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The General Theory of Superoscillations and Supershifts in Several Variables

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Abstract
In this paper we describe a general method to generate superoscillatory functions of several variables starting from a superoscillating sequence of one variable. Our results are based on the study of suitable infinite order differential operators acting on holomorphic functions with growth conditions of exponential type. Additional constraints are required when dealing with infinite order differential operators whose symbol is a function that is holomorphic in some open set, but not necessarily entire. The results proved for superoscillating sequences in several variables are extended to sequences of supershifts in several variables.

Keywords General superoscillatory functions · Supershifts in several variables · Infinite order differential operators

Mathematics Subject Classification 26A09 · 41A60

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1 Introduction

Superoscillating functions are band-limited functions that can oscillate faster than their fastest Fourier component. Physical phenomena associated with superoscillatory functions have been known for a long time for example in antennas theory, see [33], and in the context of weak values in quantum mechanics, see [1]. In more recent years there has been a wide interest in the mathematical theory of superoscillating functions and of supershifts. This notion, that generalizes the one of superoscillations, was introduced in order to study the evolution of superoscillations as initial data of the Schrödinger equation and of other field equations, like Dirac or Klein-Gordon equations.

The literature on superoscillations is quite large, and without claiming completeness we mention below some of the most relevant (and recent) results. Papers [2–7, 12, 15, 27, 30] and [31] deal with the issue of permanence of superoscillatory behavior when evolved under a suitable Schrödinger equation; papers [19–21, 28, 29] and [32] are mostly concerned with the physical nature of superoscillations, while papers [10, 11, 13, 14, 22–26] develop in depth the mathematical theory of superoscillations. Finally, [7] is a good reference for the state of the art on the mathematics of superoscillations until 2017, and the *Roadmap on Superoscillations* [18] contains the most recent advances in superoscillations theory, and their applications to technology explained by the leading experts in this field.

We note that superoscillatory functions have been considered also in several variables. They have been rigorously defined and studied in [6] and in [9] where we have initiated also the theory of supershifts in more then one variable. The aim of this paper is to further improve the results in [6, 9] and to obtain a very general theory of superoscillations and supershifts.

The results in this paper are directed to a general audience of mathematicians, physicists and engineers, and our main tool is the theory of infinite order differential operators acting on spaces of holomorphic functions.

More precisely, in this paper we consider analytic functions in one variable $G_1, \ldots, G_d$, $d \geq 2$, whose Taylor series at zero has radius of convergence greater than 1 and, possibly, less than ∞. This is a novelty with respect to [9] where it was considered only the case in which $G_1, \ldots, G_d$ are entire functions. Thus we define general superoscillating functions of several variables as expressions of the form

$$F_n(x_1, x_2, \ldots, x_d) := \sum_{j=0}^{n} Z_j(n,a) e^{ix_1 G_1(h_j(n))} e^{ix_2 G_2(h_j(n))} \ldots e^{ix_d G_d(h_j(n))}$$

where $Z_j(n,a)$, for $j = 0, \ldots, n$, $n \in \mathbb{N}$ are suitable coefficients of a superoscillating function in one variable as we will see in the sequel. We will give conditions on the functions $G_1, \ldots, G_d$ in order that

$$\lim_{n\to\infty} F_n(x_1, x_2, \ldots, x_d) = e^{ix_1 G_1(a)} e^{ix_2 G_2(a)} \ldots e^{ix_d G_d(a)},$$

so that, when $|G_\ell(a)| > 1$, $F_n(x_1, x_2, \dots, x_d)$ is superoscillating. Moreover, we shall also treat the case of sequences that admit a supershift in $d \geq 2$ variables.

The paper is organized in four sections including the introduction. Section 2 contains the preliminary material on superoscillations, the relevant function spaces and their topology, and the study of the continuity of some infinite order differential operators acting on such spaces. Section 3 is the main part of the paper and contains the definition of superoscillating functions in $d \geq 2$ variables as well as some results. Section 4 discusses the notion of supershift in this framework.

2 Preliminary Results on Infinite Order Differential Operators

We begin this section with some preliminary material on superoscillations and supershifts in one variable. Then we introduce some infinite order differential operators that will be of crucial importance to define and study superoscillations and supershifts in several variables.

Definition 2.1 We call *generalized Fourier sequence* a sequence of the form

$$f_n(x) := \sum_{j=0}^{n} Z_j(n,a) e^{ih_j(n)x}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}, \tag{1}$$

where a belongs to an open subset of $\mathbb{R}$, $Z_j(n,a)$ and $h_j(n)$ are complex and real valued functions of the variables n, a and n, respectively. The sequence (1) is said to be *a superoscillating sequence* if $\sup_{j,n} |h_j(n)| \leq 1$ and there exists an open subset of $\mathbb{R}$, which will be called *a superoscillation set*, on which $f_n(x)$ converges locally uniformly to $e^{ig(a)x}$, where g is a continuous real valued function in an open subset of $\mathbb{R}$ such that $|g(a)| > 1$.

Remark 2.2 In many applications, as we show in the next example, the index n of a superoscillating sequence may not reach the value 0. In this case we write $n \in \mathbb{N}_0 := \mathbb{N} \setminus \{0\}$.

The classical Fourier expansion is obviously not a superoscillating sequence since its frequencies are not, in general, bounded.

Example 2.3 The most important example of superoscillating sequence is

$$f_n(x) := \left(\cos\left(\frac{x}{n}\right) + ia\sin\left(\frac{x}{n}\right)\right)^n = \sum_{j=0}^{n} C_j(n,a) e^{i(1-2j/n)x}, \quad n \in \mathbb{N}_0 \text{ and } x \in \mathbb{R},$$

where $a > 1$ and the coefficients $C_j(n,a)$ are given by

$$C_j(n,a) = \binom{n}{j} \left(\frac{1+a}{2}\right)^{n-j} \left(\frac{1-a}{2}\right)^{j}.$$

If we fix $x \in \mathbb{R}$ and we let n go to infinity, we obtain that

$$\lim_{n\to\infty} f_n(x) = e^{iax},$$

and the limit is uniform on the compact sets of the real line. From this example it is possible to construct other examples, like e.g.

$$f_n(x) := \sum_{j=0}^{n} C_j(n,a)e^{i(1-2j/n)^m x}, \quad n \in \mathbb{N}_0 \text{ and } x \in \mathbb{R},$$

where $a > 1$ and $m \in \mathbb{N}$. If we fix $x \in \mathbb{R}$ and we let n go to infinity, we obtain that

$$\lim_{n\to\infty} f_n(x) = e^{ia^m x}$$

where the limit is uniform on the compact sets of the real line.

Our approach to the study of superoscillatory functions in one or several variables makes use of infinite order differential operators. Such operators naturally act on spaces of holomorphic functions. This is the reason for which we consider the holomorphic extension to entire functions of the sequence $f_n(x)$ defined in (2.1) by replacing the real variable x with the complex variable ξ. For the sequences of entire functions we shall consider, a natural notion of convergence is the convergence in the space A_1 or in the space $A_{1,B}$ for some real positive constant B (see the following definition and considerations). These spaces are recalled below:

Definition 2.4 The space A_1 is the complex algebra of entire functions such that there exists $B > 0$ such that

$$\sup_{\xi\in\mathbb{C}} \big(|f(\xi)| \exp(-B|\xi|)\big) < +\infty. \tag{2}$$

The space A_1 has a rather complicated topology, see e.g. [17], since it is a linear space obtained via an inductive limit. For our purposes, it is enough to consider, for any fixed $B > 0$, the set $A_{1,B}$ of functions f satisfying (2), and to observe that

$$\|f\|_B := \sup_{\xi\in\mathbb{C}} \big(|f(\xi)| \exp(-B|\xi|)\big)$$

defines a norm on $A_{1,B}$, called the B-norm. One can prove that $A_{1,B}$ is a Banach space with respect to this norm.

