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ON LATTICES OF z-IDEALS OF FUNCTION RINGS

Themba Dube — Oghenetega Ighedo

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ABSTRACT. An ideal I of a ring A is a z-ideal if whenever $a, b \in A$ belong to the same maximal ideals of A and $a \in I$, then $b \in I$ as well. On the other hand, an ideal J of A is a d-ideal if $\mathrm{Ann}^2(a) \subseteq J$ for every $a \in J$. It is known that the lattices $\mathsf{Z}(L)$ and $\mathsf{D}(L)$ of the ring $\mathcal{R}L$ of continuous real-valued functions on a frame L, consisting of z-ideals and d-ideals of $\mathcal{R}L$, respectively, are coherent frames. In this paper we characterize, in terms of the frame-theoretic properties of L (and, in some cases, the algebraic properties of the ring $\mathcal{R}L$), those L for which $\mathsf{Z}(L)$ and $\mathsf{D}(L)$ satisfy the various regularity conditions on algebraic frames introduced by Martínez and Zenk [20]. Every frame homomorphism $h\colon L \to M$ induces a coherent map $\mathsf{Z}(h)\colon \mathsf{Z}(L) \to \mathsf{Z}(M)$. Conditions are given of when this map is closed, or weakly closed in the sense Martínez [19]. The case of openness of this map was discussed in [11]. We also prove that, as in the case of the ring $C(X)$, the sum of two z-ideals of $\mathcal{R}L$ is a z-ideal.

1. Introduction

In [20] and [22], Martínez and Zenk show that, for any Tychonoff space X, the lattices $\mathscr{C}_z(X)$ and $\mathscr{C}_d(X)$ of z-ideals and d-ideals, respectively, of the ring $C(X)$ are coherent frames. In fact, they are more than that; they are actually normal coherent Yosida frames. This they do by viewing $C(X)$ as a lattice-ordered group, and realizing the aforementioned lattices as some quotients of the frame of convex ℓ-ideals of $C(X)$.

In proving similar results (in [10]) for the lattices $\mathsf{Z}(L)$ and $\mathsf{D}(L)$ of z-ideals and d-ideals of the ring $\mathcal{R}L$ of continuous real-valued functions on a frame L, we took a slightly different route. We worked with the frame $\mathrm{Rad}(\mathcal{R}L)$ of radical ideals of $\mathcal{R}L$, and realized $\mathsf{Z}(L)$ as a quotient of $\mathrm{Rad}(\mathcal{R}L)$, and then $\mathsf{D}(L)$ as a quotient of $\mathsf{Z}(L)$.

In their study of regularity of algebraic frames, Martínez and Zenk [20] introduce four properties that they designate as $\mathrm{Reg}(i)$, for $i = 1, 2, 3, 4$, which an algebraic frame can have. They show that Reg(2) and Reg(3) are equivalent. One of our aims in this paper is to characterize, for each of these properties, those L for which $\mathsf{Z}(L)$ and $\mathsf{D}(L)$ satisfy a given one. In fact, regarding Reg(1), we showed in [10] that $\mathsf{Z}(L)$ satisfies Reg(1) precisely when L is a P-frame. Regarding Reg(2), we showed in [9] that $\mathsf{Z}(L)$ satisfies Reg(2) precisely when L is basically disconnected, and $\mathsf{D}(L)$ satisfies Reg(2) precisely when L is cozero complemented.

Thus, we need to investigate Reg(4) for $\mathsf{Z}(L)$, and Reg(1) and Reg(4) for $\mathsf{D}(L)$. This we do in Section 3. We show that $\mathsf{Z}(L)$ satisfies Reg(4) if and only if L is an F-frame (Theorem 3.1). Regarding $\mathsf{D}(L)$, it turns out that this algebraic frame satisfies Reg(1) if and only if it satisfies

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Reg(2); that is, precisely when L is cozero complemented (Proposition 3.2). What about Reg(4) for $\mathsf{D}(L)$?

Call an f-ring A feebly complemented if whenever $ab = 0$ in A, there exist positive $c, d \in A$ such that $ac = 0 = bd$, and $c + d$ is a non-divisor of zero. We say L is feebly cozero complemented in case the f-ring $\mathcal{R}L$ is feebly complemented. There is a frame-theoretic equivalent condition (which we shall, in fact, adopt as the definition) from which it will be apparent why the moniker we give these frames is appropriate. We show in Theorem 3.3 that $\mathsf{D}(L)$ satisfies Reg(4) if and only if L is feebly cozero complemented.

In his study of the saturation nucleus as a coreflection, Martínez [19] introduces weakly closed maps, which generalize (and in some cases coincide with) closed maps of frames. In Section 4 we examine when the map $\mathsf{Z}(h)\colon \mathsf{Z}(L) \to \mathsf{Z}(M)$ is closed, and when it is weakly closed. If L is normal, there is a characterization (Theorem 4.1) of when $\mathsf{Z}(h)$ is closed in terms of a "closed-like" property of the map h.

Appended at the end of the paper is a proof that the sum of two z-ideals of $\mathcal{R}L$ is a z-ideal. This, of course, extends the well-known $C(X)$ result (see [14]), and answers a question that was asked by the second-named author in her Ph.D. thesis [15].

2. Preliminaries

2.1. Frames and their homomorphisms

Our general references for frames are [17] and [24], and our notation is standard. As usual, we denote by βL the Stone-Čech compactification of a completely regular frame L, and we write $j_L\colon \beta L \to L$ for the coreflection map from compact completely regular frames to L. The right adjoint of j_L is denoted by r_L. Recall that, for any $a \in L$, $r_L(a) = \{x \in L \mid x \prec\!\!\prec a\}$. We write $\operatorname{Coz} L$ for the set of cozero elements of L. For any $c, d \in \operatorname{Coz} L$, $r_L(c \vee d) = r_L(c) \vee r_L(d)$. If L is normal, then $r_L(a \vee b) = r_L(a) \vee r_L(b)$ for all $a, b \in L$. If $a \vee b = 1$ in a normal completely regular frame L, then there exist $c, d \in \operatorname{Coz} L$ such that $c \le a$, $d \le b$, and $c \vee d = 1$ (see, for instance, [2: Corollary 8.3.2]).

We denote by λL the Lindelöf coreflection of L, and we write $\lambda_L\colon \lambda L \to L$ for the coreflection map (see [18] for details). A frame homomorphism is *coz-surjective* if its restriction to cozero parts is surjective, and it is *coz-faithful* if its restriction to cozero parts is one-one. By υL we mean the realcompact coreflection of L (see [6] for details). For our purposes it suffices to know that there is a coz-surjective and coz-faithful map $\ell_L\colon \lambda L \to \upsilon L$ realizing λL as the Lindelöf coreflection of υL. We shall write $\lambda h\colon \lambda L \to \lambda M$ for the image of $h\colon L \to M$ under the functor λ, and $\upsilon h\colon \upsilon L \to \upsilon M$ for its image under the functor υ. Recall that $\operatorname{Coz}(\lambda L) = \{[\![c]\!] \mid c \in \operatorname{Coz} L\}$, where $[\![c]\!]$ denotes the principal ideal in $\operatorname{Coz} L$ generated by c.

By a *point* of a frame L we mean a meet-irreducible element. We denote by $\operatorname{Pt}(L)$ the set of points of L. We shall view the spectrum, ΣL, of any frame L as the space of points of L, so that the open sets of ΣL are the sets $\Sigma_a = \{p \in \operatorname{Pt}(L) \mid a \nleq p\}$, for $a \in L$.

