11-10-2022

An Approach to the Gaussian RBF Kernels via Fock Spaces

Daniel Alpay
Chapman University, alpay@chapman.edu

Fabrizio Colombo
Politecnico di Milano, fabrizio.colombo@polimi.it

Kamal Diki
Chapman University, diki@chapman.edu

Irene Sabadini
Politecnico di Milano, irene.sabadini@polimi.it

Recommended Citation

Daniel Alpay, Fabrizio Colombo, Kamal Diki, and Irene Sabadini, "An approach to the Gaussian RBF kernels via Fock spaces", J. Math. Phys. 63, 113506 (2022): https://doi.org/10.1063/5.0060342

Journal of Mathematical Physics

An approach to the Gaussian RBF kernels via Fock spaces

Cite as: J. Math. Phys. **63**, 113506 (2022); https://doi.org/10.1063/5.0060342
Submitted: 17 June 2021 • Accepted: 19 October 2022 • Published Online: 10 November 2022

Daniel Alpay, Fabrizio Colombo, Kamal Diki, et al.

Journal of
Mathematical Physics

ARTICLE

scitation.org/journal/jmp

An approach to the Gaussian RBF kernels via Fock spaces

Cite as: J. Math. Phys. 63, 113506 (2022); doi: 10.1063/5.0060342
Submitted: 17 June 2021 • Accepted: 19 October 2022 •
Published Online: 10 November 2022

View Online Export Citation CrossMark

Daniel Alpay,[1,a] Fabrizio Colombo,[2,b] Kamal Diki,[3,c] and Irene Sabadini[2,d]

AFFILIATIONS

[1] Faculty of Mathematics, Physics, and Computation, Schmid College of Science and Technology, Chapman University, One University Drive, Orange, California 92866, USA

[2] Politecnico di Milano, Dipartimento di Matematica, Via E. Bonardi, 920133 Milano, Italy

[3] Schmid College of Science and Technology, Chapman University, One University Drive, Orange, California 92866, USA

[a] E-mail: alpay@chapman.edu
[b] Author to whom correspondence should be addressed: fabrizio.colombo@polimi.it
[c] E-mail: diki@chapman.edu
[d] E-mail: irene.sabadini@polimi.it

ABSTRACT

We use methods from the Fock space and Segal–Bargmann theories to prove several results on the Gaussian RBF kernel in complex analysis. The latter is one of the most used kernels in modern machine learning kernel methods and in support vector machine classification algorithms. Complex analysis techniques allow us to consider several notions linked to the radial basis function (RBF) kernels, such as the feature space and the feature map, using the so-called Segal–Bargmann transform. We also show how the RBF kernels can be related to some of the most used operators in quantum mechanics and time frequency analysis; specifically, we prove the connections of such kernels with creation, annihilation, Fourier, translation, modulation, and Weyl operators. For the Weyl operators, we also study a semigroup property in this case.

I. INTRODUCTION

Positive definite functions and reproducing kernel Hilbert spaces (RKHSs) play an important role in different areas of mathematics, such as complex analysis, operator theory, and Schur analysis, among others. They are also used to define coherent states in quantum mechanics and appear in machine learning; see Refs. 1 and 2. Kernel methods and, in particular, support vector machines (SVMs) have different applications and are used in machine learning for solving practical problems of industrial and technological interest. Indeed, these methods provide techniques to process, analyze, and compare many types of data. The idea of using kernel methods in this framework goes back to the work of Aronszajn, see Ref. 3, who was among the first ones to apply them in statistics. Later, Aizerman and co-authors treated positive kernels using a dot product in another space, the so-called feature space; see Ref. 4. This idea was fully developed only in the 1990s, especially in relation to vectorial data since kernels provide a vectorial representation of the data in the feature space. A brief account of the history of these methods can be found in Chapter 1 of Ref. 5. We also refer to Refs. 6 and 7 for different connections and applications related to time series analysis, statistical communication, control theory, and statistical theory of regression analysis. For further motivations and applications of SVMs in machine learning, we refer, e.g., to Refs. 5 and 8 and also to Ref. 9 where the approach, however, is in the real case, not complex.

Among the most used kernels in machine learning algorithms and in support vector machine classification algorithms, there are the so-called Gaussian radial basis function (RBF) kernels. RBF kernels, or more in general RBF functions, also play an important role in neural networks; see Ref. 10. Actually, in Refs. 2 and 11, the reproducing kernel Hilbert spaces corresponding to the Gaussian RBF kernels were introduced and used to analyze the learning performance of SVMs.

In this framework, the main objective and novelty of this paper are to use the Fock and Bargmann transforms as a new approach to study the RBF kernels using complex analysis techniques. This approach may be further developed to prove many other results on the RBF kernels.

In particular, we prove the connections of such kernels with creation, annihilation, Fourier, translation, modulation, and Weyl operators, which may find direct applications, for example, in quantum mechanics.