Moreover, let f and a sequence $(f_n)_n$ belong to A_1; f_n converges to f in A_1 if and only if there exists B such that $f, f_n \in A_{1,B}$ and

$$\lim_{n\to\infty} \sup_{\xi\in\mathbb{C}} \big|f_n(\xi) - f(\xi)\big| e^{-B|\xi|} = 0.$$

With these notations and definitions we can make the notion of continuity in A_1 explicit (see [14]):

A linear operator $\mathcal{U} : A_1 \to A_1$ is continuous if and only if for any $B > 0$ there exists $B' > 0$ and $C > 0$ such that

$$\mathcal{U}(A_{1,B}) \subset A_{1,B'} \text{ and } \quad \|\mathcal{U}(f)\|_{B'} \leq C\|f\|_B, \quad \text{for any} f \in A_{1,B}. \tag{3}$$

before to continue, it is useful to recall that in the recent paper [8] we enlarged the class of superoscillating functions, with respect to the existing literature, by solving the following problem.

Problem 2.5 *Let $h_j(n)$ be given points in $[-1, 1]$, $j = 0, 1, \ldots, n$, for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. Determine the coefficients $X_j(n)$ of the sequence*

$$f_n(x) = \sum_{j=0}^{n} X_j(n) e^{ih_j(n)x}, \quad x \in \mathbb{R}$$

in such a way that

$$f_n^{(p)}(0) = (ia)^p, \quad \text{for} \quad p = 0, 1, \ldots, n.$$

Remark 2.6 The conditions $f_n^{(p)}(0) = (ia)^p$ mean that the functions $x \mapsto e^{iax}$ and $x \mapsto f_n(x)$ have the same derivatives at the origin, for $p = 0, 1, \ldots, n$, and therefore the same Taylor polynomial of order n.

Theorem 2.7 (Solution of Problem 2.5) *Let $h_j(n)$ be a given set of points in $[-1, 1]$, $j = 0, 1, \ldots, n$ for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. If $h_j(n) \neq h_i(n)$, for every $i \neq j$, then the coefficients $X_j(n, a)$ are uniquely determined and given by*

$$X_j(n, a) = \prod_{k=0,\ k \neq j}^{n} \Big(\frac{h_k(n) - a}{h_k(n) - h_j(n)}\Big). \tag{4}$$

As a consequence, the sequence

$$f_n(x) = \sum_{j=0}^{n} \prod_{k=0,\ k \neq j}^{n} \Big(\frac{h_k(n) - a}{h_k(n) - h_j(n)}\Big) e^{ixh_j(n)}, \quad x \in \mathbb{R}$$

solves Problem 2.5. Moreover, when the holomorphic extensions of the functions f_n converge in A_1, we have

$$\lim_{n \to \infty} f_n(x) = e^{iax}, \quad \text{for all} \quad x \in \mathbb{R}.$$

Remark 2.8 In Theorem 2.7, the hypothesis that the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ admits an extension to $\mathbb{C}$ that converges in A_1 can be removed. Indeed, by the residue theorem we have

$$f_n(x) = e^{iax} - \frac{1}{2i\pi}\int_\Gamma \left(\prod_{j=0}^{n} \frac{a-h_j(n)}{\zeta - h_j(n)}\right) \frac{e^{i\zeta x} d\zeta}{\zeta - a},$$

where Γ is any piecewise smooth simple loop which support surrounds in $\mathbb{C}$ the real segment $[-1,1]$ and the point $a \in \mathbb{R}$. Taking $\Gamma = \Gamma_a : \theta \in [0, 2\pi] \mapsto (|a|+2)e^{i\theta}$, $\xi \in \mathbb{C}$ in place of $x \in \mathbb{R}$, leads to the estimate

$$|e^{i\xi a} - f_n(\xi)| \le (|a|+2)\frac{e^{(|a|+2)|\xi|}}{2} \tag{5}$$

since $|a - h_j(n)|/|\zeta - h_j(n)| \le (|a|+1)/((|a|+2)-1) = 1$ for any $j = 0, \dots, n$ and $\zeta \in \operatorname{Supp}\Gamma$. This shows that the sequence $\{f_n(\xi)\}_{n\in\mathbb{N}}$ is bounded in $A_1(\mathbb{C})$. Since $f_n(\zeta)$ shares the same derivatives at the origin (up to order n) than $e^{i\zeta a}$, by inequality (5) it converges to $e^{i\zeta a}$ in $A_{1,B}$ for any $B > |a|+2$ and locally uniformly in a (see for the case $\prod_{\ell=0,\ell\neq j} |h_\ell(n) - h_j(n)| \ge \delta^n$ [16] while for the general case [25]). A similar formula holds for

$$f_n(x) = \sum_{j=0}^{n} \prod_{k=0,\ k\neq j}^{n} \left(\frac{h_k(n) - a}{h_k(n) - h_j(n)}\right) e^{ixG(h_j(n))}$$

where G is a holomorphic function in an open neighborhood Ω_G of $[-1,1]$. In this case we have

$$f_n(\xi) = e^{i\xi G(a)} - \frac{1}{2i\pi}\int_{\Gamma_G} \left(\prod_{j=0}^{n} \frac{a-h_j(n)}{\zeta - h_j(n)}\right) \frac{e^{i\xi G(\zeta)} d\zeta}{\zeta - a},$$

where Γ_G is a piecewise smooth curve surranding a and contained in $\Omega_G \setminus [-1,1]$.

The following result, see Lemma 2.9 in [13], gives a characterization of the functions in A_1 in terms of the coefficients appearing in their Taylor series expansion.

Lemma 2.9 *The entire function*

$$f(\xi) = \sum_{j=0}^{\infty} a_j \xi^j$$

belongs to A_1 *if and only if there exists* $C_f > 0$ *and* $b > 0$ *such that*

$$|a_j| \le C_f \frac{b^j}{\Gamma(j+1)}.$$

Remark 2.10 To say that $f \in A_1$ means that $f \in A_{1,B}$ for some $B > 0$. The computations in the proof of Lemma 2.9 given in [13], show that $b = 2eB$, and that we can choose $C_f = \|f\|_B$.

We now define two types of infinite order differential operators that will be used to study superoscillatory functions and supershifts in several variables. We shall denote by $\underline{x}$ the vector $(x_1, \dots, x_d)$ in $\mathbb{R}^d$.

Proposition 2.11 *Let d be a positive integer and let $R_\ell \in \mathbb{R}_+ \cup \{\infty\}$ for $\ell = 1, \dots, d$. Let $\{g_{1,m}\}_{m\in\mathbb{N}}, \dots, \{g_{d,m}\}_{m\in\mathbb{N}}$ be d sequences of complex numbers such that*

$$\lim \sup_{m\to\infty} |g_{\ell,m}|^{1/m} = \frac{1}{R_\ell}, \quad for \ \ \ell = 1, \dots, d. \tag{6}$$

Let $x_1, \dots, x_d \in \mathbb{R}$. Denote by $D_\xi := \frac{\partial}{\partial \xi}$ the derivative operator with respect to the auxiliary complex variable ξ. We define the formal operator:

$$\mathcal{U}(x_1, x_2, \dots, x_d, D_\xi) := \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{k=0}^{\infty} Y_{k,p} \frac{D_\xi^k}{i^k} \tag{7}$$

where we have set

$$y_m := i x_1 g_{1,m} + \dots + i x_d g_{d,m}, \quad \text{for } \ m \in \mathbb{N}.$$

and

$$Y_{k,p} := \sum_{\nu_1}^{k} \sum_{\nu_2=0}^{\nu_1} \dots \sum_{\nu_{p-1}=0}^{\nu_{p-2}} y_{\nu_{p-1}} y_{\nu_{p-2}-\nu_{p-1}} \dots y_{\nu_1-\nu_2} y_{k-\nu_1}.$$

Then, setting

$$R := \min_{\ell=1,\dots,d} R_\ell,$$

for any real value $0 < B < \frac{R}{4e}$, the operator $\mathcal{U}(x_1, \dots, x_d, D_\xi) : A_{1,B} \to A_{1,4eB}$ is continuous for all $\underline{x} \in \mathbb{R}^d$.

