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## Comments

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# Instability and quantization in quantum hydrodynamics

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## Abstract

In this short paper, we show how a quantum nonlocal effect of far-apart wavepackets in the Schrödinger picture of wavefunctions is replaced by a local instability problem when considering the hydrodynamical formulation of quantum mechanics, known as the Madelung picture. As a second result, we show how the Madelung equations describe quantized energies without any external quantization conditions.

*Keywords: current density; energy levels; Madelung equations; nonlocality; phase difference; quantum hydrodynamics; quantization*

## 1 Introduction

Nonlocality is one of the cornerstones of quantum mechanics and plays a key role in the theory of quantum interference (see, e.g., [1]). Over the years, several theories have been proposed to provide an alternative interpretation of quantum mechanics, which provides a different story about the nature of the particles. We are going to focus on one of these alternatives, namely, the theory of quantum hydrodynamics (QH) [2-7]. In this report, we show how one can reformulate the problem of nonlocal behavior of the phase difference of a particle described by far-apart wavepackets into a local problem with instability as a direct consequence of such conversion from nonlocality to locality. The Madelung equations

reformulate the Schrödinger equation into classical equations of motion, with the addition of the quantum potential. Following the Schrödinger equation with some Hamiltonian  $H$ ,  $i\partial_t\Psi = H\Psi$ , by taking the polar representation of the wavefunction  $\Psi = Re^{iS}$ , we obtain a system of coupled equations with observables  $(\rho, J)$  where  $\rho = R^2$  is the density function and  $J := \frac{\partial}{\partial x}S \cdot \rho$  being the current density, subject to some initial datum  $(\rho(x, 0), J(x, 0)) \in \mathbb{R}_+ \times \mathbb{R}$ . Since both  $\rho$  and  $J$  are gauge-invariant, there is no need to consider potentials, which play an essential role in the linear differential equations. Following the Madelung equations, the density function and the current density are considered physical quantities of the hydrodynamical description of the quantum particle. The Madelung formulation of quantum mechanics allows us to get a new perspective in understanding quantum mechanics and also its relation with classical mechanics since the Madelung equations are based on a continuity equation of a fluid and a Hamilton-Jacobi equation of such a fluid, with the addition of the quantum potential (see, [8-12]). In the following, we show new basic properties and behavior of the Madelung equations that touch on the foundations of the hydrodynamical formalism of quantum mechanics.

## 2 Instability in the Madelung formalism

Suppose that we have a superposition of two far-apart spatially separated narrow Gaussian wavepackets with width  $\Delta x \ll L$  positioned at  $-L$  and  $L$ ,  $L \gg 0$ , where the wavepacket located in  $x = L$  move towards the other wavepacket with momentum  $p_0$ , such that at the region  $[-\ell, \ell]$ ,  $0 < \ell < L$ , the density function is significantly small  $\rho(x) \approx \varepsilon^N$ ,  $N > 1$ , such that  $\varepsilon \ll \ell/\Delta x$ . In the Schrödinger picture, the wavefunction is given by

$$\Psi(x, t = 0) = \frac{1}{\sqrt{2}}\phi\left(\frac{x+L}{2}\right) + \frac{1}{\sqrt{2}}e^{ip_0x/\hbar}\phi\left(\frac{x-L}{2}\right), \quad (1)$$

where  $\phi(u) = e^{-u^2/2\sigma^2}/\sqrt{2\pi\sigma\mathcal{N}}$  is a Gaussian function with dispersion parameter  $\sigma > 0$ , and  $\mathcal{N} > 0$  is the normalizing constant of  $\Psi$ .

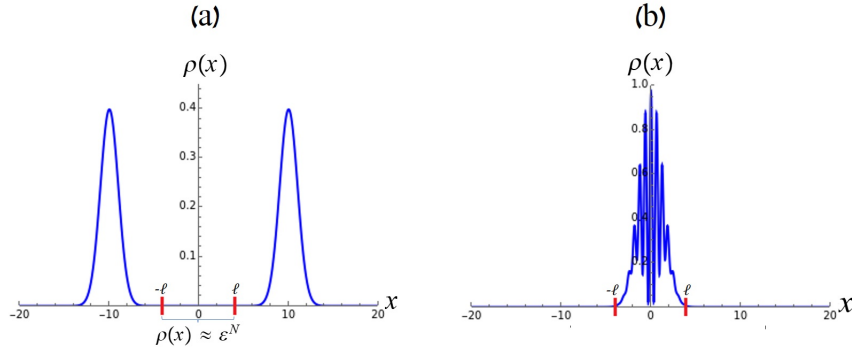


Figure 1. The pair of Gaussian wavepackets before (a) and after (b) their interference.

