]$. For $4 \leq j \leq 6$ and $k \in \mathbb{Z}$ let $\varphi_{j,k}$ be adapted to $[-2^{k+4}, 2^{k+4}]$ and assume that the function $\psi_{j,k} = \varphi_{j,k} - \varphi_{j,k-1}$ is supported in $[-2^{k+4}, 2^{k+4}]$ and vanishes on $[-2^{k-4}, 2^{k-4}]$.

Under these assumptions on $\psi_{i,k}$ and $\varphi_{j,k}$, the associated operator U_1 is bounded from $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2} \times \mathrm{L}^{p_3}$ to $\mathrm{L}^{p_4'}$.

Next we state Lemmas 9 and 10, which concern operators of the form (1.14) with one constant symbol. In particular, these results will be used in the proof of Lemma 8.

Lemma 9. *For $k \in \mathbb{Z}$ let $Q_{1,k}, Q_{2,k}, P_{5,k}$ and $P_{6,k}$ be Fourier multipliers with symbols adapted to $[-2^{k+10}, 2^{k+10}]$. Further, assume that the symbols of $Q_{1,k}, Q_{2,k}$ vanish on $[-2^{k-10}, 2^{k-10}]$. Then the trilinear operator which maps (F_1, F_2, F_3) to the two-dimensional function given by*

$$x \mapsto \sum_{(k,l)\in\mathbb{Z}^2 : k \ll l} (Q_{1,k}^{(1)} F_1)(x)\, (Q_{2,l}^{(1)} P_{5,k}^{(2)} F_2)(x)\, (P_{6,l}^{(2)} F_3)(x)$$

is bounded from $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2} \times \mathrm{L}^{p_3}$ to $\mathrm{L}^{p_4'}$ in the range $1 < p_1, p_2, p_3 < \infty$, $2 < p_4 < \infty$, $\sum_{i=1}^4 \frac{1}{p_i} = 1$, and $\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{2}$.

Proof of Lemma 9. It will be clear from the proof that the argument will not depend on the particular choice of the frequency projections satisfying the requirements from the lemma, so for simplicity of the notation, we only discuss the special cases $P_{5,k} = P_{6,k} = P_k$, $Q_{1,k} = Q_{2,k} = Q_k$.

By duality it suffices to consider the corresponding quadrilinear form

$$\sum_{(k,l)\in\mathbb{Z}^2 : k \ll l} \int_{\mathbb{R}^2} (Q_k^{(1)} F_1)\, (Q_l^{(1)} P_k^{(2)} F_2)\, (P_l^{(2)} F_3)\, F_4.$$

By the frequency localization of the functions F_i in the first fibers, to prove estimates for the quadrilinear form it suffices to study

$$\sum_{(k,l)\in\mathbb{Z}^2 : k \ll l} \mathcal{M}_{k,l} = \sum_{(k,l)\in\mathbb{Z}^2} \mathcal{M}_{k,l} - \sum_{(k,l)\in\mathbb{Z}^2 : k \sim l} \mathcal{M}_{k,l} - \sum_{(k,l)\in\mathbb{Z}^2 : k \gg l} \mathcal{M}_{k,l}, \tag{4.2}$$

where we have defined

$$\mathcal{M}_{k,l} = \int_{\mathbb{R}^2} (Q_k^{(1)} F_1)\, (Q_l^{(1)} P_k^{(2)} F_2)\, \mathcal{Q}_l^{(1)} \Big((P_l^{(2)} F_3)\, F_4 \Big).$$

Here $\mathcal{Q}_l$ is a Fourier multiplier with a symbol which is adapted to $[-2^{l+12}, 2^{l+12}]$ and vanishes on $[-2^{l-12}, 2^{l-12}]$.

Note that the sum over $k \gg l$ in (4.2) vanishes due to frequency supports. To bound the sum with $k \sim l$ we use Hölder's inequality to obtain

$$\Big|\sum_{l\in\mathbb{Z}}\sum_{s\sim 0}\mathcal{M}_{l+s,l}\Big| \leq \sum_{s\sim 0}\|Q^{(1)}_{l+s}F_1\|_{\mathrm{L}^{p_1}(\ell^\infty_l)}\|Q^{(1)}_l P^{(2)}_{l+s}F_2\|_{\mathrm{L}^{p_2}(\ell^2_l)}\Big\|\mathcal{Q}^{(1)}_l\Big((P^{(2)}_l F_3)\,F_4\Big)\Big\|_{\mathrm{L}^r(\ell^2_l)}$$

whenever $1/p_1+1/p_2+1/r=1$ and $1<p_1,p_2,r<\infty$. For a fixed s, bounds for the first two terms follow by boundedness of the maximal and square functions. The third term satisfies

$$\Big\|\mathcal{Q}^{(1)}_l\Big((P^{(2)}_l F_3)\,F_4\Big)\Big\|_{\mathrm{L}^r(\ell^2_l)} \lesssim_{p_3,p_4} \|F_3\|_{\mathrm{L}^{p_3}(\mathbb{R}^2)}\|F_4\|_{\mathrm{L}^{p_4}(\mathbb{R}^2)} \tag{4.3}$$

whenever $1<p_3,r<\infty$, $2<p_4<\infty$ and $1/r=1/p_3+1/p_4$. Indeed, this follows by Kintchine's inequality to linearize the square-sum and then use bounds for (1.9). In the end, it remains to sum in $s\sim 0$.

