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Pseudo-differential Operators on the Circle, Bernoulli Polynomials

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REGULAR PAPER

Pseudo-differential operators on the circle, Bernoulli polynomials

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Abstract We show how the classical polylogarithm function $\mathrm{Li}_s(z)$ and its relatives, the Hurwitz zeta function and the Lerch function are all of a spectral nature, and can explain many properties of the complex powers of the Laplacian on the circle and of the distribution $(x+i0)^s$. We also make a relation with a result of Keiper [Fractional Calculus and its relationship to Riemann's zeta function, Master of Science, Ohio State University, Mathematics (1975)].

Keywords Hurwitz zeta function · Polylogarithm function · Bernoulli numbers · Complex powers of operators

Mathematics Subject Classification 11M35 · 11M06 · 11B68 · 32W25

1 Introduction, notations

In his paper [26], Mazur, rightly, comments that Bernoulli polynomials are a sign of the unity of Mathematics, as they intervene in many areas: Number Theory, Stable Homotopy Theory, Differential Topology, Theory of Modular Forms, etc. The essential goal of this work is to add another example, if necessary, of spectral theoretical nature by considering certain pseudodifferential operators on the circle. We combine ideas of Morava and Epstein [28], [10] and Mikolás [25] to study the spectral zeta functions and the traces of these operators.

The salient remark, in this work, is that certain properties of the pseudodifferential operators on the circle (or more exactly of their kernels) result from certain analogies between the Riemann zeta function and the Bernoulli polynomials. In fact we have

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for $\mid z \mid < 2\pi$

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} - 2\sum_{n=0}^{\infty}(-1)^n \frac{\zeta(2n+2)}{(2\pi)^{2n+2}} z^{2n+2},$$

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

Our notations differ from those of [10]. For a related discussion see [22]. From these relations the following fundamental relations hold

$$\zeta(2n) = \pi^{2n} 2^{2n-1} \frac{(-1)^{n+1}}{(2n)!} B_{2n}, \quad n = 0, 1, 2 \ldots$$

$$\zeta(-2n+1) = -\frac{B_{2n}}{2n}, \quad n = 1, 2, 3 \ldots$$

We recall that the generating function of the Bernoulli polynomials [11], p.36

$$\frac{ze^{zx}}{e^z-1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}.$$

Almost everything in what follows is based on the functions $\mathrm{li}_0(-z) = \frac{1}{e^z-1}, \frac{z}{e^z-1}$, through convolution products. Hence the title of the present paper.

We start by very formal considerations on the Laplacian on $\mathbb{R}^n$. Consider a plane wave such as $f(x) = e^{2i\pi x \cdot \xi}$, $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$, then

$$\Delta e^{2i\pi x\cdot\xi} = -4\pi^2 \mid \xi \mid^2 e^{2i\pi x\cdot\xi}, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

By Fourier inversion we obtain, formally,

$$\begin{aligned}\Delta f(x) &= \Delta \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2i\pi x\cdot\xi}\, d\xi \\ &= \int_{\mathbb{R}^n} \hat{f}(\xi) \Delta e^{2i\pi x\cdot\xi}\, d\xi \\ &= \int_{\mathbb{R}^n} \left(-4\pi^2 \mid \xi \mid^2\right) \hat{f}(\xi) e^{2i\pi x\cdot\xi}\, d\xi.\end{aligned}$$

We may then define, again by Fourier inversion

$$\widehat{\Delta f}(\xi) = \left(-4\pi^2 \mid \xi \mid^2\right) \hat{f}(\xi),$$

and by iterating, we have for positive integer n

$$\widehat{\Delta^n f}(\xi) = \left(-4\pi^2 \mid \xi \mid^2\right)^n \hat{f}(\xi).$$

This suggests a way to develop more general powers of the Laplacian, for instance a square root

$$\widehat{\sqrt{-\Delta} f}(\xi) = \sqrt{4\pi^2 \mid \xi \mid}\, \hat{f}(\xi).$$

An important observation concerning the Dirichlet-to-Neumann map can be made using the square root of the Laplacian $\sqrt{-\Delta}$ [3]. Let $f : \mathbb{R}^n \to \mathbb{R}$ be is a smooth and bounded function. We solve the extension problem in a half-space

$$\begin{cases} \Delta u(x,y) &= 0, \quad x \in \mathbb{R}^n, y > 0 \\ u(x,0) &= f(x), \quad x \in \mathbb{R}^n \end{cases}$$

to obtain a smooth bounded solution u, then

$$-u_y(x,0) = \sqrt{-\Delta}\, f(x).$$

Hence $\sqrt{-\Delta}$ can be realized as the operator

$$T : f \longrightarrow -u_y(x,0)$$

associating the Dirichlet and Neumann conditions. In [3] and [34] an extension to a general complex power of the Laplacian is given as follows

Theorem 1 *(Caffarelli-Stinga) If $U = U(x,y)$ is the solution of the initial value problem*

$$\begin{cases} \Delta_x U + \frac{a}{y}\partial_y U + \partial_{yy} U &= 0, \quad x \in \mathbb{R}^n, \ y > 0 \\ U(x,0) &= f(x), \quad x \in \mathbb{R}^n, \end{cases}$$

then

$$\frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)}(-\Delta)^s f(x) = \lim_{y \to 0^+} -y^a U_y(x,y)$$

for $s = \frac{1-a}{2}$ and some constant C depending on n and s. Moreover, the solution $U(x,y)$ is given explicitly as

$$U(x,y) = \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/4t} e^{t\Delta} f(x) \frac{dt}{t^{1+s}}$$

$$= \frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)} \int_{\mathbb{R}^n} \frac{y^{2s}}{\left(|x-y|^2+y^2\right)^{\frac{n+2s}{2}}} f(y)\,dy.$$

One of the goals of this paper is to understand (eventually to elaborate on) a result given in [10] (proposition 3, p.9)

Proposition 1 *The function $\mathrm{li}_s(x) = \sum_{n \geq 1} \frac{e^{nx}}{n^s}$ defined for $\mathfrak{Re}s \geq 0$, $\mathfrak{Re}x < 0$, and taking values in smooth functions on the negative real line when $\mathfrak{Re}s \geq 0$, extends to an entire function of s, taking values in tempered distributions on the whole real line, satisfying the congruence*

$$li_s(x) \equiv -\frac{\Gamma(1-s)}{2}\left[e^{-i\pi s}(x+i0)^s + e^{i\pi s}(x+i0)^s\right],$$

modulo meromorphic functions with smooth coefficients.

Remark 1 We take the domain of $\Delta_{\mathbb{R}}$, on the real line, as

$$\mathcal{D}(\Delta) = \left\{u \in L^2,\ 4\pi^2 \mid \xi \mid^2 \hat{u}(\xi) \in L^2(\mathbb{R})\right\}.$$

It is known that the spectrum $\sigma(\Delta)$ of Δ coincide with $[0, \infty)$. Although we will not develop this question in detail, it is worth pointing out that the methods developed here can define - for example - the Hermite polynomials $H_n(x)$ for complex values of n. Using Rodrigues' formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = (-1)^n e^{x^2} \frac{n!}{2\pi i} \oint_\gamma \frac{e^{-z^2}}{(z-x)^{n+1}}\,dz$$

we may define for complex s

$$H_s(x) = e^{i\pi\frac{s}{2}} e^{x^2} \left(-\frac{d^2}{dx^2}\right)^{\frac{s}{2}} e^{-x^2} = e^{i\pi s} e^{x^2} \frac{\Gamma(s+1)}{2\pi i} \oint_\gamma \frac{e^{-z^2}}{(z-x)^{s+1}}\,dz.$$

In all this paper the Lerch transcendent [20] whose classical definition is

$$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s} \tag{1}$$

plays an important role, in addition to its particular value

$$\mathrm{Li}_s(z)=\Phi(z,s,1)=\sum_{n=1}^{\infty}\frac{z^n}{n^s},$$

as well as

$$\mathscr{L}_s(z)=\sum_{n=1}^{\infty}\frac{e^{2i\pi nz}}{n^s}=\mathrm{Li}_s(e^{2i\pi z}). \tag{2}$$

2 Pseudodifferential operators, complex powers

2.1 Complex powers

We follow Komatsu [19], [32] for the definition of fractional powers. Consider first a bounded linear operator A such that the resolvent set $\rho(A)$ contains the negative real axis $(-\infty,0]$. The most natural definition of the complex power A^s is given by the Dunford integral

$$A^s=\frac{1}{2i\pi}\int_{\Gamma}\zeta^s(\zeta-A)^{-1}d\zeta, \tag{3}$$

where the path Γ encircles the spectrum $\rho(A)$, counterclockwise, avoiding the negative real axis, and ζ^s takes the principal branch.

