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A Little More on Ideals Associated with Sublocales

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A little more on ideals associated with sublocales

O. Ighedo*, G.W. Kivunga, and D.N. Stephen

Dedicated to Themba Dube on the occasion of his 65^{th} birthday

Abstract. As usual, let $\mathcal{R}L$ denote the ring of real-valued continuous functions on a completely regular frame L. Let βL and λL denote the Stone-Čech compactification of L and the Lindelöf coreflection of L, respectively. There is a natural way of associating with each sublocale of βL two ideals of $\mathcal{R}L$, motivated by a similar situation in $C(X)$. In [12], the authors go one step further and associate with each sublocale of λL an ideal of $\mathcal{R}L$ in a manner similar to one of the ways one does it for sublocales of βL. The intent in this paper is to augment [12] by considering two other coreflections; namely, the realcompact and the paracompact coreflections.
We show that $\boldsymbol{M}$-ideals of $\mathcal{R}L$ indexed by sublocales of βL are precisely the intersections of maximal ideals of $\mathcal{R}L$. An $\boldsymbol{M}$-ideal of $\mathcal{R}L$ is *grounded* in case it is of the form $\boldsymbol{M}_S$ for some sublocale S of L. A similar definition is given for an $\boldsymbol{O}$-ideal of $\mathcal{R}L$. We characterise $\boldsymbol{M}$-ideals of $\mathcal{R}L$ indexed by spatial sublocales of βL, and $\boldsymbol{O}$-ideals of $\mathcal{R}L$ indexed by closed sublocales of βL in terms of grounded maximal ideals of $\mathcal{R}L$.

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1 Introduction

The $\boldsymbol{M}$- and $\boldsymbol{O}$-ideals of $C(X)$, the ring of real-valued continuous functions on a Tychonoff space X, associated with subsets of βX were introduced in [14] by Johnson and Mandelker. This is done with the aim of extending the ideals $\boldsymbol{M}^p$ and $\boldsymbol{O}^p$ for $p \in \beta X$, that were extensively studied in [13] by Gillman and Jerison.

The author in [9] extended the concept of $\boldsymbol{M}$- and $\boldsymbol{O}$-ideals of $C(X)$ associated with subsets of βX to locales. In [11], Stephen, one of the authors of the current manuscript, and Dube extended this idea to sublocales of λL, the Lindelöf regular coreflection.

The current manuscript, amongst other things, studies these ideals in terms of sublocales of υL and πL, the realcompact coreflection and the paracompact coreflection, respectively.

The paper is laid out in the following order: Following the preliminaries in Section 1, we show in Proposition 3.1 that the $\boldsymbol{M}$-ideals of $\mathcal{R}L$ are precisely the intersections of maximal ideals of $\mathcal{R}L$. This we do in terms of sublocales of βL. In Definition 2.1, grounded $\boldsymbol{M}$-ideals of $\mathcal{R}L$ are defined, and the grounded $\boldsymbol{O}$-ideals of $\mathcal{R}L$ are defined similarly. This definition paves the way for the main results, Theorems 2.6, 2.10, and 2.13, characterising $\boldsymbol{M}$-ideals of $\mathcal{R}L$ indexed by spatial sublocales of L, characterising $\boldsymbol{O}$-ideals of $\mathcal{R}L$ indexed by closed sublocales of βL, and characterising when $\boldsymbol{O}$-ideals of $\mathcal{R}L$ indexed by closed sublocales of βL are grounded, respectively.

In Section 3, $\boldsymbol{M}$- and $\boldsymbol{O}$-ideals of $\mathcal{R}L$ are associated with sublocales of the realcompact regular coreflection, denoted by υL, and sublocales of the paracompact coreflection, denoted by πL. One major and useful information that is used in this section is the fact that both υL and πL are sublocales (and not merely isomorphic to sublocales) of the Lindelöf regular coreflection, λL. If A is a sublocale of either of these, then A is also a sublocale of λL. This information is extensively used in Proposition 4.2. One of the main results in this section, Theorem 4.5, characterises when L is pseudocompact in terms of $\boldsymbol{O}$-ideals of $\mathcal{R}L$ indexed by sublocales of βL and the other coreflections. Theorem 4.7 characterises the equivalence of the realcompact coreflection and the Lindelöf regular coreflection in terms of $\boldsymbol{O}$-ideals of $\mathcal{R}L$ indexed by the respective sublocales.

In Section 4, we give a result which we have not seen in the literature on the Lindelöf regular coreflection of L. We show in Proposition 5.1 that

λL is the universal Lindelöfication of both υL and πL.

2 Preliminaries

We assume familiarity with frames and locales. Our references are [15] and [17], and our notation will be of these references, by and large. All frames in this paper are completely regular. We denote the category of completely regular frames by **CRFrm**, and the category of regular frames by **RFrm**. We refer to [4] for properties of the cozero part of a frame, and to [2] for properties of the ring of real-valued continuous functions on a frame. By a *quotient map* we will always mean an onto frame homomorphism.

2.1 The compact regular coreflection We view βL, the Stone-Čech compactification of L as the frame of strongly regular ideals of Coz L. To recall, an ideal I of Coz L is called *strongly regular* if for every $u \in I$ there exists $v \in I$ such that $u \prec\!\!\prec v$. The mapping

$$j_L \colon \beta L \to L \quad \text{defined by} \quad j_L(I) = \bigvee I$$

is a dense quotient map, and is the coreflection map to L from compact completely regular frames. We denote its right adjoint by r_L, and recall that, for any $a \in L$,

$$r_L(a) = \{c \in \mathrm{Coz}\, L \mid c \prec\!\!\prec a\}.$$

2.2 The Lindelöf regular coreflection An ideal of Coz L is called a *σ-ideal* if it is closed under countable joins. The frame of σ-ideals of Coz L is denoted by λL. It is a Lindelöf completely regular frame. The map $\lambda_L \colon \lambda L \to L$ that sends a σ-ideal to its join in L is a dense quotient map, and it is the coreflection map to L from Lindelöf completely regular frames [16]. Its right adjoint is given by

$$(\lambda_L)_*(a) = \{c \in \mathrm{Coz}\, L \mid c \leq a\}.$$

Thus, if $a \in \mathrm{Coz}\, L$, then $(\lambda_L)_*(a)$ is the principal ideal of Coz L generated by a. By a *Lindelöfication* of L is meant a Lindelöf frame M such that there is a dense quotient map $M \to L$. For this reason, the frame λL is sometimes styled the *universal Lindelöfication* of L.

2.3 The realcompact regular coreflection See [5] for the definition and properties of realcompact frames. They form a coreflective subcategory of **CRFrm**, and the coreflection is constructed as follows. For any frame M, denote by $\mathrm{Pt}(M)$ the set of points of M. Since our frames are completely regular the points are exactly the maximal elements.

The mapping $\ell\colon \lambda L \to \lambda L$ defined by

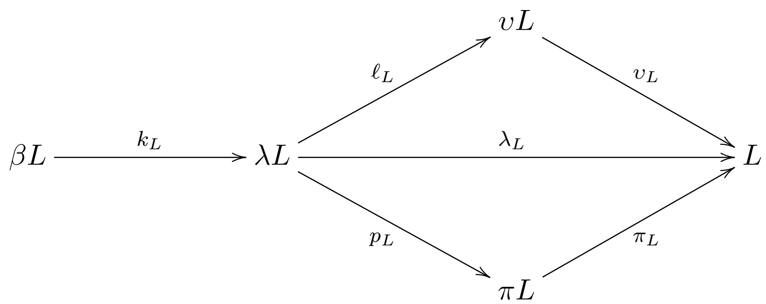
$$\ell(I) = (\lambda_L)_*\Big(\bigvee I\Big) \wedge \bigwedge\{Q \in \mathrm{Pt}(\lambda L) \mid I \leq Q\}$$

is a nucleus, and the frame $\upsilon L = \mathrm{Fix}(\ell)$ is realcompact. The induced quotient map is written $\ell_L\colon \lambda L \to \upsilon L$. The join map $\upsilon_L\colon \upsilon L \to L$ is a dense quotient map, and it is the coreflection map to L from realcompact completely regular frames. The right adjoint of υL is given by

$$(\upsilon_L)_*(a) = \{c \in \mathrm{Coz}\, L \mid c \leq a\}.$$

2.4 The paracompact coreflection One way of describing the paracompact coreflection of L, denoted πL, is given in [5, Proposition 6]. For our purposes it suffices to recall that there is a nucleus $p\colon \lambda L \to \lambda L$ such that $\pi L = \mathrm{Fix}(p)$. As in the other cases, we write $p_L\colon \lambda L \to \pi L$ for the associated quotient map. The join map $\pi_L\colon \pi L \to L$ is a dense quotient map, and it is the coreflection map to L from paracompact frames. We describe its right adjoint in the next paragraph.