Proof Let us consider $f \in A_{1,B}$; then we have

$$\begin{aligned}\mathcal{U}(x_1, \dots, x_d, D_\xi) f(\xi) &= \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{Y_{k,p}}{p!} \frac{D_\xi^k}{i^k} f(\xi) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{Y_{k,p}}{p!} \sum_{j=k}^{\infty} a_j \frac{j!}{(j-k)!} \xi^{j-k} \\ &= \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{Y_{k,p}}{p!} \sum_{j=0}^{\infty} a_{j+k} \frac{(j+k)!}{j!} \xi^j.\end{aligned}$$

Taking the modulus we get

$$|\mathcal{U}(x_1,\ldots,x_d,D_\xi)f(\xi)| \leq \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{|Y_{k,p}|}{p!}\sum_{j=0}^{\infty}|a_{j+k}|\frac{(j+k)!}{j!}|\xi|^j.$$

and Lemma 2.9 gives the estimate on the coefficients a_{j+k}

$$|a_{j+k}| \leq C_f\frac{b^{j+k}}{\Gamma(j+k+1)}.$$

where $b = 2eB$. Using the well known inequality $(a+b)! \leq 2^{a+b}a!b!$ we also have

$$(j+k)! \leq 2^{j+k}j!k!$$

so we get

$$|\mathcal{U}(x_1,\ldots,x_d,D_\xi)f(\xi)| \leq \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{|Y_{k,p}|}{p!}C_f\sum_{j=0}^{\infty}\frac{b^{j+k}}{\Gamma(j+k+1)}\frac{2^{j+k}k!j!}{j!}|\xi|^j.$$

Now we use the Gamma function estimate

$$\frac{1}{\Gamma(a+b+2)} \leq \frac{1}{\Gamma(a+1)}\frac{1}{\Gamma(b+1)} \tag{8}$$

to separate the series, and we have

$$\frac{1}{\Gamma(j-\frac{1}{2}+k_1-\frac{1}{2}+2)} \leq \frac{1}{\Gamma(j+\frac{1}{2})}\frac{1}{\Gamma(k_1+\frac{1}{2})}$$

and so

$$|\mathcal{U}(x_1,\ldots,x_d,D_\xi)f(\xi)| \leq C_f\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{|Y_{k,p}|}{p!}\frac{k!(2b)^k}{\Gamma(k+\frac{1}{2})}\sum_{j=0}^{\infty}\frac{1}{\Gamma(j+\frac{1}{2})}(2b|\xi|)^j.$$

Now observe that the latter series satisfies the estimate

$$\sum_{j=0}^{\infty}\frac{1}{\Gamma(k+\frac{1}{2})}(2b|\xi|)^j \leq Ce^{4b|\xi|}$$

where C is a positive constant, because of the properties of the Mittag-Leffler function; moreover, the series

$$\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{|Y_{k,p}|}{p!}\frac{k!(2b)^k}{\Gamma(k+\frac{1}{2})} \tag{9}$$

is convergent and it is bounded by a positive real constant $C_{\underline{x},G_1,\dots,G_d}$. In fact, using Stirling formula for the Gamma function, we have

$$k! \sim \sqrt{2\pi k}\, e^{-k} k^k, \quad \text{for} \quad k \to \infty$$

and then we deduce

$$\frac{\Gamma(k+1)}{\Gamma(k+1/2)} \sim \frac{\sqrt{2\pi\, k}\, e^{-k} k^k}{\sqrt{2\pi(k-1/2)}\, e^{-(k-1/2)}\, (k-1/2)^{(k-1/2)}} \sim \sqrt{k-1/2}, \quad \text{for} \quad k \to \infty \tag{10}$$

so that

$$\frac{k!}{\Gamma(k+\frac{1}{2})} \sim \sqrt{k-1/2}, \quad \text{for} \quad k \to \infty. \tag{11}$$

Now observe that (9) has positive coefficients and, by the definition of $Y_{k,p}$ and by (11), it converges if

$$\sum_{p=1}^{\infty} \frac{1}{p!} \sum_{k=1}^{\infty} \left(\sum_{\nu_1=0}^{k} \cdots \sum_{\nu_{p-1}=0}^{\nu_{p-2}} |y_{\nu_{p-1}}||y_{\nu_{p-2}-\nu_{p-1}}| \dots |y_{\nu_1-\nu_2}||y_{k-\nu_1}| \right) (2b)^k \sqrt{k-1/2}$$

converges. Observing that the m-th power of an absolutely convergent series $\sum_{m=0}^{\infty} a_m$ can be computed by means of the Cauchy product:

$$\left(\sum_{m=0}^{\infty} a_m \right)^p = \sum_{k=0}^{\infty} \sum_{\nu_1=0}^{k} \cdots \sum_{\nu_{p-1}=0}^{\nu_{p-2}} a_{\nu_{p-1}} a_{\nu_{p-2}-\nu_{p-1}} \dots a_{k-\nu_1}, \tag{12}$$

and, using the inequality:

$$\begin{aligned} \sqrt{k-\frac{1}{2}} &\leq k \leq \nu_{p-1} + (\nu_{p-2}-\nu_{p-1}) + \ldots + (k-\nu_1) \\ &\leq (\nu_{p-1}+2) \cdot (\nu_{p-2}-\nu_{p-1}+2) \cdot \dots \cdot (k-\nu_{p-1}+2), \end{aligned}$$

where $k \geq \nu_1 \geq \cdots \geq \nu_{p-1} \geq 0$, we deduce that there exists a positive constant $C_{\underline{x},G_1,\dots,G_d}$ such that the following chain of inequalities holds:

$$\begin{aligned} &\sum_{p=1}^{\infty} \frac{1}{p!} \sum_{k=1}^{\infty} \left(\sum_{\nu_1=0}^{k} \cdots \sum_{\nu_{p-1}=0}^{\nu_{p-2}} |y_{\nu_{p-1}}||y_{\nu_{p-2}-\nu_{p-1}}| \dots |y_{k-\nu_1}| \right) (2b)^k \sqrt{k-1/2} \\ &\leq \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{k=1}^{\infty} \left(\sum_{\nu_1=0}^{k} \cdots \sum_{\nu_{p-1}=0}^{\nu_{p-2}} |y_{\nu_{p-1}}(\nu_{p-1}+2)(2b)^{\nu_{p-1}} \right. \end{aligned}$$

$$
||y_{\nu_{p-2}-\nu_{p-1}}(\nu_{p-2}-\nu_{p-1}+2)(2b)^{\nu_{p-2}-\nu_{p-1}}|\times
$$
$$
\ldots \times |y_{k-\nu_1}(k-\nu_1+2)(2b)^{k-\nu_1}|\Big)
$$
$$
= \sum_{p=1}^{\infty} \frac{1}{(p)!} \left[\sum_{m=0}^{\infty} |y_m|(m+2)(2b)^m \right]^p \leq \sum_{p=1}^{\infty} \frac{1}{(p)!} \left[\sum_{m=1}^{\infty} |x_1|(m+2)(2b)^m |g_{1,m}| \right.
$$
$$
\left. + \ldots + |x_d|(m+2)(2b)^m |g_{d,m}| \right]^p \leq C_{\underline{x}, G_1, \ldots, G_d};
$$

note that for the equality we used (12), while the last inequality follows by the assumption

$$
B < \frac{R}{4e}
$$

which implies $2b < R$. From the previous estimate we have that the series (9) converges for all $x_1, \ldots, x_d \in \mathbb{R}$. So we finally have

$$
|\mathcal{U}(x_1, \ldots, x_d, D_\xi) f(\xi)| \leq C_f \, C_{\underline{x}, G_1, \ldots, G_d} \, C \, e^{4b|\xi|}, \quad \underline{x} \in \mathbb{R}^d, \ \ \xi \in \mathbb{C}. \tag{13}
$$