There is an another way to define the powers of the Laplacian on $\mathbb{R}^n$ using Riesz potentials [33] p. 117. For a function $u:\mathbb{R}^n\to\mathbb{R}, n\geq 1$, the fractional Laplacian $(-\Delta)^s u$, with $0<s<1$, is naturally defined using the Fourier transform as

$$\widehat{(-\Delta)^s u}(\xi)=(2\pi|\xi|)^{2s}\hat{u}(\xi),\ \ \xi\in\mathbb{R}^n.$$

Hence, in terms of the Riesz potential,

$$(-\Delta)^s u=I_{2s}(u),\quad 0<s<\frac{n}{2},$$

with

$$I_\alpha u(x)=\frac{\Gamma\left(\frac{n}{2}-\frac{\alpha}{2}\right)}{\pi^{\frac{n}{2}}2^\alpha\Gamma(\frac{\alpha}{2})}\int_{\mathbb{R}^n}|x-y|^{-n+\alpha}u(y)\,dy.$$

And in terms of the heat semigroup on $\mathbb{R}^n$ we have, in terms of principal values PV

$$(-\Delta)^s u(x)=\frac{4^s\Gamma\left(\frac{n}{2}+s\right)}{\pi^{\frac{n}{2}}}\mathrm{PV}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy,\quad y\in\mathbb{R}^n.$$

The following result contains the essential of spectral properties [32], p.93.

Theorem 2 *Let A be an elliptic differential operator of order m on a closed n-dimensional manifold M. Let $f\in\mathcal{D}'(M)$ and let $f(x)=\sum_{j=1}^{\infty}\lambda_n\varphi_n(x)$ be the Fourier series expansion of f in the eigenfunctions of the operator A. Then*

$$A^s f(x)=\sum_{j=1}^{\infty}\lambda_n^s\varphi_n(x).$$

In particular, $\varphi_n(x)$ are also the eigenfunctions of the operator A^s with eigenvalues λ_n^s.

Since infinite separable Hilbert spaces are isometric, if $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the unit circle and ℓ^2 the space of infinite sequences $(\xi_k)_{k\geq 0}$ of complex numbers that are square summable, $\sum_{k=0}^{\infty} \mid \xi_k \mid^2 < \infty$, with a norm and dot product given by

$$\|\xi\| = \left(\sum_{k=0}^{\infty} |\xi_k|^2\right)^{\frac{1}{2}}, \quad \xi \cdot \eta = \sum_{k=0}^{\infty} \xi_i \, \bar{\eta}_i,$$

the functions $\xi \mapsto \xi^k$ form an orthonormal basis of $L^2(\mathbb{S}^1)$, with an isometry

$$\mathcal{I} : L^2(\mathbb{S}^1) \longrightarrow \ell^2(\mathbb{Z})$$

$$f \longrightarrow (\xi_k), \quad \xi_k = f \cdot \xi^k = \int f(\xi)\xi^{-k} \, d\mu(\xi),$$

where

$$\int f(\xi) \, d\mu(\xi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) \, dt.$$

Remark 2 On the Hilbert space of square integrable functions on a circle $H = L^2\left(\mathbb{S}^1, \frac{d\theta}{2\pi}\right)$ an orthonormal basis is given by $\varphi_n(\theta) = e^{in\theta}$, with $n \in \mathbb{Z}$. One can introduce two sets of unitary operators on H

$$P : \mathbb{Z} \longrightarrow \mathcal{U}(H), \quad P(m)(\varphi_n) = \varphi_{n+m}$$

$$Q : \mathbb{R} \longrightarrow \mathcal{U}(H), \quad Q(k)(\varphi_n) = e^{ink}\varphi_n.$$

satisfying a sort of canonical commutation rules:

$$P(m) \, Q(k) = e^{imk} Q(k) \, P(m).$$

Up to homeomorphisms, the circle $\mathbb{S}^1$ is the only compact connected 1-dimensional manifold. We identify $\mathbb{S}^1$ with the interval $[0, \ 2\pi]$ (resp. $[0, \ 1]$), with periodic conditions, and then, as 1-dimensional manifold it is the quotient of the real line $\mathbb{R}$ by the subgroup of integers $2\pi\mathbb{Z}$ (resp. $\mathbb{Z}$). From this point of view, $\mathbb{S}^1$ can be thought of as being an abelian group, isomorphic to $SO(2, \mathbb{R})$, or to the 1-dimensional torus. The circle can also be considered as the real projective line $\mathbb{RP}^1$, consisting of lines in $\mathbb{R}^2$ going through the origin (identified with $\mathbb{R} \cup \{\infty\}$ by taking the slope of a line).

The spectrum of the Laplacian $\Delta = -\frac{d^2}{dx^2}$ on the circle $\mathbb{S}^1$ is given by

$$\lambda_0 = 0, \quad \lambda_{2k-1} = \lambda_{2k} = k^2, \ k \geq 1,$$

corresponding to the eigenfuctions

$$f_{2k-1}(x) = \sin(kx), \quad f_{2k}(x) = \cos(kx), \quad k \geq 1.$$

We now introduce the heat flow on $\mathbb{S}^1$ and $\mathbb{R}$. Given a initial heat distribution $f(t, \theta)$ on $\mathbb{S}^1$, the heat distribution $f(t, \theta)$ at the time t is the solution of the initial value problem

$$(\partial_t + \Delta) f(t, \theta) = 0.$$

So, by separation of variables,

$$f(t, \theta) = \sum_{n \in \mathbb{Z}} a_n e^{-n^2 t} e^{in\theta},$$

where a_n is the n-th Fourier coefficient of $f(t, \theta)$. In particular

$$f(t, \theta) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \sum_{n \in \mathbb{Z}} e^{-n^2 t} e^{in\theta} \ \overline{e^{in\psi}} d\psi.$$

We will call

$$e_{\mathbb{S}^1}(\theta,\psi)=\sum_{n\in\mathbb{Z}} e^{-n^2 t} e^{in\theta}\ \overline{e^{in\psi}}=\sum_{n\in\mathbb{Z}} e^{-n^2 t} e^{in(\theta-\psi)} \tag{4}$$

the heat flow on $\mathbb{S}^1$. It is a smooth function on $(0,\infty)\times\mathbb{S}\times\mathbb{S}^1$. The integral kernel for the heat flow on the real line $\mathbb{R}$ is

$$e_{\mathbb{R}}(t,x,y)=\frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{(x-y)^2}{4t}},\quad t>0.$$

Naturally we have the periodization relation

$$e_{t,\mathbb{S}^1}(\theta,\psi)=\sum_{n\in\mathbb{Z}} e_{\mathbb{R}}(t,\theta,\psi+2\pi n).$$

We denote by $e^{-t\Delta}$ the operator

$$e^{-t\Delta}: L^2\longrightarrow L^2,$$

taking the heat distribution at time $t=0$ to the heat distribution $f(t,\theta)$ at time t. The notation is suggested by the observation that $e^{-t\Delta}$ acts by multiplication by e^{-tn^2} on the n^2-eigenspace of Δ. The heat operator for the circle $\mathbb{S}^1$ is defined as

$$H_t=e^{-t\Delta},\quad t>0,$$

and the spectral zeta function is

$$\zeta_\Delta(s)=\operatorname{Tr}\left(\Delta^{-s}\right),$$

which is related to the heat operator by the Mellin transform

$$\operatorname{Tr}\left(\Delta^{-s}\right)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\operatorname{Tr}\left(e^{-t\Delta}\right)dt.$$

More explicitly,

$$\operatorname{Tr}\left(e^{-t\Delta}\right)=\sum_{n\in\mathbb{Z}} e^{-n^2 t}=\int_{\mathbb{S}^1} e_{\mathbb{S}^1}(t,\theta,\theta)\,d\theta.$$

For short time, the trace of the heat kernel for the circle looks like the trace of the heat kernel for the line, in the sense that

$$\sum_{n\in\mathbb{Z}} e^{-n^2 t}=\int_{\mathbb{S}^1} e_{\mathbb{S}^1}(t,\theta,\theta)\,d\theta=\int_{-\pi}^{\pi} e_{\mathbb{R}}(t,x,x)ds+R(t),$$

where $R(t)$, the remainder term, behaves near the origin $t=0$ like $e^{-\frac{\alpha}{t}}$, $\alpha>0$. For long time we may use the Poisson summation formula

$$\sqrt{t}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}=\sum_{n\in\mathbb{Z}} e^{-\frac{\pi n^2}{t}},\quad t>0$$

to obtain

$$\sqrt{t}\ \operatorname{Tr}\left(e^{-t\Delta}\right)=\sqrt{\pi}\ \operatorname{Tr}\left(e^{-\frac{\pi^2}{t}\Delta}\right).$$

If we identify functions on $\mathbb{S}^1=\mathbb{R}/\mathbb{Z}$ with periodic functions on the interval [0, 1], we see that

$$-\Delta_{\mathbb{S}^1}\left(e^{2i\pi kx}\right)=(2\pi k)^2 e^{2i\pi kx},\quad \Delta_{\mathbb{S}^1}=\frac{d^2}{dx^2}.$$

If

$$u(t) = \sum_{k\in\mathbb{Z}} (2\pi|k|)^{-2s} c_k(u) e^{2i\pi kx}$$

with $\int_{\mathbb{S}^1} u(t)dt = 0$ and $\Re\mathfrak{e} s > 0$, we get by Theorem (2)

$$\begin{aligned}\left(-\Delta_{\mathbb{S}^1}\right)^s u(x) &= \sum_{k\in\mathbb{Z}\setminus\{0\}} (2\pi|k|)^{-2s} c_k(u) e^{2i\pi kx} \\ &= \sum_{k\in\mathbb{Z}\setminus\{0\}} (2\pi|k|)^{-2s} \int_0^1 u(y) e^{2i\pi k(x-y)}\, dy \\ &= \int_0^1 K_{-2s}(x-y) u(y)\, dy,\end{aligned}$$

where the kernel is given by a polylogarithm

$$K_{-2s}(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{2i\pi kx}}{(2\pi|k|)^{2s}}. \tag{5}$$