The structure and main results of the paper are as follows: in Sec. II, we briefly recall the preliminary results on the RBF kernels and spaces and some material that we need in the sequel. In Sec. III, we write the Gaussian RBF kernel in terms of a special Fock kernel and provide an isomorphism between RBF and Fock spaces and we extend various results from the Fock to RBF spaces; in particular, we discuss the reproducing kernel property and we provide the characterization and estimates of elements in the RBF space. Section IV deals with a Segal–Bargmann-type transform in the setting of RBF spaces. We present two approaches to study it, which are, in principle, different. Indeed, we first consider the Hermite generating function approach and we then use the RBF diagram approach. We show that both approaches coincide. In Sec. V, we study how the RBF spaces can be connected to different operators that appear in quantum mechanics and time-frequency analysis. More precisely, we discuss the links with creation, annihilation, and Weyl operators. In Sec. VI, we provide connections between RBF spaces and the Fourier transform.

II. PRELIMINARIES

In this section, we review the classical notions of feature map, feature space, Fock and RBF kernels, and their RKHSs. For more details, see Refs. 2, 11, and 12.

Definition 2.1. Let X be a non-empty set. Then, a function $k : X \times X \longrightarrow \mathbb{C}$ is called a kernel on X if there exists a $\mathbb{C}$-Hilbert space H with an inner product $\langle \cdot, \cdot \rangle_H$ and a map $\Psi : X \longrightarrow H$ such that we have

$$k(x, x') = \langle \Psi(x'), \Psi(x) \rangle_H \text{ for any } x, x' \in H.$$

Moreover, the space H is called the feature space and Ψ is called a feature map.

Definition 2.2 (Fock space). Let $\alpha > 0$; an entire function $f : \mathbb{C} \longrightarrow \mathbb{C}$ belongs to the Fock space, denoted by $\mathcal{F}_\alpha(\mathbb{C})$ (or simply $\mathcal{F}_\alpha$), if we have

$$\|f\|^2_{\mathcal{F}_\alpha} := \left(\frac{\alpha}{\pi}\right) \int_{\mathbb{C}} |f(z)|^2 \exp(-\alpha |z|^2) dA(z) < \infty, \tag{2.1}$$

where $dA(z) = dxdy$ is the Lebesgue measure with respect to the variable $z = x + iy$.

Remark 2.3. The Fock space $\mathcal{F}_\alpha$ is a reproducing kernel Hilbert space with the reproducing kernel defined by

$$F_\alpha(z, w) := \exp(\alpha z \overline{w}) \qquad \text{for any } z, w \in \mathbb{C}. \tag{2.2}$$

Moreover, we have the reproducing kernel property that can be expressed in terms of this integral representation,

$$f(w) = \int_{\mathbb{C}} f(z) \overline{F_\alpha(z, w)} dA_\alpha(z), \tag{2.3}$$

where we have set $dA_\alpha(z) = \left(\frac{\alpha}{\pi}\right) \exp(-\alpha |z|^2) dA(z)$.

Let $w \in \mathbb{C}$ be fixed; the normalized Fock kernel is given by the formula

$$f^\alpha_w(z) := \frac{F_\alpha(z, w)}{\sqrt{F_\alpha(w, w)}}, \quad z \in \mathbb{C}. \tag{2.4}$$

In particular, we have

$$f^\alpha_w(z) := \exp\left(\alpha \left(z\overline{w} - \frac{|w|^2}{2}\right)\right), \quad z, w \in \mathbb{C}. \tag{2.5}$$

We recall the Weyl operators on the Fock spaces (see Refs. 12 and 13)

Definition 2.4 (Weyl operator). Let $\alpha > 0$ and $a \in \mathbb{C}$. Then, the Weyl operator is defined and denoted by $\mathcal{W}^\alpha_a : \mathcal{F}_\alpha \longrightarrow \mathcal{F}_\alpha$, with

$$\mathcal{W}^\alpha_a f(z) := f(z - a) f^\alpha_a(z), \quad f \in \mathcal{F}_\alpha, z, a \in \mathbb{C}. \tag{2.6}$$

It is known that we have the semi-group property given by

$$\mathcal{W}^\alpha_a \mathcal{W}^\alpha_b = \exp(-\alpha i \operatorname{Im}(a\bar{b})) \mathcal{W}^\alpha_{a+b}, \quad a, b \in \mathbb{C}. \tag{2.7}$$

The RBF kernel and associated reproducing kernel Hilbert spaces in the complex variable case were first introduced in Ref. 11; see also Ref. 2. Indeed, we briefly review these notions here since they are relevant in the sequel.

Definition 2.5 (RBF kernel). Let $\gamma > 0$, $z \in \mathbb{C}$, and $w \in \mathbb{C}$. The function defined by

$$K_\gamma(z,w) = \exp\left(-\frac{(z-\overline{w})^2}{\gamma^2}\right) \tag{2.8}$$

is called the Gaussian RBF kernel with width $\frac{1}{\gamma}$.