Summarizing what is mentioned above, let $k_L\colon \beta L \to \lambda L$ be the map given by $J \mapsto \langle J\rangle_\sigma$, where $\langle J\rangle_\sigma$ designates the σ-ideal of $\mathrm{Coz}\, L$ generated by J. Then k_L is a dense quotient map. We therefore have the commutative diagram



(‡)

where each morphism is a dense quotient map. Since the composite of any two dense homomorphisms is dense, the composite of any composable maps in diagram (‡) is a dense quotient map. From the commutativity of the lower triangle in this diagram, we deduce that

$$(\lambda_L)_* = (p_L)_* \circ (\pi_L)_*.$$

Since p_L arises from a nucleus, its right adjoint is the identical embedding $\pi L \rightarrowtail \lambda L$. A consequence of this is that

$$(\pi_L)_* = (\lambda_L)_* = (\upsilon_L)_*.$$

2.5 Sublocales Throughout, we use the terminology and notation of [17] regarding sublocales. We thus denote by $\mathcal{S}(L)$ the co-frame of sublocales of L. Whenever we speak of a join of sublocales, the join will be meant in this lattice. Let us highlight just a few matters that are of relevance for our purposes in this paper.

The open (resp. closed, resp. Boolean) sublocale of L associated to an element a of L is given by

$$\mathfrak{o}_L(a) = \{a \to x \mid x \in L\} = \{x \mid x = a \to x\},$$

$$\mathfrak{c}_L(a) = {\uparrow}a = \{x \in L \mid x \geq a\} \quad \text{and} \quad \mathfrak{b}_L(a) = \{a \to x \mid x \in L\};$$

with the subscript dropped if the context is clear. Let us recall that $\mathfrak{b}(a)$ is the smallest sublocale of L containing a.

As usual, by a *point* of L we mean an element $p \in L$ such that $p < 1$ and, for all $x, y \in L$,

$$x \wedge y \leq p \quad \Longrightarrow \quad x \leq p \ \text{ or } \ y \leq p.$$

Equivalently, the inequalities above can be replaced with equalities. Points are also called *prime elements*. The set of all points of L will be denoted by $\mathrm{Pt}(L)$. A *one-point sublocale* of L is a sublocale of the form $\mathfrak{b}(p) = \{p, 1\}$, for $p \in \mathrm{Pt}(L)$.

A *maximal element* of L is an element which is maximal in the poset $L \smallsetminus \{1\}$. Maximal elements are points in any frame. In regular frames maximal elements are exactly the points of the frame. A consequence of this is that, if L is a regular frame, then for any $p \in \mathrm{Pt}(L)$, $\mathfrak{b}(p) = \mathfrak{c}(p)$.

We write S° for the *interior* of a sublocale S, and $\overline{S}$ for the closure of S. We recall that, for any $a \in L$,

$$\mathfrak{c}(a)^\circ = \mathfrak{o}(a^*) \qquad \text{and} \qquad \overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*).$$

A frame is called *spatial* if it is isomorphic to $\Omega(X)$, the frame of open subsets of X, for some topological space X. An internal characterization is that each element is a meet of points. To say a sublocale is spatial means that it is spatial as a frame. Because of the way joins are calculated in $\mathcal{S}(L)$; namely,

$$\bigvee_{i\in I} S_i = \left\{ \bigwedge M \mid M \subseteq \bigcup_{i\in I} S_i \right\},$$

and because the points of a sublocale are exactly the points of the containing frame that belong to the sublocale, that is, for any $S \in \mathcal{S}(L)$,

$$\mathrm{Pt}(S) = S \cap \mathrm{Pt}(L),$$

it is easy to see that the join of spatial sublocales is a spatial sublocale.

2.6 The ring $\mathcal{R}L$ The ring $\mathcal{R}L$ has as its elements frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, where $\mathfrak{L}(\mathbb{R})$ denotes the frame of reals. We refer to [2] concerning the ring $\mathcal{R}L$, the cozero map $\mathrm{coz}\colon \mathcal{R}L \to L$, and the properties of the cozero map. As in this reference, we will denote its elements with lower case Greek letters. These rings are also discussed in Chapter XIV of [17]. We recall, in particular, that a frame homomorphism $h\colon M \to L$ induces a ring homomorphism $\mathcal{R}h\colon \mathcal{R}M \to \mathcal{R}L$ given by $\mathcal{R}h(\alpha) = h \circ \alpha$, and for which $\mathrm{coz}(\mathcal{R}h(\alpha)) = h(\mathrm{coz}\,\alpha)$.

Concerning rings generally, our usage of the term "ideal" does not exclude the entire ring. We write $\mathrm{Idl}(A)$ for the poset of ideals of a ring A ordered by inclusion.

3 Ideals from sublocales

We now come to the main theme of this paper; namely, ideals of $\mathcal{R}L$ associated with sublocales of βL. These ideals were introduced by Dube [9] as a generalization of similarly defined ideals of $C(X)$. We recall how they are defined.

For each sublocale S of βL, the ideals $\boldsymbol{M}^S$ and $\boldsymbol{O}^S$ of $\mathcal{R}L$ are defined by

$$\boldsymbol{M}^S = \{\alpha \in \mathcal{R}L \mid S \subseteq \mathfrak{c}_{\beta L}(r_L(\mathrm{coz}\,\alpha))\}$$

and

$$\boldsymbol{O}^S = \{\alpha \in \mathcal{R}L \mid S \subseteq \mathrm{int}\,\mathfrak{c}_{\beta L}(r_L(\mathrm{coz}\,\alpha))\}.$$

In light of the description of interiors of closed sublocales, and taking into account the fact that r_L commutes with pseudocomplementation, that is, $r_L(a)^* = r_L(a^*)$ for all $a \in L$, we have the equality

$$\boldsymbol{O}^S = \{\alpha \in \mathcal{R}L \mid S \subseteq \mathfrak{o}_{\beta L}(r_L(\mathrm{coz}\,\alpha)^*)\}.$$

In the event that S is a closed sublocale, say $S = \mathfrak{c}_{\beta L}(I)$ for some $I \in \beta L$, then

$$\boldsymbol{M}^{\mathfrak{c}_{\beta L}(I)} = \{\alpha \in \mathcal{R}L \mid r_L(\mathrm{coz}\,\alpha) \subseteq I\}$$

and

$$\boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)} = \{\alpha \in \mathcal{R}L \mid r_L(\mathrm{coz}\,\alpha) \prec\!\!\prec I\}.$$

If A is a sublocale of L (and hence $r_L[A]$ is a sublocale of βL), the ideals $\boldsymbol{M}^{r_L[A]}$ and $\boldsymbol{O}^{r_L[A]}$ are denoted by $\boldsymbol{M}_A$ and $\boldsymbol{O}_A$, respectively, and, as observed in [9], are expressible only in terms of L without invoking βL as follows:

$$\boldsymbol{M}_A = \{\alpha \in \mathcal{R}L \mid A \subseteq \mathfrak{c}_L(\mathrm{coz}\,\alpha)\} \text{ and } \boldsymbol{O}_A = \{\alpha \in \mathcal{R}L \mid A \subseteq \mathfrak{o}_L((\mathrm{coz}\,\alpha)^*)\}.$$