Remark 3 In the Physics literature the Fermi-Dirac and Bose-Einstein integrals are defined, respectively, by the Mellin transform

$$F_s(x) = \frac{1}{\Gamma(s+1)} \int_0^{+\infty} \frac{t^s}{e^{t-x}+1}\, dt, \quad G_s(x) = \frac{1}{\Gamma(s+1)} \int_0^{+\infty} \frac{t^s}{e^{t-x}-1}\, dt.$$

They are related to the polylogarithm function $\mathrm{Li}_s(z) = \Phi(z, s, 1)$ (1), written as Mellin transform

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - z}\, dt,$$

by

$$F_s(x) = -\mathrm{Li}_{s+1}(-e^x), \quad G_s(x) = \mathrm{Li}_{s+1}(e^x).$$

2.2 Determinants

This section uses the relation between finite dimensional trace and determinant. Consider the Laplace operator Δ on a closed manifold M. It has a discrete spectrum consisting of eigenvalues

$$0 \le \lambda_1 \le \lambda_2 \cdots,$$

repeated with multiplicity. We omit the zero eigenvalue and assume $\lambda_1 > 0$. Formally, the determinant of Δ is, in principle

$$\det \Delta = \prod_{n=1}^{\infty} \lambda_n.$$

This infinite product diverges. For example, if $M = \mathbb{S}^1$ and Δ is the usual scalar Laplace operator, then up to a constant the infinite product is

$$\prod_{n=1}^{\infty} n^2$$

after omitting the zero eigenvalue, and this is divergent. We attribute a value to these divergent infinite products by a regularization method, due to Ray and Singer [30]. We assume that the zeta function of Δ converges for large $\Re\mathfrak{e} s$, so that in some right half plane

$$-\zeta'_\Delta(s) = \sum_{n=1}^{\infty} \lambda_n^{-s} \log \lambda_n,$$

and then use the regularity of the analytic continuation of ζ_Δ at $s = 0$ to define

$$\det \Delta = e^{-\zeta'_\Delta(0)}.$$

Remark 4 The definition is motivated by the fact that for a finite dimensional case, if a matrix A admits the eigenvalues $\lambda_n,\ 1 \le n \le N$,

$$\det A = \prod_{n=1}^{N} \lambda_n = \exp\left(-\frac{d}{ds}\Big|_{s=0} \sum_{n=1}^{N} \frac{1}{\lambda_n^s} \right).$$

For the case of the circle $\mathbb{S}^1$, we have, ζ being the Riemann zeta function,

$$\zeta_\Delta(s) = 2\zeta(2s), \quad \zeta'_\Delta(0) = 4\zeta'(0) = -2\log(2\pi), \quad \det \Delta = (2\pi)^2.$$

2.3 Symbols

According to Seeley [31], if A is an elliptic invertible pseudo-differential operator of order m, and has a ray of minimal growth, then along this ray, the L^2 operator norm of the resolvent verifies

$$||(A-\lambda)^{-1}|| = O\left(\frac{1}{|\lambda|}\right).$$

We can define, for $\Re\mathfrak{e} s < 0$

$$A^s = \frac{i}{2\pi} \int_\Gamma \lambda^s (A-\lambda)^{-1} d\lambda,$$

where Γ is a curve, disjoint from the spectrum of A, beginning at ∞, passing along the ray of minimal growth to a small circle about the origin, then clockwise about the circle, and back to ∞ along the ray.

If a_m is the top term of the symbol of A, then the symbol of A is

$$\sigma(A) = \sum_{j=0}^{\infty} a_{m-j}.$$

With $B(\lambda) = (A-\lambda)^{-1}$ we have

$$\sigma(B)\sigma(A-\lambda) = 1.$$

In particular

$$b_{-m}(a_m - \lambda) = 1$$

$$b_{-m-l}(a_m - \lambda) + \sum (\partial/\partial\xi)^\alpha b_{-m-j} D_x^\alpha a_{m-k}/\alpha! = 0, \quad l > 0,$$

where the sum is taken for $j < l,\ j + k + |\alpha| = l$. According to Seeley [31], p. 290, the symbol of the s-power A^s is then

$$\sigma(A^s) = \sum_{j=0}^{\infty} \frac{i}{2\pi} \int_\Gamma \lambda^s b_{-m-j}\, d\lambda.$$

3 The spectrum of positive elliptic operators and periodic bicharacteristics

There is a far reaching extension of the Poisson summation formula due to Duistermaat-Guillemin [8], Chazarain [4] and Colin de Verdière [5]. Consider a compact n-dimensional manifold X and a scalar elliptic pseudodifferential operator P of order 1 on X which is positive and selfadjoint. The spectrum of this operator is a discrete set

$$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_j \leq \cdots \longrightarrow +\infty.$$

This defines a distribution

$$\sigma_P = \sum_{j=0}^{\infty} \delta_{\lambda_j}$$

The principal symbol p of P defines a Hamiltonian vector field H_p on $T^*X \setminus X$, the cotangent bundle with the zero section removed. The Laplacian Δ is defined with respect to a Riemannian metric on X. A standard example is to consider, for a suitable constant c, the operator $c - \Delta$, which is positive and selfadjoint. In this situation we can evoke the spectral theorem to define a positive square root $P = \sqrt{c - \Delta}$, which is pseudodifferential operator of order 1 and whose principal symbol is given by $p(x, \xi) = \|\xi\|$ for $x \in X$ and $\xi \in T_x X$. The Hamiltonian flow of p is the geodesic spray of X (at least on the unit sphere bundle). The Fourier transform of the spectral distribution σ_P is

$$\hat{\sigma}_P(t) = \mathcal{F}\sigma_P = \sum_{j=0}^{\infty} e^{-i\lambda_j t}. \tag{6}$$

This can be seen as the distributional trace of the unitary operator e^{-itP}, which is a Fourier integral operator. A preliminary result says that $\hat{\sigma}_P$ is a tempered distribution, and therefore so is σ_P. In particular the eigenvalue counting function satisfies Weyl's law

$$N_P(\lambda) = \sharp\left\{j,\, \lambda_j \leq \lambda\right\} \sim \frac{\omega_n}{(2\pi)^n} \mathrm{Vol}(M) \lambda^{\frac{n}{2}}, \quad \lambda \to \infty,$$

where ω_n denotes the volume of the n-dimensional unit ball. The celebrated antecedent of results of this type is due to H. Weyl who showed (for D a plane domain and the Dirichlet problem) that the spectrum of the Laplacian determines the volume. A clear confirmation of Weyl's law can be provided by the unit circle. If $n = 1$, the unit ball is $[-1, 1]$ of volume 2, and $\mathrm{Vol}\left(\mathbb{S}^1\right) = 2\pi$, so

$$\frac{\omega_n}{(2\pi)^n} \mathrm{Vol}(M) \lambda^{\frac{n}{2}} = 2\lambda^{\frac{1}{2}}.$$

On the other hand, for λ large

$$N(\lambda) = 2\lfloor \lambda^{\frac{n}{2}} \rfloor - 1 \sim 2\lambda^{\frac{1}{2}}.$$

The fundamental result is the connection between periods and eigenvalues

Theorem 3 *The distribution $\hat{\sigma}_P$ is $\mathcal{C}^\infty$ outside the set of periods of periodic trajectories of the hamiltonian H_P*

The classical Poisson summation formula

$$\sum_{k\in\mathbb{Z}} e^{ikt} = 2\pi \sum_{k\in\mathbb{Z}} \delta_{2\pi k}$$

is a particular case of this theorem. On the flat torus $\mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ we have on the one hand the spectrum $\lambda_k = k^2,\ k = 0, 1, 2, \cdots$ of the laplacian $-\dfrac{d^2}{dx^2}$, and on the other hand the lengths $2k\pi,\ k = 0, 1, 2, \cdots$ of the closed periodic geodesics. This corresponds to the pairwise orthogonal subspaces decomposition

$$L^2(\mathbb{S}^1) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(\mathbb{S}^1),$$

where $\mathcal{H}_0(\mathbb{S}^1) = \mathbb{R}$ is the set of constant functions, and $\mathcal{H}_k(\mathbb{S}^1)$ is the two-dimensional space spanned by the functions $\cos k\theta$ and $\sin k\theta,\ k \geq 1$.