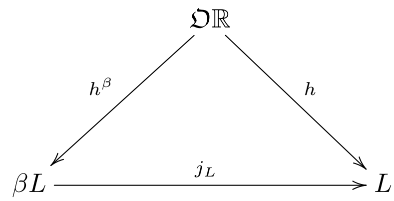
2.2. The ring $\mathcal{R}L$ and some of its ideals

Our approach to the ring $\mathcal{R}L$ follows that of [2]. Thus, its elements are frame homomorphisms $\mathfrak{O}\mathbb{R} \to L$. We could have taken the approach in [3], in which case members of $\mathcal{R}L$ would be frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, where $\mathfrak{L}(\mathbb{R})$ denotes the frame of reals, but the former approach is more convenient for our purposes, as will be apparent in Section 5.

In the abstract we recalled the algebraic definitions of z-ideal and d-ideal. In $\mathcal{R}L$ there are purely frame-theoretic characterizations. Namely, an ideal Q of $\mathcal{R}L$ is a z-ideal if and only if, for any $\alpha, \beta \in \mathcal{R}L$, $\operatorname{coz}\alpha \leq \operatorname{coz}\beta$ and $\beta \in Q$ imply $\alpha \in Q$. The inequality may be replaced with an equality. On the other hand, Q is a d-ideal if and only if, for any $\alpha, \beta \in \mathcal{R}L$, $\operatorname{coz}\alpha \leq (\operatorname{coz}\beta)^{**}$ and $\beta \in Q$ imply $\alpha \in Q$. For any $a \in L$, the ideal $\boldsymbol{M}_a$ is defined by

$$\boldsymbol{M}_a = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \leq a\}.$$

Clearly, $\boldsymbol{M}_a$ is a z-ideal. Recall that L is a C^**-quotient* of βL, which is to say for every bounded $h \in \mathcal{R}L$, there is a (necessarily unique) $h^\beta \in \mathcal{R}(\beta L)$ such that the triangle below commutes.



2.3. Algebraic frames

We will denote algebraic frames by $A, B, \ldots$ instead of $L, M, \ldots$, which we will reserve for completely regular frames. We write $\mathfrak{k}(A)$ for the set of compact elements of A. If $\mathfrak{k}(A)$ generates L, in the sense that every element of A is the join of compact elements below it, then A is said to be *algebraic*. An algebraic frame A is said to have the *finite intersection property* (FIP) if $a \wedge b \in \mathfrak{k}(A)$ for all $a, b \in \mathfrak{k}(A)$. A compact algebraic frame with FIP is called *coherent*, as is a frame homomorphism $\phi\colon A \to B$ between coherent frames that takes compact elements to compact elements.

When we are dealing with coherent frames we shall denote the pseudocomplement of an element a by $a^\perp$, and refer to $a^\perp$ as the *polar* of a. The d-nucleus (see [20]) on an algebraic frame A with FIP is defined by

$$d(a) = \bigvee\{c^{\perp\perp} \mid c \in \mathfrak{k}(A), c \leq a\}.$$

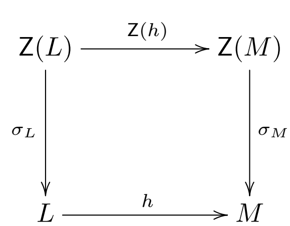
We write dA for the frame $\operatorname{Fix}(d)$, and denote by $d_A\colon A \to dA$ the dense onto frame homomorphism it induces. It is shown in [20] that $d_A(c) = c^{\perp\perp}$ for any $c \in \mathfrak{k}(A)$, and that $\mathfrak{k}(dA) = \{c^{\perp\perp} \mid c \in \mathfrak{k}(A)\}$. Also, $x^\perp \in dA$ for any $x \in A$, an upshot of which is that the polar of any $a \in dA$, considered in dA, is precisely the polar $a^\perp$ of a as an element of A. Elements of dA are called *d-elements* of A.

2.4. Frames of d- and z-ideals

Throughout this subsection L and M stand for completely regular frames. We summarize some results from [9], [10] and [11] that we shall need. The lattice of z-ideals of $\mathcal{R}L$ will be denoted by $\mathsf{Z}(L)$, and the lattice of d-ideals of $\mathcal{R}L$ by $\mathsf{D}(L)$. Both these lattices are normal coherent frames (and more). Their compact elements are given by

$$\mathfrak{k}(\mathsf{Z}(L)) = \{\boldsymbol{M}_c \mid c \in \operatorname{Coz} L\} \qquad \text{and} \qquad \mathfrak{k}(\mathsf{D}(L)) = \{\boldsymbol{M}_{c^{**}} \mid c \in \operatorname{Coz} L\}.$$

It is shown in Proposition 4.1 of [9] that $\mathsf{D}(L) = d(\mathsf{Z}(L))$, where d denotes the d-nucleus on $\mathsf{Z}(L)$. We shall frequently denote the bottom of $\mathsf{Z}(L)$ and $\mathsf{D}(L)$ by $\perp$, and the top by $\top$. A frame homomorphism $h\colon L \to M$ induces a coherent map $\mathsf{Z}(h)\colon \mathsf{Z}(L) \to \mathsf{Z}(M)$ such that the square



commutes. Explicitly, for any $Q \in \mathsf{Z}(L)$, $\sigma_L(Q) = \bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\}$, and

$$\mathsf{Z}(h)(Q) = \bigvee_{\mathsf{Z}(M)} \{\boldsymbol{M}_{h(\operatorname{coz}\alpha)} \mid \alpha \in Q\} = \bigcup\{\boldsymbol{M}_{h(\operatorname{coz}\alpha)} \mid \alpha \in Q\}.$$

In particular, for any $c \in \operatorname{Coz} L$, $\mathsf{Z}(h)(\boldsymbol{M}_c) = \boldsymbol{M}_{h(c)}$. For any $b \in M$, $\mathsf{Z}(h)_*(b) = \boldsymbol{M}_{h_*(b)}$. The map σ_L is a dense onto frame homomorphism. In fact, for any $a \in L$, $\sigma_L(M_a) = a$. Finally, it is shown in [10: Lemma 3.6] that if $c, d \in \operatorname{Coz} L$, then $\boldsymbol{M}_c \vee \boldsymbol{M}_d = \boldsymbol{M}_{c\vee d}$.

3. Concerning the Reg-properties

Here are the regularity properties of Martínez and Zenk [20] that an algebraic frame A can have, listed from strongest to weakest. When we say an element $a \in A$ is *regular* we mean that $a = \bigvee\{x \in A \mid x \prec a\}$.

Reg(1) A is regular.

Reg(2) Each d-element of A is regular.

Reg(3) Each polar of A is regular.

Reg(4) Each $c^\perp$, with c compact, is regular.

As already mentioned, Reg(2) and Reg(3) are equivalent. Also, Reg(4) is equivalent to the property that if $c, d \in \mathfrak{k}(A)$ are such that $c \wedge d = 0$, then $c^\perp \vee d^\perp = 1$.

Recall that a completely regular frame L is called a *P-frame* if $c \vee c^* = 1$ for every $c \in \operatorname{Coz} L$. If $c^* \vee c^{**} = 1$ for every $c \in \operatorname{Coz} L$, then L is said to be *basically disconnected.* As mentioned in the Introduction, we showed in [10: Proposition 3.10] that

$\mathsf{Z}(L)$ *satisfies* Reg(1) *if and only if L is a P-frame*,

and in [9: Proposition 5.4] we showed that

$\mathsf{Z}(L)$ *satisfies* Reg(2) *if and only if L is basically disconnected.*

We shall now characterize when $\mathsf{Z}(L)$ satisfies Reg(4).