Remark 2.6. If $x, x' \in \mathbb{R}$, we have that

$$K_\gamma(x,x') = \exp\left(-\frac{(x-x')^2}{\gamma^2}\right)$$

is the standard real valued RBF kernel, which is used in (SVM) kernel methods.

The RKHSs associated with the complex RBF kernels K_γ were first introduced in Refs. 2 and 11. We revise this notion in the next definition

Definition 2.7 (RBF space). Let $\gamma > 0$; an entire function $f : \mathbb{C} \longrightarrow \mathbb{C}$ belongs to the RBF space, denoted by $\mathcal{H}_\gamma^{RBF}(\mathbb{C})$ (or simply $\mathcal{H}_\gamma$), if we have

$$\|f\|^2_{\mathcal{H}_\gamma} := \left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}} |f(z)|^2 \exp\left(\frac{(z-\overline{z})^2}{\gamma^2}\right) dA(z) < \infty, \tag{2.9}$$

where $dA(z) = dxdy$ is the Lebesgue measure with respect to the variable $z = x + iy$.

Finally, we recall the scalar product on the standard Hilbert space $L^2(\mathbb{R})$, which is given by

$$\langle \phi, \psi \rangle_{L^2(\mathbb{R})} := \int_{\mathbb{R}} \overline{\phi(x)}\psi(x)dx.$$

III. FROM FOCK TO RBF KERNELS

In this section, we apply some well-known results on the Fock spaces in order to develop further the RBF kernels and associated Hilbert spaces. First, we can write the RBF kernel using the classical Fock kernel as follows.

Proposition 3.1. Let $\gamma > 0$, $z \in \mathbb{C}$, and $w \in \mathbb{C}$. Then, we have

$$K_\gamma(z,w) = \exp\left(-\frac{(z^2+\overline{w}^2)}{\gamma^2}\right) F_{\frac{2}{\gamma^2}}(z,w). \tag{3.10}$$

For all $\alpha > 0$, we also have

$$F_\alpha(z,w) = \exp\left(\alpha\frac{(z^2+\overline{w}^2)}{2}\right) K_{\sqrt{\frac{2}{\alpha}}}(z,w). \tag{3.11}$$

Proof. Let $\gamma > 0$ and $z, w \in \mathbb{C}$. We develop simple calculations using the RBF and Fock kernel definitions to get

$$\begin{aligned} K_\gamma(z,w) &= \exp\left(-\frac{(z-\overline{w})^2}{\gamma^2}\right) \\ &= \exp\left(-\frac{(z^2+\overline{w}^2)}{\gamma^2}\right)\exp\left(\frac{2}{\gamma^2}z\overline{w}\right) \\ &= \exp\left(-\frac{(z^2+\overline{w}^2)}{\gamma^2}\right) F_{\frac{2}{\gamma^2}}(z,w). \end{aligned}$$

We set $\alpha = \frac{2}{\gamma^2}$ and apply formula (3.10) with some easy calculations; we obtain

$$F_\alpha(z,w) = \exp\left(\alpha\frac{(z^2+\overline{w}^2)}{2}\right) K_{\sqrt{\frac{2}{\alpha}}}(z,w).$$

□

Theorem 3.2 *(RBF-Fock isomorphism). Let $\gamma > 0$; an entire function $f : \mathbb{C} \longrightarrow \mathbb{C}$ belongs to the RBF space $\mathcal{H}_\gamma$ if and only if there exists a unique function g in the Fock space $\mathcal{F}_{\frac{2}{\gamma^2}}$ such that*

$$f(z) = \exp\left(-\frac{z^2}{\gamma^2}\right)g(z) \quad \text{for any } z \in \mathbb{C}.$$

Moreover, there exists an isometric isomorphism between the RBF and Fock spaces given by the multiplication operator $\mathcal{M}_{RBF}^{\gamma^2} : \mathcal{H}_\gamma \longrightarrow \mathcal{F}_{\frac{2}{\gamma^2}}$ defined by

$$\mathcal{M}_{RBF}^{\gamma^2}[f](z) := \mathcal{M}_{\exp\left(\frac{z^2}{\gamma^2}\right)}[f](z) = \exp\left(\frac{z^2}{\gamma^2}\right)f(z) \quad \text{for any } f \in \mathcal{H}_\gamma, z \in \mathbb{C}. \tag{3.12}$$