Therefore, it suffices to bound the form in (4.2) with $(k,l)\in\mathbb{Z}^2$. In this case we note that the third factor in $\mathcal{M}_{k,l}$ does not depend on k. This allows to apply Cauchy-Schwarz in l and Hölder's inequality in the integration, yielding

$$\Big|\sum_{(k,l)\in\mathbb{Z}^2}\mathcal{M}_{k,l}\Big| \leq \Big\|\sum_{k\in\mathbb{Z}}(Q^{(1)}_k F_1)\,(Q^{(1)}_l P^{(2)}_k F_2)\Big\|_{\mathrm{L}^s(\ell^2_l)}\Big\|\mathcal{Q}^{(1)}_l\Big((P^{(2)}_l F_3)\,F_4\Big)\Big\|_{\mathrm{L}^r(\ell^2_l)}$$

whenever $1/r+1/s=1$, $1<r,s<\infty$. The second factor on the right-hand side equals (4.3). From the known bounds, we obtain the condition $1/p_3+1/p_4=1/r$ and $1<p_3,r<\infty$, $2<p_4<\infty$. The first factor maps $\mathrm{L}^{p_1}(\mathbb{R}^2)\times\mathrm{L}^{p_2}(\ell^2)\to\mathrm{L}^s(\ell^2)$ whenever $1/p_1+1/p_2=1/s$ and $1<p_1,p_2<\infty$, $1/s>1/2$. This follows from vector-valued estimates for (1.9) as discussed in the display following (2.5). □

Lemma 10. *For $k\in\mathbb{Z}$, let $Q_{1,k}, Q_{2,k}$ be Fourier multipliers with symbols $\sigma_{1,k},\sigma_{2,k}$ which are functions supported in $[-2^{k+10},2^{k+10}]$ and vanish on $[-2^{k-10},2^{k-10}]$. For $k\in\mathbb{Z}$ let $P_{5,k},P_{6,k}$ be Fourier multipliers with symbols $\rho_{5,k},\rho_{6,k}$, which are functions adapted to $[-2^{k+10},2^{k+10}]$.*

Assume that for each $i\in\{1,2,5,6\}$, these functions are of the form

$$\sigma_{i,k}=\rho_{i,k}-\rho_{i,k-1},$$

where $\sigma_{4,k},\sigma_{5,k}$ are further functions supported in $[-2^{k+10},2^{k+10}]$, which vanish on $[-2^{k-10},2^{k-10}]$, while $\rho_{1,k},\rho_{2,k}$ are functions adapted to $[-2^{k+10},2^{k+10}]$. Then the trilinear operator which maps (F_1,F_2,F_3) to the two-dimensional function given by

$$x\mapsto\sum_{(k,l)\in\mathbb{Z}^2:k\gg l}(Q^{(1)}_{1,k}F_1)(x)\,(Q^{(1)}_{2,l}P^{(2)}_{5,k}F_2)(x)\,(P^{(2)}_{6,l}F_3)(x) \tag{4.4}$$

is bounded from $\mathrm{L}^{p_1}\times\mathrm{L}^{p_2}\times\mathrm{L}^{p_3}$ to $\mathrm{L}^{p_4'}$ in the range $1<p_1,p_2,p_3<\infty$, $2<p_4<\infty$, $\sum_{i=1}^4\frac{1}{p_i}=1$, and $\frac{1}{p_2}+\frac{1}{p_3}>\frac{1}{2}$.

Proof of Lemma 10. Let us augment the definitions in the statement of the lemma by setting $Q_{i,k}$ and $P_{i,k}$ to be Fourier multipliers with the respective symbols $\sigma_{i,k}$ and $\rho_{i,k}$ for any $i\in\{1,2,5,6\}$. Observe that because of the condition on the bump functions we have the telescoping identity

$$\sum_{k=k_0}^{k_1}P_{i,k}f\,Q_{j,k}g+\sum_{k=k_0}^{k_1}Q_{i,k}f\,P_{j,k-1}g=P_{i,k_1}f\,P_{j,k_1}g-P_{i,k_0-1}f\,P_{j,k_0-1}g \tag{4.5}$$

for any $f,g\in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ and $i,j\in\{1,2,5,6\}$. Also note that letting $k_0\to-\infty$ and $k_1\to\infty$, the right-hand side becomes the pointwise product fg.

Our goal is to deduce the claim from Lemma 9 by two applications of the telescoping identity (4.5). We write the sum over $k\gg l$ in (4.4) as $\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}:l\ll k}$ and use (4.5) in l. This gives

$$(4.4) = -\sum_{(k,l)\in\mathbb{Z}^2:k\gg l}(Q^{(1)}_{1,k}F_1)\,(P^{(1)}_{2,l-1}P^{(2)}_{5,k}F_2)\,(Q^{(2)}_{6,l}F_3) \tag{4.6}$$

$$+\sum_{k\in\mathbb{Z}}(Q^{(1)}_{1,k}F_1)\,(P^{(1)}_{2,k-49}P^{(2)}_{5,k}F_2)\,(P^{(2)}_{6,k-49}F_3) \tag{4.7}$$

Indeed, (4.7) is the boundary term at $l=k-49$, while the term at $-\infty$ vanishes.

Let us consider (4.7). We dualize it and write the corresponding form up to a constant as

$$\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^2}\mathcal{Q}^{(1)}_k\Big((Q^{(1)}_{1,k}F_1)\,(P^{(1)}_{2,k-49}P^{(2)}_{5,k}F_2)\Big)\,(P^{(2)}_{6,k-49}F_3)\,F_4,$$

where $\mathcal{Q}_k$ has a symbol adapted to $[-2^{k+12},2^{k+12}]$ which vanishes on $[-2^{k-12},2^{k-12}]$. Note that by the support assumption on $\sigma_{2,k}$ we necessary have that $\rho_{2,k}(0)$ is the same constant $c_0\in\mathbb{R}$ for each $k\in\mathbb{Z}$. Then we write $P_{2,k-49}=c_0I+(P_{2,k-49}-c_0I)$ and split the operator accordingly. By the frequency support information in the first fibers it suffices to bound the terms

$$\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^2}c_0\,\mathcal{Q}^{(1)}_k\Big((Q^{(1)}_{1,k}F_1)\,(P^{(2)}_{5,k}F_2)\Big)\,(P^{(2)}_{6,k-9}F_3)\,F_4 \quad\text{and} \tag{4.8}$$

$$\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^2}\mathcal{Q}^{(1)}_k\Big((Q^{(1)}_{1,k}F_1)\,(\widetilde{\mathcal{Q}}^{(1)}_kP^{(2)}_{5,k}F_2)\Big)\,(P^{(2)}_{6,k-9}F_3)\,F_4, \tag{4.9}$$

where the symbol of $\widetilde{\mathcal{Q}}_k$ is adapted in $[-2^{k+100},2^{k+100}]$ and vanishes at the origin. The desired bounds for the first term (4.8) follow from