Recall that L is an *F-frame* if the open quotient of each cozero element of L is a C^*-quotient. A characterization we shall use (Proposition 8.4.10 in [2]) is that L is an F-frame if and only if for all $a, b \in \operatorname{Coz} L$ such that $a \wedge b = 0$, there exist $c, d \in \operatorname{Coz} L$ such that $c \vee d = 1$ and $a \wedge c = 0 = b \wedge d$.

Observe that, for any $a, b \in \operatorname{Coz} L$, $a \wedge b = 0$ if and only if $\boldsymbol{M}_a \wedge \boldsymbol{M}_b = \bot$ in $\mathsf{Z}(L)$. Also, $(\boldsymbol{M}_a)^\perp = \boldsymbol{M}_{a^*}$.

Theorem 3.1. *For any completely regular frame L, $\mathsf{Z}(L)$ satisfies* Reg(4) *if and only if L is an F-frame.*

P r o o f. Assume $\mathsf{Z}(L)$ satisfies Reg(4). Let $a \wedge b = 0$ in $\operatorname{Coz} L$. Then $\boldsymbol{M}_a \wedge \boldsymbol{M}_b = \bot$ in $\mathsf{Z}(L)$, so that, by Reg(4), $(\boldsymbol{M}_a)^\perp \vee (\boldsymbol{M}_b)^\perp = \top$, that is, $\boldsymbol{M}_{a^*} \vee \boldsymbol{M}_{b^*} = \top$. Using the fact that $\mathsf{Z}(L)$ is an algebraic frame whose compact elements are the ideals $\boldsymbol{M}_c$, for $c \in \operatorname{Coz} L$, the equality $\boldsymbol{M}_{a^*} \vee \boldsymbol{M}_{b^*} = \top$ implies

$$\bigvee\{\boldsymbol{M}_s \mid s \in \operatorname{Coz} L, s \le a^*\} \vee \bigvee\{\boldsymbol{M}_t \mid t \in \operatorname{Coz} L, t \le b^*\} = \top,$$

which, by compactness of the frame $\mathsf{Z}(L)$, implies that there exist $c, d \in \operatorname{Coz} L$ such that $c \le a^*$, $d \le b^*$, and $\boldsymbol{M}_c \vee \boldsymbol{M}_d = \top$. Applying the homomorphism $\sigma_L \colon \mathsf{Z}(L) \to L$ to this last equality gives $c \vee d = 1$. But now the comparisons $c \le a^*$ and $d \le b^*$ imply $a \wedge c = 0 = b \wedge d$. Therefore L is an F-frame.

Conversely, assume L is an F-frame. Consider any $a,b \in \operatorname{Coz} L$ with $\boldsymbol{M}_a \wedge \boldsymbol{M}_b = \bot$. Then $a \wedge b = 0$, and so, in light of L being an F-frame, there exist $c,d \in \operatorname{Coz} L$ such that $c \vee d = 1$ and $a \wedge c = 0 = b \wedge d$. The latter implies $c \le a^*$ and $d \le b^*$, so that $\boldsymbol{M}_c \le (\boldsymbol{M}_a)^{\perp}$ and $\boldsymbol{M}_d \le (\boldsymbol{M}_b)^{\perp}$. Since c and d are cozero elements, the equality $c \vee d = 1$ implies $\boldsymbol{M}_c \vee \boldsymbol{M}_d = \top$, whence $(\boldsymbol{M}_a)^{\perp} \vee (\boldsymbol{M}_b)^{\perp} = \top$. Therefore $\mathsf{Z}(L)$ satisfies Reg(4). □

Corollary 3.1.1. *$\mathscr{C}_z(X)$ satisfies* Reg(4) *if and only if X is an F-space.*

We now turn to $\mathsf{D}(L)$. In Proposition 5.5 of [9] it is shown that $\mathsf{D}(L)$ satisfies Reg(2) if and only if L is cozero complemented. Proposition 3.1 in [13] states that, for a completely regular frame L, every prime d-ideal of $\mathcal{R}L$ is minimal prime if and only if L is cozero complemented. Applying Proposition 5.2 in [20] we see that the frame $\mathsf{D}(L)$ is regular precisely when every prime d-ideal of $\mathcal{R}L$ is minimal prime. Putting all these together we can then state the following result.

Proposition 3.2. *The following conditions are equivalent for a completely regular frame L.*

(1) $\mathsf{D}(L)$ *satisfies* Reg(1).
(2) $\mathsf{D}(L)$ *satisfies* Reg(2).
(3) $\mathsf{D}(L)$ *satisfies* Reg(3).
(4) *L is cozero complemented.*

We shall now give a necessary and sufficient condition for $\mathsf{D}(L)$ to satisfy Reg(4). We will call frames with the property in question "feebly cozero complemented". Before we define these frames let us explain our choice of terminology. Recall that a ring R (throughout, by "ring" we mean a commutative ring with identity) is called *complemented* if for every $a \in R$ there exists $b \in R$ such that $ab = 0$ and $a + b$ is a non-divisor of zero. A Tychonoff space X is called *cozero complemented* if for every cozero-set U of X there is a cozero-set V such that $U \cap V = \emptyset$ and $U \cup V$ is dense. Then X is cozero complemented precisely when $C(X)$ is complemented.

In [16] the authors call a ring R *weakly complemented* if whenever $ab = 0$ in R there exist finitely generated ideals I and J of R such that $a \in I$, $b \in J$, $IJ = 0$, and $I + J$ contains a non-divisor of zero. In [23] McGovern calls a Tychonoff space X *weakly cozero complemented* if for each pair of disjoint cozero sets C_1, C_2 there exists a pair of disjoint cozero-sets T_1, T_2 such that $C_i \subseteq T_i$ for $i = 1, 2$, and $T_1 \cup T_2$ is dense in X. As stated in Theorem 5.4 of [16], X is weakly cozero complemented if and only if $C(X)$ is weakly complemented.

In [8: Theorem 2.7], Bhattacharjee and McGovern prove that the total ring of quotients, $q(A)$, of a ring A is a PF-ring (which is to say every principal ideal of $q(A)$ is a flat $q(A)$-module) if and only if for every $a, b \in A$ with $ab = 0$, there exists $x, y \in A$ such that $ay = 0 = bx$ while $x + y$ is a non-divisor of zero. We tweak this condition slightly to formulate the following definition in f-rings. When we say an element a of an f-ring is *positive*, we mean that $a \ge 0$.

Definition 3.1. An f-ring R is *feebly complemented* if whenever $ab = 0$ in R there exist positive $c, d \in R$ such that $ac = 0 = bd$ and $c + d$ is a non-divisor of zero.

The frame property we seek will turn out to be such that, when applied to spaces, X has the topological counterpart precisely when $C(X)$ is feebly complemented. It is on the basis of this that we formulate the following definition.

Definition 3.2. A completely regular frame L is *feebly cozero complemented* if for any $a, b \in \operatorname{Coz} L$ with $a \wedge b = 0$, there exist $c, d \in \operatorname{Coz} L$ such that $a \wedge c = 0 = b \wedge d$ and $c \vee d$ is dense. We use the same term for Tychonoff spaces defined similarly.

Remark 1. The condition in this definition is a "cozero version" of the condition Bhattacharjee [7: Theorem 4.9] uses to characterize when the inverse topology on the set of minimal prime elements of an algebraic frame with the FIP is Hausdorff. It is also a formal weakening (extended to frames) of condition (c) in Theorem 3.1 of [8]. We thank the referee for drawing our attention to this.