Proof. We set $g(z) = \exp(\frac{z^2}{\gamma^2})f(z)$ for every $z \in \mathbb{C}$. Then, we just need to show that g belongs to $\mathcal{F}_{\frac{2}{\gamma^2}}$. First, it is clear that g is an entire function as multiplication of two entire functions. Now, we compute the norm of g with respect to the Fock space $\mathcal{F}_{\frac{2}{\gamma^2}}$. Indeed, we have

$$\begin{aligned}
\|g\|^2_{\mathcal{F}_{\frac{2}{\gamma^2}}} &= \left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}}|g(z)|^2 \exp(-2\gamma^2|z|^2)dA(z)\\
&= \left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}}|f(z)|^2 \exp\left(\frac{z^2+\overline{z}^2}{\gamma^2}\right)\exp(-2\gamma^2|z|^2)dA(z)\\
&= \left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}}|f(z)|^2 \exp\left(\frac{(z-\overline{z})^2}{\gamma^2}\right)dA(z)\\
&= \|f\|^2_{\mathcal{H}_\gamma} < \infty.
\end{aligned}$$

In particular, the previous computations show the isometry property of the multiplication operator $\mathcal{M}_{RBF}^{\gamma^2}$, that is, we have

$$\|\mathcal{M}_{RBF}^{\gamma^2}[f]\|_{\mathcal{F}_{\frac{2}{\gamma^2}}} = \|f\|_{\mathcal{H}_\gamma} \quad \text{for any } f \in \mathcal{H}_\gamma.$$

It was proved in Refs. 2 and 11 that the family of functions given by

$$e_n^\gamma(z) = \sqrt{\frac{2^n}{\gamma^{2n}n!}}z^n \exp\left(-\frac{z^2}{\gamma^2}\right) \tag{3.13}$$

form an orthonormal basis of the RBF space $\mathcal{H}_\gamma$. Furthermore, we have

$$\mathcal{M}_{RBF}^{\gamma^2}(e_n^\gamma)(z) = \sqrt{\frac{2^n}{\gamma^{2n}n!}}z^n \quad \text{for any } z \in \mathbb{C}. \tag{3.14}$$

Then, the multiplication operator $\mathcal{M}_{RBF}^{\gamma^2}$ maps an orthonormal basis of the RBF space $\mathcal{H}_\gamma$ onto an orthonormal basis of the Fock space $\mathcal{F}_{\frac{2}{\gamma^2}}$. Thus, $\mathcal{M}_{RBF}^{\gamma^2}$ is a surjective isometric operator from $\mathcal{H}_\gamma$ onto $\mathcal{F}_{\frac{2}{\gamma^2}}$. Hence, the RBF spaces and Fock spaces are isometrically isomorphic to each other according to some specific choices of the width parameter $\gamma > 0$. □

Theorem 3.3. *Let $\gamma > 0$; then, the inverse operator of $\mathcal{M}_{RBF}^{\gamma^2}$ is also its adjoint. It is given by the operator*

$$\left(\mathcal{M}_{RBF}^{\gamma^2}\right)^{-1} : \mathcal{F}_{\frac{2}{\gamma^2}} \longrightarrow \mathcal{H}_\gamma,$$

which can be computed using the equalities

$$\left(\mathcal{M}_{RBF}^{\gamma^2}\right)^{-1} = \left(\mathcal{M}_{RBF}^{\gamma^2}\right)^{*} = \mathcal{M}_{RBF}^{-\gamma^2}. \tag{3.15}$$

Proof. We note that $\mathcal{M}^{\gamma^2}_{RBF}$ is an isometric isomorphism by Theorem 3.2. Thus, it defines an unitary operator between RBF and Fock spaces; thus, its inverse coincides with its adjoint operator. Hence, the inverse and adjoint operators of $\mathcal{M}^{\gamma^2}_{RBF}$ are given by

$$\left(\mathcal{M}^{\gamma^2}_{RBF}\right)^{-1} : \mathcal{F}_{\frac{2}{\gamma^2}} \longrightarrow \mathcal{H}_\gamma,$$

which is obtained using the following formula:

$$\left(\mathcal{M}^{\gamma^2}_{RBF}\right)^{-1} = \mathcal{M}^{-\gamma^2}_{RBF}. \tag{3.16}$$

In particular, we have the following identity:

$$\langle \mathcal{M}^{\gamma^2}_{RBF} f, g\rangle_{\mathcal{F}_{\frac{2}{\gamma^2}}} = \langle f, \mathcal{M}^{-\gamma^2}_{RBF} g\rangle_{\mathcal{H}_\gamma} \quad \textit{for any } f \in \mathcal{H}_\gamma, g \in \mathcal{F}_{\frac{2}{\gamma^2}}. \tag{3.17}$$

□

Theorem 3.4 (RBF kernel reproducing property). *Let $\gamma > 0$; the RBF Hilbert space $\mathcal{H}_\gamma$ is a reproducing kernel Hilbert space whose reproducing kernel is given by the RBF kernel $K_\gamma(z, w)$. Moreover, we have the reproducing property, which is given by the following integral representation:*

$$f(w) = \left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}} f(z)\overline{K_\gamma(z,w)}\,\exp\left(\frac{(z-\bar{z})^2}{\gamma^2}\right)dA(z), \quad f \in \mathcal{H}_\gamma, w \in \mathbb{C}. \tag{3.18}$$