Remark 5 For $M = \mathbb{R}^n/L$ a n-torus, L is a lattice of $\mathbb{R}^n$, the eigenfunctions of the Laplacian are $e^{2i\pi x\cdot l'}$, $l' \in L'$, the dual lattice of all vectors $x \in \mathrm{span}(L)$ such that $\langle x, y\rangle$ is an integer for all $y \in L$. The eigenvalues are $4\pi^2\|l'\|^2$ and the trace is given by

$$\mathrm{Tr}\left(e^{-t\Delta}\right) = \sum_{\omega\in L'} e^{-4\pi^2 t\|\omega\|^2} = \frac{\mathrm{Vol}(M)}{(4\pi t)^{\frac{n}{2}}} \sum_{\omega\in L} e^{-\frac{\|l\|^2}{4t}}.$$

The last equality is obtained by the Poisson summation formula. The closed geodesics on M lift to the line segments from 0 to $l \in L$, on $\mathbb{R}^n$. So the length spectrum, i.e., the lengths of closed geodesies, is the set $\{\|l\|,\ l \in L\}$.

From this point of view, an extension of the polylogarithm function, is the Epstein zeta function (in its simplest form) given by the Dirichlet series

$$F(x_1, \cdots, x_n) = {\sum_{N\in L}}' \frac{e^{N\cdot x}}{\|N\|^{2s}}, \quad x = (x_1, \cdots, x_n), \tag{7}$$

where

$$L = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n, \quad a_i = (0, \cdots, 2\pi, 0, \cdots, 0), \quad \|N\|^{2s} = (n_1^2 + n_2^2 + \cdots + n_n^2)^s.$$

The exponential function $e^{N\cdot x}$ is an eigenfunction of the initial value problem

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \lambda u = 0$$

in the parallelotope $D = \{0 \le x_1, \cdots, x_n \le 2\pi\}$, with Dirichlet conditions

$$u(x_1, \cdots x_{i-1}, 0, x_{i+1}, \cdots, x_n) = u(x_1, \cdots x_{i-1}, 2\pi, x_{i+1}, \cdots, x_n), \quad 1 \le i \le n.$$

The associated eigenvalues are $\|N\|^2$. If we arrange the eigenfunctions into a sequence $\omega_n(x)$, with λ_n as in [27], the series (7) may be written as

$$\sum_{\lambda_n\neq 0} \frac{\omega_n(x)\overline{\omega(y)}}{\lambda_n^s}.$$

4 The spectral zeta function in quantum mechanics

The goal of this section is to point out that the polylogarithm function is a particular example of a large construction, of a spectral nature. We define the quantum zeta function as a formal sum over the non-zero eigenvalues λ of a Hamiltonian (with the boundary conditions, etc. taken into account), when it exists [6]

$$Z(s) = \sum_{\lambda\neq 0} \frac{1}{\lambda^s}. \tag{8}$$

We also define the associated function, the parity zeta function

$$Y(s) = \sum_{\lambda\neq 0} \frac{(-1)^n}{\lambda^s}. \tag{9}$$

When the ordered energy eigenvalues λ are discrete, non-vanishing, and sufficiently divergent, we assume that these Dirichlet series converge for sufficiently large $\mathfrak{Re}\, s$.

Now we consider a real-valued, one-dimensional, potential $V(x)$ and introduce

$$-\frac{1}{2}\frac{\partial^2 G}{\partial x^2} + V(x)G - \lambda G = -\delta_{x_0}.$$

The standard Green function is expressed in terms of orthonormal eigenfunctions ψ_n as

$$G(x, x_0, \lambda) = \sum_n \frac{\psi_n(x)\psi_n^*(x_0)}{\lambda - \lambda_n}, \tag{10}$$

which stands as a particular solution. Based on of the orthonormality of the wave functions the relation (10) gives

$$Z(1) = -\int G(x, x, 0)\, dx.$$

More generally

$$Z(n) = -\int G^{(n-1)}(x, x, 0)\, dx,$$

where $G^{(m)}$ denotes the m-th partial derivative of G with respect to the energy argument. Another representation for $Z(n),\ n \in \mathbb{Z}_+$ is [6], [17]

$$Z(n) = (-1)^n \int G(x_1, x_2, 0)G(x_2, x_3, 0)\cdots G(x_n, x_1, 0)dx_1dx_2\cdots dx_n. \tag{11}$$

For symmetric potentials V the parity zeta function enjoys the formal relation

$$Y(1) = -\int G(x, -x, 0)dx.$$

The eigenstate representation of the propagator is

$$K(x, t; x_0, 0) = \sum_n \psi_n(x)\psi_n^*(x_0)e^{-i\lambda_n t}, \tag{12}$$

which leads to a Mellin transform

$$Z(s) = \frac{i^s}{\Gamma(s)} \int_0^\infty t^{s-1} \int K(x, t; x, 0)dx\, dt.$$

This representation is often referred to as the heat-kernel form of the zeta function. It is of interest that the inverse Mellin transform yields the quantum partition function

$$\sum_n e^{-\lambda_n t} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} Z(s)\Gamma(s)t^{-s}\, ds$$

4.1 Minakshisundaram-Pleijel zeta function

There are several possible extension of the polylogarithm function, since the latter is intimately linked to the circle. For a compact Riemannian manifold M of dimension N with eigenvalues

$\lambda_1, \lambda_2, \ldots$

of the Laplace-Beltrami operator Δ, the zeta function is given for $\mathfrak{Re}(s)$ sufficiently large by

$$Z(s) = \mathrm{Tr}(\Delta^{-s}) = \sum_{n=1}^{\infty} |\lambda_n|^{-s},$$

(where if an eigenvalue is zero it is omitted in the sum). The manifold may have a boundary, in which case one has to prescribe suitable boundary conditions, such as Dirichlet or Neumann boundary conditions. More generally one can define

$$Z(P, Q, s) = \sum_{n=1}^{\infty} \frac{f_n(P)f_n(Q)}{\lambda_n^s}$$

for P and Q on the manifold, where the f_n are normalized eigenfunctions. This can be analytically continued to a meromorphic function of complex values of s, and is holomorphic for $P \neq Q$. The only possible poles are simples, and are at

$$s = \frac{N}{2}, \; \frac{N}{2} - 1, \; \frac{N}{2} - 2, \ldots, \frac{1}{2}, \; -\frac{1}{2}, \; -\frac{1}{2}, \ldots$$

for N odd, while for even N at

$$s = \frac{N}{2}, \; \frac{N}{2} - 1, \; \frac{N}{2} - 2, \ldots, 2, 1, \cdots.$$

If N is odd then $Z(P, P, s)$ vanishes at $s = 0, -1, -2, \ldots$. If N is even, the residues at the poles can be explicitly found in terms of the metric of M. The Wiener-Ikehara Tauberian theorem gives Carleman's asymptotic formula

$$\sum_{\lambda_n < T} f_n(P)^2 \sim \frac{T^{N/2}}{(2\sqrt{\pi})^N \Gamma(N/2+1)}.$$

If the manifold is a circle of dimension $N = 1$, then the eigenvalues of the Laplacian are n^2 for integers n, and the zeta function

$$Z(s) = \sum_{n \neq 0} \frac{1}{(n^2)^s} = 2\zeta(2s),$$

where ζ is the Riemann zeta function.

4.2 Second emergence of the polylogarithm function

Let $f(x) \in L^1([0, 1])$ be a real or complex function having the period 1. We start from the Fourier series, converging almost everywhere

$$f(x) \sim \alpha_0 + 2\sum_{n=0}^{\infty} (\alpha_n \cos 2n\pi x + \beta_n \sin 2n\pi x) \tag{13}$$

and integrate (formally) p-times to obtain the series

$$H_p(x) + \sum_{n=0}^{\infty} \frac{2}{(2n\pi)^p} \left\{ \alpha_n \cos\left(2n\pi x - \frac{p\pi}{2}\right) + \beta_n \sin\left(2n\pi x - \frac{p\pi}{2}\right) \right\}, \tag{14}$$

where $H_p(x)$ is a polynomial of degree p such that

$$\frac{d^p}{dx^p} H_p(x) = \alpha_0. \tag{15}$$

The series (14) converges for every integer $p \geq 1$ to a function $F_p(x)$, with

$$\frac{d^p}{dx^p} F_p(x) = f(x), \quad a.e$$

Since $H_p(x)$ is a polynomial, the relation (14) does not define a periodic function for $p \geq 1$. We replace $H_p(x)$ by the suitable function $\tilde{H}_p(x)$ having the period 1, represented by the Fourier expansion over $(0, 1)$ of $H_p(x)$, that is we consider $\tilde{H}_p(x) = H_p(x - \lfloor x \rfloor)$. If, moreover, we require that the constant term of this development to vanish

$$\int_0^1 H_p(x) dx = 0, \quad p \geq 1,$$

then $\tilde{H}_p(x)$ is uniquely determined by the former conditions

$$\begin{cases} \tilde{H}_1'(x) & = \alpha_0 \\ \frac{1}{p+1}\, \tilde{H}_{p+1}'(x) & = \tilde{H}_p(x), \quad 0 < x < 1. \end{cases}.$$

We then have $\tilde{H}_p(x) = \alpha_0 B_b(x)$, where $B_b(x)$ is the Bernoulli function. This shows that for every x and every integer $p \geq 1$

$$\tilde{H}_p(x) = -\alpha_0 \sum_{n=1}^{\infty} \frac{2}{(2n\pi)^p} \cos\left(2n\pi x - \frac{p\pi}{2}\right). \tag{16}$$

By inserting (16) in (14) we obtain

$$\mathfrak{F}_p(x) = \alpha_0 \sum_{n=1}^{\infty} \frac{2}{(2n\pi)^p} \left[(\alpha_n - \alpha_0) \cos\left(2n\pi x - \frac{p\pi}{2}\right) + \beta_n \sin\left(2n\pi x - \frac{p\pi}{2}\right)\right]. \tag{17}$$

It is very natural to consider, more generally, the series

$$\mathfrak{F}_s(x) == \alpha_0 \sum_{n=1}^{\infty} \frac{2}{(2n\pi)^s} \left[(\alpha_n - \alpha_0) \cos\left(2n\pi x - \frac{\pi s}{2}\right) + \beta_n \sin\left(2n\pi x - \frac{\pi s}{2}\right)\right]. \tag{18}$$

Since $f \in L^1(0, 1)$, α_n and β_n tend to 0 at infinity, so the series (18) converges absolutely and uniformly on any compact subset of the right half-plane $\{\mathfrak{Re}s > 1\}$. It also converges for $s = 1$ by partial summation.