Recall that every cozero element in L is a cozero of some positive member of $\mathcal{R}L$, and that any $h \in \mathcal{R}L$ is a non-divisor of zero precisely when $\operatorname{coz} h$ is dense. For use in the upcoming proof we write d_L for the frame homomorphism $\mathsf{Z}(L) \to \mathsf{D}(L)$ induced by the d-nucleus on $\mathsf{Z}(L)$. Recall that if $s \in \operatorname{Coz} L$, then $d_L(\boldsymbol{M}_s) = (\boldsymbol{M}_s)^{\perp\perp} = \boldsymbol{M}_{s^{**}}$. We write $\sqcup$ for the join in $\mathsf{D}(L)$.

Theorem 3.3. *The following conditions are equivalent for a completely regular frame L.*

(1) $\mathsf{D}(L)$ *satisfies* Reg(4).

(2) *L is feebly cozero complemented.*

(3) *$\mathcal{R}L$ is a feebly complemented f-ring.*

Proof. (1) ⇒ (2): Let $a \wedge b = 0$ in $\operatorname{Coz} L$. Then $\boldsymbol{M}_{a^{**}}$ and $\boldsymbol{M}_{b^{**}}$ are compact elements of $\mathsf{D}(L)$ with $\boldsymbol{M}_{a^{**}} \wedge \boldsymbol{M}_{b^{**}} = \perp$ since $a \wedge b = 0$ implies $a^{**} \wedge b^{**} = 0$. Since $\mathsf{D}(L)$ satisfies Reg(4),

$$(\boldsymbol{M}_{a^{**}})^{\perp} \sqcup (\boldsymbol{M}_{b^{**}})^{\perp} = \boldsymbol{M}_{a^{*}} \sqcup \boldsymbol{M}_{b^{*}} = \top,$$

so that, by compactness of the algebraic frame $\mathsf{D}(L)$, we can find $c, d \in \operatorname{Coz} L$ such that $\boldsymbol{M}_{c^{**}} \le \boldsymbol{M}_{a^{*}}$, $\boldsymbol{M}_{d^{**}} \le \boldsymbol{M}_{b^{*}}$, and $\boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{d^{**}} = \top$. Now, as shown in [11: Lemma 3.2], $\boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{d^{**}} = \boldsymbol{M}_{(c\vee d)^{**}}$. Consequently, $\boldsymbol{M}_{(c\vee d)^{**}} = \boldsymbol{M}_1$, which implies $(c \vee d)^{**} = 1$, that is, $c \vee d$ is dense. The inequalities $\boldsymbol{M}_{c^{**}} \le \boldsymbol{M}_{a^{*}}$ and $\boldsymbol{M}_{d^{**}} \le \boldsymbol{M}_{b^{*}}$ imply $c^{**} \le a^{*}$ and $d^{**} \le b^{*}$, whence we deduce that $a \wedge c = 0 = b \wedge d$. Therefore L is feebly cozero complemented.

(2) ⇒ (1): Let S and T be disjoint compact elements of $\mathsf{D}(L)$. Pick $a, b \in \operatorname{Coz} L$ such that $S = \boldsymbol{M}_{a^{**}}$ and $T = \boldsymbol{M}_{b^{**}}$. Then $\boldsymbol{M}_{a^{**}} \wedge \boldsymbol{M}_{b^{**}} = \perp$, which implies $a^{**} \wedge b^{**} = 0$, and hence $a \wedge b = 0$. Since L is feebly cozero complemented, there exist $c, d \in \operatorname{Coz} L$ such that $a \wedge c = 0 = b \wedge d$ and $c \vee d$ is dense. The equalities $a \wedge c = 0 = b \wedge d$ imply $c \le a^{*}$ and $d \le b^{*}$. Now

$$\begin{aligned}
(\boldsymbol{M}_{a^{**}})^{\perp} \sqcup (\boldsymbol{M}_{b^{**}})^{\perp} = \boldsymbol{M}_{a^{*}} \sqcup \boldsymbol{M}_{b^{*}} &= d_L\big(\boldsymbol{M}_{a^{*}} \vee \boldsymbol{M}_{b^{*}}\big) \\
&\ge d_L\big(\boldsymbol{M}_{c} \vee \boldsymbol{M}_{d}\big) \\
&= d_L\big(\boldsymbol{M}_{c\vee d}\big) \qquad \text{since } c, d \in \operatorname{Coz} L \\
&= (\boldsymbol{M}_{c\vee d})^{\perp\perp} = \boldsymbol{M}_{(c\vee d)^{**}} = \top,
\end{aligned}$$

the last step since $(c \vee d)^{**} = 1$. Therefore $\mathsf{D}(L)$ satisfies Reg(4).

(2) ⇒ (3): Let $g, h \in \mathcal{R}L$ be such that $gh = 0$. Then $\operatorname{coz} g \wedge \operatorname{coz} h = 0$. Since L is feebly cozero complemented, there exist positive $k, l \in \mathcal{R}L$ such that

$$\operatorname{coz} g \wedge \operatorname{coz} k = 0 = \operatorname{coz} h \wedge \operatorname{coz} l \quad \text{ and } \quad \operatorname{coz} k \vee \operatorname{coz} l \text{ is dense.}$$

The latter implies $\operatorname{coz}(k + l)$ is dense, and hence $k + l$ is a non-divisor of zero. Thus, $\mathcal{R}L$ is feebly complemented because $gk = 0 = hl$.

(3) ⇒ (2): This is proved similarly to the foregoing implication. □

Corollary 3.3.1. *For any Tychonoff space X, $\mathscr{C}_d(X)$ satisfies* Reg(4) *if and only if X is feebly cozero complemented if and only if $C(X)$ is a feebly complemented f-ring.*

Now that we have had to define feebly cozero complemented frames, we may as well indicate how they are related to other frames of a similar kind. Adapting McGovern's definition from spaces, we say a frame L is *weakly cozero complemented* if it satisfies the following property:

(W-frm) For any $a, b \in \operatorname{Coz} L$ with $a \wedge b = 0$, there exist $c, d \in \operatorname{Coz} L$ such that $a \le c$, $b \le d$, $c \wedge d = 0$, and $c \vee d$ is dense.

Notice that in this foregoing property $a \wedge d = 0 = b \wedge d$.

Observation 1. Any F-frame is feebly cozero complemented, and every weakly cozero complemented frame is feebly cozero complemented. Note though that $\mathfrak{O}\mathbb{R}$ is a feebly cozero complemented frame that is not an F-frame. In fact, any frame L in which $a^* \in \operatorname{Coz} L$ for every $a \in L$ (these are called Oz-frames) is feebly cozero complemented, however an Oz-frame is an F-frame precisely when it is extremally disconnected (see [4: 4.1]).

In the proof of the upcoming result we will use the fact that a dense onto frame homomorphism commutes with pseudocomplements, and hence it preserves and reflects the density of elements.