Proof. We insert formula (3.10) of Proposition 3.1 in the right-hand side of (3.18) and get

$$\begin{aligned}
&\left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}} f(z)\overline{K_\gamma(z,w)}e^{\frac{(z-\bar{z})^2}{\gamma^2}}dA(z)\\
&\qquad = \left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}} f(z)\exp\left(-\frac{(\bar{z}^2+w^2)}{\gamma^2}\right)\overline{F_{\frac{2}{\gamma^2}}(z,w)}e^{\frac{(z-\bar{z})^2}{\gamma^2}}dA(z)\\
&\qquad = \exp\left(-\frac{w^2}{\gamma^2}\right)\left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}} f(z)e^{\frac{z^2}{\gamma^2}}\overline{F_{\frac{2}{\gamma^2}}(z,w)}e^{-\frac{2}{\gamma^2}|z|^2}dA(z)\\
&\qquad = \exp\left(-\frac{w^2}{\gamma^2}\right)\left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}} \mathcal{M}^{\gamma^2}_{RBF}[f](z)\overline{F_{\frac{2}{\gamma^2}}(z,w)}e^{-\frac{2}{\gamma^2}|z|^2}dA(z).
\end{aligned}$$

However, we already know by Theorem 3.2 that $\mathcal{M}^{\gamma^2}_{RBF}$ is an isometric isomorphism between the Fock and RBF spaces. Thus, it is clear that $\mathcal{M}^{\gamma^2}_{RBF}(f) \in \mathcal{F}_{\frac{2}{\gamma^2}}$ since we have $f \in \mathcal{H}_\gamma$. We use the classical Fock reproducing kernel property and the explicit expression of $\mathcal{M}^{\gamma^2}_{RBF}(f)$ to get

$$\begin{aligned}
\left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}} \mathcal{M}^{\gamma^2}_{RBF}[f](z)\overline{F_{\frac{2}{\gamma^2}}(z,w)}\,\exp\left(-\frac{2}{\gamma^2}|z|^2\right)dA(z) &= \mathcal{M}^{\gamma^2}_{RBF}[f](w)\\
&= \exp\left(\frac{w^2}{\gamma^2}\right)f(w).
\end{aligned}$$

Therefore, we insert this in the previous calculations and obtain

$$\left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}} f(z)\overline{K_\gamma(z,w)}\,\exp\left(\frac{(z-\bar{z})^2}{\gamma^2}\right)dA(z) = f(w).$$

□

Inspired from the case treated in Refs. 2 and 11, we prove the next result.

Proposition 3.5. For a fixed $w \in \mathbb{C}$, we denote by K^w_γ the function defined as

$$K^w_\gamma(z) := K_\gamma(z, w).$$

Then, the following holds:

1. $K_\gamma(z,w) = \sum_{n=0}^{\infty} e_n(z)e_n(\overline{w})$ *for any* $z,w \in \mathbb{C}$.
2. $\langle K_\gamma^w, K_\gamma^z\rangle_{\mathcal{H}_\gamma} = K_\gamma(z,w)$ *for any* $z,w \in \mathbb{C}$.

Proof.

1. Let $z, w \in \mathbb{C}$; we make the following calculations:

$$\sum_{n=0}^{\infty} e_n^\gamma(z)e_n^\gamma(\overline{w}) = \left(\sum_{n=0}^{\infty} \frac{2^n}{\gamma^{2n}n!} z^n\overline{w}^n\right)\exp\left(-\frac{(z^2+\overline{w}^2)}{2}\right)$$
$$= \exp\left(\frac{2}{\gamma^2}z\overline{w}\right)\exp\left(-\frac{(z^2+\overline{w}^2)}{2}\right)$$
$$= \mathfrak{F}_{\frac{2}{\gamma^2}}(z,w)\exp\left(-\frac{(z^2+\overline{w}^2)}{2}\right).$$

Then, we apply Proposition 3.1 and get

$$\sum_{n=0}^{\infty} e_n^\gamma(z)e_n^\gamma(\overline{w}) = K_\gamma(z,w).$$

2. For a fixed $z, w \in \mathbb{C}$, it is clear that the function K_γ^w belongs to the RBF space $\mathcal{H}_\gamma$. Thus, using the reproducing kernel property proved in Theorem 3.4, we have

$$\langle K_\gamma^w, K_\gamma^z\rangle_{\mathcal{H}_\gamma} = K_\gamma^w(z)$$
$$= K_\gamma(w,z).$$

□

Remark 3.6. In analogy with the classical notion of Fock coherent states that appear in quantum mechanics, the kernel functions K_γ^w will be called the RBF coherent states.

We can control functions of the RBF spaces as it is described in the next result.