$$\Big|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^2}(Q^{(1)}_{1,k}F_1)\,(P^{(2)}_{5,k}F_2)\,\mathcal{Q}^{(1)}_k\Big((P^{(2)}_{6,k-49}F_3)\,F_4\Big)\Big|$$

$$\leq \|(Q^{(1)}_{1,k}F_1)\,(P^{(2)}_{5,k}F_2)\|_{\mathrm{L}^r(\ell^2_k)}\Big\|\mathcal{Q}^{(1)}_k\Big((P^{(2)}_{6,k-49}F_3)\,F_4\Big)\Big\|_{\mathrm{L}^s(\ell^2_k)} \tag{4.10}$$

whenever $1/r+1/s=1$ and $1<r,s<\infty$. For the first term in (4.10) we use Hölder's inequality to obtain

$$\|(Q^{(1)}_{1,k}F_1)\,(P^{(2)}_{5,k}F_2)\|_{\mathrm{L}^r(\ell^2_k)}\leq\|Q^{(1)}_{1,k}F_1\|_{\mathrm{L}^{p_1}(\ell^2_k)}\|P^{(2)}_{5,k}F_2\|_{\mathrm{L}^{p_2}(\ell^\infty_k)}\lesssim_{p_1,p_2}\|F_1\|_{\mathrm{L}^{p_1}(\mathbb{R}^2)}\|F_2\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)}.$$

whenever $1/r=1/p_1+1/p_2$, $1<p_1,p_2<\infty$. For the last inequality we have used bounds on the maximal and square functions. For the second term in (4.10) we use Kintchine's inequality to linearize the square-sum. Then we apply bounds for the corresponding operator (1.9), yielding

$$\Big\|\mathcal{Q}^{(1)}_k\Big((P^{(2)}_{6,k-49}F_3)\,F_4\Big)\Big\|_{\mathrm{L}^s(\ell^2_k)}\lesssim_{p_3,p_4}\|F_3\|_{\mathrm{L}^{p_3}(\mathbb{R}^2)}\|F_4\|_{\mathrm{L}^{p_4}(\mathbb{R}^2)}$$

whenever $1/s=1/p_3+1/p_4$, $1<p_3<\infty$, and $2<p_4<\infty$. This yields the desired estimate.

For (4.9) we proceed in the analogous way, this time estimating $\widetilde{\mathcal{Q}}^{(1)}_kP^{(2)}_{5,k}F_2$ by the Hardy-Littlewood maximal function.

It remains to estimate (4.6), for which we still want to switch the projections in k. Now we write the double sum over $k \gg l$ as $\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}:k\gg l}$ and telescope (4.6) in k, giving

$$(4.6) = \sum_{(k,l)\in\mathbb{Z}^2:k\gg l} (P^{(1)}_{1,k-1}F_1)\,(P^{(1)}_{2,l-1}Q^{(2)}_{5,k}F_2)\,(Q^{(2)}_{6,l}F_3), \tag{4.11}$$

$$+ \sum_{l\in\mathbb{Z}} (P^{(1)}_{1,l+51}F_1)\,(P^{(1)}_{2,l-1}P^{(2)}_{5,l+51}F_2)\,(Q^{(2)}_{6,l}F_3), \tag{4.12}$$

$$- \sum_{l\in\mathbb{Z}} (F_1)\,(P^{(1)}_{2,l-1}F_2)\,(Q^{(2)}_{6,l}F_3). \tag{4.13}$$

The term (4.13) is the boundary term at $k=\infty$, which is just a pointwise product of F_1 with an operator of the form (1.9). We obtain $\mathrm{L}^{p_1}\times\mathrm{L}^{p_2}\times\mathrm{L}^{p_3}\to\mathrm{L}^{p_4'}$ estimates in the range $1<p_1,p_2,p_3,p_4<\infty$, $1/p_2+1/p_3>1/2$, whenever the exponents satisfy the Hölder scaling. The term (4.11) follows from Lemma 9 and symmetry considerations, i.e. after interchanging the role of F_1 and F_3 and the role of the first and second fibers. We obtain estimates in the range $1<p_1,p_2,p_3<\infty$, $2<p_4<\infty$, $1/p_2+1/p_3>1/2$. Finally, note that

$$(4.12) = \sum_{l\in\mathbb{Z}} (P^{(1)}_{1,l+51}F_1)\,(P^{(1)}_{2,l-1}P^{(2)}_{5,l-49}F_2)\,(Q^{(2)}_{6,l}F_3) \tag{4.14}$$

$$+ \sum_{l\in\mathbb{Z}} (P^{(1)}_{1,l+51}F_1)\,(P^{(1)}_{2,l-1}\mathcal{Q}^{(2)}_{l}F_2)\,(Q^{(2)}_{6,l}F_3), \tag{4.15}$$

where the symbol of $\mathcal{Q}_l$ is up to a constant adapted in $[-2^{l+51},2^{l+51}]$ and it vanishes near the origin. The term (4.15) is bounded by Hölder's inequality and bounds on the maximal and square functions in the full range. For (4.14) one can proceed analogously as for (4.7), with the roles of the first and second fiber interchanged. This gives bounds in the range $1<p_1,p_2,p_3<\infty$, $2<p_4<\infty$ and Hölder scaling. □

Now we are ready to prove Lemma 8.

Proof of Lemma 8. For simplicity of notation, we shall assume that $P_{1,k}=P_{2,k}=P_k$ and $Q_{1,k}=Q_{2,k}=Q_k$, but as it will be clear from the proof, the arguments will work for general Fourier multiplier operators with symbols satisfying the above assumptions.

The proof will be based on analyzing the ordering of the parameters k,l, and m. Note that by symmetry we may assume $k\le l\le m$. If, in addition, it holds that $k\sim l$ and $l\sim m$, then all frequencies are comparable up to an absolute constant and the claim follows by Hölder's inequality.