As above, if A is a closed sublocale of L, say $A = \mathfrak{c}_L(a)$ for some $a \in L$, then

$$\boldsymbol{M}_{\mathfrak{c}_L(a)} = \{\alpha \in \mathcal{R}L \mid \mathrm{coz}\,\alpha \le a\} \quad \text{and} \quad \boldsymbol{O}_{\mathfrak{c}_L(a)} = \{\alpha \in \mathcal{R}L \mid \mathrm{coz}\,\alpha \prec a\}.$$

It is clear that for any sublocale S of βL,

$$\boldsymbol{O}^S \subseteq \boldsymbol{M}^S = \boldsymbol{M}^{\overline{S}}.$$

Equally clear is that for any two sublocales $S \subseteq T$ of βL,

$$\boldsymbol{O}^T \subseteq \boldsymbol{O}^S \qquad \text{and} \qquad \boldsymbol{M}^T \subseteq \boldsymbol{M}^S.$$

For use below, let us recall from [12, Proposition 3.4(b)] that if A and B are sublocales of βL with $\boldsymbol{O}^A \subseteq \boldsymbol{M}^B$, then $\overline{B} \subseteq \overline{A}$. A consequence of this is that if A and B are *closed* sublocales of βL, with $\boldsymbol{M}^A \subseteq \boldsymbol{M}^B$, then $B \subseteq A$. Hence, for closed sublocales A and B of βL,

$$\boldsymbol{M}^A = \boldsymbol{M}^B \quad \text{iff } A = B.$$

Let us introduce the following notation of convenience. For any frame L, we set

$$\mathrm{Idl}_{\boldsymbol{M}}(\mathcal{R}L) = \{\boldsymbol{M}^A \mid A \in \mathcal{S}(\beta L)\} \text{ and } \mathrm{Idl}_{\boldsymbol{O}}(\mathcal{R}L) = \{\boldsymbol{O}^A \mid A \in \mathcal{S}(\beta L)\}.$$

Since $\boldsymbol{M}^A = \boldsymbol{M}^{\overline{A}}$ for any sublocale A of βL, we see that

$$\mathrm{Idl}_{\boldsymbol{M}}(\mathcal{R}L) = \{\boldsymbol{M}^F \mid F \text{ is a closed sublocale of } \beta L\}.$$

3.1 $\boldsymbol{M}$-ideals as intersections of maximal ideals It is known that in spaces the $\boldsymbol{M}$-ideals of $C(X)$ are precisely the intersections of maximal ideals of $C(X)$. This is so because, for any $A \subseteq \beta(X)$

$$\boldsymbol{M}^A = \bigcap_{p \in A} \boldsymbol{M}^p,$$

and the set of maximal ideals of $C(X)$ is

$$\mathrm{Max}(C(X)) = \{\boldsymbol{M}^p \mid p \in \beta X\}.$$

Regarding pointfree function rings, it is nowhere recorded anywhere in the literature (so far as we are aware) whether or not the $\boldsymbol{M}$-ideals of $\mathcal{R}L$ are also exactly the intersections of maximal ideals. One of our main goals in this section is to show that they are. We will also show that every $\boldsymbol{M}$-ideal indexed by a spatial sublocale is contained in a maximal $\boldsymbol{M}$-ideal. The latter we will show by first characterizing the maximal $\boldsymbol{M}$-ideals.

To get started, let us recall from [10] that the set of maximal ideals of $\mathcal{R}L$ is given by

$$\mathrm{Max}(\mathcal{R}L) = \{\boldsymbol{M}^{\mathfrak{c}_{\beta L}(p)} \mid p \in \mathrm{Pt}(\beta L)\} = \{\boldsymbol{M}^{\mathfrak{b}_{\beta L}(p)} \mid p \in \mathrm{Pt}(\beta L)\}.$$

We also need to recall that for any sublocale $A \subseteq \beta L$, $\boldsymbol{M}^A = \boldsymbol{M}^{\overline{A}}$. Therefore there is no loss of generality in considering the ideals $\boldsymbol{M}^A$ only for the closed sublocales A of βL.

Proposition 3.1. *The $\boldsymbol{M}$-ideals of $\mathcal{R}L$ are precisely the intersections of maximal ideals of $\mathcal{R}L$.*

Proof. Let A be a closed sublocale of βL. Since every complemented sublocale of a spatial frame is spatial, $A = \bigvee\{\mathfrak{b}_{\beta L}(p) \mid p \in \mathrm{Pt}(A)\}$.

Therefore

$$\boldsymbol{M}^A = \boldsymbol{M}^{\bigvee\{\mathfrak{b}_{\beta L}(p) \mid p\in \mathrm{Pt}(A)\}} = \bigcap_{p\in\mathrm{Pt}(A)} \boldsymbol{M}^{\mathfrak{b}_{\beta L}(p)},$$

showing that every $\boldsymbol{M}$-ideal is an intersection of maximal ideals.

The other inclusion follows from the fact that for any collection $\{S_i\}$ of sublocales, we have the equality $\bigcap_i \boldsymbol{M}^{S_i} = \boldsymbol{M}^T$, where $T = \bigvee_i S_i$. □

Strictly speaking, L is not a sublocale of βL, but can be viewed as a sublocale of βL by identifying it with the sublocale $r_L[L]$ of βL, via the localic isomorphism

$$a \mapsto r_L(a)\colon L \to r_L[L].$$

Thus viewed, we can then say among the $\boldsymbol{M}$-ideals of $\mathcal{R}L$ are those which are indexed by sublocales of L. More precisely, as was recalled above, they are the $\boldsymbol{M}$-ideals of $\mathcal{R}L$ of the form $\boldsymbol{M}^{r_L[S]}$, for $S \in \mathcal{S}(L)$. Our abridged notation and description for them (again as recalled above) is

$$\boldsymbol{M}_S = \boldsymbol{M}^{r_L[S]} = \{\alpha \in \mathcal{R}L \mid S \subseteq \mathfrak{c}_L(\mathrm{coz}\,\alpha)\}.$$

Collectively, we give them the following name.

Definition 3.2. We say an $\boldsymbol{M}$-ideal of $\mathcal{R}L$ is *grounded* in case it is of the form $\boldsymbol{M}_S$ for some sublocale S of L. We denote by $\mathrm{Idl}_{g\boldsymbol{M}}(\mathcal{R}L)$ the set of grounded $\boldsymbol{M}$-ideals of $\mathcal{R}L$.

We wish to explore some properties of grounded $\boldsymbol{M}$-ideals. We shall start by showing that if $\mathrm{Pt}(L) \neq \emptyset$, then among the maximal ideals of $\mathcal{R}L$ there are grounded ones, and they are precisely those indexed by the one-point sublocales of L. Intuitively, this should be so because (viewing L as a sublocale of βL) the points of L are exactly the points of βL that belong to L. A rigorous proof is however needed. To present it, we need some preliminary observations, which we record in the following lemma. They might very well be known, but we shall present proofs since we do not have a reference for them.

Lemma 3.3. *Let $f\colon L \to M$ be a localic map. Then:*

(a) *For any $S \in \mathcal{S}(L)$,*

$$\overline{f[S]} = \mathfrak{c}_M\big(f\big(\bigwedge S\big)\big).$$

(b) *If f is one-one and $a \in L$ is such that $f(a) \in \mathrm{Pt}(M)$, then $a \in \mathrm{Pt}(L)$.*

Proof. (a) Using the formula for calculating the closure of a sublocale, and using the fact that a localic map preserves meets, we see that

$$\overline{f[S]} = \mathfrak{c}_M\Big(\bigwedge f[S]\Big) = \mathfrak{c}_M\Big(f\Big(\bigwedge S\Big)\Big);$$

as claimed.

(b) Consider any $x, y \in L$ with $x \wedge y \leq a$. Then, $f(x) \wedge f(y) \leq f(a)$. Since $f(a)$ is a point, we have $f(x) \leq f(a)$ or $f(y) \leq f(a)$, which implies $x \leq a$ or $y \leq a$ since f is one-one. Therefore a is a point in L. □

The result in part (a) of this lemma enables us to give an alternative description of the grounded $\boldsymbol{M}$-ideals of $\mathcal{R}L$.