Theorem 4 *(Mikolas) We have*

$$\mathfrak{F}_s(x) = \int_0^1 f(x - t) \left(\mathfrak{l}_s(t) - \mathfrak{l}_s(x)\right) dt,$$

where

$$\mathfrak{l}_s(u) = \sum_{n=1}^{\infty} \frac{2}{(2n\pi)^s} \cos\left(2n\pi u - \frac{\pi s}{2}\right).$$

This function is actually related to a zeta function. The Hurwitz zeta function is defined, for $\mathfrak{Re}s > 1$, by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}, \quad a \in \mathbb{C} \setminus \left\{0, -1, -2, \cdots\right\}.$$

We have $\zeta(s, 1) = \zeta(s)$, the Riemann zeta function, and even for general values of a, $\zeta(s, a)$ and $\zeta(s)$ share many properties. For instance we have the contour integral representation

$$2i\pi\zeta(s, a) = -\Gamma(1 - s) \int_{\infty}^{(0^+)} (-t)^{s-1} e^{-at} (1 - e^{-t})^{-1} \, dt, \quad \mathfrak{Re}a > 0, \quad |\arg(-t)| \leq \pi. \tag{19}$$

By classical methods of contours deformation we get Hurwitz's formula [11]

$$\zeta(s, a) = 2(2\pi)^{s-1} \Gamma(1 - s) \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \sin\left(2\pi n a + \frac{\pi s}{2}\right). \tag{20}$$

It verifies the difference-differential equation, similar to the one satisfied by Bernoulli polynomials $B_p(u)$. In fact, for $s = p \geq 1$ integer, we have

$$\mathfrak{l}_p(u) = \frac{1}{(p - 1)!} \zeta(1 - p, u) = -B_p(u).$$

Furthermore the **F**-equation holds

$$\frac{\partial}{\partial a} \zeta(s, a) = -s\zeta(s + 1, a). \tag{21}$$

We refer to [21] for a exposition on the Bernoulli zeta function.

The Hurwitz zeta function satisfies an identity which generalizes the functional equation of the Riemann zeta function

$$\zeta(1-s,a)=\frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi is/2}\sum_{n=1}^{\infty}\frac{e^{2\pi ina}}{n^s}+e^{\pi is/2}\sum_{n=1}^{\infty}\frac{e^{-2\pi ina}}{n^s}\right),$$

valid for $\mathfrak{Re}(s)>1$ and $0<a\leq 1$. The Riemann zeta functional equation is a special case, corresponding to $a=1$

$$\zeta(1-s)=\frac{2\Gamma(s)}{(2\pi)^s}\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

Hurwitz's formula can also be expressed as

$$\zeta(s,a)=\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left(\sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi na)}{n^{1-s}}+\cos\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{\sin(2\pi na)}{n^{1-s}}\right)$$

The Lerch zeta function is defined by

$$L(t,x,s)=\sum_{n=0}^{\infty}\frac{e^{2i\pi nt}}{(n+x)^s},\quad \mathfrak{Re}(s)>1,\quad \mathfrak{Re}(x)>0,\quad \mathfrak{Im}(t)\geq 0.$$

Remark 6 The function

$$p_\lambda(u)=\sum_{n=1}^{\infty}\frac{\sin 2n\pi u}{n^\lambda},\quad \lambda\geq 1$$

satisfies Franel type integral, which could suggest a role of non-trivial zeros of the Riemann zeta function [14]

$$\int_0^1 p_\lambda(at)p_\lambda(bt)\,dt=\frac{\zeta(2\lambda)}{2}\frac{\gcd(a,b)}{a^\lambda b^\lambda}.$$

In fact, if 2λ is a zero of $\zeta(s)$, then the left side vanishes for every a and b.

The previous formula for $\zeta(s,a)$ is analog to

$$\frac{1}{n!}B_n(x)=-\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{e^{2i\pi kx}}{(2i\pi k)^n}=-\frac{2}{(2\pi)^n}\sum_{k=1}^{\infty}\frac{\cos\left(2\pi kx-\frac{n\pi}{2}\right)}{k^n},\tag{22}$$

valid for $n=1, 0<x<1$ and for $n>1, 0\leq x\leq 1$. We deduce, from this expansion, by using the classical integral

$$\int_0^{\infty}t^n e^{-at}\,dt=\frac{n!}{a^{n+1}},\quad \mathfrak{Re}a>0,\ \ n=0,1,\cdots,$$

that

$$B_n(x)=-\frac{-n}{(-2i\pi)^n}\left\{\int_0^{\infty}t^{n-1}\frac{e^{-2i\pi x}}{e^t-e^{-2i\pi x}}\,dt+(-1)^n\int_0^{\infty}t^{n-1}\frac{e^{2i\pi x}}{e^t-e^{2i\pi x}}\,dt\right\}\tag{23}$$

The relation (22) is the discrete version of (5). The Bernoulli polynomials verify also an **F**-equation, similar to (21) and (29)

$$B_n'(x)=nB_{n-1}(x).$$

4.3 The polylogarithm function and tempered distributions

We define

$$\mathrm{li}_s(x)=\sum_{n=1}^{\infty}\frac{e^{nx}}{n^s}$$

for which the following relation of Lindelöf [35] holds

$$\mathrm{li}_s(x) = \Gamma(1-s)(-x)^{s-1} + \sum_{k=0}^{\infty} \zeta(s-k)\frac{x^k}{k!}. \tag{24}$$

Another interesting relation, due to Wirtinger [35], is

$$\mathrm{li}_s(x) = \Gamma(1-s)\sum_{-\infty}^{\infty} (-2i\pi x + 2in\pi)^{s-1}. \tag{25}$$

In particular for $\mathfrak{Re}s < 1$

$$\lim_{x\to 0} (-2i\pi x)^{1-s}\mathrm{li}_s(x) = \Gamma(1-s). \tag{26}$$

We define the distribution

$$\gamma_+^s(x) = \frac{x_+^{s-1}}{\Gamma(s)}, \qquad x_+ = \max(x, 0),$$

where $x^c = e^{c\log x}$ for $x > 0, c \in \mathbb{C}$. The function $\gamma_+^s(x)$ is locally integrable on $\mathbb{R}$ if and only if $\mathfrak{Re}s > 0$. It defines a distribution on $\mathbb{R}$ in this case. Then [10]

$$\mathrm{li}_s(x) = \left(\gamma_+^s(x) * \mathrm{li}_0\right)(x), \tag{27}$$

where li_0 is defined by

$$\mathrm{li}_0(x) = \frac{1}{e^{-x}-1} = -\frac{1}{x}\sum_{n=0}^{\infty} B_n\frac{x^n}{n!}, \quad |\, x \,| < 2\pi.$$

Remark 7 We will need the following relation, which results from the partial fractions decomposition of the hyperbolic cotangent.

$$\frac{1}{e^{-x}-1} = -\frac{1}{x} - \frac{1}{2} - 2\sum_{k=1}^{\infty}\frac{x}{x^2+4\pi^2k^2}, \quad x \notin 2i\pi\mathbb{Z}. \tag{28}$$