Proposition 3.4. *The following conditions are equivalent for a completely regular frame L.*

(1) *L is feebly cozero complemented.*

(2) *βL is feebly cozero complemented.*

(3) *λL is feebly cozero complemented.*

(4) *υL is feebly cozero complemented.*

Proof. To prove all equivalences it suffices to show that if $h\colon L \to M$ is a dense coz-onto homomorphism, then L is feebly cozero complemented if and only if M is feebly cozero complemented. Assume L is feebly cozero complemented. Let $u \wedge v = 0$ in $\operatorname{Coz} M$. Since h is coz-onto, there exist $a, b \in \operatorname{Coz} L$ such that $h(a) = u$ and $h(b) = v$. Since h is dense, $a \wedge b = 0$. Since L is feebly cozero complemented, there exist $c, d \in \operatorname{Coz} L$ such that $c \vee d$ is dense, and $a \wedge c = 0 = b \wedge d$. Then $h(c)$ and $h(d)$ are cozero elements of M such that $h(c) \vee h(d)$ is dense, and $u \wedge h(c) = 0 = v \wedge h(d)$. Therefore M is feebly cozero complemented. A similar argument shows the converse. □

4. When induced maps are closed

In [11] we showed that for certain types of morphisms $h\colon L \to M$ in CRFrm, the induced coherent map $\mathsf{Z}(h)\colon \mathsf{Z}(L) \to \mathsf{Z}(M)$ is open if and only if h is open. Here we shall consider a similar situation for closed maps and their weaker variants. Recall that a frame homomorphism $h\colon L \to M$ is *closed* if for any $a \in L$ and $z \in M$,

$$h_*(h(a) \vee z) = a \vee h_*(z).$$

Equivalently, h is closed in case, for any $a, b \in L$ and $z \in M$,

$$h(a) \le h(b) \vee z \implies a \le b \vee h_*(z).$$

Closed frame maps generalize closed continuous functions. In [19], Martínez weakens the concept of closed frame maps by defining a homomorphism $h\colon L \to M$ to be *weakly closed* in case, for any $a \in L$ and $z \in M$,

$$h(a) \vee z = 1 \implies a \vee h_*(z) = 1.$$

We aim to examine which morphisms $h\colon L \to M$ in CRFrm induce closed or weakly closed coherent maps $\mathsf{Z}(h)\colon \mathsf{Z}(L) \to \mathsf{Z}(M)$. We start with lemmas which show that when dealing with coherent frames, to test closedness or weak closedness it suffices to restrict to compact elements.

Lemma 4.1. *Let $\phi\colon A \to B$ be a coherent map between algebraic frames.*

(a) *ϕ is closed if and only if for any $c, d \in \mathfrak{k}(A)$ and $u \in \mathfrak{k}(B)$,*

$$\phi(c) \le \phi(d) \vee u \implies c \le d \vee \phi_*(u).$$

(b) *ϕ is weakly closed if and only if for any $c \in \mathfrak{k}(A)$ and $u \in \mathfrak{k}(B)$,*

$$\phi(c) \vee u = 1 \implies c \vee \phi_*(u) = 1.$$

Proof. (a) The condition is obviously necessary. To show that it is sufficient, consider any $a, b \in A$ and $z \in B$ such that $\phi(a) \le \phi(b) \vee z$. Let c be a compact element in A below a. Then

$$\phi(c) \le \phi(b) \vee z = \bigvee\{\phi(t) \mid t \in \mathfrak{k}(A), t \le b\} \vee \bigvee\{v \in \mathfrak{k}(B) \mid v \le z\}.$$

Since $\phi(c)$ is compact, there exist $d \in \mathfrak{k}(A)$ and $u \in \mathfrak{k}(B)$ such that $d \le b$, $u \le z$, and $\phi(c) \le \phi(d) \vee u$. Then, by the stated condition, $c \le d \vee \phi_*(u)$, which implies $c \le b \vee \phi_*(z)$. Since a is the join of compact elements below it, it follows that $a \le b \vee \phi_*(z)$. Therefore ϕ is closed.

(b) This is proved similarly, except that here use is made of the compactness of B. □

We know that if $c, d \in \operatorname{Coz} L$, then $\boldsymbol{M}_{c \vee d} = \boldsymbol{M}_c \vee \boldsymbol{M}_d$. In the lemma that follows, which we need for use below, we remove the restriction that c and d be cozero elements, but with the penalty that we must impose normality. Our proof piggybacks on $C(X)$. Let us expatiate.

Let X be a Tychonoff space. With every closed set $A \subseteq X$ is associated an ideal M_A of $C(X)$ defined by

$$M_A = \{f \in C(X) \mid A \subseteq Z(f)\}.$$

In [1: Lemma 2.8], the authors prove that if X is normal, then for any closed sets $A, B \subseteq X$, $M_{A \cap B} = M_A + M_B$. We will use this result in our proof. Let us remind the reader that if L is a normal frame, then $r_L(a \vee b) = r_L(a) \vee r_L(b)$ for every $a, b \in L$.

Lemma 4.2. *If L is a normal frame, then $\boldsymbol{M}_a \vee \boldsymbol{M}_b = \boldsymbol{M}_{a \vee b}$, for all $a, b \in L$.*

Proof. Consider any normal Tychonoff space X, and let $\phi\colon C(X) \to \mathcal{R}(\mathfrak{O}X)$ be the ring isomorphism that sends $f \in C(X)$ to the frame homomorphism $\mathfrak{O}f$. It is clear that for any $U \in \mathfrak{O}X$, the image of the ideal $M_{X \smallsetminus U}$ of $C(X)$ under ϕ is $\boldsymbol{M}_U$. Consequently, if K is any normal spatial frame, and $s, t \in K$, then $\boldsymbol{M}_{s \vee t} = \boldsymbol{M}_s + \boldsymbol{M}_t$. But $\boldsymbol{M}_s + \boldsymbol{M}_t$ is a z-ideal since $\mathcal{R}K$ is isomorphic to some $C(X)$ (see also the Appendix), so $\boldsymbol{M}_s + \boldsymbol{M}_t$ is the join of the set $\{\boldsymbol{M}_s, \boldsymbol{M}_t\}$ in the frame $\mathsf{Z}(\mathcal{R}K)$. That is,

$$\boldsymbol{M}_s \vee \boldsymbol{M}_t = \boldsymbol{M}_{s \vee t}. \tag{†}$$

Now consider the map $\mathsf{Z}(j_L)\colon \mathsf{Z}(\beta L) \to \mathsf{Z}(L)$ induced by $j_L\colon \beta L \to L$. We claim that $\mathsf{Z}(j_L)(\boldsymbol{M}_{r_L(w)}) = \boldsymbol{M}_w$, for any $w \in L$. Since $j_L r_L(w) = w$, it is immediate that $\mathsf{Z}(j_L)(\boldsymbol{M}_{r_L(w)}) \le \boldsymbol{M}_w$. Let c be a cozero element below w. Pick a bounded $h \in \mathcal{R}L$ such that $\operatorname{coz} h = w$. Then, for the function $h^\beta \in \mathcal{R}(\beta L)$ with $j_L \cdot h^\beta = h$, we have

$$\operatorname{coz} h = \operatorname{coz}(j_L \cdot h^\beta) = j_L(\operatorname{coz} h^\beta),$$

whence $\operatorname{coz} h^\beta \le r_L(\operatorname{coz} h)$. Now,

$$\begin{aligned}
\boldsymbol{M}_w &= \bigvee_{\mathsf{Z}(L)} \{\boldsymbol{M}_{\operatorname{coz} k} \mid k \in \mathcal{R}^*L \text{ and } \operatorname{coz} k \le w\} \\
&= \bigvee_{\mathsf{Z}(L)} \{\boldsymbol{M}_{j_L(\operatorname{coz} k^\beta)} \mid k \in \mathcal{R}^*L \text{ and } \operatorname{coz} k \le w\} \\
&= \bigvee_{\mathsf{Z}(L)} \{\mathsf{Z}(j_L)(\boldsymbol{M}_{\operatorname{coz} k^\beta}) \mid k \in \mathcal{R}^*L \text{ and } \operatorname{coz} k \le w\} \\
&= \mathsf{Z}(j_L)\Big(\bigvee_{\mathsf{Z}(\beta L)} \{\boldsymbol{M}_{\operatorname{coz} k^\beta} \mid k \in \mathcal{R}^*L \text{ and } \operatorname{coz} k \le w\}\Big) \\
&\le \mathsf{Z}(j_L)(\boldsymbol{M}_{r_L(w)}),
\end{aligned}$$