Corollary 3.4. *For any L, $\mathrm{Idl}_{g\boldsymbol{M}}(\mathcal{R}L) = \big\{\boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L(a))} \mid a \in L\big\}$.*

Proof. This is so because for any $S \in \mathcal{S}(L)$, $\boldsymbol{M}_S = \boldsymbol{M}_{\overline{S}}$, and for any $x \in L$,

$$\boldsymbol{M}_{\mathfrak{c}_L(x)} = \boldsymbol{M}^{r_L[\mathfrak{c}_L(x)]} = \boldsymbol{M}^{\overline{r_L[\mathfrak{c}_L(x)]}} = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L(x))};$$

the last equality holding in view of Lemma 3.3(a). □

Recall the description of maximal ideals of $\mathcal{R}L$ that is recited in the displayed equalities just before the statement of Proposition 3.1.

Theorem 3.5. *If L has points, then among the maximal ideals of $\mathcal{R}L$ there are grounded ones. Furthermore, they are precisely those indexed by the one-point sublocales of L.*

Proof. For any $q \in \mathrm{Pt}(L)$,

$$\boldsymbol{M}_{\mathfrak{c}_L(q)} = \boldsymbol{M}^{r_L[\mathfrak{c}_L(q)]} = \boldsymbol{M}^{\overline{r_L[\mathfrak{c}_L(q)]}} = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L(q))};$$

the last equality holding in view of Lemma 3.3(a) since $\bigwedge \mathfrak{c}_L(q) = q$. Therefore $\boldsymbol{M}_{\mathfrak{c}_L(q)}$ is a maximal ideal because $r_L(q)$ is a point of βL. This proves the first assertion in the statement of the theorem.

To prove the second assertion, we must show that any grounded maximal ideal is indexed by a one-point sublocale of L. Suppose, then, that J is a grounded maximal ideal of $\mathcal{R}L$. Then there exists a point $p \in \beta L$ and a sublocale S of L such that

$$J = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(p)} = \boldsymbol{M}_S.$$

Thus,

$$\boldsymbol{M}^{\mathfrak{c}_{\beta L}(p)} = \boldsymbol{M}^{r_L[S]} = \boldsymbol{M}^{\overline{r_L[S]}} = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L(\bigwedge S))};$$

the last equality in light of Lemma 3.3(a). Since $\mathfrak{c}_{\beta L}(p)$ and $\mathfrak{c}_{\beta L}(r_L(\bigwedge S))$ are closed sublocales, they coincide, and hence $r_L(\bigwedge S) = p$, which implies that $\bigwedge S$ is point in L by Lemma 3.3(b) since r_L is one-one. Now, a calculation as in the displayed string of equalities in the first sentence of the proof of the first part shows that $J = \boldsymbol{M}_{\mathfrak{c}_L(\bigwedge S)}$, which is a grounded maximal ideal indexed by the one-point sublocale $\mathfrak{c}_L(\bigwedge S)$ of L. □

We denote by $\mathrm{Max}_g(\mathcal{R}L)$ the set of all grounded maximal ideals of $\mathcal{R}L$. Thus,

$$\mathrm{Max}_g(\mathcal{R}L) = \{\boldsymbol{M}_{\mathfrak{c}_L(q)} \mid q \in \mathrm{Pt}(L)\}.$$

Since a sublocale of L may fail to have any point (for instance if $L = \mathfrak{B}(\mathfrak{O}\mathbb{R})$), we see that a grounded $\boldsymbol{M}$-ideal need not be a non-void intersection of grounded maximal ideals. Naturally, one asks if for $S \subseteq L$ spatial, $\boldsymbol{M}_S$ is an intersection of grounded maximal ideals. This is indeed the case, and more, as we show in the following theorem.

Theorem 3.6. *The $\boldsymbol{M}$-ideals of $\mathcal{R}L$ indexed by the spatial sublocales of L are precisely the intersections of grounded maximal ideals of $\mathcal{R}L$.*

Proof. Suppose, first, that S is a spatial sublocale of L. Then S is a join of its one-point sublocales. That is,

$$S = \bigvee \{\mathfrak{b}(q) \mid q \in \mathrm{Pt}(S)\}.$$

Therefore,

$$\begin{aligned}
\boldsymbol{M}_S &= \bigcap\{\boldsymbol{M}_{\mathfrak{b}_L(q)} \mid q \in \mathrm{Pt}(S)\} \\
&= \bigcap\{\boldsymbol{M}^{r_L[\mathfrak{b}_L(q)]} \mid q \in \mathrm{Pt}(S)\} \\
&= \bigcap\{\boldsymbol{M}^{\overline{r_L[\mathfrak{b}_L(q)]}} \mid q \in \mathrm{Pt}(S)\} \\
&= \bigcap\{\boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L(q))} \mid q \in \mathrm{Pt}(S)\} \quad \text{since } \bigwedge \mathfrak{b}_L(q) = q \\
&= \bigcap\{\boldsymbol{M}_{\mathfrak{c}_L(q)} \mid q \in \mathrm{Pt}(S)\}.
\end{aligned}$$

This shows that $\boldsymbol{M}_S$ is an intersection of grounded maximal ideals.

On the other hand, let $\{q_i \mid i \in I\}$ be a collection of points of L, and consider the intersection $\bigcap_{i\in I} \boldsymbol{M}_{\mathfrak{c}_L(q_i)}$ of grounded maximal ideals. We have the equalities

$$\bigcap_{i\in I} \boldsymbol{M}_{\mathfrak{c}_L(q_i)} = \bigcap_{i\in I} \boldsymbol{M}_{\mathfrak{b}_L(q_i)} = \boldsymbol{M}_{\bigvee_{i\in I} \mathfrak{b}_L(q_i)}.$$

So it suffices to show that the sublocale $\bigvee_{i\in I} \mathfrak{b}_L(q_i)$ of L is spatial. Since each $\mathfrak{b}_L(q_i) = \{1, q_i\}$, if $x \in \bigvee \mathfrak{b}_L(q_i)$ and $x \neq 1$, there is a subset $K \subseteq I$ and indices i_k, for $k \in K$, such that for each $k, q_{i_k} \neq 1$ and $x = \bigwedge_k q_{i_k}$. This shows that x is a meet of points of L, and hence points of $\bigvee_{i\in I} \mathfrak{b}_L(q_i)$. Therefore $\bigvee_{i\in I} \mathfrak{b}_L(q_i)$ is spatial, as required. □

Remark 3.7. In the last part of the proof, that $\bigvee_{i\in I} \mathfrak{b}_L(q_i)$ is a spatial sublocale could also have been deduced from the fact that the join of any family of spatial sublocales is spatial, and that, in a regular frame, each one-point sublocale is the two-element chain, and hence spatial.

Since the mapping

$$a \mapsto \mathfrak{c}_{\beta L}(r_L(a)) \colon L \to \mathcal{S}(\beta L)$$

is one-one (because both r_L and $\mathfrak{c}_{\beta L}(-)$ are one-one), we see from Corollary 3.4 that the grounded $\boldsymbol{M}$-ideals of $\mathcal{R}L$ are in a bijective correspondence with the elements of L. Since the $\boldsymbol{M}$-ideals are exactly the ideals $\boldsymbol{M}^F$, for F a closed sublocale of βL, it follows from Corollary 3.4 that every $\boldsymbol{M}$-ideal is grounded if and only if

$$(\forall I \in \beta L)(\exists a \in L) \text{ such that } \boldsymbol{M}^{\mathfrak{c}_{\beta L}(I)} = \boldsymbol{M}^{\mathfrak{c}_{\beta L}(r_L(a))}.$$

This, in turn, holds if and only if every element of βL equals $r_L(x)$ for some $x \in L$. This says the localic map $r_L \colon L \to \beta L$ must be onto. Since it is always one-one, this says it must be an isomorphism in **Loc**. In all, then, we deduce the following.