Proposition 3.7 (RBF estimate). Let $\gamma > 0$ *and* $f \in \mathcal{H}_\gamma$*. Then, we have*

$$|f(z)| \le \exp\left(\frac{2}{\gamma^2}y^2\right)\|f\|_{\mathcal{H}_\gamma} \quad \text{for any } z = x+iy \in \mathbb{C}. \tag{3.19}$$

In particular, if f is restricted to the real line, we have

$$|f(x)| \le \|f\|_{\mathcal{H}_\gamma}, \quad x \in \mathbb{R}.$$

Proof. We know by the reproducing kernel property proved in Theorem 3.4 that

$$f(z) = \langle f, K_\gamma^z\rangle_{\mathcal{H}_\gamma}.$$

Thus, using the Cauchy–Schwartz inequality, we have

$$|f(z)| \le \|K_\gamma^z\|_{\mathcal{H}_\gamma}\|f\|_{\mathcal{H}_\gamma}. \tag{3.20}$$

However, it is clear by the reproducing property that we have

$$\|K_\gamma^z\|_{\mathcal{H}_\gamma}^2 = K_\gamma^z(z) = K_\gamma(z,z).$$

Therefore, making some simple calculations, we obtain

$$\|K_\gamma^z\|_{\mathcal{H}_\gamma}^2 = \exp\left(\frac{4}{\gamma^2}y^2\right).$$

Finally, we insert the previous calculation in inequality (3.20) and conclude the proof of the first part of the statement. If f is restricted to the real line, we just take $y = 0$ in the RBF kernel estimate (3.19). □

Remark 3.8. The RBF kernel estimate that we got in the previous result can be directly deduced from Theorem 3.2 combined with the classical Fock kernel estimate in complex analysis.

Let $\gamma > 0$, and set

$$e_n^{\gamma}(z) = \sqrt{\frac{2^n}{\gamma^{2n} n!}} z^n \exp\left(-\frac{z^2}{\gamma^2}\right), \quad z \in \mathbb{C}. \tag{3.21}$$

Then, as a direct consequence of the previous result, we have the following.

Corollary 3.9. For any $w \in \mathbb{C}$ and $n \geq 0$, we have

$$e_n^{\gamma}(w) = \left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}} e_n^{\gamma}(z)\overline{K_{\gamma}(z,w)} \exp\left(\frac{(z-\overline{z})^2}{\gamma^2}\right) dA(z).$$

In particular, using formula (3.21), it holds that

$$w^n \exp\left(-\frac{w^2}{\gamma^2}\right) = \left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}} z^n \exp\left(-\frac{z^2}{\gamma^2}\right)\overline{K_{\gamma}(z,w)} \exp\left(\frac{(z-\overline{z})^2}{\gamma^2}\right) dA(z).$$

Proof. We just need to apply Theorem 3.4 to the orthonormal basis functions e_n^{γ}. □

Theorem 3.10 *(sequential characterization). An entire function $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to the RBF space $\mathcal{H}_{\gamma}$ if and only if it holds that*

$$\sum_{k=0}^{\infty} \frac{k!\gamma^{2k}}{2^k} \left| \sum_{j=0}^{[\frac{k}{2}]} \frac{a_{k-2j}}{\gamma^{2j} j!} \right|^2 < \infty. \tag{3.22}$$

Proof. We note that thanks to Theorem 3.2, we know that f belongs to the RBF space $\mathcal{H}_{\gamma}$ if and only if there exists a unique function $g \in \mathcal{F}_{\frac{2}{\gamma^2}}$ such that we have

$$f(z) = \exp\left(-\frac{z^2}{\gamma^2}\right) g(z), \quad \forall z \in \mathbb{C}. \tag{3.23}$$

We can write $g(z) = \sum_{k=0}^{\infty} b_k z^k$ that belongs to $\mathcal{F}_{\frac{2}{\gamma^2}}$ so that we have the growth condition given by

$$\sum_{k=0}^{\infty} \frac{k!\gamma^{2k}}{2^k} |b_k|^2 < \infty. \tag{3.24}$$

We observe that using the Cauchy product, we have

$$\begin{aligned} g(z) &= \exp\left(\frac{z^2}{\gamma^2}\right) f(z) \\ &= \left(\sum_{n=0}^{\infty} \frac{z^{2n}}{\gamma^{2n} n!}\right)\left(\sum_{n=0}^{\infty} a_n z^n\right) \\ &= \sum_{k=0}^{\infty} \beta_k z^k, \end{aligned}$$

where we have set $\beta_k = \sum_{j=0}^{k} s_j a_{k-j}$ with $s_j = \frac{1}{\gamma^{2m} m!}$ for $j = 2m$ and $s_j = 0$ for j odd. As a consequence, we note that for any $k \geq 0$, we have

$$\beta_k = \sum_{j=0}^{k} s_j a_{k-j} = \sum_j s_{2j} a_{k-2j} + \sum_j s_{2j+1} a_{k-(2j+1)} = \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{a_{k-2j}}{\gamma^{2j} j!}.$$