Consider now the case with two dominating frequencies when $k\ll l$ and $l\sim m$, i.e. $l=m+s$ for some $-50\le s\le 0$. Then we need to estimate

$$\sum_{m\in\mathbb{Z}}\ \sum_{-50\le s\le 0}\ \sum_{\substack{k\in\mathbb{Z}:\\ k\ll m+s}} (Q^{(1)}_{k}P^{(2)}_{m}F_1)\,(Q^{(1)}_{m+s}P^{(2)}_{k}F_2)\,(Q^{(1)}_{m}P^{(2)}_{m+s}F_3).$$

By Cauchy-Schwarz in m and Hölder's inequality, its $\mathrm{L}^{p_4'}$ norm is bounded by

$$\sum_{-50\le s\le 0}\Big\|\sum_{\substack{k\in\mathbb{Z}:\\ k\ll m+s}} (Q^{(1)}_{k}P^{(2)}_{m}F_1)(Q^{(1)}_{m+s}P^{(2)}_{k}F_2)\Big\|_{\mathrm{L}^{p_3'}(\ell^2_m)}\|Q^{(1)}_{m}P^{(2)}_{m+s}F_3\|_{\mathrm{L}^{p_3}(\ell^2_m)}$$

whenever $1<p_3<\infty$ and $1/p_4'=1/p_3+1/p_3'$. For a fixed s, the second term is a square function, bounded in the full range. For the first term we use Kintchine's inequality, which reduces to showing

$$\Big\| \sum_{|m|\leq M} a_m \sum_{\substack{k\in\mathbb{Z}:\\ k\ll m+s}} (Q_k^{(1)} P_m^{(2)} F_1)(Q_{m+s}^{(1)} P_k^{(2)} F_2)\Big\|_{\mathrm{L}^{p_3'}(\mathbb{R}^2)} \lesssim_{p_1,p_2} \|F_1\|_{\mathrm{L}^{p_1}(\mathbb{R}^2)} \|F_2\|_{\mathrm{L}^{p_2}(\mathbb{R}^2)}$$

uniformly in $M>0$, where $|a_m|\leq 1$. This estimate holds when $1<p_1,p_2,p_3,p_4<\infty$, $1/p_1+1/p_2=1/p_3'$, $1/p_1+1/p_2>1/2$. Indeed, this follows from Theorem 1 after normalizing the symbols of P_m and Q_{m+s}. In the end, it remains to sum in $-50\leq s\leq 0$.

Now we consider the $k\leq l\ll m$ with one dominating frequency. We dualize and consider the corresponding trilinear form. By the frequency localizations it suffices to bound

$$\sum_{(k,l,m)\in\mathbb{Z}^3: k\leq l\ll m} \int_{\mathbb{R}^2} (Q_k^{(1)} P_m^{(2)} F_1)\,(Q_l^{(1)} P_k^{(2)} F_2)\,(Q_m^{(1)} P_l^{(2)} F_3)\,(\mathcal{Q}_m^{(1)} \mathcal{P}_m^{(2)} F_4),$$

where $\mathcal{Q}_m$, $\mathcal{P}_m$ are adapted to $[-2^{m+6},2^{m+6}]$ and $\mathcal{Q}_m$ vanishes in $[-2^{m-6},2^{m-6}]$. As in the proof of Lemma 10, we note that $\varphi_m(0)=c_0$ is a constant with $|c_0|\leq 1$ for each $m\in\mathbb{Z}$. We write $P_m=c_0I+(P_m-c_0I)$ for the projection acting on the second fiber of F_1. Then this reduces to

$$\sum_{(k,l,m)\in\mathbb{Z}^3: k\leq l\ll m} \mathcal{M}_{k,l,m} \quad + \quad \sum_{(k,l,m)\in\mathbb{Z}^3: k\leq l\ll m} \mathcal{E}_{k,l,m},$$

where we have set

$$\mathcal{M}_{k,l,m} = \int_{\mathbb{R}^2} (Q_k^{(1)} F_1)\,(Q_l^{(1)} P_k^{(2)} F_2)\,(Q_m^{(1)} P_l^{(2)} F_3)\,(\mathcal{Q}_m^{(1)} \mathcal{P}_m^{(2)} F_4), \tag{4.16}$$

$$\mathcal{E}_{k,l,m} = \int_{\mathbb{R}^2} (Q_k^{(1)} \widetilde{\mathcal{Q}}_m^{(2)} F_1)\,(Q_l^{(1)} P_k^{(2)} F_2)\,(Q_m^{(1)} P_l^{(2)} F_3)\,(\mathcal{Q}_m^{(1)} \mathcal{P}_m^{(2)} F_4). \tag{4.17}$$

Here $\widetilde{\mathcal{Q}}_m$ is associated with a bump function, which is up to a constant multiple, adapted to $[-2^{m+10},2^{m+10}]$. Moreover, the bump function vanishes at the origin. We have redefined $\mathcal{P}_m$ by subsuming c_0 into its definition.

First we consider (4.16). We write the sum over $(k,l,m)\in\mathbb{Z}^3$ with $k\leq l\ll m$ as

$$\sum_{(k,l,m)\in\mathbb{Z}^3: k\leq l\ll m} \mathcal{M}_{k,l,m} = \Big(\sum_{(k,l,m)\in\mathbb{Z}^3: k\leq l\ll m} \mathcal{M}_{k,l,m} - \sum_{(k,l,m)\in\mathbb{Z}^3: k\leq l} \mathcal{M}_{k,l,m}\Big) \tag{4.18}$$

$$+ \sum_{(k,l,m)\in\mathbb{Z}^3: k\leq l} \mathcal{M}_{k,l,m}. \tag{4.19}$$

Let us first focus on (4.19). We split the summation into the sums over m and (k,l), apply Cauchy-Schwarz in m and Hölder's inequality in the integration. This yields

$$\Big| \sum_{(k,l,m)\in\mathbb{Z}^3: k\leq l} \mathcal{M}_{k,l,m}\Big|$$

$$\leq \Big\| \sum_{(k,l)\in\mathbb{Z}^2: k\leq l} (Q_k^{(1)} F_1)\,(Q_l^{(1)} P_k^{(2)} F_2)\,(Q_m^{(1)} P_l^{(2)} F_3)\Big\|_{\mathrm{L}^{p_4'}(\ell^2_m)} \|\mathcal{Q}_m^{(1)} \mathcal{P}_m^{(2)} F_4\|_{\mathrm{L}^{p_4}(\ell^2_m)}. \tag{4.20}$$