Recalling that $b = 2eB$, the estimate (13) implies that $\mathcal{U}(x_1, \ldots, x_d,) f \in A_{1,8eB}$, in fact

$$
|\mathcal{U}(x_1, \ldots, x_d, D_\xi) f(\xi)| \, e^{-8eB|\xi|} \leq C_f \, C_{\underline{x}, G_1, \ldots, G_d} \, C \quad \underline{x} \in \mathbb{R}^d, \ \ \xi \in \mathbb{C}.
$$

Moreover, we deduce that the $8eB$-norm satisfies the estimate

$$
\|\mathcal{U}(x_1, \ldots, x_d, D_\xi) f\|_{8eB} \leq C_f \, C_{\underline{x}, G_1, \ldots, G_d} \, C = C_{\underline{x}, G_1, \ldots, G_d} \, C \|f\|_B.
$$

Thus $\mathcal{U}(x_1, \ldots, x_d, D_\xi) : A_{1,B} \to A_{1,8eB}$ is continuous for all $\underline{x} \in \mathbb{R}^d$. □

Remark 2.12 Whenever we fix a compact subset $K \subset \mathbb{R}^d$, we have that, for any $\underline{x} \in K$, the constants $C_{\underline{x}, G_1, \ldots, G_d}$ appearing in the proof of the previous theorem are bounded by a constant which depends only on K and $G_1, \ldots, G_d$. Moreover, if $R_\ell = \infty$ for any $\ell = 1, \ldots, d$, the continuity of the operator $\mathcal{U}(x_1, \ldots, x_d, D_\xi)$ holds for any $B > 0$ and the proof shows that the operator $\mathcal{U}(x_1, \ldots, x_d, D_\xi)$ acts continuously from A_1 to itself.

Proposition 2.13 *Let d be a positive integer and let $R_\ell \in \mathbb{R}_+ \cup \{\infty\}$ for any $\ell = 1, \ldots, d$. Let $(g_{1,m}), \ldots, (g_{d,m})$ be d sequences of complex numbers such that*

$$
\limsup_{m \to \infty} |g_{\ell,m}|^{1/m} = \frac{1}{R_\ell}, \quad \textit{for} \ \ \ell = 1, \ldots, d. \tag{14}
$$

We define the formal operator

$$\mathcal{V}(x_1, \ldots, x_d, D_\xi) := \sum_{k=0}^{\infty} Y_k' i^k D_\xi^k, \tag{15}$$

where

$$Y_k' := \sum_{\nu_1=0}^{k} \sum_{\nu_2=0}^{\nu_1} \cdots \sum_{\nu_{d-1}=0}^{\nu_{d-2}} g_{1,\nu_{d-1}} x_1^{\nu_{d-1}} g_{2,\nu_{d-2}-\nu_{d-1}} x_2^{\nu_{d-2}-\nu_{d-1}} \cdots g_{d,k-\nu_1} x_d^{k-\nu_1}$$

and $x_1, \ldots, x_d \in \mathbb{R}$, $\xi \in \mathbb{C}$. *Then, for any real value* $B > 0$, *the operator*

$$\mathcal{V}(x_1, \ldots, x_d, D_\xi) : A_{1,B} \to A_{1,8eB}$$

is continuous whenever $|x_\ell| < \frac{R}{4eB}$ *for any* $\ell = 1, \ldots, d$ *where* $R := \min_{\ell=1,\ldots,d} R_\ell$.

Proof We apply the operator $\mathcal{V}(x_1, \ldots, x_d, D_\xi)$ to a function f in $A_{1,B}$ for $|\underline{x}| < \frac{R}{4eB}$. We have

$$\begin{aligned}\mathcal{V}(x_1, \ldots, x_d, D_\xi) f(\xi) &= \sum_{k=0}^{\infty} \frac{Y_k'}{i^k} D_\xi^k f(\xi) = \sum_{k=0}^{\infty} \frac{Y_k'}{i^k} D_\xi^k \sum_{j=0}^{\infty} a_j \xi^j \\ &= \sum_{k=0}^{\infty} \frac{Y_k'}{i^k} \sum_{j=k}^{\infty} a_j \frac{j!}{(j-k)!} \xi^{j-k} \\ &= \sum_{k=0}^{\infty} \frac{Y_k'}{i^k} \sum_{j=0}^{\infty} a_{k+j} \frac{(k+j)!}{j!} \xi^j.\end{aligned}$$

We then have

$$|\mathcal{V}(x_1, \ldots, x_d, D_\xi) f(\xi)| \le \sum_{k=0}^{\infty} |Y_k'| \sum_{j=0}^{\infty} |a_{k+j}| \frac{(k+j)!}{j!} |\xi|^j$$

and using the estimate in Lemma 2.9

$$|a_{k+j}| \le C_f \frac{b^{k+j}}{\Gamma(k+j+1)},$$

where $b = 2eB$, we get

$$|\mathcal{V}(x_1, \ldots, x_d, D_\xi) f(\xi)| \le \sum_{k=0}^{\infty} |Y_k'| C_f \sum_{j=0}^{\infty} \frac{b^{k+j}}{\Gamma(k+j+1)} \frac{(k+j)!}{j!} |\xi|^j.$$

With the estimates: $(k+j)! \leq 2^{k+j} k! j!$, and

$$\frac{1}{\Gamma(k-\frac{1}{2}+j-\frac{1}{2}+2)} \leq \frac{1}{\Gamma(k+\frac{1}{2})} \frac{1}{\Gamma(j+\frac{1}{2})}$$

we separate the series

$$|\mathcal{V}(x_1,\ldots,x_d,D_\xi)f(\xi)| \leq \sum_{k=0}^{\infty} |Y_k'| \sum_{j=0}^{\infty} C_f b^{k+j} \frac{1}{\Gamma(k+\frac{1}{2})} \frac{1}{\Gamma(j+\frac{1}{2})} \frac{2^{k+j} k! j!}{j!} |\xi|^j.$$

Furthermore, we get

$$|\mathcal{V}(x_1,\ldots,x_d,D_\xi)f(\xi)| \leq C_f \sum_{k=0}^{\infty} |Y_k'| (2b)^k \frac{k!}{\Gamma(k+\frac{1}{2})} \sum_{j=0}^{\infty} \frac{1}{\Gamma(j+\frac{1}{2})} (2b|\xi|)^j.$$

Using (10) we have

$$\frac{k!}{\Gamma(k+\frac{1}{2})} \sim \sqrt{k-1/2}, \quad \text{for} \quad k \to \infty.$$