However, we know that

$$g(z) = \sum_{k=0}^{\infty} b_k z^k = \sum_{k=0}^{\infty} \beta_k z^k.$$

Thus, if we identify the coefficients, we obtain that for any $k \geq 0$,

$$b_k = \beta_k = \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{a_{k-2j}}{\gamma^{2j} j!}.$$

Finally, we replace in (3.24) and get the condition

$$\sum_{k=0}^{\infty} \frac{k! \gamma^{2k}}{2^k} \left| \sum_{j=0}^{\left[\frac{k}{2}\right]} \frac{a_{k-2j}}{\gamma^{2j} j!} \right|^2 < \infty.$$

□

IV. THE RBF SEGAL–BARGMANN TRANSFORM

In this section, we will use the Segal–Bargmann transform and its different properties in order to study the notions of feature map and feature spaces associated with the RBF kernels. In order to introduce the RBF version of the Segal–Bargmann transform, we will follow two possible approaches that we can compare later.

A. The Hermite generating function approach

We denote by $\mathcal{A}_{SB}^{\alpha}(z,x)$ the classical Segal–Bargmann kernel corresponding to the Fock space $\mathcal{F}_\alpha$. More precisely, we have (see Ref. 14)

$$\mathcal{A}_{SB}^{\alpha}(z,x) = \exp\left(-\frac{\alpha}{2}(z^2 + x^2) + \sqrt{2}\alpha z x\right), \quad \forall z \in \mathbb{C}, x \in \mathbb{R}. \tag{4.25}$$

Then, the classical Segal–Bargmann transform $B_\alpha : L^2(\mathbb{R}) \longrightarrow \mathcal{F}_\alpha(\mathbb{C})$ is defined for any $\varphi \in L^2(\mathbb{R})$ by the following expression:

$$B_\alpha[\varphi](z) = \int_{\mathbb{R}} \mathcal{A}_{SB}^{\alpha}(z,x)\varphi(x)dx. \tag{4.26}$$

In this paper, we will take $\alpha = \frac{2}{\gamma^2}$ with $\gamma > 0$. Now, we consider the RBF Segal–Bargmann kernel given by the generating function,

$$\mathcal{A}_{RBF}^{\gamma}(z,x) := \sum_{n=0}^{\infty} e_n^{\gamma}(z)\psi_n^{\alpha}(x), \quad z \in \mathbb{C}, x \in \mathbb{R}, \tag{4.27}$$

where ψ_n^α are the normalized weighted Hermite functions with the parameter $\alpha = \frac{2}{\gamma^2}$ and e_n^γ is an orthonormal basis of the RBF space $\mathcal{H}_\gamma$ that was introduced before in Sec. III. This allows us to introduce the map $\Phi : \mathbb{C} \longrightarrow L^2(\mathbb{R})$, defined by

$$\Phi(z) := \mathcal{A}_{RBF}^{\gamma}(z,\cdot), \quad \forall z \in \mathbb{C}. \tag{4.28}$$

We will study the RBF Segal–Bargmann transform of form (I) defined by

$$\mathfrak{B}_{RBF}^{\gamma}[\psi](z) := \langle \Phi(\bar{z}), \psi \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \mathcal{A}_{RBF}^{\gamma}(z,x)\psi(x)dx. \tag{4.29}$$

Now, we can prove the following.

Proposition 4.1. Let $\gamma > 0$. Then, it holds that

$$\mathcal{A}^{\gamma}_{RBF}(z,x) = \exp\left(-\frac{z^2}{\gamma^2}\right)\mathcal{A}^{\alpha}_{SB}(z,x) \quad \textit{for any } z \in \mathbb{C}, x \in \mathbb{R}$$

with $\alpha = \frac{2}{\gamma^2}$.

Proof. It is well-known that the classical Segal–Bargmann kernel can be obtained as the generating function associated with the normalized weighted Hermite functions. Indeed, for any $z \in \mathbb{C}$ and $x \in \mathbb{R}$, we have

$$\mathcal{A}^{\alpha}_{SB}(z,x) = \sum_{n=0}^{\infty}\sqrt{\frac{\alpha^n}{n!}}z^n\psi^{\alpha}_n(x).$$

We set $\alpha = \frac{2}{\gamma^2}$ and continue the calculations to get

$$\begin{aligned}\mathcal{A}^{\gamma}_{RBF}(z,x) &= \sum_{n=0}^{\infty} e^{\gamma}_n(z)\psi^{\alpha}_n(x)\\ &= \exp\left(-\frac{z^2}{\gamma^2}\right)\sum_{n=0}^{\infty}\sqrt{\frac{\alpha^n}{n!}}z^n\psi^{\alpha}_n(x)\\ &= \exp\left(-\frac{z^2}{\gamma^2}\right)\mathcal{A}^{\alpha}_{SB}(z,x).\end{aligned}$$