Corollary 3.8. *Every $\boldsymbol{M}$-ideal of $\mathcal{R}L$ is grounded iff L is compact.*

Up to this point in this section we have so far spoken only about the $\boldsymbol{M}$-ideals. We shall now address the $\boldsymbol{O}$-ideals. To set the tone, we recall from [11, Lemma 2.1] that, for any closed sublocale A of βL, $\boldsymbol{O}^A = m\boldsymbol{M}^A$. That is, the $\boldsymbol{O}$-ideals indexed by the closed sublocales of βL are precisely the pure parts of the $\boldsymbol{M}$-ideals. Having observed that the $\boldsymbol{M}$-ideals are precisely the intersections of the maximal ideals, it is reasonable to expect the $\boldsymbol{O}$-ideals to be exactly the intersections of the pure parts of the maximal ideals. We shall show that this is indeed the case.

For use in the upcoming proof, let us state as a lemma the following result which is proved in [10, page 12].

Lemma 3.9. *Let I be an element of βL. Then*

$$\boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)} = \bigcap\{\boldsymbol{O}^{\mathfrak{c}_{\beta L}(p)} \mid p \in \mathrm{Pt}(\beta L) \text{ and } p \geq I\}.$$

Here is the theorem we have been aiming for.

Theorem 3.10. *The $\boldsymbol{O}$-ideals of $\mathcal{R}L$ indexed by the closed sublocales of βL are precisely the intersections of the pure parts of the maximal ideals of $\mathcal{R}L$.*

Proof. Let A be a closed sublocale of βL. Then $A = \mathfrak{c}_{\beta L}(I)$ for some $I \in \beta L$. Since, for any $p \in \mathrm{Pt}(\beta L)$, $\boldsymbol{O}^{\mathfrak{c}_{\beta L}(p)}$ is the pure part of the maximal ideal $\boldsymbol{M}^{\mathfrak{c}_{\beta L}(p)}$, it follows from Lemma 3.9 that $\boldsymbol{O}^A$ is the intersection of the pure parts of some maximal ideals of $\mathcal{R}L$.

Conversely, let $\{p_i \mid i \in J\}$ be a collection of points of βL, and set $I = \bigwedge p_i$. Observe that the set $\{p_i \mid i \in I\}$ is exactly the set of all the points of βL that are above I, because if q is any point of βL with $q \geq I$, then for any $i \in J$, $q \geq p_j$, which implies $q = p_j$ since p_j is a maximal element. Therefore Lemma 3.9 yields

$$\bigcap_{i \in I} \boldsymbol{O}^{\mathfrak{c}_{\beta L}(p_i)} = \boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)};$$

and this completes the proof. □

When we were dealing with $\boldsymbol{M}$-ideals above, we used extensively the fact that it sufficed to limit to closed sublocales of βL. This is not the case with $\boldsymbol{O}$-ideals, for the following reasons. Recall that a frame L is called *basically disconnected* if $c^* \vee c^{**} = 1$ for every $c \in \operatorname{Coz} L$.

(a) The pure ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{O}^F$ for F a closed sublocale of L (see [7, Proposition 4.3]).

(b) Every $\boldsymbol{O}$-ideal of $\mathcal{R}L$ is pure if and only if L is basically disconnected (see [11, Theorem 3.1]).

We will prove the result concerning groundedness of the $\boldsymbol{O}$-ideals associated with closed sublocales of βL. For that we will need the following preliminary results.

Lemma 3.11. *For any sublocale A of βL, $\boldsymbol{O}^A$ is pure iff $\boldsymbol{O}^A = \boldsymbol{O}^{\overline{A}}$.*

We also need the following lemma which, incidentally, will also be useful later on. We recall from [12, Proposition 3.4(b)] that if A and B are sublocales of βL with $\boldsymbol{O}^A \subseteq \boldsymbol{M}^B$, then $\overline{B} \subseteq \overline{A}$.

Lemma 3.12. *For any sublocales A and B of βL, if $\boldsymbol{O}^A = \boldsymbol{O}^B$, then $\overline{A} = \overline{B}$. In particular, if A and B are closed, then $A = B$.*

Proof. Since $\boldsymbol{O}^A = \boldsymbol{O}^B \subseteq \boldsymbol{M}^B$, the result cited above from [12] yields $\overline{B} \subseteq \overline{A}$. By symmetry, we also have the reverse inclusion, hence the claimed equality. □

Definition 3.13. An $\boldsymbol{O}$-ideal of $\mathcal{R}L$ is *grounded* if it is of the form $\boldsymbol{O}_S$ for some $S \in \mathcal{S}(L)$. We denote the set of all grounded $\boldsymbol{O}$-ideals of $\mathcal{R}L$ by $\operatorname{Idl}_{g\boldsymbol{O}}(\mathcal{R}L)$.

Lemma 3.14. *For any L, $\operatorname{Idl}_{g\boldsymbol{O}}(\mathcal{R}L) = \left\{\boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(a))} \mid a \in L\right\}$.*

Proof. Let P be a grounded $\boldsymbol{O}$-ideal of $\mathcal{R}L$. Then there exists $I \in \beta L$ and a sublocale S of L such that

$$P = \boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)} = \boldsymbol{O}_S.$$

Thus

$$\boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)} = \boldsymbol{O}^{r_L(S)} = \boldsymbol{O}^{\overline{r_L[S]}} = \boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(\bigwedge S))}.$$

The middle equalities are obtained using Lemma 3.11, and since $\mathfrak{c}_{\beta L}(I)$ and $\mathfrak{c}_{\beta L}(r_L(\bigwedge S))$ are closed sublocales of βL, Lemma 3.12 implies that they are equal. Now, for any $J \in \beta L$ with

$$\boldsymbol{O}^{\mathfrak{c}_{\beta L}(I)} = \boldsymbol{O}^{\mathfrak{c}_{\beta L}(J)},$$

there is some sublocale T of L such that

$$\boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(\bigwedge S))} = \boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(\bigwedge T))}.$$

This, in turn, implies that

$$I = r_L(\bigwedge S) = r_L(\bigwedge T) = J.$$

Since r_L is one-to-one, $\bigwedge S = \bigwedge T$ must be an element of L. So that $\boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(\bigwedge S))} = \boldsymbol{O}^{\mathfrak{c}_{\beta L}(r_L(a))}$ for some $a \in L$. □

3.2 A word on maximality Given a sublocale A of βL, we say $\boldsymbol{O}^A$ is a *maximal* $\boldsymbol{O}$*-ideal* if, for any sublocale B of βL, the containments $\boldsymbol{O}^A \subseteq \boldsymbol{O}^B \subset \mathcal{R}L$ imply that $\boldsymbol{O}^A = \boldsymbol{O}^B$. A *maximal* $\boldsymbol{M}$*-ideal* is defined similarly. Thus, an $\boldsymbol{O}$-ideal is a maximal $\boldsymbol{O}$-ideal if it is a maximal element in the poset of $\boldsymbol{O}$-ideals which are proper ideals, ordered by inclusion.

Let us recall from [12, Proposition 3.4(b)] that if A and B are sublocales of βL with $\boldsymbol{O}^A \subseteq \boldsymbol{M}^B$, then $\overline{B} \subseteq \overline{A}$. Consequently, if $\boldsymbol{O}^A \subseteq \boldsymbol{O}^B$, then also $\overline{B} \subseteq \overline{A}$ because of the containments $\boldsymbol{O}^A \subseteq \boldsymbol{O}^B \subseteq \boldsymbol{M}^B$.

Since the maximal ideals of $\mathcal{R}L$ are precisely the $\boldsymbol{M}$-ideals associated to one-point sublocales of βL, every $\boldsymbol{M}$-ideal is contained in a maximal $\boldsymbol{M}$-ideal. It is thus natural to ask if every $\boldsymbol{O}$-ideal is contained in a maximal $\boldsymbol{O}$-ideal. We show below that each $\boldsymbol{O}$-ideal associated to a sublocale that has points is contained in a maximal $\boldsymbol{O}$-ideal.