Since the second term is a square function, it remains to prove $\mathrm{L}^{p_1}(\mathbb{R}^2)\times\mathrm{L}^{p_2}(\mathbb{R}^2)\times\mathrm{L}^{p_3}(\ell^2)\to\mathrm{L}^{p_4'}(\ell^2)$ vector-valued estimates for the trilinear operator

$$\sum_{(k,l)\in\mathbb{Z}^2: k\leq l} (Q_k^{(1)} F_1)\,(Q_l^{(1)} P_k^{(2)} F_2)\,(P_l^{(2)} F_3). \tag{4.21}$$

These estimates will again follow by freezing the functions F_1 and F_2 and using Marcinkiewicz-Zygmund inequalities, provided we show scalar-valued boundedness of (4.21). To show this, we split the summation in (4.21) into regions where $k\sim l$, $k\ll l$, or $k\gg l$. The case $k\sim l$ is bounded by Hölder's inequality. The remaining two cases follow by Lemmas 9 and 10 above. Summarizing, we obtain estimates for (4.20) when $1 < p_1, p_2, p_3 < \infty$, $2 < p_4 < \infty$, $1/p_1 + 1/p_2 > 1/2$, $1/p_2 + 1/p_3 > 1$, and $\sum_{i=1}^{4} 1/p_i = 1$.

It remains to estimate (4.18), which splits further into several terms. We fix $k \leq l$ and then we consider several cases depending on the size of m relative to k and l. If for any triple $(j_1, j_2, j_3) \in \{k, l, m\}^3$ with all three distinct entries it holds $j_1 \sim j_2$ and $j_2 \sim j_3$, then the claim follows by Hölder's inequality. If $l \gg \max(k, m)$, then the corresponding term is zero due to frequency supports. Therefore, the remaining two cases with $k \leq l$ are: $k \sim l$ and $m \ll k$, or $k \ll l$ and $m \sim l$.

Consider first the case when $k \sim l$ and $m \ll k$. Then we have two Q-type projections in each parameter. Bounds are deduced by two applications of Hölder's inequality as follows. Taking the triangle inquality, enlarging the sum in m to be over the whole $\mathbb{Z}$ and applying Hölder's inequality in l, we obtain

$$\Big|\sum_{-50\leq s\leq 0}\sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}:m\ll l+s}\mathcal{M}_{l+s,l,m}\Big|$$
$$\leq \sum_{-50\leq s\leq 0}\int_{\mathbb{R}^2}\sum_{m\in\mathbb{Z}}\|Q^{(1)}_{l+s}F_1\|_{\ell^2_l}\,\|Q^{(1)}_{l}P^{(2)}_{l+s}F_2\|_{\ell^2_l}\,\|Q^{(1)}_{m}P^{(2)}_{l}F_3\|_{\ell^\infty_l}\,|\mathcal{Q}^{(1)}_{m}\mathcal{P}^{(2)}_{m}F_4|$$

By Cauchy-Schwarz in m and Hölder in the integration, we bound the term for a fixed s by

$$\|Q^{(1)}_{l+s}F_1\|_{\mathrm{L}^{p_1}(\ell^2_l)}\,\|Q^{(1)}_{l}P^{(2)}_{l+s}F_2\|_{\mathrm{L}^{p_2}(\ell^2_l)}\,\|Q^{(1)}_{m}P^{(2)}_{l}F_3\|_{\mathrm{L}^{p_3}(\ell^2_m(\ell^\infty_l))}\,\|\mathcal{Q}^{(1)}_{m}\mathcal{P}^{(2)}_{m}F_4\|_{\mathrm{L}^{p_4}(\ell^2_m)}. \tag{4.22}$$

Each of the terms is bounded in full range. The third term, square-maximal function term is bounded in by the Fefferman-Stein inequality, see [22]. The remaining terms are square functions.

Next we consider the case $k \ll l$ and $m \sim l$. We split

$$\sum_{\substack{(k,l,m)\in\mathbb{Z}^3:\\ k\ll l,m\sim l}}\mathcal{M}_{k,l,m} = \sum_{\substack{(k,l,m)\in\mathbb{Z}^3:\\ m\sim l}}\mathcal{M}_{k,l,m} - \sum_{\substack{(k,l,m)\in\mathbb{Z}^3:\\ l-50\leq k\leq l+100,m\sim l}}\mathcal{M}_{k,l,m} - \sum_{\substack{(k,l,m)\in\mathbb{Z}^3:\\ k\gg l+50,m\sim l}}\mathcal{M}_{k,l,m}.$$

Note that bounds for the second sum follow by Hölder's inequality, while the third sum is zero due to frequency supports. Thus, it remains to consider the case $m \sim l$. We again split the sum over $(k, l) \in \mathbb{Z}^2$ and write $m = l + s$, $s \sim 0$. Now we use Hölder's inequality in l to bound

$$\sum_{s\sim 0}\Big|\sum_{(k,l)\in\mathbb{Z}^2}\mathcal{M}_{k,l,l+s}\Big|$$
$$\leq \sum_{s\sim 0}\Big\|\sum_{k\in\mathbb{Z}}(Q^{(1)}_{k}F_1)\,(Q^{(1)}_{l}P^{(2)}_{k}F_2)\Big\|_{\mathrm{L}^{r}(\ell^2_l)}\|Q^{(1)}_{l+s}P^{(2)}_{l}F_3\|_{\mathrm{L}^{p_3}(\ell^2_l)}\|Q^{(1)}_{l+s}P^{(2)}_{l+s}F_3\|_{\mathrm{L}^{p_4}(\ell^\infty_l)} \tag{4.23}$$

where $1/r + 1/p_3 + 1/p_4 = 1$. The second and third term are maximal and square functions, respectively, while bounds for the first term follow by $\mathrm{L}^p(\mathbb{R}^2) \times \mathrm{L}^q(\ell^2) \to \mathrm{L}^r(\ell^2)$ vector-valued estimates for the operator (1.9). Summing in s, we obtain estimates in the range $1 < p_1, p_2, p_3, p_4 < \infty$, $1/p_1 + 1/p_2 > 1/2$, and $\sum_{i=1}^{4} 1/p_i = 1$.