As illustrated in Figure 1, the initial state shows that the far-apart (Gaussian) wavepackets do not reveal their relative phase, and so any measurement of one of the wavepackets will not reveal the relative phase. Only after the wavepackets approach close enough to each other we have an interference pattern. This basic quantum effect has a profoundly different essence when dealing with the hydrodynamical description of quantum mechanics, as will soon be discussed.

Let us now observe the effect in quantum hydrodynamics when considering  $(\rho, J)$  as the description of the quantum system in the form of a fluid.

At the region  $[-\ell, \ell]$ , the phase difference is given by the integral of  $J/\rho$  at that region,

$$S^\ell(t) = \int_{-\ell}^{\ell} \frac{J(x, t)}{\rho(x, t)} dx. \quad (2)$$

Suppose now that we have a small change  $\varepsilon$  in the initial condition of  $J(x, 0)$ . Then, the phase difference at the interval  $[-\ell, \ell]$  is given by

$$S_{+\varepsilon}^\ell(t=0) = \int_{-\ell}^{\ell} \frac{J(x, 0) + \varepsilon}{\rho(x, 0)} dx = S^\ell(t=0) + \varepsilon \int_{-\ell}^{\ell} \frac{1}{\rho(x, 0)} dx \quad (3)$$

so for  $\rho(x) \approx \varepsilon^N$  at the region  $[-\ell, \ell]$ , we have

$$S_{+\varepsilon}^\ell(t=0) \approx S^\ell(t=0) + \frac{2\ell}{\varepsilon^{N-1}}, \quad (4)$$

with  $2\ell/\varepsilon^{N-1} \gg 1$ . As an illustrative example of the effect, in the case of (1), we have

$$\begin{aligned} S_{+\varepsilon}^\ell(t=0) &\approx \arctan\left(\frac{\sin(p_0 x/\hbar)}{\phi\left(\frac{x+L}{2}\right)/\phi\left(\frac{x-L}{2}\right) + \cos(p_0 x/\hbar)}\right) (|_{x=\ell} - |_{x=-\ell}) + \frac{2\ell}{\varepsilon^{N-1}} \\ &\approx \frac{2\ell}{\varepsilon^{N-1}}. \end{aligned} \quad (5)$$

The instability effect does not depend on the shape of the wavefunction as long as the given conditions are fulfilled, e.g., having separated wavepackets with  $\rho(x) \approx \varepsilon^N$  at the region  $[-\ell, \ell]$  for some arbitrary small  $\varepsilon$  and some integer  $N > 1$ . Moreover, the growth rate of the initial phase difference is also the same for every shape of the wavefunction (as long as the wavefunction does not depend on  $\varepsilon$  explicitly). In general, we have

$$S_{+\varepsilon}^\ell(t=0) \approx \frac{1}{\varepsilon^N} \int_{-\ell}^{\ell} J(x, 0) dx + \frac{2\ell}{\varepsilon^{N-1}}. \quad (6)$$

The instability effect can be seen as an *amplification effect* where small changes in the environment of the wavepackets will be revealed as a large change in their observed phase difference. We identified that when the local formulation

of quantum mechanics is considered, there is a natural occurrence of instability in the relation between the current density and the density function, causing a dramatic change when the initial current density possesses a significantly small change  $\varepsilon$ . This instability seems to be a compensation that comes with the local formulation, which is switched off when dealing with nonlocal pictures such as the Schrödinger or the Heisenberg formulations of quantum mechanics.

For future research, we suggest looking at more complex quantum systems and exploring them following quantum hydrodynamics with finding traces of instabilities that emerged from such a formulation.

### 3 Madelung equations without quantization conditions

Soon after the Schrödinger equation was introduced to the world by Erwin Schrödinger, the hydrodynamic formulation of quantum mechanics was proposed by Erwin Madelung. A fundamental question is whether Madelung's equations genuinely describe the same physics as in the Schrödinger equation or not.