A straightforward calculation shows that, for any sublocale $A \subseteq \beta L$,

$$\boldsymbol{O}^A = \mathcal{R}L \quad \text{iff} \quad \boldsymbol{M}^A = \mathcal{R}L \quad \text{iff} \quad A = \mathsf{O}.$$

We shall use this fact below.

Theorem 3.15. *If $A \in \mathcal{S}(\beta L)$ has at least one point, then $\boldsymbol{O}^A$ is contained in a maximal $\boldsymbol{O}$-ideal.*

Proof. Let us show first that if S is a one-point sublocale of βL, then $\boldsymbol{O}^S$ is a maximal $\boldsymbol{O}$-ideal of $\mathcal{R}L$. Pick a $p \in \mathrm{Pt}(\beta L)$ such that $S = \mathfrak{c}_{\beta L}(p)$. Consider any $T \in \mathcal{S}(\beta L)$ with $\boldsymbol{O}^S \subseteq \boldsymbol{O}^T \subset \mathcal{R}L$. Then $T \neq \mathsf{O}$. By what we observed above, $\overline{T} \subseteq \overline{S} = S$, which implies that $T \subseteq S$. Since T is not the void-sublocale and S is a one-point sublocale, it follows that $T = S$, whence $\boldsymbol{O}^S = \boldsymbol{O}^T$. Therefore $\boldsymbol{O}^S$ is a maximal $\boldsymbol{O}$-ideal.

Now, given that A has at least one point, let p be a point of βL belonging to A. Then $\mathfrak{c}_{\beta L}(p) \subseteq A$, and so $\boldsymbol{O}^A$ is contained in the maximal $\boldsymbol{O}$-ideal $\boldsymbol{O}^{\mathfrak{c}_{\beta L}(p)}$. □

Call a pure ideal of a ring *maximal pure* if it is maximal among pure ideals that are proper ideals. Since the pure ideals of $\mathcal{R}L$ are exactly the $\boldsymbol{O}$-ideals associated with closed sublocales of βL, and since complemented (and hence closed) sublocales of a spatial frame are spatial (and hence have points if they are non-void), we deduce from Theorem 3.15 the following result.

Corollary 3.16. *Every proper pure ideal of $\mathcal{R}L$ is contained in a maximal pure ideal.*

We apply this to $C(X)$. Recall that for any Tychonoff space X, the rings $C(X)$ and $\mathcal{R}(\Omega(X))$ are isomorphic. Every ring isomorphism preserves (under direct image) purity; so

> *Every proper pure ideal of $C(X)$ is contained in a maximal pure ideal.*

4 Ideals associated with sublocales of λL

In [12], the authors considered one of these types of ideals (the $\boldsymbol{O}$-ideals) associated with sublocales of λL. The reason for concentrating only on just the one type of ideals, and only for λL, is apparent when one reads the mentioned paper. To recall, for any sublocale A of λL, the ideal $\boldsymbol{N}^A$ of $\mathcal{R}L$ (as defined in [12]) is the set

$$\boldsymbol{N}^A = \{\alpha \in \mathcal{R}L \mid A \subseteq \mathfrak{o}_{\lambda L}(\varrho_L(\operatorname{coz}\alpha)^*)\}.$$

Without introducing too many symbols, we wish to have a uniform notation that will enable us to define most economically $\boldsymbol{O}$- and $\boldsymbol{M}$-ideals of

$\mathcal{R}L$ indexed by sublocales of λL, υL, and πL, respectively. So let γ denote any of the coreflectors above; that is, let $\gamma \in \{\beta, \lambda, \upsilon, \pi\}$. Then γL has the obvious meaning. Since each of the coreflection maps $\gamma L \to L$ is a join map, we shall denote each by $j_\gamma \colon \gamma L \to L$, and the right adjoint by r_γ. Recall from the previous section that $r_\lambda = r_\upsilon = r_\pi$.

Definition 4.1. For any sublocale A of γL, the ideals $\boldsymbol{O}_\gamma^A$ and $\boldsymbol{M}_\gamma^A$ are defined by

$$\boldsymbol{O}_\gamma^A = \{\alpha \in \mathcal{R}L \mid A \subseteq \mathfrak{o}_{\gamma L}(r_\gamma(\operatorname{coz}\alpha)^*)\}$$

and

$$\boldsymbol{M}_\gamma^A = \{\alpha \in \mathcal{R}L \mid A \subseteq \mathfrak{c}_{\gamma L}(r_\gamma(\operatorname{coz}\alpha))\}.$$

That each of these sets is indeed an ideal of $\mathcal{R}L$ is verified routinely using the properties of the cozero map $\operatorname{coz}\colon \mathcal{R}L \to L$. A few comments before we proceed are in order.

(a) For any $a \in L$, write $[a]$ for the set $\{c \in \operatorname{Coz} L \mid c \leq a\}$ – a notation which is sometimes used in the context of λL. Then, recalling that all the r_γ coincide (except for $\gamma = \beta$), we observe that, for any $\alpha \in \mathcal{R}L$,

$$\alpha \in \boldsymbol{O}_\gamma^A \quad\Longleftrightarrow\quad A \subseteq \mathfrak{o}_{\gamma L}([(\operatorname{coz}\alpha)^*]),$$

and similarly for $\boldsymbol{M}_\gamma^A$.

(b) Unadorned, $\boldsymbol{O}^A$ and $\boldsymbol{M}^A$ will have the meanings we have recited above from [9].

(c) Although the descriptions in Definition 4.1 indicate that the ring whose ideal is under consideration is $\mathcal{R}L$, our notation suppresses the role of L, and may thus be ambiguous if there are two or more function rings under consideration. In such instances, we shall write, for instance, $\boldsymbol{O}_{(L,\gamma)}^A$ and $\boldsymbol{M}_{(L,\gamma)}^A$.

We now want to find some relations between these ideals. Since both υL and πL are sublocales (and not merely isomorphic to sublocales) of λL, if A is a sublocale of either of these, then A is also a sublocale of λL. Recall that if A is a sublocale of L and $a \in A$, then for the open sublocale $\mathfrak{o}_A(a)$ of A we have

$$\mathfrak{o}_A(a) = A \cap \mathfrak{o}_L(a),$$

and similarly for the closed sublocale because the associated quotient map $\nu_A \colon L \to A$ fixes elements of A.

Proposition 4.2. *For any completely regular frame L we have the following.*

(a) *For any sublocale A of υL, $\boldsymbol{O}^A_\upsilon = \boldsymbol{O}^A_\lambda$ and $\boldsymbol{M}^A_\upsilon = \boldsymbol{M}^A_\lambda$.*

(b) *For any sublocale B of πL, $\boldsymbol{O}^B_\pi = \boldsymbol{O}^B_\lambda$ and $\boldsymbol{M}^B_\pi = \boldsymbol{M}^B_\lambda$.*

Proof. (a) We prove the result only for the $\boldsymbol{O}$-ideals as the other result can be proved similarly. Let $\alpha \in \boldsymbol{O}^A_\upsilon$. Since υL is a dense sublocale of λL, the pseudocomplement of any element of υL taken in υL is exactly its pseudocomplement taken in λL. Consequently, in view of the fact that $r_\lambda = r_\upsilon$, the relation $\alpha \in \boldsymbol{O}^A_\upsilon$ implies that

$$A \subseteq \mathfrak{o}_{\upsilon L}(r_\upsilon(\operatorname{coz}\alpha)^*) = \upsilon L \cap \mathfrak{o}_{\lambda L}(r_\lambda(\operatorname{coz}\alpha)^*) \subseteq \mathfrak{o}_{\lambda L}(r_\lambda(\operatorname{coz}\alpha)^*),$$

showing that $\alpha \in \boldsymbol{O}^A_\lambda$, whence $\boldsymbol{O}^A_\upsilon \subseteq \boldsymbol{O}^A_\lambda$.