Moreover, since $\sqrt{k-1/2} \leq k \leq (\nu_{d-1}+2)(\nu_{d-2}-\nu_{d-1}+2)\cdots(k-\nu_1)$, we also obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} |Y_k'| (2b)^k \frac{k!}{\Gamma(k+\frac{1}{2})} &\leq C \sum_{k=0}^{\infty} \sum_{\nu_1=0}^{k} \sum_{\nu_2=0}^{\nu_1} \\
&\cdots \sum_{\nu_{d-1}}^{\nu_{d-2}} |g_{1,\nu_{d-1}}||x_1|^{\nu_{d-1}} (\nu_{d-1}+2)(2b)^{\nu_{d-1}} \\
&\cdot |g_{2,\nu_{d-2}-\nu_{d-1}}||x_2|^{\nu_{d-2}-\nu_{d-1}} (\nu_{d-2}-\nu_{d-1}+2)(2b)^{\nu_{d-2}-\nu_{d-1}} \\
&\cdots |g_{d,k-\nu_1}||x_d|^{\nu_{k-\nu_1}} (k-\nu_1+2)(2b)^{k-\nu_1} \\
&\leq C \prod_{\ell=1}^{d} \left(\sum_{m=0}^{\infty} |g_{\ell,m}||x_\ell|^m m (2b)^m \right).
\end{aligned}$$

Since $|x_\ell| < \frac{R}{4eB}$ for any $\ell = 1,\ldots,d$ and $b = 2eB$, the series $\sum_{m\ell=1}^{\infty} m|g_{\ell,m}|(2b|x_\ell|)^m$, converges to a constant κ_{x_ℓ} which depends on $x_\ell \in \mathbb{R}$. Thus there exist constants C_{x_ℓ}, $\ell = 1,\ldots,d$ such that

$$|\mathcal{V}(x_1,\ldots,x_d,D_\xi)f(\xi)| \leq C_f C_{x_1} \ldots C_{x_d} (2b|\xi|) e^{2b|\xi|} \leq C_f C_{x_1,\ldots,x_d} e^{4b|\xi|}$$

from which, recalling that $C_f = \|f\|_B$, we deduce

$$\|\mathcal{V}(x_1,\ldots,x_d,D_\xi)f\|_{8eB} \leq C_{x_1,\ldots,x_d} \|f\|_B,$$

for some $C_{x_1,\ldots,x_d}$. We conclude that the operator $\mathcal{V}(x_1, \ldots, x_d, D_\xi) : A_{1,B} \to A_{1,8eB}$ is continuous. □

Remark 2.14 Whenever we fix a compact subset

$$K \subset \left\{ \underline{x} \in \mathbb{R}^d \,:\, |x_\ell| < \frac{R}{4eB} \text{ for any } \ell = 1, \ldots, d \right\},$$

we have that, for any $\underline{x} \in K$, the constants C_{x_ℓ}'s, appearing in the proof of the previous theorem are bounded by a constant which depends only on K. Moreover, if $R_\ell = \infty$ for any $\ell = 1, \ldots, d$, the continuity of the operator $\mathcal{V}(x_1, \ldots, x_d, D_\xi)$ holds to be true for any $\underline{x} \in \mathbb{R}^d$ and the proof of the previous theorem shows that $\mathcal{V}(x_1, \ldots, x_d, D_\xi)$ satisfies the conditions in (3). Thus we conclude that the operator $\mathcal{V}(x_1, \ldots, x_d, D_\xi)$ acts continuously from A_1 to itself.

3 Superoscillating Functions in Several Variables

We recall some preliminary definitions related to superoscillating functions in several variables.

Definition 3.1 (Generalized Fourier sequence in several variables) For $d \in \mathbb{N}$ such that $d \geq 2$, we assume that $(x_1, \ldots, x_d) \in \mathbb{R}^d$. Let $\{h_{j,\ell}(n)\}$, $j = 0, \ldots, n$ for $n \in \mathbb{N}$, be real-valued sequences for $\ell = 1, \ldots, d$. We call *generalized Fourier sequence in several variables* a sequence of the form

$$F_n(x_1, \ldots, x_d) = \sum_{j=0}^{n} c_j(n) e^{ix_1 h_{j,1}(n)} e^{ix_2 h_{j,2}(n)} \ldots e^{ix_d h_{j,d}(n)}, \tag{16}$$

where $\{c_j(n)\}_{j,n}$, for $j = 0, \ldots, n$ and $n \in \mathbb{N}$, is a complex-valued sequence.

Definition 3.2 (Superoscillating sequence) A generalized Fourier sequence in several variables $F_n(x_1, \ldots, x_d)$, with $d \in \mathbb{N}$ such that $d \geq 2$, is said to be *a superoscillating sequence* if

$$\sup_{j=0,\ldots,n,\ n\in\mathbb{N}} |h_{j,\ell}(n)| \leq 1, \quad \text{for } \ell = 1, \ldots, d,$$

and there exists an open subset of $\mathbb{R}^d$, which will be called *a superoscillation set*, on which $F_n(x_1, \ldots, x_d)$ converges locally uniformly to $e^{ix_1 G_1(a)} e^{ix_2 G_2(a)} \ldots e^{ix_d G_d(a)}$, where a belongs to an open subset U of $\mathbb{R}$, G_ℓ's are continuous functions of real variable whose domain contains U and $|G_\ell(a)| > 1$ for $\ell = 1, \ldots, d$ and $a \in U$.

In the paper [6] we studied the function theory of superoscillating functions in several variables under the additional hypothesis that there exist $r_\ell \in \mathbb{N}$, such that

$$p = r_1 q_1 + \ldots + r_d q_d. \tag{17}$$

In that case, we proved that for $p, q_\ell \in \mathbb{N}$, $\ell = 1, \ldots, d$, $n \in \mathbb{N}_0$, the function

$$F_n(x, y_1, \ldots, y_d) = \sum_{j=0}^{n} C_j(n, a) e^{ix(1-2j/n)^p} e^{iy_1(1-2j/n)^{q_1}} \ldots e^{iy_d(1-2j/n)^{q_d}}$$

is superoscillating when $|a| > 1$, where $C_j(n, a)$ are suitable coefficients. In the paper [9], we were able to remove the condition (17), while here we will show that it is possible to replace the functions $(1 - 2j/n)^p$ in the exponent of $e^{ix(1-2j/n)^p}$ with more general holomorphic functions. As we shall see, different function spaces are involved in the proofs according to the fact that the holomorphic functions are entire or not.

Theorem 3.3 (The general case of $d \geq 2$ variables) *Let d be a positive integer and let $R_\ell \in \mathbb{R}_+ \cup \{\infty\}$ be such that $R_\ell > 1$ for any $\ell = 1, \ldots, d$. Let $G_1, \ldots, G_d$ be holomorphic functions whose series expansion at zero is given by*

$$G_\ell(\lambda) = \sum_{m_\ell=0}^{\infty} g_{\ell,m} \lambda^{m_\ell}, \quad \forall \ell = 1, \ldots, d \tag{18}$$

and, moreover, the sequences $\{g_{\ell,m}\}_{m\in\mathbb{N}}$ satisfy the condition

$$\lim \sup_{m\to\infty} |g_{\ell,m}|^{1/m} = \frac{1}{R_\ell}, \quad \forall \ell = 1, \ldots, d.$$

Let $a \in \mathbb{R}$ and

$$\left\{ f_n : x \in \mathbb{R} \mapsto \sum_{j=0}^{n} Z_j(n, a) e^{ih_j(n)x} \right\}_{n\in\mathbb{N}}, \tag{19}$$

be a generalized Fourier sequence as in Definition 2.1, with in addition $h_j(n) \in [-1, 1]$ for any $0 \leq j \leq n$. Suppose also that the sequence of entire extensions

$$\left\{ f_n : \xi \in \mathbb{C} \mapsto \sum_{j=0}^{n} Z_j(n, a) e^{ih_j(n)\xi} \right\}_{n\in\mathbb{N}},$$

converges to $\xi \mapsto e^{ia\xi}$ in $A_{1,B}$ for some positive real value $0 < B < \frac{R}{4e}$, where $R := \min_{\ell=1,\ldots,d} R_\ell$. Let also

$$F_n : (x_1, \ldots, x_d) \in \mathbb{R}^d \mapsto \sum_{j=0}^{n} Z_j(n, a) e^{ix_1 G_1(h_j(n))} e^{ix_2 G_2(h_j(n))} \ldots e^{ix_d G_d(h_j(n))}.$$

Then, whenever $|a| < R$ and $a \in \mathbb{R}$, we have

$$\lim_{n\to\infty} F_n(x_1, x_2, \dots, x_d) = e^{ix_1 G_1(a)} e^{ix_2 G_2(a)} \dots e^{ix_d G_d(a)},$$

uniformly on compact subsets of $\mathbb{R}^d$. In particular, the sequence $\{F_n(x_1, x_2, \dots, x_d)\}_{n\geq 0}$ is superoscillating according to Definition (3.2) when $|G_\ell(a)| > 1$ for any $|a| < R$.