the last step since $\operatorname{coz} k^\beta \le r_L(\operatorname{coz} k) \le r_L(w)$ for each k involved in the previous join. This establishes the claimed equality. Thus,

$$\begin{aligned} \boldsymbol{M}_a \vee \boldsymbol{M}_b &= \mathsf{Z}(j_L)(\boldsymbol{M}_{r_L(a)}) \vee \mathsf{Z}(j_L)(\boldsymbol{M}_{r_L(b)}) \\ &= \mathsf{Z}(j_L)\big(\boldsymbol{M}_{r_L(a)} \vee \boldsymbol{M}_{r_L(b)}\big) \\ &= \mathsf{Z}(j_L)\big(\boldsymbol{M}_{r_L(a)\vee r_L(b)}\big) \quad \text{by } (\dagger) \text{ since } \beta L \text{ is normal and spatial} \\ &= \mathsf{Z}(j_L)\big(\boldsymbol{M}_{r_L(a\vee b)}\big) \quad \text{since } L \text{ is normal} \\ &= \boldsymbol{M}_{a\vee b}, \end{aligned}$$

which completes the proof. □

Let us say a morphism $h\colon L \to M$ in CRFrm is *coz-closed* in case, for any $c, d \in \operatorname{Coz} L$ and $u \in \operatorname{Coz} M$, the inequality $h(c) \le h(d) \vee u$ implies $c \le d \vee h_*(u)$. This is of course the definition of closedness restricted to cozero elements. In fact, if M is Lindelöf, then $h\colon L \to M$ is closed if and only if it is coz-closed. To see the non-trivial implication, assume h is coz-closed, and consider any $a, b \in L$ and $w \in M$ such that $h(a) \le h(b) \vee w$. Let $c \le a$ be a cozero element in L. Then

$$h(c) \le \bigvee\{h(d) \mid d \in \operatorname{Coz} L, d \le b\} \vee \bigvee\{s \in \operatorname{Coz} M \mid s \le w\}.$$

By [5: Corollary 4], $h(c)$ is a Lindelöf element. So, in view of the fact that the join of countably many cozero elements is a cozero element, there exist $d \in \operatorname{Coz} L$ and $s \in \operatorname{Coz} M$ such that $d \le b$, $s \le w$, and $h(c) \le h(d) \vee s$. By coz-closedness, $c \le d \vee h_*(s)$, which implies $c \le b \vee h_*(w)$, and hence $a \le b \vee h_*(w)$ by complete regularity.

Lemma 4.3. *Let $h\colon L \to M$ be a morphism in* CRFrm. *In* Figure 1 *below every quadrilateral commutes.*

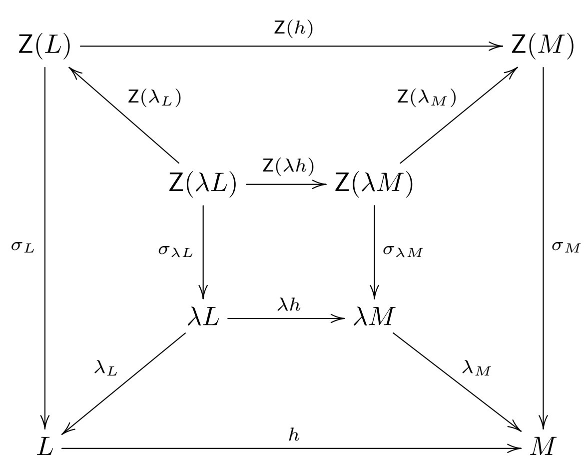


Figure 1.

P r o o f. The only quadrilateral whose commutativity is not already known is the upper trapezium (or trapezoid, in American English). Since all the four objects and four morphisms making up the trapezium are coherent, it suffices to chase the diagram, starting with a compact element $\boldsymbol{M}_{[\![c]\!]}$ in $\mathsf{Z}(\lambda L)$. Indeed, we have

$$\mathsf{Z}(h)\mathsf{Z}(\lambda_L)(\boldsymbol{M}_{[\![c]\!]}) = \boldsymbol{M}_{h(c)} = \mathsf{Z}(\lambda_M)\mathsf{Z}(\lambda h)(\boldsymbol{M}_{[\![c]\!]}),$$

which proves the result. □

Theorem 4.1. *Let $h\colon L \to M$ be a morphism in* CRFrm. *Consider the following statements:*

(1) $\mathsf{Z}(h)$ *is closed.*
(2) $\mathsf{Z}(\lambda h)$ *is closed.*
(3) $\mathsf{Z}(\upsilon h)$ *is closed.*
(4) λh *is coz-closed.*
(5) λh *is closed.*
(6) h *is coz-closed.*

We have the following implications: (1) $\iff$ (2) $\iff$ (3) $\implies$ (4) $\iff$ (5) $\implies$ (6). *If L is normal, then all the statements are equivalent.*

P r o o f. We shall make reference to the diagram in Figure 1.

(1) ⇔ (2): It is shown in [11: Proposition 5.8] that, for any morphism $g\colon K \to N$ in CRFrm, the map $\mathsf{Z}(g)\colon \mathsf{Z}(K) \to \mathsf{Z}(N)$ is an isomorphism if and only if g is coz-faithful and coz-surjective. Since, for any completely regular frame K, $\lambda_K\colon \lambda K \to K$ is coz-faithful and coz-surjective, each of the maps $\mathsf{Z}(\lambda_L)$ and $\mathsf{Z}(\lambda_M)$ is an isomorphism. Now, composites of closed maps are closed maps, and isomorphisms are closed maps, so the commutativity of the upper trapezium in Figure 1 shows that $\mathsf{Z}(h)$ is closed if and only if $\mathsf{Z}(\lambda h)$ is closed.

(2) ⇔ (3): This is proved similarly to the foregoing equivalence because there is commutative square

$$\begin{array}{ccc} \mathsf{Z}(\lambda L) & \xrightarrow{\mathsf{Z}(\lambda h)} & \mathsf{Z}(\lambda M) \\ {\scriptstyle \mathsf{Z}(\ell_L)}\downarrow & & \downarrow{\scriptstyle \mathsf{Z}(\ell_M)} \\ \mathsf{Z}(\upsilon L) & \xrightarrow{\mathsf{Z}(\upsilon h)} & \mathsf{Z}(\upsilon M) \end{array}$$

where the downward morphisms are isomorphisms.

(2) ⇒ (4): We show that, in general, for any morphism $g\colon K \to N$ in CRFrm, if $\mathsf{Z}(g)\colon \mathsf{Z}(K) \to \mathsf{Z}(N)$ is closed, then g is coz-closed. Consider any $c, d \in \operatorname{Coz} K$ and $u \in \operatorname{Coz} N$ such that $g(c) \le g(d) \vee u$. Since c, d, $g(c)$, and u are cozero elements, this implies

$$\mathsf{Z}(g)(\boldsymbol{M}_c) = \boldsymbol{M}_{g(c)} \le \boldsymbol{M}_{g(d)\vee u} = \boldsymbol{M}_{g(d)} \vee \boldsymbol{M}_u = \mathsf{Z}(g)(\boldsymbol{M}_d) \vee \boldsymbol{M}_u,$$

so that, in light of $\mathsf{Z}(g)$ being closed,

$$\boldsymbol{M}_c \le \boldsymbol{M}_d \vee \mathsf{Z}(g)_*\big(\boldsymbol{M}_u\big) = \boldsymbol{M}_d \vee \boldsymbol{M}_{g_*(u)}.$$

On applying the frame homomorphism $\sigma_L\colon \mathsf{Z}(L) \to L$, we obtain $c \le d \vee g(u)$, hence g is coz-closed.