To reverse the inclusion, let $\alpha \in \boldsymbol{O}^A_\lambda$. Then $A \subseteq \mathfrak{o}_{\lambda L}(r_\lambda(\operatorname{coz}\alpha)^*)$, which implies

$$\begin{aligned} A = \upsilon L \cap A \subseteq \upsilon L \cap \mathfrak{o}_{\lambda L}(r_\lambda(\operatorname{coz}\alpha)^*) &= \upsilon L \cap \mathfrak{o}_{\lambda L}(r_\upsilon(\operatorname{coz}\alpha)^*) \\ &= \mathfrak{o}_{\upsilon L}(r_\upsilon(\operatorname{coz}\alpha)^*). \end{aligned}$$

Therefore $\alpha \in \boldsymbol{O}^A_\upsilon$, as desired.

(b) The proof is similar to the one above. □

Remark 4.3. We deliberately elected to give a direct proof for this result. We could also have proved it using [12, Lemma 5.1], which states that:

> *If $h\colon M \to L$ is a surjective frame homomorphism, A is a sublocale of L, and $a \in L$, then $A \subseteq \mathfrak{o}_L(a)$ iff $h_*[A] \subseteq \mathfrak{o}_M(h_*(a))$.*

Applying this lemma to the homomorphism $\ell_L\colon \lambda L \to \upsilon L$, for any sublocale A of υL and any $a \in \upsilon L$, we have

$$A \subseteq \mathfrak{o}_{\upsilon L}(a) \quad \text{iff} \quad (\ell_L)_*[A] \subseteq \mathfrak{o}_{\lambda L}((\ell_L)_*(a)) \quad \text{iff} \quad A \subseteq \mathfrak{o}_{\lambda L}(a)$$

because $(\ell_L)_*$ is the identical embedding $\upsilon L \rightarrowtail \lambda L$. The proposition would then follow from this.

In [12, Proposition 5.2], it is shown that (in our notation), for any sublocale A of λL, $\boldsymbol{O}_\lambda^A = \boldsymbol{O}^{(k_L)_*[A]}$. The argument hinges on the lemma from [12] that we recited in Remark 4.3 and the fact that $k_L \circ r_L = (\lambda_L)_*$. The latter holds simply because k_L is onto.

Now, since $\ell_L \circ k_L$ and $p_L \circ k_L$ are onto, we have

$$(\ell_L \circ k_L) \circ r_L = (\upsilon_L)_* \quad \text{and} \quad (\ell_L \circ p_L) \circ r_L = (\pi_L)_*,$$

and a calculation identical to that in the proof of [12, Proposition 5.2] yields the following result.

Proposition 4.4. *Let L be a completely regular frame.*

(a) *For any sublocale A of υL, $\boldsymbol{O}_\upsilon^A = \boldsymbol{O}^{(k_L)_*[A]}$.*

(b) *For any sublocale B of πL, $\boldsymbol{O}_\pi^B = \boldsymbol{O}^{(k_L)_*[B]}$.*

In [12, Theorem 5.3], the authors prove that L is pseudocompact if and only if, in our notation,

$$\{\boldsymbol{O}^A \mid A \in \mathcal{S}(\beta L)\} = \{\boldsymbol{O}_\lambda^B \mid B \in \mathcal{S}(\lambda L)\}.$$

The proof uses the fact that a frame is pseudocompact precisely when λL is compact. Now, we also have that:

> *A frame L is pseudocompact iff υL is compact* ([5, Proposition 4]) *iff πL is compact* ([6, Proposition 3]).

Minor modifications to the proof of [12, Theorem 5.3] yield the following characterizations. We include the one from [12].

Theorem 4.5. *The following are equivalent for a completely regular frame L.*

(1) *L is pseudocompact.*

(2) $\{\boldsymbol{O}^A \mid A \in \mathcal{S}(\beta L)\} = \{\boldsymbol{O}_\lambda^B \mid B \in \mathcal{S}(\lambda L)\}$.

(3) $\{\boldsymbol{O}^A \mid A \in \mathcal{S}(\beta L)\} = \{\boldsymbol{O}_\upsilon^B \mid B \in \mathcal{S}(\upsilon L)\}$.

(4) $\{\boldsymbol{O}^A \mid A \in \mathcal{S}(\beta L)\} = \{\boldsymbol{O}_\pi^B \mid B \in \mathcal{S}(\pi L)\}$.

A little care is needed in reading this theorem. For instance, looking at conditions (2) and (3), one may erroneously say from it we can deduce that L is pseudocompact if and only if $\{\boldsymbol{O}_\lambda^B \mid B \in \mathcal{S}(\lambda L)\} = \{\boldsymbol{O}_\upsilon^B \mid B \in \mathcal{S}(\upsilon L)\}$. That, however, is not the case. We shall see that the coincidence of these two sets characterizes different types of frames. Of course we always have the containment

$$\{\boldsymbol{O}_\upsilon^B \mid B \in \mathcal{S}(\upsilon L)\} \subseteq \{\boldsymbol{O}_\lambda^B \mid B \in \mathcal{S}(\lambda L)\}$$

in view of Proposition 4.2(a).

We shall need a lemma to prove the theorem that we are aiming for. We recall from [12, Proposition 3.4(b)] that if A and B are sublocales of βL with $\boldsymbol{O}^A \subseteq \boldsymbol{M}^B$, then $\overline{B} \subseteq \overline{A}$.

Lemma 4.6. *For any sublocales A and B of βL, if $\boldsymbol{O}^A = \boldsymbol{O}^B$, then $\overline{A} = \overline{B}$.*

Proof. Since $\boldsymbol{O}^A = \boldsymbol{O}^B \subseteq \boldsymbol{M}^B$, the result cited above yields $\overline{B} \subseteq \overline{A}$. By symmetry, we also have the reverse inclusion, hence the claimed equality. □

Although the elements of λL and υL are ideals (and hence sets) we shall write them as lower case letters in the following proof. Recall that in **CRFrm** a homomorphism is one-one if and only if it is *codense*, meaning that it is only the top element that it sends to the top element. As is well known, this holds already in **RFrm**. Let us also remind the reader that, as observed in [12], if A and B are closed sublocales of βL and $\boldsymbol{O}^A = \boldsymbol{O}^B$, then $A = B$.

Theorem 4.7. *The following are equivalent for a completely regular frame L.*

(1) $\{\boldsymbol{O}_\lambda^B \mid B \in \mathcal{S}(\lambda L)\} = \{\boldsymbol{O}_\upsilon^A \mid A \in \mathcal{S}(\upsilon L)\}$.
(2) $\{\boldsymbol{O}_\lambda^B \mid B \in \mathcal{S}(\lambda L)$ *and B is closed*$\} = \{\boldsymbol{O}_\upsilon^A \mid A \in \mathcal{S}(\upsilon L)\}$.
(3) $\upsilon L = \lambda L$.

Proof. That condition (1) implies (2) is trivial, and it should be obvious that condition (3) implies (1). So, to be done, we must show that condition (2) implies (3). Suppose, then, that condition (2) holds. We prove (3) by showing that ℓ_L is codense, which will make it one-one, whence the result will follow. Suppose, then, that a is an element of λL with $\ell_L(a) = 1_{\upsilon L}$.