To bound (4.17), we proceed analogously as for (4.16), except that in the analogue of (4.21) we use $\mathrm{L}^{p_1}(\ell^2) \times \mathrm{L}^{p_2}(\mathbb{R}^2) \times \mathrm{L}^{p_3}(\ell^2) \to \mathrm{L}^{p_4}(\ell^2)$ vector-valued estimates for that operator and in the display analogous to (4.22) we now have the maximal-square function

$$\|Q^{(1)}_{l+s}\widetilde{\mathcal{Q}}^{(2)}_{m}F_1\|_{\mathrm{L}^{p_1}(\ell^\infty_m(\ell^2_l))},$$

which is also bounded in the full range by the Fefferman-Stein inequality. The analogue for the vector-valued estimates used in (4.23) are now $\mathrm{L}^p(\ell^2) \times \mathrm{L}^q(\ell^2) \to \mathrm{L}^r(\ell^2)$ vector-valued estimates for the operator of the form (1.9). □

Finally, we are ready to tackle the operators that result after the cone decomposition. Recall that we need to prove bounds for U_1 and U_2 under the assumptions that for $1 \leq i \leq 3$, $\psi_{i,k}$ is adapted to $[-2^{k+1}, 2^{k+1}]$, and for $4 \leq j \leq 6$, $\varphi_{j,k}$ is adapted to $[-2^{k+1}, 2^{k+1}]$. Moreover, each $\psi_{i,k}$ vanishes on $[-2^{k-1}, 2^{k-1}]$ and $\varphi_{j,k}(0) = \varphi_{j,0}(0)$.

4.1. **Completing the proof of Theorem 4.** Let ϕ_k be a function adapted in $[-2^{k+1}, 2^{k+1}]$, which satisfies $\phi_k(0) = \phi_0(0)$ for each $k \in \mathbb{Z}$. We write

$$\phi_k = c_0^{-1}(c_0\phi_k - \widetilde{\phi}_k) + c_0^{-1}\widetilde{\phi}_k, \tag{4.24}$$

where $\widetilde{\phi}_k$ is constantly equal to $c_0\phi_0(0)$ on $[-2^{k-1}, 2^{k-1}]$, and $|c_0| \leq 1$ is such that $\widetilde{\phi}_k$ is a function adapted in $[-2^{k+1}, 2^{k+1}]$. Note that the function $c_0\phi_k - \widetilde{\phi}_k$ vanishes at the origin.

Consider U_1 and U_2 which resulted from the cone decomposition. Writing each of the symbols of the P-type operators as in (4.24), it suffices to bound the operators under the following assumptions. For each $1 \leq i \leq 3$, $\psi_{i,k}$ is adapted in $[-2^{k+1}, 2^{k+1}]$ and vanishes on $[-2^{k-1}, 2^{k1}]$. For each $4 \leq j \leq 5$, $\varphi_{j,k}$ is supported in $[-2^{k+1}, 2^{k+1}]$ and exactly one of the following holds:

1. For each $4 \leq j \leq 6$, the function $\varphi_{j,k}$ is constantly equal to $\varphi_{j,0}(0)$ on $[-2^{k-1}, 2^{k-1}]$.
2. There is an index $4 \leq j_0 \leq 6$, such that $\varphi_{j_0,k}(0) = 0$ for each k. Moreover, for $j \neq j_0$, $\varphi_{j,k}$ is constantly equal to $\varphi_{j,0}(0)$ on $[-2^{k-1}, 2^{k-1}]$.
3. There is an index $4 \leq j_0 \leq 6$, such that $\varphi_{j_0,k}$ is constantly equal to $\varphi_{j_0,0}(0)$ on $[-2^{k-1}, 2^{k-1}]$. Moreover, for $j \neq j_0$, it holds $\varphi_{j,k}(0) = 0$ for each k.
4. For each $4 \leq j \leq 6$, it holds $\varphi_{j,k}(0) = 0$ for each k.

We will analyze each of these cases for the operators of the form U_1 and U_2 and we start with U_1. Our first aim is to reduce considerations to the case when the symbols $\psi_{i,k}$ are supported in $[-2^{k+3}, 2^{k+3}]$, vanish on $[-2^{k-3}, 2^{k-3}]$, and are of the form $\varphi_{i,k} - \varphi_{i,k-1}$ for a function $\varphi_{i,k}$ supported in $[-2^{k+3}, 2^{k+3}]$. The argument that follows is along the lines of an argument used in Section 6 of [14] when transitioning from the dyadic to the continuous setting.

Let us denote by P_φ the one-dimensional Fourier multiplier with symbol φ, i.e. $P_\varphi f = f * \check{\varphi}$. As before we shall denote its fiber-wise action on a two-dimensional function with a superscript. Let ϕ be a non-negative smooth function supported in $[-2^{-0.4}, 2^{-0.4}]$ and such that ϕ is constantly equal to one on $[-2^{-0.6}, 2^{-0.6}]$. For $a \in \mathbb{R}$ define ϑ_a and ρ_a by

$$\vartheta_a(\xi) = \phi(2^{-a-1}\xi) - \phi(2^{-a}\xi),$$
$$\rho_a(\xi) = \phi(2^{-a-0.6}\xi) - \phi(2^{-a-0.5}).$$

Note that ϑ_a is supported in $[-2^{a+0.6}, 2^{a+0.6}]$ and vanishes on $[-2^{a-0.6}, 2^{a-0.6}]$. Moreover, ϑ_a is constantly equal to one on $[2^{a-0.4}, 2^{a+0.4}]$ and $[-2^{a+0.4}, -2^{a-0.4}]$. Finally, ρ_a is supported in $[-2^{a+0.2}, 2^{a+0.2}]$ and vanishes on $[-2^{a-0.1}, 2^{a-0.1}]$. In particular, we have $\vartheta_a = 1$ on the support of ρ_a. Moreover, for $k \in \mathbb{Z}$ and $1 \leq i \leq 3$ we have

$$\textstyle\sum_{l=-20}^{20} \rho_{k+0.1l} = 1 \quad \text{on} \quad \operatorname{supp}(\psi_{i,k}), \tag{4.25}$$

$$\textstyle\sum_{l=-20}^{20} \rho_{k+0.1l} = 0 \quad \text{on} \quad \operatorname{supp}(\psi_{i,k'}) \quad \text{if} \quad |k'-k| \geq 10. \tag{4.26}$$

Let $n \in \mathbb{Z}$, $0 \leq s \leq 9$. Note that due to (4.26) and (4.25) we may write for $\xi, \eta \in \mathbb{R}$

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \psi_{i,k}(\xi)\varphi_{j,k}(\eta) &= \sum_{s=0}^{9}\sum_{n\in\mathbb{Z}} \psi_{i,10n+s}(\xi)\varphi_{j,10n+s}(\eta)\\ &= \sum_{s=0}^{9}\sum_{l=-20}^{20}\sum_{n\in\mathbb{Z}} \rho_{10n+s+0.1l}(\xi)\Psi_{i,s}(\xi)\varphi_{j,10n+s}(\eta),\end{aligned}$$

where $\Psi_{i,s} = \sum_{k\in\mathbb{Z}} \psi_{i,10k+s}$.