Proof Since $R_\ell \geq 1$ for any $\ell = 1, \dots, d$ and $|h_j(n)| \leq 1$, using (12) we have for any $(x_1, \dots, x_d) \in \mathbb{R}^d$ the chain of equalities

$$\begin{aligned}
F_n(x_1, \dots, x_d) &= \sum_{j=0}^{n} Z_j(n, a) \exp\left(\sum_{m=0}^{\infty} y_m (h_j(n))^m\right) \\
&= \sum_{j=0}^{n} Z_j(n, a) \left(\sum_{p=0}^{\infty} \frac{\left(\sum_{m=0}^{\infty} y_m \left(h_j(n)\right)^m\right)^p}{p!}\right) \\
&= \sum_{p=0}^{\infty} \frac{1}{p!} \left(\sum_{k=0}^{\infty} Y_{k,p} \left(\sum_{j=0}^{n} (h_j(n))^k Z_j(n, a)\right)\right) \\
&= \sum_{p=0}^{\infty} \frac{1}{p!} \left(\sum_{k=0}^{\infty} Y_{k,p} \left(\sum_{j=0}^{n} \left[\frac{D_\xi^k}{i^k} \left(Z_j(n, a) e^{ih_j(n)\xi}\right)\right]_{\xi=0}\right)\right) \\
&= \left[\mathcal{U}_g(x_1, \dots, x_d, D_\xi)(f_n(\xi))\right]_{\xi=0}
\end{aligned}$$

where, for any $m, k \in \mathbb{N}$,

$$\begin{aligned}
y_m &= ix_1 g_{1,m} + \cdots + ix_d g_{d,m} \\
Y_{k,p} &= \sum_{\nu_1=0}^{k} \sum_{\nu_2=0}^{\nu_1} \cdots \sum_{\nu_{p-1}=0}^{\nu_{p-2}} y_{\nu_{p-1}} y_{\nu_{p-2}-\nu_{p-1}} \cdots y_{\nu_1-\nu_2} y_{k-\nu_1}
\end{aligned}$$

and $\mathcal{U}_g$ is the continuous operator from $A_{1,B}$ to $A_{1,8eB}$, see Proposition 2.11, defined by

$$\mathcal{U}_g(x_1, \dots, x_d, D_\xi) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{Y_{k,p}}{p!} \left(\frac{D_\xi}{i}\right)^k. \tag{20}$$

The explicit computation of the term $\mathcal{U}_g(x_1, \dots, x_d, D_\xi) e^{i\xi a}$ gives

$$\mathcal{U}_g(x_1, \dots, x_d, D_\xi) e^{i\xi a} = \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{k=0}^{\infty} Y_{k,p} \frac{D_\xi^k}{i^k} e^{i\xi a} = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \frac{Y_{k,p}}{p!} a^k e^{i\xi a}.$$

By the continuity of $\mathcal{U}_g$, we can exchange the operator $\mathcal{U}_g$ with the limit and we finally get

$$
\begin{aligned}
&\lim_{n\to\infty} F_n(x_1,\dots,x_d) = \lim_{n\to\infty} \left[\mathcal{U}_g(x_1,\dots,x_d,D_\xi)(f_n(\xi))\right]_{\xi=0} \\
&= \left[\mathcal{U}_g(x_1,\dots,x_d,D_\xi)e^{i\xi a}\right]_{\xi=0} \\
&= \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{k=0}^{\infty} Y_{k,p}a^k e^{i\xi a}\Big|_{\xi=0} \\
&= \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{k=0}^{\infty}\left(\sum_{\nu_1=0}^{k}\dots\sum_{\nu_k=0}^{\nu_{k-1}}(y_{\nu_{p-1}}a^{\nu_{p-1}})(y_{\nu_{p-2}-\nu_{p-1}}a^{\nu_{p-2}-\nu_{p-1}})\right. \\
&\quad \left.\dots(y_{\nu_1-\nu_2}a^{\nu_1-\nu_2})(y_{k-\nu_1}a^{k-\nu_1})\right) \\
&= \sum_{p=0}^{\infty}\frac{1}{p!}\left(\sum_{m=1}^{\infty} y_m a^m\right)^p = \sum_{p=0}^{\infty}\frac{1}{p!}(ix_1G_1(a)+\dots+ix_dG_d(a))^p \\
&= e^{ix_1G_1(a)+\dots+ix_dG_d(a)},
\end{aligned}
$$

where the third equality is due to the formula (12) and the fourth equality holds because we are assuming $|a| < R$. The previous limit is uniform over the compact subset of $\mathbb{R}^d$ because of Remark 2.12. □

Remark 3.4 From the inspection of the proof we observe that:

(I) The space of the entire functions on which the operator $\mathcal{U}(x_1,\dots,x_d,D_\xi)$ acts is the space $A_{1,B}$ in one complex variable, for some positive real value $0 < B < \frac{R}{4e}$.

(III) The variables $(x_1, x_1,\dots,x_d)$ become the coefficients of the infinite order differential operator $\mathcal{U}(x_1,x_2,\dots,x_d,D_\xi)$, defined in (20), that still acts on the space $A_{1,B}$.

Example 3.5 Using Theorem 3.3 and Remark 2.8, we can construct a superoscillating sequence of d variables. In fact, consider the sequence

$$
\left\{ f_n : \xi \in \mathbb{C} \mapsto \sum_{j=0}^{n} X_j(n,a)e^{ih_j(n)\xi} \right\}_{n\in\mathbb{N}}
$$

where $X_j(n,a)$'s are defined as in (4) and $h_j(n) \in [-1,1]$ are fixed. By Remark 2.8 we know that this sequence converges in A_1 to $e^{i\xi a}$. Let $G_1,\dots,G_d$ be holomorphic functions whose Taylor series centered at zero have rays of convergence equal respectively to $R_1,\dots,R_d$ such that

$$\min_{\ell=1,\ldots,d} R_\ell/4e > 3.$$

We can choose $a \in \mathbb{R}$ such that $|a| > 1$ and

$$|a| + 2 \leq \min_{\ell=1,\ldots,d} R_\ell.$$

If $|G_\ell(a)| > 1$ for all $\ell = 1, \ldots, d$, then by Theorem 3.3 we have that the sequence

$$\{F_n\}_{n\in\mathbb{N}} = \left\{ (x_1, \ldots, x_d) \mapsto \sum_{j=0}^{n} X_j(n, a) \prod_{\ell=1}^{d} e^{ix_\ell G_\ell(a)} \right\}_{n\in\mathbb{N}}$$

is superoscillating and it converges on any compact subset of $\mathbb{R}^d$ towards

$$(x_1, \ldots, x_d) \mapsto \prod_{\ell=1}^{d} e^{ix_\ell G_\ell(a)}.$$

4 Supershifts in Several Variables

The procedure to define superoscillating functions can be extended to the case of supershift. Recall that the supershift property of a function extends the notion of superoscillation and that this concept, that we recall below in the case of one variable, turned out to be a crucial ingredient for the study of the evolution of superoscillatory functions as initial conditions of the Schrödinger equation.