For complex value z

$$\mathrm{li}_s(z) = \Gamma(1-s)(-z)^{s-1} + \sum_{k=0}^{\infty}\frac{\zeta(s-k)}{k!}z^k,$$

with

$$\lim_{s\to k+1}\left\{\frac{\zeta(s-k)}{k!}z^k + \Gamma(1-s)(-z)^{s-1}\right\} = \frac{z^k}{k!}\left\{\sum_{h=1}^{k}\frac{1}{h} - \ln(-z)\right\},$$

where the sum over h vanishes if $k = 0$. So, for positive integer orders and for $|\, z \,| < 2\pi$ we have the series expansion

$$\mathrm{li}_n(z) = \frac{z^{n-1}}{(n-1)!}\{H_{n-1} - \ln(-z)\} + \sum_{\substack{k=0\\k\neq n-1}}^{\infty}\frac{\zeta(n-k)}{k!}z^k,$$

where H_n denotes the n-th harmonic number

$$H_n = \sum_{k=1}^{n}\frac{1}{k}, \qquad H_0 = 0.$$

5 The distributions $(x + i0)^s$ and $(x - i0)^s$

5.1 The distribution t_+^λ and summation formulas

We review some classical facts on distributions theory that we need to complete our analysis of the Proposition (1). This concerns Hadamard regularization of a distribution T, also called Hadamard finite part, and denoted by T_{reg}. Our principal reference is [15] and [36]. The map (Mellin transform)

$$\lambda \longrightarrow \langle t_+^{\lambda-1}, \phi\rangle = \int_0^\infty t^{\lambda-1}\phi(t)\,dt, \quad \phi \in \mathscr{S}(\mathbb{R}),$$

defined on the open half-plane $\{\lambda \in \mathbb{C},\ \mathfrak{Re}\lambda > 0\}$, defines a distribution on $\mathbb{R}$, with support in $[0,\ \infty)$. The map

$$\lambda \longrightarrow \gamma_\lambda^+ = \frac{1}{\Gamma(\lambda)} t_+^{\lambda-1} \in \mathscr{S}'(\mathbb{R})$$

verifies

$$\gamma_\lambda^+ = {\gamma_{\lambda+k}^+}^{(k)}, \quad k \in \mathbb{Z}.$$

So it can be analytically extended to a holomorphic function on the punctured complex plane $\mathbb{C}\backslash\{-1, -2, \ldots\}$. Explicitly, choose $k = 1, 2, \ldots$ and the complex number λ with $\mathfrak{Re}\lambda > -k$ and $\lambda = -1, -2, \cdots, -k+1$, then

$$t_+^\lambda = \sum_{r=1}^{k} \frac{(-1)^{r-1}}{(\lambda+r)(r-1)!}\delta^{(r-1)} + \left(t_+^\lambda\right)_{reg,\,k}.$$

Here, for all test functions $\phi \in \mathscr{S}(\mathbb{R})$, the regular part is defined by

$$\langle \left(t_+^\lambda\right)_{reg,\,k},\ \phi\rangle = \int_0^1 t^\lambda \left(\phi(t) - \sum_{r=0}^{k-1} \frac{\phi^{(r)}(0)}{r!}\right) dt + \int_1^\infty t^\lambda \phi(t)\,dt.$$

We say that the map $\lambda \longrightarrow t_+^\lambda$ is a meromorphic function on the complex plane $\mathbb{C}$, with values in the space $\mathscr{S}'(\mathbb{R})$ of tempered distributions. This function has simple poles at the points $\lambda = -1, -2, \cdots$, with the residues

$$\text{Res}_{\lambda=-k}\left(t_+^\lambda\right) = \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}, \quad k = 1, 2, \cdots$$

Parallel to t_+ we can define t_-

$$t_- = \begin{cases} 0 & \text{if} \quad t \geq 0 \\ |\,t\,| & \text{if} \quad t \leq 0, \end{cases}$$

or

$$\langle t_-,\ \phi\rangle = \int_{-\infty}^{\infty} t_- \phi(t)\,dt = \int_{-\infty}^{0} |\,t\,|\,\phi(t)\,dt.$$

Similarly, let $\phi \in \mathscr{S}(\mathbb{R})$, the function $\lambda \mapsto t_-^\lambda(\phi)$, defined on the open half-plane $\{\lambda \in \mathbb{C},\ \mathfrak{Re}\lambda > 0\}$ can be analytically extended to a holomorphic function on the punctured complex plane $\mathbb{C} \setminus \{-1, -2, \cdots\}$. Explicitly, choose $k = 1, 2, \cdots$ and the complex number λ with $\mathfrak{Re}\lambda > -k$ and $\lambda = -1, -2, \cdots, -k+1$, then

$$t_-^\lambda = \sum_{r=1}^{k} \frac{1}{(\lambda+r)(r-1)!}\delta^{(r-1)} + \left(t_-^\lambda\right)_{reg,\,k}.$$

Here, for all test functions $\phi \in \mathscr{S}(\mathbb{R})$, the regular part is defined by

$$\langle \left(t_-^\lambda\right)_{reg,\,k},\ \phi\rangle = \int_{-1}^0 |\,t\,|^\lambda \left(\phi(t) - \sum_{r=0}^{k-1} \frac{\phi^{(r)}(0)}{r!}\right) dt + \int_{-\infty}^{-1} |\,t\,|^\lambda\ \phi(t)\,dt.$$

We say that the map $\lambda \longrightarrow t_+^\lambda$ is a meromorphic function on the complex plane $\mathbb{C}$, with values in the space $\mathscr{S}'(\mathbb{R})$ of tempered distributions. This function has simple poles at the points $\lambda = -1, -2, \cdots$ with the residues

$$\mathrm{Res}_{\lambda=-k}\left(t_+^\lambda\right) = \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}, \quad k = 1, 2, \cdots$$

Remark 8 The tempered distribution $T = \frac{1}{2}(t_+ + t_-)$ is a fundamental solution of the differential operator $\frac{d^2}{dx^2}$, that is $\frac{d^2}{dx^2}T = \delta$, since for the second derivatives we have

$$\ddot{t}_+ = \delta_0, \quad \ddot{t}_- = \delta_0.$$

Furthermore, let $\mathscr{D}'(\mathbb{R})_+$ be the space of distributions $u \in \mathscr{D}'(\mathbb{R})$ such that there exists an $l \in \mathbb{R}$ with $\mathrm{supp}\, u \subset [l, \infty)$. For $u, v \in \mathscr{D}'(\mathbb{R})_+$, the convolution product $u * v$ is well defined and defines an element of $\mathscr{D}'(\mathbb{R})_+$. Each $u \in \mathscr{D}'(\mathbb{R})_+$ possesses a k-th order antiderivative in $\mathscr{D}'(\mathbb{R})_+$ given by

$$v = u * \gamma_+^k.$$

By analytic continuation we obtain

$$\partial\gamma_+^{s+1} = \gamma_+^s, \quad x\gamma_+^s = s\gamma_+^{s+1},$$

and similarly to (21), the **F**-equation

$$x\partial\gamma_+^s = (s-1)\gamma_+^s \tag{29}$$

holds.

The distributions $(x+i0)^\lambda$ and $(x-i0)^\lambda$ are well known [15], p.59, and are defined by

$$(x+i0)^\lambda = \begin{cases} e^{i\lambda\pi} \mid x \mid^\lambda & \text{for} \quad x < 0 \\ x^\lambda & \text{for} \quad x > 0 \end{cases} ; \; (x-i0)^\lambda = \begin{cases} e^{-i\lambda\pi} \mid x \mid^\lambda & \text{for} \quad x < 0 \\ x^\lambda & \text{for} \quad x > 0 \end{cases}$$

or, in a more concise form

$$(x+i0)^\lambda = x_+^\lambda + e^{i\lambda\pi}x_-^\lambda, \qquad (x-i0)^\lambda = x_+^\lambda + e^{-i\lambda\pi}x_-^\lambda.$$

The distributions $(x+i0)^\lambda$, $(x-i0)^\lambda$ are entire functions of λ [15], p. 94. Moreover, as distributions we have

$$\frac{d}{dx}(x+i0)^\lambda = \lambda(x+i0)^{\lambda-1}, \qquad \frac{d}{dx}(x-i0)^\lambda = \lambda(x-i0)^{\lambda-1}.$$

Remark 9 For each $\epsilon > 0$, the function $(x \pm i\epsilon)^\lambda$ defines a tempered distribution, and we have in $\mathscr{S}'$

$$(x \pm i0)^\lambda = \lim_{\epsilon \to 0}(x \pm i\epsilon)^\lambda.$$

Let

$$\mathbf{H}_\pm = \{z = x + iy \in \mathbb{C}, x \in \mathbb{R}, \pm y > 0\}$$

The function $f(z) = \frac{1}{z}$, defined on $\mathbf{H}_\pm$, has a limit in $\mathscr{D}'(\mathbb{R})$, when y tends to 0

$$\lim_{y\to 0,\, y>0} \frac{1}{x \pm iy} = \frac{1}{x \pm i0},$$

with

$$\frac{1}{x+i0} - \frac{1}{x+i0} = -2i\pi\delta, \qquad \frac{1}{x+i0} + \frac{1}{x+i0} = 2\mathrm{VP}\frac{1}{x}. \tag{30}$$

It is important to observe that the first relation implies the Poisson summation formula. In fact the two functions

$$f_+(z)=\sum_{n=0}^{\infty}e^{2i\pi nz},\qquad f_-(z)=\sum_{-\infty}^{-1}e^{2i\pi nz}=\sum_{1}^{\infty}e^{-2i\pi nz}. \tag{31}$$

are holomorphic in $\mathbf{H}_+$ and $\mathbf{H}_-$ respectively, with

$$f_+(z)=\frac{1}{1-e^{2i\pi z}},\ \ z\in\mathbf{H}_+;\qquad f_-(z)=\frac{e^{-2i\pi z}}{1-e^{-2i\pi z}}=-\frac{1}{1-e^{2i\pi z}},\ \ z\in\mathbf{H}_-.$$