Let us now consider U_1 with bump functions satisfying (Q) and (P). Applying the above considerations in each parameter k, l, m, it suffices to prove bounds for

$$\sum_{(n_1,n_2,n_3)\in\mathbb{Z}^3} \Big((P^{(1)}_{\rho_{10n_1+s_1+0.1l_1}} P^{(2)}_{\varphi_{4,10n_3+s_3}} G_{1,s_1})\, (P^{(1)}_{\rho_{10n_2+s_2+0.1l_2}} P^{(2)}_{\varphi_{5,10n_1+s_1}} G_{2,s_2}) \, (P^{(1)}_{\rho_{10n_3+s_3+0.1l_3}} P^{(2)}_{\varphi_{6,10n_2+s_2}} G_{3,s_3}) \Big)$$

for fixed $0 \le s_1, s_2, s_3 \le 9$ and $-20 \le l_1, l_2, l_3 \le 20$, where $G_{i,s} = P^{(1)}_{\Psi_{i,s}} F_i$. The Mikhlin-Hörmander theorem in one variable gives

$$\|G_{i,s}\|_{\mathrm{L}^{p_i}(\mathbb{R}^2)} = \|P^{(1)}_{\Psi_{i,s}} F_i\|_{\mathrm{L}^{p_i}(\mathbb{R}^2)} \lesssim_s \|F_i\|_{\mathrm{L}^{p_i}(\mathbb{R}^2)}.$$

Now, for $b \in \mathbb{R}$ and $0 \le s \le 9$ define $G_{i,s,b} = \sum_{n\in\mathbb{Z}} P^{(1)}_{\rho_{10n+s+b}} G_{i,s}$. By support considerations,

$$P^{(1)}_{\vartheta_{k+b}} G_{i,s,b} = P^{(1)}_{\rho_{10n+s+b}} G_{i,s} \quad \text{if} \quad k = 10n + s \in 10\mathbb{Z} + s$$

and $P^{(1)}_{\vartheta_{k+b}} G_{i,s,b} = 0$ if $k \notin 10n + s$. The Littlewood-Paley inequality in one variable gives

$$\|G_{i,s,b}\|_{\mathrm{L}^{p_i}(\mathbb{R}^2)} \lesssim_{p_i,s,b} \|G_{i,s}\|_{\mathrm{L}^{p_i}(\mathbb{R}^2)}.$$

Thus, it suffices to bound

$$\sum_{(k_1,k_2,k_3)\in\mathbb{Z}^3} (P^{(1)}_{\vartheta_{k_1+b_1}} P^{(2)}_{\varphi_{4,k_3}} G_{1,s_1,b_1})\, (P^{(1)}_{\vartheta_{k_2+b_2}} P^{(2)}_{\varphi_{5,k_1}} G_{2,s_2,b_2})\, (P^{(1)}_{\vartheta_{k_3+b_3}} P^{(2)}_{\varphi_{6,k_2}} G_{3,s_3,b_3}) \tag{4.27}$$

for each fixed $b_i = 0.1l_i$, $-20 \le l_i \le 20$ and $0 \le s_i \le 9$, $1 \le i \le 3$.

Note that ϑ_b is supported in $[-2^3, 2^3]$ and vanishes on $[-2^{-3}, 2^{-3}]$. Moreover, $(\vartheta_b)_k$ equals $(\phi_b)_k - (\phi_b)_{k-1}$, where $\phi_b = \phi(2^{-b-1}\cdot)$ is supported in $[-2^3, 2^3]$.

Now we are ready to distinguish further cases depending on the form of $\varphi_{j,k}$.

Case (1) of U_1. Bounds for (4.27) follow from Lemma 8, applied with $\psi_{i,k}$ being a constant multiple of ϑ_{k+b_i} for $1 \le i \le 3$.

Case (2) of U_1. Without loss of generality, we may assume $j_0 = 4$. Then we dualize and write the quadrilinear form in question up to a constant as

$$\int_{\mathbb{R}^2} \sum_{(k,m)\in\mathbb{Z}^2} (Q^{(1)}_{1,k} P^{(2)}_{4,m} F_1)(x)\, \mathcal{Q}^{(1)}_{1,k} \Big(\sum_{l\in\mathbb{Z}} (Q^{(1)}_{2,l} P^{(2)}_{5,k} F_2)(x)\, (Q^{(1)}_{3,m} P^{(2)}_{6,l} F_3)(x) F_4(x) \Big) dx,$$

where $\mathcal{Q}_{1,k}$ is a constant multiple of a function adapted in $[-2^4, 2^4]$, which vanishes on $[-2^{-4}, 2^{-4}]$. By Cauchy-Schwarz in k, m, and Hölder's inequality in the integration, it suffices to bound a fiber-wise square function and

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \Big| \mathcal{Q}^{(1)}_{1,k} \sum_{l\in\mathbb{Z}} (Q^{(1)}_{2,l} P^{(2)}_{5,k} F_2)(x)\, (Q^{(1)}_{3,m} P^{(2)}_{6,l} F_3)(x) F_4(x) \Big|^2 \Big)^{1/2} \Big\|_{\mathrm{L}^{p_1'}(\ell^2_m)}.$$

Dualizing with a function $\widetilde{F}_1$ and using Kintchine's inequality we see that it suffices to prove $\mathrm{L}^{p_1} \times \mathrm{L}^{p_2} \times \mathrm{L}^{p_3} \to \mathrm{L}^{p_4'}$ bounds for the operator

$$\sum_{(k,l)\in\mathbb{Z}} a_k (\mathcal{Q}^{(1)}_{1,k} \widetilde{F}_1)(x) (Q^{(1)}_{2,l} P^{(2)}_{5,k} F_2)(x)\, (P^{(2)}_{6,l} F_3)(x).$$

Now we perform the analogous averaging argument as the one that led to (4.27). Then it suffices to consider the case when $a_k = 1$ and $\mathcal{Q}_{1,k}$ has a symbol supported in $[-2^9, 2^9]$, which vanishes on $[-2^{-9}, 2^{-9}]$, and is of the form $\theta_k - \theta_{k-1}$ for a suitable function θ. The desired bounds then follow by Lemmas 9 and 10.