Definition 4.1 (Supershift) Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval with $[-1, 1] \subset \mathcal{I}$ and let $\varphi : \mathcal{I} \times \mathbb{R} \to \mathbb{R}$, be a continuous function on $\mathcal{I}$. We set

$$\varphi_h(x) := \varphi(h, x), \quad h \in \mathcal{I}, \quad x \in \mathbb{R}$$

and we consider a sequence of points $(h_j(n))$ such that

$$h_j(n) \in [-1, 1] \ \text{ for } \ j = 0, \ldots, n \ \text{ and } \ n \in \mathbb{N}.$$

We define the functions

$$\psi_n(x) = \sum_{j=0}^{n} c_j(n) \varphi_{h_j(n)}(x), \tag{21}$$

where $(c_j(n))$ is a sequence of complex numbers for $j = 0, \ldots, n$ and $n \in \mathbb{N}$. If

$$\lim_{n\to\infty} \psi_n(x) = \varphi_a(x)$$

for some $a \in \mathcal{I}$ with $|a| > 1$, we say that the function $\psi_n(x)$, for $x \in \mathbb{R}$, admits a supershift.

Remark 4.2 The term supershift comes from the fact that the interval $\mathcal{I}$ can be arbitrarily large (it can also be $\mathbb{R}$) and that the constant a can be arbitrarily far away from the interval $[-1, 1]$ where the functions $\varphi_{h_{j,n}}(\cdot)$ are indexed, see (21).

Problem 2.5, for the supershift case, is formulated as follows.

Problem 4.3 *Let $h_j(n)$ be a given points in $[-1, 1]$, $j = 0, 1, \dots, n$, for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. Suppose that for every $x \in \mathbb{R}$ the function $h \mapsto G(hx)$ extends to a holomorphic and entire function in h. Consider the functions*

$$f_n(x) = \sum_{j=0}^{n} Y_j(n, a) G(h_j(n)x), \quad x \in \mathbb{R}$$

where $h \mapsto G(hx)$ depends on the parameter $x \in \mathbb{R}$. Determine the coefficients $Y_j(n)$ in such a way that

$$f_n^{(p)}(0) = (a)^p G^{(p)}(0) \quad for \quad p = 0, 1, \dots, n. \tag{22}$$

The solution of Problem 4.3, obtained in [8], is summarized in the following theorem.

Theorem 4.4 *Let $h_j(n)$ be a given set of points in $[-1, 1]$, $j = 0, 1, \dots, n$ for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. If $h_j(n) \neq h_i(n)$ for every $i \neq j$ and $G^{(p)}(0) \neq 0$ for all $p = 0, 1, \dots, n$, then there exists a unique solution $Y_j(n, a)$ of the linear system* (22) *and it is given by*

$$Y_j(n, a) = \prod_{k=0,\ k\neq j}^{n} \Big(\frac{h_k(n) - a}{h_k(n) - h_j(n)} \Big),$$

so that

$$f_n(x) = \sum_{j=0}^{n} \prod_{k=0,\ k\neq j}^{n} \Big(\frac{h_k(n) - a}{h_k(n) - h_j(n)} \Big) G(h_j(n)x), \quad x \in \mathbb{R}.$$

Remark 4.5 By the residue theorem, the functions f_n's admit the following integral representation formula

$$f_n(\xi) = G(a\xi) - \frac{1}{2\pi} \int_\Gamma \left(\prod_{j=0}^{n} \frac{a - h_j(n)}{\zeta - h_j(n)} \right) \frac{G(\xi\zeta) d\zeta}{\zeta - a},$$

where Γ is a simple loop which support surrounds $[-1, 1]$. If $G \in A_1$, that is $|G(\xi)| \leq Ae^{B|\xi|}$ for some $A,\ B \geq 0$, it follows from the integral representation formula that

$$|G(a\xi) - f_n(\xi)| \leq A(|a|+2)\frac{e^{(|a|+2)B|\xi|}}{2} \quad \forall \xi \in \mathbb{C},\ \forall n \in \mathbb{N}.$$

Since $\xi \mapsto G(a\xi)$ and f_n share the same Taylor coefficients at the origin up to order n, the convergence of the sequence $\{f_n\}_{n\in\mathbb{N}}$ towards $\xi \mapsto G(a\xi)$ holds in $A_{1,B'}$, and thus in A_1, provided $B' > (|a|+2)B$.

We can now extend the notion of supershift of a function in several variables.

Definition 4.6 (Supershifts in several variables) Let $|a| > 1$. For $d \in \mathbb{N}$ with $d \geq 2$, we assume that $(x_1, \ldots, x_d) \in \mathbb{R}^d$. Let $\{h_{j,\ell}(n)\}$, $j = 0, \ldots, n$ for $n \in \mathbb{N}$, be real-valued sequences for $\ell = 1, \ldots, d$ such that

$$\sup_{j=0,\ldots,n,\ n\in\mathbb{N}} |h_{j,\ell}(n)| \leq 1, \quad \text{for } \ell = 1, \ldots, d.$$

Let $G_\ell(\lambda)$, for $\ell = 1, \ldots, d$, be entire holomorphic functions. We say that the sequence

$$F_n(x_1, \ldots, x_d) = \sum_{j=0}^{n} c_j(n) G_1(x_1 h_{j,1}(n)) G_2(x_2 h_{j,2}(n)) \ldots G_d(x_d h_{j,d}(n)), \tag{23}$$

where $\{c_j(n)\}_{j,n}$, $j = 0, \ldots, n$, for $n \in \mathbb{N}$ is a complex-valued sequence, admits the supershift property if

$$\lim_{n\to\infty} F_n(x_1, \ldots, x_d) = G_1(x_1 a) G_2(x_2 a) \ldots G_d(x_d a).$$

Theorem 4.7 (The case of $d \geq 2$ variables) *Let $|a| > 1$ and let*

$$f_n(x) := \sum_{j=0}^{n} Z_j(n, a) e^{ih_j(n)x}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}, \tag{24}$$

be a superoscillating function as in Definition 2.1 and assume that its holomorphic extension to the entire functions $f_n(\xi)$ converges to $e^{ia\xi}$ in the space $A_{1,B}$ for some positive real value B. Let d be a positive integer and let $R_\ell \in \mathbb{R}_+ \cup \{\infty\}$ for any $\ell = 1, \ldots, d$. Let $G_1, \ldots, G_d$ be holomorphic functions whose series expansion at zero is given by

$$G_\ell(\lambda) = \sum_{m_\ell=0}^{\infty} g_{\ell,m} \lambda^{m_\ell}, \quad \forall \ell = 1, \ldots, d. \tag{25}$$

Moreover, we suppose the sequences $\{g_{l,m}\}$*'s satisfy the condition*

$$\limsup_{m\to\infty} |g_{\ell,m}|^{1/m} = \frac{1}{R_\ell}, \quad \forall \ell = 1, \ldots, d.$$

We define

$$F_n(x_1, \ldots, x_d) = \sum_{j=0}^{n} Z_j(n,a) G_1(x_1 h_j(n)) \cdots G_d(x_d h_j(n)),$$

where $Z_j(n,a)$ *are given as in* (24)*. Then,* $F_n(x_1, \ldots, x_d)$ *admits the supershift property that is*

$$\lim_{n\to\infty} F_n(x_1, \ldots, x_d) = G_1(x_1 a) \cdots G_d(x_d a)$$

uniformly on compact subsets of $\{\underline{x} \in \mathbb{R}^d : |x_\ell| < R' \text{ for any } \ell = 1, \ldots, d\}$ *where*

$$R' := \min\left(\frac{R}{|a|}, \frac{R}{4eB}, R\right) \qquad \text{and } R := \min_{\ell=1,\ldots,d} R_\ell.$$