In a neighborhood U of the $(-1,\ 1)$ we have

$$f_+(z)=-\frac{1}{z}+h_+(z),\quad f_-(z)=\frac{1}{z}+h_-(z),$$

with $h_+,\ h_-$ holomorphic in $(-1,\ 1)$. Hence, as a hyperfunction on $(-1,\ 1)$

$$\begin{aligned}\sum_{-\infty}^{\infty}e^{2i\pi nx}&=\frac{1}{1-e^{2i\pi(x+i0)}}-\frac{1}{1-e^{2i\pi(x-i0)}}\\&=-\frac{1}{2i\pi}\left(\frac{1}{x+i0}-\frac{1}{x-i0}\right)=\delta_0=\sum_{k\in\mathbb{Z}}\delta_n|_{(-1,1)}.\end{aligned}$$

Since the left side is 1-periodic, we obtain the Poisson summation formula, as equality between two hyperfunctions.

$$\sum_{-\infty}^{\infty}e^{2i\pi nx}=\sum_{-\infty}^{\infty}\delta_n.$$

But actually the summation formula is also valid in the sense of distributions. We need only to make explicit the functions h_+, h_- on $(-1,1)$. We use (28) in the form

$$\frac{1}{1-e^{-z}}=\frac{1}{z}+\frac{1}{2}+2\sum_{k=1}^{\infty}\frac{z}{z^2+4\pi^2k^2},\quad z\notin 2i\pi\mathbb{Z},$$

which is a variant of the self-reciprocal relation

$$2\int_0^{\infty}\sin xt\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}\right)dt=\frac{1}{e^x-1}-\frac{1}{x}.$$

Hence, in $\mathscr{D}'\left(\mathbb{S}^1\right)$

$$\sum_{-\infty}^{\infty}e^{2i\pi nx}=\frac{1}{1-e^{2i\pi(x+i0)}}-\frac{1}{1-e^{2i\pi(x-i0)}}=-\frac{1}{2i\pi}\left(\frac{1}{x+i0}-\frac{1}{x-i0}\right)=\delta_0.$$

or, in $\mathscr{D}'(\mathbb{R})$

$$\sum_{-\infty}^{\infty}e^{2i\pi nx}=\sum_{-\infty}^{\infty}\delta_n.$$

5.2 The operator $-i\dfrac{d}{dx}$ on the circle.

We would like to look at a variant of the antecedent ideas, which amounts to comparing the functions cot and coth. It is to balance the role played by the two (partial fractions decompositions of the) functions

$$\pi\cot\pi x=1+2x\sum_{n=1}^{\infty}\frac{1}{x^2-n^2},\quad x\in\mathbb{R}\setminus\mathbb{Z},$$

and

$$\coth x = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{1}{x^2 + \pi^2 n^2}, \quad x \in \mathbb{R} \setminus i\pi\mathbb{Z}.$$

Given a trigonometric series

$$S = \sum_{-\infty}^{\infty} a_n e^{inx},$$

its conjugate is defined as

$$T = -i \sum_{-\infty}^{\infty} (\operatorname{sg} n) a_n e^{inx},$$

where $\operatorname{sg} n = 1$ if $n > 0$, $\operatorname{sg} n = -1$ if $n < 0$ and $\operatorname{sg} 0 = 0$. Formally we have

$$S + iT = a_0 + 2 \sum_{n=1}^{\infty} a_n e^{inx}.$$

The space $H = L^2(\mathbb{S}^1, \mathbb{C})$ admits the decomposition

$$H = H^+ \oplus H^-,$$

where

$$H^+ = \Big\{ f \in H,\ f(x) = \sum_{n=0}^{\infty} a_n e^{inx} \Big\}, \quad H^- = \Big\{ f \in H,\ f(x) = \sum_{n<0} a_n e^{inx} \Big\}.$$

Every element of H is the boundary value of an holomorphic function in the open unit disk. A major result in Fourier Analysis is that the conjugate function $J(f)$ of $f \in H^+$ is a singular integral. In other words, the operator defining the decomposition $H = H^+ \oplus H^-$ is the singular operator integral [16], (Theorem 16, p.122) and [9] (section 13.9)

$$f \longrightarrow J(f)(x) = \frac{1}{2\pi} \mathrm{PV} \int_0^{2\pi} K(x,t) f(t)\, dt,$$

where, formally,

$$K(x,t) = \sum_{n=0}^{\infty} e^{in(x-t)} - \sum_{n<0} e^{in(x-t)} = 1 + i \cot \frac{1}{2}(x-t).$$

The hyperfunction interpretation is as follows: We replace $x - t$ in $K(x,t)$ by x and we use the twin functions (31) and the second relation (30) to obtain on $(-\pi, \pi)$

$$\sum_{n=0}^{\infty} e^{inx} - \sum_{n<0} e^{inx} = -\frac{1}{i} \left(\frac{1}{x+i0} + \frac{1}{x-i0} \right) = 2i\, \mathrm{PV} \frac{1}{x}.$$

5.3 Formal considerations

It is maybe not so surprising that the functions cos and cosh play parallel roles. For a real k the cosine Fourier transform of $f(x) = e^{-\frac{1}{2}x^2} \cos kx$ is $F_c(x) = e^{-\frac{1}{2}(k^2+x^2)} \cosh kx$, which amounts to the following extremely elegant formula linking the two functions, valid for $\alpha > 0,\ \beta > 0,\ \alpha\beta = 2\pi$

$$\sqrt{\alpha} \left(\frac{1}{2} + \sum_{n=1}^{\infty} e^{-\frac{1}{2}(\alpha^2+n^2)} \cos k\alpha n \right) = \sqrt{\beta} e^{-\frac{1}{2}k^2} \left(\frac{1}{2} + \sum_{n=1}^{\infty} e^{-\frac{1}{2}\beta^2 n^2} \cosh k\beta n \right),$$

which is a direct application of the Poisson summation formula. A second formal consideration concerns the calculations of the previous subsection, which can be linked by the classical limit relations

$$\lim_{t\to\infty} e^{itx}\,\mathrm{PV}\frac{1}{x} = \delta_0.$$

Finally a third formal consideration is that many of the formulas obtained could be explained by pullbacks and pushforwards of distributions and some nonlinear transformations of the Dirca δ-function. Suppose that X and Y are open subsets of $\mathbb{R}^n$ and $\Phi : X \longrightarrow Y$ a C^∞ diffeomorphism. Then Φ is proper and the pushforward $\Phi_* : \mathscr{D}'(X) \longrightarrow \mathscr{D}'(Y)$ is a sequentially continuous linear mapping. The pullback $\Phi^* : \mathscr{D}'(Y) \longrightarrow \mathscr{D}'(X)$ is a continuous linear mapping. In particular

$$\Phi^*\delta_{\Phi(x)} = \frac{1}{|\det D\Phi(x)|}\delta_x,$$

where $\det D\Phi(x)$ is the determinant of the Jacobian $D\Phi(x)$ of Φ at x. A formal extension, chiefly used in the Physics literature [24] [Eq. (A.18)] is, in the one dimensional case and for simple zeros,

$$\delta_{\Phi(x)} = \sum_m \frac{1}{|\Phi'(x_m)|}\delta_{x_m}, \quad \Phi(x_m) = 0, \quad \Phi'(x_m) \neq 0. \tag{32}$$

With this formula, the Poisson summation formula takes the form

$$\sum_{-\infty}^{\infty}\delta_n = \sum_{-\infty}^{\infty} e^{2i\pi nx} = \pi\,\delta_{\tan\pi x}.$$

6 On the Kieper theorem

We have seen the relations between the logarithmic derivatives of the Γ-function and the Hurwitz zeta function

$$\psi^{(n)}(z) = \frac{d^{n+1}}{dz^{n+1}}\log\Gamma(z) = (-1)^{n+1}n!\zeta(n+1, z).$$

The goal of this section is to give a similar relation for complex derivative of the digamma function $\psi(z)$. The Laurent series expansion can be used to define generalized Stieltjes constants that occur in the series

$$\zeta(s, a) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(a)(s-1)^n.$$

In particular, the constant term is given by

$$\lim_{s\to 1}\left[\zeta(s, a) - \frac{1}{s-1}\right] = -\frac{\Gamma'(a)}{\Gamma(a)} = -\psi(a).$$

As a special case, $\gamma_0(1) = -\psi(1) = \gamma_0 = \gamma$, the classical Euler's constant.