We show how the Madelung equations indeed describe quantized physics without any quantization conditions. In particular, we show that when quantized energy levels are solutions of the Schrödinger equations, they are also solutions of the Madelung one – they are a complete set of solutions. We start by observing that the Madelung equations consider two physical quantities,  $\rho$  and  $J$ , instead of one, the wavefunction. We recall that  $\rho$  describes the density function, and thus, it is assumed to be integrable, normalized, and satisfy

$$\rho(x) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty. \quad (7)$$

We now show how the property (7) imposes quantized energy levels of the Schrödinger equation. First, we consider some potential  $V(x)$ , with the spatial part of the Schrödinger equation  $-\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$ . Following the Madelung representation and taking the real part, we have

$$-R''(x) + S'^2 R(x) - (E - V(x))R(x) = 0 \quad (8)$$

where  $S' = v$  is the hydrodynamical flow velocity of the particle. We assume that the solution  $R$  takes the form

$$R(x) = \sum_{j=0}^{\infty} \alpha_j x^j \cdot h(x), \quad (9)$$

for some constant  $\alpha_j$  and a twice-differentiable function,  $h$ , that represents the behavior of  $\rho$  about its tails. Then, we have

$$\frac{d}{dx}R(x) = \sum_{j=0}^{\infty} j\alpha_j x^{j-1} h(x) + \sum_{j=0}^{\infty} \alpha_j x^j h'(x), \quad (10)$$

and

$$\frac{d^2}{dx^2}R(x) = \sum_{j=0}^{\infty} (j+1)(j+2)\alpha_{j+2}x^j h(x) + \sum_{j=0}^{\infty} \alpha_j x^j h''(x) + 2 \sum_{j=0}^{\infty} j\alpha_j x^{j-1} h'(x). \quad (11)$$

Substituting (9), (10) and (11) in (8), we get

$$\sum_{j=0}^{\infty} [(j+1)(j+2)\alpha_{j+2}h(x) + \alpha_j h''(x) + 2j\alpha_j h'(x)/x + (E - V(x) - S'^2)\alpha_j h(x)] x^j = 0.$$

We decompose the constant part and the part that depends on  $x$ , for each of the components in the coefficients of  $x^j$ ,

$$\begin{aligned} h(x) &= C^{(0)} + Q^{(0)}(x), \quad h'(x)/x = C^{(1)} + Q^{(1)}(x), \\ h''(x) &= C^{(2)} + Q^{(2)}(x), \quad V(x) + S'^2 = C^{(4)} + Q^{(4)}(x). \end{aligned}$$

In order to satisfy equation (8), the terms that depend on  $x$  have to be equal to some real constant  $b_j$ . So

$$\sum_{j=0}^{\infty} [(j+1)(j+2)\alpha_{j+2}C^{(0)} + \alpha_j C^{(2)} + 2j\alpha_j C^{(1)} - \alpha_j (E - C^{(4)})C^{(0)} + b_j] x^j = 0$$

Now, for achieving the above equality, the coefficients of  $x^j$  should be zero. Then, after some algebraic calculation, we have

$$\frac{\alpha_{j+2}}{\alpha_j} = -\frac{C^{(2)} + 2jC^{(1)} - (E - C^{(4)})C^{(0)} + b_j/\alpha_j}{(j+1)(j+2)C^{(0)}}. \quad (12)$$

Since  $\alpha_{j+2}/\alpha_j$  implies, in general, on a divergence sum (9), for having convergence, we impose the condition that the nominator of the RHS of (12) is zero. This produces our energy levels

$$E = C^{(4)} + \frac{1}{C^{(0)}} \left( C^{(2)} + \frac{b_j}{\alpha_j} + 2C^{(1)} \cdot j \right), \quad j = 0, 1, 2, \dots \quad (13)$$

The proposed approach is not unique for the 1D case, and in fact, it holds for a particle in arbitrary  $N$  spatial dimensions  $\mathbf{x} = (x_1, x_2, \dots, x_N)^T$ .

### 3.1 Quantization of the angular momentum

The Schrödinger equation shows that angular momentum takes discrete values, i.e., it is quantized. But do the Madelung equations describe quantized angular momentum? In the following, we show that by paying attention to the properties of the probability density function  $\rho$ , the quantization of the

angular momentum naturally appears in the proposed local geometrical formalism of quantum mechanics. Let us consider the angular momentum operator  $\widehat{L}^2$  in the polar coordinates  $(r, \theta, \phi) \in \mathbb{R}_{\geq} \times [0, \pi] \times [0, 2\pi]$ . We consider the separation of variables  $\sqrt{\rho(r, \theta, \phi)} = \overline{R}(r) \cdot \mathcal{R}(\theta, \phi)$ , and consider the angular momentum operator in the angle coordinates  $(\theta, \phi)$ . We start with  $\widehat{L}^2 = -\left[\frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \phi^2}\right]$ , and obtain the real and imaginary parts

$$\sin(\theta) (\cos \theta \mathcal{R}_\theta + \sin(\theta) (\mathcal{R}_{\theta\theta} - S_\theta^2 \mathcal{R})) + (\mathcal{R}_{\phi\phi} - S_\phi^2 \mathcal{R}) + \lambda \sin^2(\theta) \mathcal{R} = 0. \quad (14)$$