Proof Since $|x_\ell| < R$ for any $\ell = 1, \ldots, d$, we have

$$\begin{aligned} F_n(x_1, \ldots, x_d) &= \sum_{j=0}^{n} Z_j(n,a) G_1(x_1 h_j(n)) \ldots G_d(x_d h_j(n)) \\ &= \sum_{j=0}^{n} Z_j(n,a) \sum_{m_1=0}^{\infty} g_{m_1} \cdots \sum_{m_d=0}^{\infty} g_{m_d} x_1^{m_1} \cdots x_d^{m_d} (h_j(n))^{m_1+\cdots+m_d} \\ &= \sum_{j=0}^{n} Z_j(n,a) \sum_{k=0}^{\infty} \sum_{\nu_1=0}^{k} \sum_{\nu_2=0}^{\nu_1} \cdots \sum_{\nu_{d-1}=0}^{\nu_{d-2}} g_{1,\nu_{d-1}} x_1^{\nu_{d-1}} g_{2,\nu_{d-2}-\nu_{d-1}} x_2^{\nu_{d-2}-\nu_{d-1}} \\ &\quad \cdots g_{d,k-\nu_1} x_d^{k-\nu_1} (h_j(n))^k \\ &= \sum_{j=0}^{n} Z_j(n,a) \sum_{k=0}^{\infty} Y_k'(h_j(n))^k, \end{aligned}$$

where

$$Y_k' = \sum_{\nu_1=0}^{k} \sum_{\nu_2=0}^{\nu_1} \cdots \sum_{\nu_{d-1}=0}^{\nu_{d-2}} g_{1,\nu_{d-1}} x_1^{\nu_{d-1}} g_{2,\nu_{d-2}-\nu_{d-1}} x_2^{\nu_{d-2}-\nu_{d-1}} \cdots g_{d,k-\nu_1} x_d^{k-\nu_1}.$$

We now consider the auxiliary complex variable ξ and we note that

$$\lambda^\ell = \frac{1}{i^\ell} D_\xi^\ell e^{i\xi\lambda}\Big|_{\xi=0} \quad \text{for} \quad \lambda \in \mathbb{C}, \quad \ell \in \mathbb{N}, \tag{26}$$

where D_ξ is the derivative with respect to ξ and $|_{\xi=0}$ denotes the restriction to $\xi = 0$. We have

$$F_n(x_1,\ldots,x_d) = \sum_{j=0}^{n} Z_j(n,a) \sum_{k=0}^{\infty} Y'_k[h_j(n)]^k = \sum_{j=0}^{n} Z_j(n,a) \sum_{k=0}^{\infty} Y'_k \frac{1}{i^k} D_\xi^k e^{i\xi h_j(n)}\Big|_{\xi=0}$$
$$= \sum_{k=0}^{\infty} Y'_k \frac{1}{i^k} D_\xi^k \sum_{j=0}^{n} Z_j(n,a) e^{i\xi h_j(n)}\Big|_{\xi=0}.$$

We define the operator

$$\mathcal{V}(x_1,\ldots,x_d,D_\xi) := \sum_{k=0}^{\infty} \frac{Y'_k}{i^k} D_\xi^k$$

so that we can write

$$F_n(x_1,\ldots,x_d) = \mathcal{V}(x_1,\ldots,x_d,D_\xi) \sum_{j=0}^{n} Z_j(n,a) e^{i\xi h_j(n)}\Big|_{\xi=0}.$$

Since $|x_\ell| < \frac{R}{4eB}$ for any $\ell = 1,\ldots,d$, we can use Proposition 2.13 to compute the following limit

$$\lim_{n\to\infty} F_n(x_1,\ldots,x_d) = \mathcal{V}(x_1,\ldots,x_d,D_\xi) \lim_{n\to\infty} \sum_{j=0}^{n} Z_j(n,a) e^{i\xi h_j(n)}\Big|_{\xi=0}$$
$$= \mathcal{V}(x_1,\ldots,x_d,D_\xi) e^{i\xi a}\Big|_{\xi=0}$$
$$= \sum_{k=0}^{\infty} \sum_{\nu_2=0}^{\nu_1} \cdots \sum_{\nu_{d-1}=0}^{\nu_{d-2}} g_{1,\nu_{d-1}} x_1^{\nu_{d-1}} g_{2,\nu_{d-2}-\nu_{d-1}} x_2^{\nu_{d-2}-\nu_{d-1}}$$
$$\cdots g_{d,k-\nu_1} x_d^{k-\nu_1} \frac{1}{i^k} D_\xi^k e^{i\xi a}\Big|_{\xi=0}$$
$$= \sum_{k=0}^{\infty} \sum_{\nu_2=0}^{\nu_1} \cdots \sum_{\nu_{d-1}=0}^{\nu_{d-2}} g_{1,\nu_{d-1}} x_1^{\nu_{d-1}} g_{2,\nu_{d-2}-\nu_{d-1}} x_2^{\nu_{d-2}-\nu_{d-1}} \cdots g_{d,k-\nu_1} x_d^{k-\nu_1} a^k$$
$$= \prod_{\ell=1}^{d} \left(\sum_{m=0}^{\infty} g_{\ell,m} x_\ell^m a^m \right) = G_1(ax_1)\cdots G_d(ax_d)$$

where the last equality holds because we are assuming $|x_\ell| < \frac{R}{|a|}$ for any $\ell = 1,\ldots,d$. The previous limit is uniform over the compact subset of $\{\underline{x} \in \mathbb{R}^d : |x_\ell| < R' \text{ for any } \ell = 1,\ldots,d\}$ because of Remark 2.14. □

Remark 4.8 A special case of the previous theorem occurs when the holomorphic functions G_ℓ's are entire functions. Moreover, differently from Theorem 3.3, in Theorem

4.7 the parameters x_ℓ appear in the arguments of the functions G_ℓ's. This implies that the hypothesis of Theorem 4.7 impose more constraints on the parameters x_ℓ's, namely $|x_\ell| < R'$ for any $\ell = 1, \dots, d$.

Example 4.9 Applying Theorem 4.7, it is possible to construct examples of sequences admitting the supeshift property. Suppose $G_1, \dots, G_d$ satisfy the hypothesis of Theorem 4.7. We can consider the superoscillating sequences

$$f_{1,n}(x) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1+a}{2}\right)^{n-j} \left(\frac{1-a}{2}\right)^{j} e^{i(1-2j/n)x} \quad n \in \mathbb{N}_0,$$

and

$$f_{2,n}(x) = \sum_{j=0}^{n} \prod_{k=0,\ k\neq j}^{n} \left(\frac{h_k(n)-a}{h_k(n)-h_j(n)}\right) e^{ixh_j(n)} \quad n \in \mathbb{N},$$

which are such that the sequences $\{\xi \in \mathbb{C} \mapsto f_{i,n}(\xi)\}$ for $i = 1, 2$ converge to $\xi \mapsto e^{ia\xi}$ in $A_{1,B}$ where $B > |a|+1$ (see [26]), respectively $B > |a|+2$ (see Remark 2.8). Thus the sequences

$$F_{1,n}(x_1, \dots, x_d) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1+a}{2}\right)^{n-j} \left(\frac{1-a}{2}\right)^{j} G_1(x_1(1-2j/n)) \cdots G_d(x_d(1-2j/n))$$

and

$$F_{2,n}(x_1, \dots, x_d) = \sum_{j=0}^{n} \prod_{k=0,\ k\neq j}^{n} \left(\frac{h_k(n)-a}{h_k(n)-h_j(n)}\right) G_1(x_1 h_j(n)) \cdots G_d(x_d h_j(n))$$

admit the supershift property.

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