Case (3) of U_1. Without loss of generality, we may assume $j_0 = 6$. Then we write the operator in question as

$$\sum_{(k,m)\in\mathbb{Z}^2} (Q^{(1)}_{1,k} P^{(2)}_{4,m} F_1)(x) \Big(\sum_{l\in\mathbb{Z}} (Q^{(1)}_{2,l} P^{(2)}_{5,k} F_2)(x)\,(Q^{(1)}_{3,m} P^{(2)}_{6,l} F_3)(x) \Big).$$

By the Cauchy-Schwarz inequality in k and m, this reduces to vector-valued estimates for the operator (1.9) and a fiber-wise square function. We obtain $\mathrm{L}^{p_1}\times\mathrm{L}^{p_2}\times\mathrm{L}^{p_3}\to\mathrm{L}^{p_4'}$ estimates with Hölder scaling and in the range $1<p_1,p_2,p_3,p_4<\infty$, $1/p_2+1/p_3>1/2$.

Case (4) of U_1. This case easily follows from three applications of Hölder's inequality, reducing to fiber-wise square functions.

It remains to treat U_2. By the analogous argument as leading to (4.27), it suffices to consider the case when for $1\le i\le 3$, the symbols $\psi_{i,k}$ are supported in $[-2^{k+3},2^{k+3}]$, vanish on $[-2^{k-3},2^{k-3}]$, and are of the form $\varphi_{i,k}-\varphi_{i,k-1}$ for a function $\varphi_{i,k}$ supported in $[-2^{k+3},2^{k+3}]$. Then we distinguish further cases depending on the form of $\varphi_{j,k}$.

Case (1) of U_2. Let $\widetilde{Q}_{5,k}$ and $\widetilde{P}_{1,k}$ be associated with $\widetilde{\psi}_{5,k}$ and $\widetilde{\varphi}_{1,k}$, respectively, where

$$\widetilde{\psi}_{5,k} = \varphi_{5,k} - \varphi_{5,k-1}$$

and $\widetilde{\varphi}_{1,k} = \varphi_{1,k-1}$ (with $\varphi_{1,k}$ defined in (4.1)). Observe that $\widetilde{\psi}_{5,k}$ is supported in $[-2^{k+3},2^{k+3}]$ and vanishes on $[-2^{k-3},2^{k-3}]$. Moreover, $\widetilde{\varphi}_{1,k}$ is supported in $[-2^{k+4},2^{k+4}]$ and observe that $\widetilde{\varphi}_{1,k}-\widetilde{\varphi}_{1,k-1}=\varphi_{1,k-1}-\varphi_{1,k-2}=\psi_{1,k-1}$ is supported in $[-2^{k+4},2^{k+4}]$ and vanishes on $[-2^{k-4},2^{k-4}]$. An application of the telescoping identity (4.5) in k yields that U_2 then equals

$$-\sum_{(k,l,m)\in\mathbb{Z}^3} (\widetilde{Q}^{(1)}_{5,k} P^{(2)}_{4,m} F_1)\,(Q^{(1)}_{2,l} \widetilde{P}^{(2)}_{1,k} F_2)\,(Q^{(1)}_{3,m} P^{(2)}_{6,l} F_3) \tag{4.28}$$

$$+\sum_{(l,m)\in\mathbb{Z}^2} (P^{(2)}_{4,m} F_1)\,(Q^{(1)}_{2,l} F_2)\,(Q^{(1)}_{3,m} P^{(2)}_{6,l} F_3) \tag{4.29}$$

Bounds for the first term follow from Lemma 8, applied with $\psi_{1,k}$ being $\widetilde{\psi}_{5,k}$ and $\varphi_{5,k}=\widetilde{\varphi}_{1,k}$. The desired bounds for the second term follow from Lemmas 9 and 10, and by Hölder's inequality used for the portion of the sum when $l\sim m$.

Case (2) of U_2. Let $j_0=6$. Performing the analogous steps as in Case (2) of U_1, we see that it suffices to show estimates for

$$\sum_{(k,m)\in\mathbb{Z}^2} a_m (P^{(1)}_{5,k} P^{(2)}_{4,m} F_1)\,(Q^{(2)}_{1,k} F_2)\,(\mathcal{Q}^{(1)}_{3,m} \widetilde{F}_3),$$

where $\mathcal{Q}_{3,m}$ is as in Case (2) of U_1. By an averaging argument (as leading to (4.27)) it suffices to consider the case when $a_k=1$ and $\mathcal{Q}_{3,m}$ has a symbol supported in $[-2^9,2^9]$, which vanishes on $[-2^{-9},2^{-9}]$, and is of the form $\theta_k-\theta_{k-1}$ for a suitable function θ. The clam then follows by the telescoping identity (4.5) in m, Lemmas 9 and 10, and bounds for (1.9). If $j_0=5$, we can proceed as in Case (2) of U_1. Then we need to show estimates for

$$\sum_{(l,m)\in\mathbb{Z}^2} a_l (P^{(2)}_{4,m} F_1)\,(\mathcal{Q}^{(1)}_{2,l} \widetilde{F}_2)\,(Q^{(1)}_{3,m} P^{(2)}_{6,l} F_3).$$

If $j_0=4$, we first use the telescoping identity (4.5) in k, giving terms of the form (4.28) and (4.29). For (4.29), we apply the Cauchy-Schwarz inequality in m, which leads to vector-valued estimates for the operator (1.9). For (4.28), we can proceed as in Case (2) of U_1, up to obvious modifications.

Cases (3) and (4) of U_2. These two cases are analogous to Cases (3) and (4) of U_1. □

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