(4) ⇔ (5): We observed above that closedness and coz-closedness are equivalent for a morphism in CRFrm whose codomain is Lindelöf.

(5) ⇒ (6): Let c, d be cozero elements in L, and u be a cozero element in M such that $h(c) \le h(d) \vee u$. Then, in the lattice $\operatorname{Coz}(\lambda M)$, we have

$$[\![h(c)]\!] \le [\![h(d) \vee u]\!] = [\![h(d)]\!] \vee [\![u]\!].$$

Recalling that, for any $s \in \operatorname{Coz} L$, and $t \in \operatorname{Coz} M$, $(\lambda h)([\![s]\!]) = [\![h(s)]\!]$, and $(\lambda h)_*([\![t]\!]) = [\![h_*(t)]\!]$ – for the latter see [12] – we have $(\lambda h)([\![c]\!]) \le (\lambda h)([\![d]\!]) \vee [\![u]\!]$, which, by closedness of λh, implies

$$[\![c]\!] \le [\![d]\!] \vee (\lambda h)_*([\![u]\!]) = [\![d]\!] \vee [\![h_*(u)]\!].$$

Applying the homomorphism λ_L, we obtain $c \le d \vee h_*(u)$, and therefore h is coz-closed. This finishes the proof of the first assertion.

For the second part, it suffices to show that (6) implies (1) if L is normal. So assume that L is normal and h is coz-closed. To show that $\mathsf{Z}(h)$ is closed, it is enough to test the defining condition on compact elements (recall Lemma 4.1). Consider therefore $c, d \in \operatorname{Coz} L$ and $u \in \operatorname{Coz} M$ such that $\mathsf{Z}(h)(\boldsymbol{M}_c) \le \mathsf{Z}(h)(\boldsymbol{M}_d) \vee \boldsymbol{M}_u$. This implies $\boldsymbol{M}_{h(c)} \le \boldsymbol{M}_{h(d)} \vee \boldsymbol{M}_u$, so that $h(c) \le h(d) \vee u$, and hence $c \le d \vee h_*(u)$ by coz-closedness of h. Therefore, in view of Lemma 4.2,

$$\boldsymbol{M}_c \le \boldsymbol{M}_{d \vee h_*(u)} = \boldsymbol{M}_d \vee \boldsymbol{M}_{h_*(u)} = \boldsymbol{M}_d \vee \mathsf{Z}(h)_*(\boldsymbol{M}_u),$$

which shows that $\mathsf{Z}(h)$ is closed. □

Conspicuous by its absence in the previous theorem is a statement about the closedness of the map $\mathsf{Z}(\beta h)\colon \mathsf{Z}(\beta L) \to \mathsf{Z}(\beta M)$. The reason is that this map is actually always closed, as we prove next. Recall that any frame homomorphism with a regular domain and compact codomain is closed.

Proposition 4.2. *For any morphism $h\colon L \to M$ in* CRFrm*, $\mathsf{Z}(\beta h)\colon \mathsf{Z}(\beta L) \to \mathsf{Z}(\beta M)$ is closed.*

Proof. Let $I, J \in \operatorname{Coz}(\beta L)$ and $U \in \operatorname{Coz}(\beta M)$ be such that

$$\mathsf{Z}(\beta h)(\boldsymbol{M}_I) \le \mathsf{Z}(\beta h)(\boldsymbol{M}_J) \vee \boldsymbol{M}_U.$$

This implies $\boldsymbol{M}_{(\beta h)(I)} \le \boldsymbol{M}_{(\beta h)(J)} \vee \boldsymbol{M}_U$, so that $(\beta h)(I) \le (\beta h)(J) \vee U$. Since βh is closed (as it maps out of a regular frame into a compact one), $I \le J \vee (\beta h)_*(U)$. Consequently, in light of Lemma 4.2 (which applies since βL is normal),

$$\boldsymbol{M}_I \le \boldsymbol{M}_{J \vee (\beta h)_*(U)} = \boldsymbol{M}_J \vee \boldsymbol{M}_{(\beta h)_*(U)} = \boldsymbol{M}_J \vee \mathsf{Z}(\beta h)_*(\boldsymbol{M}_U),$$

which shows that $\mathsf{Z}(\beta h)$ is closed. □

We close the section by remarking that Theorem 4.1 is valid with "closed" replaced with "weakly closed".

5. Appendix: Sums of z-ideals in $\mathcal{R}L$

In any ring $C(X)$, the sum of two z-ideals is a z-ideal. This is proved in [14] using properties of the Stone-Čech compactification. In [25], Rudd proves this result without invoking the Stone-Čech compactification. Our proof will piggyback on that of Gillman and Jerison [14]. We have tried, without success, to find a purely frame-theoretic proof; hence our resorting to massaging the point-sensitive one. We shall at times (for the sake of clarity) write $j \cdot k$ for the composite jk.

We first establish a lemma. Given a bounded function $k \in \mathcal{R}L$, let k^β be the unique function in $\mathcal{R}(\beta L)$ with $k = j_L \cdot k^\beta$ (recall the diagram in Subsection 2.2). The map $k \mapsto k^\beta$ is a ring isomorphism $\mathcal{R}^*L \to \mathcal{R}(\beta L)$. Recall from Subsection 2.1 how we regard the spectrum of a frame M. In what follows the set-theoretic complement of Σ_a (for $a \in M$) will be denoted by Σ'_a, so that

$$\Sigma'_a = \{p \in \operatorname{Pt}(M) \mid a \le p\}.$$

Recall that $\Sigma k^\beta \colon \Sigma\beta L \to \Sigma\mathfrak{O}\mathbb{R}$ is the continuous function defined by

$$\Sigma k^\beta(I) = k^\beta_*(I) = \mathbb{R} \smallsetminus \{p_I\},$$

for some $p_I \in \mathbb{R}$ uniquely determined by I. We now define a continuous map $\hat{k}\colon \Sigma\beta L \to \mathbb{R}$ by $\hat{k}(I) = p_I$. Write $Z(\hat{k})$ for the zero-set of $\hat{k}$.