The Stieltjes constants γ_n are the Laurent coefficients in the expansion of the Riemann zeta function at $s = 1$

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n\,(s-1)^n.$$

The generalized Stieltjes constants, denoted as $\gamma_n(q)$, are the coefficients of the Laurent expansion of the Hurwitz zeta function at $s = 1$

$$\zeta(s, a) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(a)\,(s-1)^n.$$

Hence

$$\lim_{s\to 1}(\zeta(s, a) - \zeta(s)) = \gamma_0(a) - \gamma_0.$$

According to Franel [13], Blagoushine [1]

$$\gamma_n = -\frac{2\pi}{n+1}\int_{-\infty}^{\infty}\frac{\left\{\log\left(\frac{1}{2}+iu\right)\right\}^{n+1}}{\left(e^{\pi u}+e^{-\pi u}\right)^2}\,du \tag{33}$$

The Hurwitz zeta function also has the following integral representation

$$\zeta(s,\ a) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}e^{-ax}}{1-e^{-x}}\,dx,$$

where $\mathfrak{Re}(s) > 1$ and $0 < a \leq 1$. According to [2] we have

$$\gamma_n(a) = \lim_{m\to\infty}\left(\sum_{k=0}^{m}\frac{\log^n(k+a)}{k+a} - \frac{\log^{n+1}(m+a)}{n+1}\right),$$

$$\gamma_n(a) = -\frac{2\pi}{n+1}\int_{-\infty}^{\infty}\frac{\left\{\log\left(a-\frac{1}{2}+iu\right)\right\}^{n+1}}{\left(e^{\pi u}+e^{-\pi u}\right)^2}\,du$$

which generalizes the classical formula

$$\gamma_n = \lim_{m\to\infty}\left(\sum_{k=1}^{m}\frac{\log^n k}{k} - \frac{\log^{n+1} m}{n+1}\right),$$

and also (33)

$$\log(\Gamma(s)) = -\gamma_0(s-1) + \sum_{n=2}^{\infty}(-1)^n\zeta(n)(s-1)^n.$$

We have the relation

$$D_a\psi(a) = \frac{1}{a^2} + \sum_{n=1}^{\infty}\frac{1}{(a+n)^2} = \zeta(2,a), \quad D_a = \frac{d}{da}.$$

The function ψ is related to zeta functions and the Euler's constant γ by

$$\lim_{s\to 1}\left\{\zeta(s) - \zeta(s,a)\right\} = \psi(a) + \gamma.$$

In this section we define fractional integrals and derivatives by, respectively,

$${}_cT_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_c^t (t-u)^{\alpha-1} f(u)du, \qquad {}_cD_t^{\alpha} f(t) = \frac{d^n}{dt^n}\left({}_cT_t^{n-\alpha} f(t)\right). \tag{34}$$

We unify our notations by setting

$${}_cD_t^{-\alpha} f(t) = {}_cI_t^{\alpha} f(t)$$

to get a function ${}_cD_t^{\alpha} f(t)$ analytic in $\alpha \in \mathbb{C}$. Some classical examples are

$${}_0D_t^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \quad {}_{\infty}D_t^{\alpha}(e^{kt}) = k^{\alpha}e^{kt}.$$

Theorem 5 *(Kieper) For complex values of argument s, the Hurwitz zeta $\zeta(s,a)$ function and the digamma function $\psi(z)$ are related by*

$$\zeta(s,a) = \frac{(-1)^s}{\Gamma(s)}\,{}_{\infty}D_a^{s-1}\psi(a), \quad \mathfrak{Re}s > 1.$$

In particular

$$\zeta(s) = \frac{(-1)^s}{\Gamma(s)}\,{}_{\infty}D_a^{s-1}\psi(a)_{|a=1}, \quad \mathfrak{Re}s > 1.$$

The Lerch transcendent (1)

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

under the conditions

$$|z| \le 1,\ a \notin \{0, -1, -2, \cdots\},\ \ \mathfrak{Res} > 1$$

verifies

$$\Phi(z, 0, a) = \frac{1}{1-z}, \quad \Phi(z, s-1, a) = \left(a + z\frac{\partial}{\partial z}\right)\Phi(z, s-1, a).$$

The fundamental relation of Lerch is

$$\frac{(2\pi)^s z^a}{\Gamma(s)}\Phi(z, 1-s, a) = e^{\frac{i\pi s}{2}}\Phi\left(e^{-2ia\pi}, s, \frac{\log z}{2i\pi}\right) + e^{2i\pi a - \frac{i\pi s}{2}}\Phi\left(e^{2ia\pi}, s, 1 - \frac{\log z}{2i\pi}\right).$$

Moreover

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{t^{s-1}e^{-at}}{1 - ze^{-t}}\, dt.$$

Taking $z = e^x$ and defining the distribution

$$\gamma_{a,+}(x) = \frac{x_+^{s-1}e^{ax}}{\Gamma(s)},$$

we obtain, as an extension of (27), the convolution relation

$$\Phi(e^x, s, a) = \gamma_{a,+} * \mathrm{li}_0(x).$$

7 Other operators

Given $a \in (0, 1)$, we introduce a new operator on the circle

$$\Delta_a = -\frac{d^2}{d\theta^2} + 2ia\frac{d}{d\theta} + a^2 = \left(i\frac{d}{d\theta} + a\right)^2.$$

The eigenvalues are $\lambda_n = (n-a)^2,\ n = 0, 1, \cdots$. The spectral zeta function of Δ_a, expressed in terms of the Hurwitz zeta function, is

$$\zeta_{\Delta_a}(s) = \sum_{n\in\mathbb{Z}} \frac{1}{(n-a)^{2s}} = \zeta(2s,\ 1-a) + \zeta(2s,\ a).$$

By using the Lerch identity

$$\zeta'(0,\ a) = \log\Gamma(a) - \frac{1}{2}\log(2\pi)$$

we obtain

$$\zeta'(0,\ a) = 2\log\Gamma(a)\Gamma(1-a) - 2\log(2\pi).$$

By the complement formula of the Γ-function, the determinant of Δ_a is

$$\det\Delta_a = e^{-\zeta'_{\Delta_a}(0)} = \frac{(2\pi)^2}{(\Gamma(a)\Gamma(1-a))^2} = 4\sin^2(\pi a).$$

8 A strange formula

In this section we give the polylogarithm function a Lambert series expansion. Lambert series are sums of the form

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}, \quad a_n \in \mathbb{R}.$$

They were considered in connection with the convergence of power series. We refer to [29] (p.125) for the following properties. If the series $\sum_{n=1}^{\infty} a_n$ converges, then the Lambert series converges for all real values of x except at $x = \pm 1$, otherwise it converges for those values of x for which the series $\sum_{n=1}^{\infty} a_n x^n$ converges.

We will need some facts on the convolution of arithmetical sequences, or functions. Let f and g be arithmetic functions with associated Dirichlet series $F(s)$ and $G(s)$

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}.$$

Let $h = f * g$ be the Dirichlet convolution of f and g

$$h(n) = \sum_{d|n} f(d) g(\frac{n}{d}),$$

and let $H(s)$ be the associated Dirichlet series. If $F(s)$ and $G(s)$ converge absolutely at some point s, then so does $H(s)$, and the equality $H(s) = F(s)G(s)$ holds.

We have, from the definition, that if $(a_n)_{n\geq 1}$ is a given sequence, and $\mathbf{1}$ is the constant sequence, $\mathbf{1}(n) = 1, n \geq 1$ the general term of $\mathbf{1} * a$ is $A_n = (\mathbf{1} * a)(n) = \sum_{d|n} a_d$, and

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n} = \sum_{m=1}^{\infty} a_m (x^m + x^{2m} + + x^{3m} + \cdots) = \sum_{n=1}^{\infty} \left(\sum_{d|n} a_d \right) x^n = \sum_{n=1}^{\infty} A_n x^n.$$

This is equivalent to

$$\varphi(s)\zeta(s) = \psi(s), \quad \varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \psi(s) = \sum_{m=1}^{\infty} \frac{A_n}{n^s}, \tag{35}$$

where $\zeta(s)$ is the Riemann zeta function. From

$$\varphi(s) = \frac{1}{\zeta(s)} \psi(s), \quad \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where $\mu(n)$ is the Möbius arithmetical function, we have the inversion formula

$$a_n = \sum_{k|n} \mu(k) A_{\frac{n}{k}}, \quad n \geq 1. \tag{36}$$

If k is a positive integer, the Jordan totient function $J_k(n)$ is defined by the number of ordered k-tuples of integers $x_1, x_2, \cdots, x_k,\ 1 \leq x_i \leq n$, such that

$$\gcd(x_1, x_2, \cdots, x_k, n) = 1.$$

It coincides with Euler's totient function when $k = 1$, and satisfies the following properties

$$\sum_{d|n} J_k(d) = n^k, \tag{37}$$

or, in terms of Dirichlet convolutions, as [23]

$J_k(n) * 1 = n^k,$

By Möbius inversion we obtain

$$J_k(n) = \mu(n) \star n^k = n^k \sum_{d|n} \mu(d) d^{-k}.$$

It also can be given by

$$J_k(n) = n^k \prod_{p|n} \left(1 - \frac{1}{p^k}\right).$$

This last expression allows us to extend the definition to complex values of s

$$J_s(n) = n^s \prod_{p|n} \left(1 - \frac{1}{p^s}\right), \qquad J_s(n) = \mu(n) \star n^s, \quad s \in \mathbb{C}.$$

This an entire function of s. Similarly (37) extends to

$$\sum_{d|n} J_s(d) = n^s, \quad s \in \mathbb{C}. \tag{38}$$

As a consequence, the polylogarithm function admits a Lambert series expansion

$$\sum_{n=1}^{\infty} J_{-s}(n) \frac{z^n}{1 - z^n} = \mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad |z| < 1, \ \Re\mathfrak{e} s \geq 1.$$

Taking into account (35), we see that the previous relation is just another form of the following equality

$$\sum_{n \geq 1} \frac{J_{-s}(n)}{n^z} = \frac{\zeta(z+s)}{\zeta(z)}, \quad \Re\mathfrak{e} z > 1, \quad s \in \mathbb{C}.$$

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