and

$$\sin(\theta) (\cos \theta S_\theta + \sin \theta (S_{\theta\theta} \mathcal{R} + 2S_\theta \mathcal{R}_\theta)) + S_{\phi\phi} \mathcal{R} + 2S_\phi \mathcal{R}_\phi = 0, \quad (15)$$

where  $f_x := df/dx$ ,  $f_{xy} := d^2 f/dx dy$ . Let us focus now on the real part (14). By taking  $u = \cos(\theta)$ ,  $-1 \leq u \leq 1$ , eq. (14) takes the form

$$\sin(\theta) (-2 \sin(\theta) \cos \theta \mathcal{R}_u + \sin(\theta) (\sin^2(\theta) \mathcal{R}_{uu} - S_\theta^2 \mathcal{R})) + (\mathcal{R}_{\phi\phi} - S_\phi^2 \mathcal{R}) + \lambda \sin^2(\theta) \mathcal{R} = 0. \quad (16)$$

Dividing by  $\sin^2(\theta)$ , we have

$$-2u \mathcal{R}_u + (1 - u^2) \mathcal{R}_{uu} - S_\theta^2 \mathcal{R} + \frac{1}{1 - u^2} (\mathcal{R}_{\phi\phi} - S_\phi^2 \mathcal{R}) + \lambda \mathcal{R} = 0.$$

We now consider a power sum for  $\mathcal{R}$ ,  $\mathcal{R} = \sum_{j,k=0}^{\infty} a_{jk} u^j \phi^k$ . Then, we have

$$\begin{aligned} & \sum_{j,k=0}^{\infty} [(j+1)(j+2) a_{(j+2)k} - j(j+1) a_{jk} - S_\theta^2 \mathcal{R} \\ & + \frac{1}{1-u^2} ((k+1)(k+2) a_{j(k+2)} - S_\phi^2 a_{jk}) + \lambda a_{jk}] u^j \phi^k \\ & = 0. \end{aligned}$$

For this equation to hold, we assume that all the parts in the coefficient of  $u^j \phi^k$  that are not constants cancel each other. We thus have

$$\sum_{j,k=0}^{\infty} [(j+1)(j+2) a_{(j+2)k} - j(j+1) a_{jk} + \lambda a_{jk}] u^j \phi^k = 0.$$

Then, for this equation to hold, the coefficients should sum up to zero,

$$(j+1)(j+2) a_{(j+2)k} - j(j+1) a_{jk} + \lambda a_{jk} = 0,$$

and after some calculations, we obtain

$$\frac{a_{(j+2)k}}{a_{jk}} = \frac{-j(j+1) + \lambda}{(j+1)(j+2)}. \quad (17)$$

Now, in the limit  $j \rightarrow \infty$  we get, in general, a divergence of the density function. Thus, we should only consider a finite power sum of  $\mathcal{R}$  with respect to  $u$ , and



so we finally obtain the quantized angular momentum by taking the numerator at the RHS to be zero, and we get  $\lambda = j(j+1)$ . Following the angular momentum operator  $\widehat{L}^2 = -\left(r^2 \frac{d^2}{dr^2} - \frac{d}{dr} r^2 \frac{d}{dr}\right)$ , the eigenvalue equation gives us the real and imaginary parts of the Madelung equations as a system of decoupled equations

$$2rR(r)' = \lambda R(r) \implies R(r) = c_1 r^{\lambda/2}, \quad (18)$$

$$rS(r)' = 0. \quad (19)$$

Substituting  $\lambda = j(j+1)$ , the first equation gives the radial part of the quantized angular momentum,  $R(r) = c_1 r^{j(j+1)/2}$ . We note that in the cylindrical coordinate system  $(z, \rho, \phi)$  it seems, at first glance, that we have to use an external quantum condition for getting quantized angular momentum. However, as we observe that  $L_z$  is still a component of the angular momentum, it must have integer values following a similar approach given above for the polar coordinate system. Another point we would like to emphasize is that in two dimensions, one cannot obtain quantization using the Madelung equations only and must have an external quantization condition. For example, suppose we have a central potential in the form of a harmonic oscillator  $V(r) = \frac{1}{2}\kappa r^2$ , with the underlying polar coordinates  $(r, \phi)$ . Then, it is straightforward that the solutions of the Schrödinger equation  $R(r) e^{im\phi}$  cannot be quantized in the representation of  $\rho$  and  $J$ , i.e., non-integer  $m$  is still a valid solution in the picture of the Madelung equations.

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