Lemma 5.1. *Let k be a bounded function in $\mathcal{R}L$. For any $a \in L$,*

$$Z(\hat{k}) \supseteq \Sigma'_{r_L(a)} \implies \operatorname{coz} k \leq a.$$

Proof. Let $I \in \mathrm{Pt}(\beta L)$ be such that $r_L(a) \leq I$. We aim to show that $\operatorname{coz}(k^\beta) \leq I$. By hypothesis, the inequality $r_L(a) \leq I$ implies $\hat{k}(I) = 0$, which, in turn, implies $k^\beta_*(I) = \mathbb{R} \smallsetminus \{0\}$. Thus,

$$\operatorname{coz}(k^\beta) = k^\beta(\mathbb{R} \smallsetminus \{0\}) = k^\beta(k^\beta_*(I)) \leq I.$$

Since βL is spatial, so that $r_L(a) = \bigwedge\{J \in \mathrm{Pt}(\beta L) \mid r_L(a) \leq J\}$, it follows that $\operatorname{coz}(k^\beta) \leq r_L(a)$. Consequently,

$$\operatorname{coz} k = \operatorname{coz}(j_L k^\beta) = j_L(\operatorname{coz}(k^\beta)) = \bigvee \operatorname{coz}(k^\beta) \leq \bigvee r_L(a) = a,$$

which proves the claim. □

Proposition 5.1. *The sum of two z-ideals in $\mathcal{R}L$ is a z-ideal.*

Proof. Let Q_1 and Q_2 be z-ideals in $\mathcal{R}L$. Let $h \in \mathcal{R}L$ be such that $\operatorname{coz} h \leq \operatorname{coz}(g_1 + g_2)$, with $g_i \in Q_i$, for $i = 1, 2$. We must show that $h \in Q_1 + Q_2$. Since the function $g = \frac{h}{1+|h|}$ is bounded and $\operatorname{coz} g = \operatorname{coz} h$, we may assume, without loss of generality, that h is bounded. For brevity, write $c_i = \operatorname{coz}(g_i)$. Since r_L preserves binary joins of cozero elements, $r_L(\operatorname{coz} h) \leq r_L(c_1) \vee r_L(c_2)$. Let $\hat{h}\colon \Sigma\beta L \to \mathbb{R}$ be the map as described above. We define a map $t\colon \Sigma'_{r_L(c_1)} \cup \Sigma'_{r_L(c_2)} \to \mathbb{R}$ as follows:

$$t(I) = \begin{cases} 0 & \text{if } I \in \Sigma'_{r_L(c_1)}, \\ \hat{h}(I) & \text{if } I \in \Sigma'_{r_L(c_2)}. \end{cases}$$

In order that t is well defined, we need to check that if $I \in \Sigma'_{r_L(c_1)} \cap \Sigma'_{r_L(c_2)}$, then $\hat{h}(I) = 0$. Observe that if $I \in \Sigma'_{r_L(c_1)} \cap \Sigma'_{r_L(c_2)}$ then $r_L(c_1) \vee r_L(c_2) \leq I$. Now,

$$\bigvee h^\beta(\mathbb{R} \smallsetminus \{0\}) = j_L h^\beta(\mathbb{R} \smallsetminus \{0\}) = h(\mathbb{R} \smallsetminus \{0\}) = \operatorname{coz} h,$$

which implies

$$h^\beta(\mathbb{R} \smallsetminus \{0\}) \leq r_L(\operatorname{coz} h) \leq r_L(c_1) \vee r_L(c_2) \leq I,$$

so that $\mathbb{R} \smallsetminus \{0\} \leq h^\beta_*(I)$, and hence $h^\beta_*(I) = \mathbb{R} \smallsetminus \{0\}$ since both $\mathbb{R} \smallsetminus \{0\}$ and $h^\beta_*(I)$ are points in $\mathfrak{O}\mathbb{R}$. Thus, $\Sigma h^\beta(I) = \mathbb{R} \smallsetminus \{0\}$, whence $\hat{h}(I) = 0$, as required. Therefore t is a continuous function on the closed set $\Sigma'_{r_L(c_1)} \cup \Sigma'_{r_L(c_2)}$ of the normal space $\Sigma\beta L$. By Tietze's extension theorem, t has an extension to a continuous function $\bar{t}\colon \Sigma\beta L \to \mathbb{R}$. The map $\bar{t}$ is zero everywhere where t is zero. Therefore $Z(\bar{t}) \supseteq \Sigma'_{r_L(c_1)}$. Let $\tau\colon \mathfrak{O}\mathbb{R} \to \beta L$ be the frame homomorphism $\tau = \mu \cdot \mathfrak{O}\bar{t}$, where $\mu\colon \mathfrak{O}\Sigma\beta L \to \beta L$ is the isomorphism given by $\Sigma_J \mapsto J$, for each $J \in \beta L$. Since $Z(\bar{t}) \supseteq \Sigma'_{r_L(c_1)}$, we have $\operatorname{coz} \tau \leq r_L(c_1)$, as a straightforward calculation shows, and so the homomorphism $j_L\tau\colon \mathfrak{O}\mathbb{R} \to L$ has the property that

$$\operatorname{coz}(j_L\tau) = \bigvee \operatorname{coz} \tau \leq \bigvee r_L(c_1) = c_1 = \operatorname{coz}(g_1).$$

Since Q_1 is a z-ideal in $\mathcal{R}L$, it follows that $j_L\tau \in Q_1$.

We now argue that $h - j_L\tau \in Q_2$. Let us write $g = j_L\tau$. By Lemma 5.1, it suffices to show that $\Sigma'_{r_L(c_2)} \subseteq Z(\widehat{h-g})$. We claim that if $I \in \Sigma'_{r_L(c_2)}$, then $g^\beta_*(I) = h^\beta_*(I)$. Observe that

$$g^\beta_*(I) = \tau_*(I) = (\mathfrak{O}\bar{t})_*\mu_*(I) = (\mathfrak{O}\bar{t})_*(\Sigma_I) = (\mathfrak{O}\bar{t})_*[\Sigma\beta L \smallsetminus \{I\}].$$

Since $\bar{t}(I) = p_I$, as $I \in \Sigma'_{r_L(c_2)}$, it follows that

$$\bar{t}^{-1}[\mathbb{R} \smallsetminus \{p_I\}] = \{J \in \Sigma\beta L \mid \bar{t}(J) \in \mathbb{R} \smallsetminus \{p_I\}\} \subseteq \Sigma\beta L \smallsetminus \{I\},$$

that is, $\mathfrak{O}\bar{t}(h_*^\beta(I)) \subseteq \Sigma\beta L \smallsetminus \{I\}$, so that

$$h_*^\beta(I) \le (\mathfrak{O}\bar{t})_*(\Sigma\beta L \smallsetminus \{I\}) = \tau_*(I) = g_*^\beta(I),$$

and hence the claimed equality since these are both points in $\mathfrak{O}\mathbb{R}$. Next, we show that $(h-g)_*^\beta(I) = \mathbb{R} \smallsetminus \{0\}$, which will prove that $\widehat{(h-g)}(I) = 0$, as desired. Pick $r \in \mathbb{R}$ such that $h_*^\beta(I) = g_*^\beta(I) = \mathbb{R} \smallsetminus \{r\}$. Then, by [2: Lemma 3.2.1],

$$\operatorname{coz}(h^\beta - r) = h^\beta(\mathbb{R} \smallsetminus \{r\}) = h^\beta h_*^\beta(I) \le I.$$

Similarly, $\operatorname{coz}(r - k^\beta) \le I$, and therefore

$$\begin{aligned}(h-g)^\beta(\mathbb{R} \smallsetminus \{0\}) = \operatorname{coz}((h-g)^\beta) &= \operatorname{coz}(h^\beta - g^\beta)\\ &= \operatorname{coz}\big((h^\beta - r) + (r - g^\beta)\big)\\ &\le I \vee I = I.\end{aligned}$$

This implies $\mathbb{R} \smallsetminus \{0\} \le (h-g)_*^\beta(I)$, and hence equality as these are both points in $\mathfrak{O}\mathbb{R}$. Thus, $\widehat{(h-g)}(I) = 0$, and therefore $\Sigma'_{r_L(c_2)} \subseteq Z(\widehat{h-g})$, whence $\operatorname{coz}(h-g) \le c_2$, implying $h - g \in Q_2$. Consequently, $h \in Q_1 + Q_2$, which proves that $Q_1 + Q_2$ is a z-ideal. □

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