By the hypothesis in condition (2), for the closed sublocale $\mathfrak{c}_{\lambda L}(a)$, there is a sublocale A of υL such that $\boldsymbol{O}_{\lambda}^{\mathfrak{c}_{\lambda L}(a)} = \boldsymbol{O}_{\upsilon}^{A}$. In light of Proposition 4.2 above and [12, Proposition 5.2], this implies

$$\boldsymbol{O}^{(k_L)_*[\mathfrak{c}_{\lambda L}(a)]} = \boldsymbol{O}^{(k_L)_*[A]},$$

and hence, with the closure taken in βL,

$$\boldsymbol{O}^{\overline{(k_L)_*[\mathfrak{c}_{\lambda L}(a)]}} = \boldsymbol{O}^{\overline{(k_L)_*[A]}},$$

by virtue of Lemma 4.6. Therefore $\overline{(k_L)_*[\mathfrak{c}_{\lambda L}(a)]} = \overline{(k_L)_*[A]}$. Calculating these closures, keeping in mind the fact that localic maps preserve meets and that $(k_L)_*$ is one-one, we obtain

$$\begin{aligned}\bigwedge (k_L)_*[\mathfrak{c}_{\lambda L}(a)] = \bigwedge (k_L)_*[A] \quad &\Longrightarrow \quad (k_L)_*\Big(\bigwedge \mathfrak{c}_{\lambda L}(a)\Big) = (k_L)_*\Big(\bigwedge A\Big)\\ &\Longrightarrow \quad (k_L)_*(a) = (k_L)_*\Big(\bigwedge A\Big)\\ &\Longrightarrow \quad a = \bigwedge A.\end{aligned}$$

Consequently, $a \in A$, and so $a \in \upsilon L$ since $A \subseteq \upsilon L$. But ℓ_L fixes elements of υL because $\upsilon L = \mathrm{Fix}(\ell)$, so we deduce that $a = 1_{\lambda L}$, which proves the result. □

It is proper that we point the reader to the article [8] where other characterizations of the frames L with $\lambda L = \upsilon L$ are presented. Since πL is a sublocale of λL whose elements are those fixed by the nucleus $p\colon \lambda L \to \lambda L$ alluded to in Subsection 2.4, exactly the proof we have presented (with obvious modifications) yields the following result.

Proposition 4.8. *The following are equivalent for a completely regular frame L.*

(1) $\{\boldsymbol{O}_{\lambda}^{B} \mid B \in \mathcal{S}(\lambda L)\} = \{\boldsymbol{O}_{\pi}^{B} \mid B \in \mathcal{S}(\pi L)\}$.

(2) $\{\boldsymbol{O}_{\lambda}^{B} \mid B \in \mathcal{S}(\lambda L)$ *and B is closed*$\} = \{\boldsymbol{O}_{\pi}^{B} \mid B \in \mathcal{S}(\pi L)\}$.

(3) $\pi L = \lambda L$.

We close this section with the following two remarks.

Remark 4.9. All the results in the theorems and propositions presented above hold with the "operator" $\boldsymbol{O}^{(-)}$ replaced by $\boldsymbol{M}^{(-)}$, with exactly the same proofs.

Remark 4.10. In Lemma 4.6 we saw that if two sublocales of βL index the same $\boldsymbol{O}$-ideal, then they have the same closure. This actually also holds for the other coreflectors discussed here. We show it for υ. Suppose that $\boldsymbol{O}_{\upsilon}^{A} = \boldsymbol{O}_{\upsilon}^{B}$ for some sublocales A and B of υL. Then, by Proposition 4.4, $\boldsymbol{O}^{(k_L)_*[A]} = \boldsymbol{O}^{(k_L)_*[B]}$, and so, by Lemma 4.6, $\boldsymbol{O}^{\overline{(k_L)_*[A]}} = \boldsymbol{O}^{\overline{(k_L)_*[B]}}$, so that $\overline{(k_L)_*[A]} = \overline{(k_L)_*[B]}$, and hence, by a calculation as in the proof of Theorem 4.7, $\bigwedge A = \bigwedge B$, which implies $\mathrm{cl}_{\upsilon L}A = \mathrm{cl}_{\upsilon L}B$, as claimed.

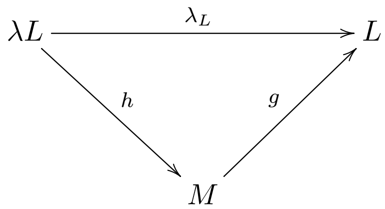
5 Appendix: A little more on λL

In this appendix we aim to record a result about λL that does not seem to have hitherto been mentioned in the literature, so far as we have been able to determine. This is what has prompted us to pursue this matter.

In light of [7, Lemma 2.12], it turns out that (up to isomorphism), βL is the Stone-Čech compactification of λL, of υL, and of πL. A glance at diagram (‡) suggests that (up to isomorphism) λL is the universal Lindelöfication of both υL and πL. We prove that this is indeed the case. Our proof mirrors that of the proof of [7, Lemma 2.12]. Let us set up the ingredients.

Recall that if $h\colon L \to M$ is a dense frame homomorphism, then $h_*(h(x)) \leq y$ whenever $x \prec y$ (see, for instance, the proof of [17, Lemma V.6.6.1]). Next, from [3, Proposition 4] we recall that every countable cover of a frame by cozero elements has a *shrinking*. That is, if $\bigvee_{n\in\mathbb{N}} c_n = 1$ for some sequence $(c)_{n\in\mathbb{N}}$ in Coz L, then there exists, for each n, an element $b_n \in \mathrm{Coz}\, L$ such that $b_n \prec\!\!\prec c_n$ and $\bigvee_{n\in\mathbb{N}} b_n = 1$. Finally, we recall from [1, Corollary 8.2.13] that λL is the unique (up to isomorphism) Lindelöfication of $h\colon M \to L$ of L such that h_* takes countable cozero covers to covers.

Proposition 5.1. *Suppose that the homomorphism $\lambda_L\colon \lambda L \to L$ factorizes* *as*



with h onto. Then $h\colon \lambda L \to M$ is, up to isomorphism, the universal Lindelöfication of M.

Proof. Simple calculation reveals that h is dense, so that $h\colon \lambda L \to M$ is a Lindelöfication of M. To show that it is the universal one, consider a sequence $(c_n)_{n\in\mathbb{N}}$ in Coz M with $\bigvee_{n\in\mathbb{N}} c_n = 1_M$. Let $(d_n)_{\mathbb{N}}$ be a shrinking of $(c_n)_{n\in\mathbb{N}}$. Then $\bigvee_{n\in\mathbb{N}} g(d_n) = 1_L$, and since each $g(d_n)$ is a cozero element of L and $(\lambda_L)_* = h_* \circ g_*$, we have $\bigvee_{n\in\mathbb{N}} h_*(g_*(g(d_n))) = 1_{\lambda L}$. Since $g_*(g(d_n)) \le c_n$ for every n, it follows that $\bigvee_{n\in\mathbb{N}} h_*(c_n) = 1_M$, thus proving the result by [1, Corollary 8.2.13]. □

As an immediate corollary we have the following results, one of which can also be deduced from the fact that $\mathrm{Coz}(\upsilon L) = \mathrm{Coz}(\lambda L)$.

Corollary 5.2. *For any completely regular frame L, $\ell_L\colon \lambda L \to \upsilon L$ is the universal Lindelöfication of υL and $p_L\colon \lambda L \to \pi L$ is the universal Lindelöfication of πL.*

Let us take a closer look at the result that $p_L\colon \lambda L \to \pi L$ is the universal Lindelöfication of πL and see what further information can be extracted from it. The first one is about $\mathcal{R}(\pi L)$. It is well documented that the rings $\mathcal{R}L$, $\mathcal{R}(\upsilon L)$ and $\mathcal{R}(\lambda L)$ are all isomorphic. In light of Corollary 5.2, we therefore have the ring isomorphisms

$$\mathcal{R}L \cong \mathcal{R}(\lambda L) \cong \mathcal{R}(\upsilon L) \cong \mathcal{R}(\pi L).$$

Next, in the language of [1], every frame is a dense C-quotient of its universal Lindelöfication. Therefore, in light of [1, Theorem 8.2.12], we have the σ-frame isomorphisms (including the ones that are already well known)

$$\mathrm{Coz}\, L \cong \mathrm{Coz}(\lambda L) \cong \mathrm{Coz}(\upsilon L) \cong \mathrm{Coz}(\pi L).$$

In [1], Ball and Walters-Wayland initiated the study of "disconnectivity" properties in frames. These include being extremally disconnected, basically disconnected, a P-frame, an almost P-frame, an F-frame, a quasi-F-frame. Each of these frames has been characterized ring-theoretically by a number of authors. Virtually everywhere where this has been done, it has been mentioned that L satisfies one or the other disconnectivity property if and only if λL does if and only if υL does. Now πL can also be added to these lists.

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