□

Corollary 4.2. Let $\gamma > 0$ and $\varphi \in L^2(\mathbb{R})$, and set $\alpha = \frac{2}{\gamma^2}$. Then, we have

$$\mathfrak{B}^{\gamma}_{RBF}[\varphi](z) = \mathcal{M}^{-\gamma^2}_{RBF}[B_{\alpha}(\varphi)](z) \quad \textit{for any } z \in \mathbb{C}.$$

Proof. To prove the result, we only need to write the integral representation of the transform $\mathfrak{B}^{\gamma}_{RBF}$ and apply Proposition 4.1. □

Proposition 4.3. Let $\gamma > 0$. Then, we have the explicit expression given by

$$\mathcal{A}^{\gamma}_{RBF}(z,x) = \exp\left(-\frac{(x-\sqrt{2}z)^2}{\gamma^2}\right) \quad \textit{for any } z \in \mathbb{C}, x \in \mathbb{R}.$$

Proof. We know by Proposition 4.1 that we have

$$\mathcal{A}^{\gamma}_{RBF}(z,x) = \exp\left(-\frac{z^2}{\gamma^2}\right)\mathcal{A}^{\alpha}_{SB}(z,x) \text{ for any } z \in \mathbb{C}, x \in \mathbb{R}$$

with $\alpha = \frac{2}{\gamma^2}$. Then, we insert formula (4.25) and obtain

$$\begin{aligned}\mathcal{A}^{\gamma}_{RBF}(z,x) &= \exp\left(-\frac{z^2}{\gamma^2}\right)\exp\left(-\frac{(z^2+x^2)}{\gamma^2} + 2\frac{\sqrt{2}}{\gamma^2}zx\right)\\ &= \exp\left(-\frac{1}{\gamma^2}(2z^2+x^2-2\sqrt{2}zx)\right)\\ &= \exp\left(-\frac{(x-\sqrt{2}z)^2}{\gamma^2}\right).\end{aligned}$$

□

Remark 4.4. We have

$$\mathcal{A}^{\gamma}_{RBF}(0,x) = e^{-\frac{x^2}{\gamma^2}}, \quad \forall x \in \mathbb{R}.$$

We introduce the map $\Phi : \mathbb{C} \longrightarrow L^2(\mathbb{R})$, defined by

$$\Phi(z) := \mathcal{A}^{\gamma}_{RBF}(z,\cdot), \quad \forall z \in \mathbb{C}. \tag{4.30}$$

In the sense that for any fixed $z \in \mathbb{C}$, we have

$$\Phi(z)(x) = \mathcal{A}^{\gamma}_{RBF}(z,x), \quad \forall x \in \mathbb{R}.$$

Proposition 4.5. Let $z, w \in \mathbb{C}$. Then, we have

$$\langle \Phi(z), \Phi(w) \rangle_{L^2(\mathbb{R})} = \gamma\sqrt{\frac{\pi}{2}} K_\gamma(w,z).$$

Moreover, for any $z \in \mathbb{C}$, we have

$$\|\Phi(z)\|_{L^2(\mathbb{R})} = \sqrt{\gamma\sqrt{\frac{\pi}{2}}}\,\exp\left(-\frac{(z-\bar{z})^2}{2\gamma^2}\right).$$

Proof. Let $z, w \in \mathbb{C}$ be fixed. Then, using Proposition 4.3, we have

$$\begin{aligned}
\langle \Phi(z), \Phi(w) \rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \overline{\Phi(z)(x)}\Phi(w)(x)dx \\
&= \int_{\mathbb{R}} \overline{\mathcal{A}^{\gamma}_{RBF}(z,x)}\mathcal{A}^{\gamma}_{RBF}(w,x)dx \\
&= \int_{\mathbb{R}} \exp\left(-\frac{(x-\sqrt{2}\bar{z})^2}{\gamma^2}\right)\exp\left(-\frac{(x-\sqrt{2}w)^2}{\gamma^2}\right)dx \\
&= \exp\left(-\frac{2}{\gamma^2}(\bar{z}^2 + w^2)\right)\int_{\mathbb{R}} \exp\left(-\frac{2}{\gamma^2}x^2 + \frac{2\sqrt{2}}{\gamma^2}(\bar{z}+w)x\right)dx.
\end{aligned}$$

At this stage, we can use the well-known Gaussian integral formula given by

$$\int_{\mathbb{R}} e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}, \quad a > 0, b \in \mathbb{C}.$$

Indeed, we set $a = \frac{2}{\gamma^2}$ and $b = \frac{2\sqrt{2}}{\gamma^2}(\bar{z}+w) \in \mathbb{C}$, which leads to

$$\frac{b^2}{4a} = \frac{(\bar{z}+w)^2}{\gamma^2}.$$

Therefore, we obtain

$$\begin{aligned}
\langle \Phi(z), \Phi(w) \rangle_{L^2(\mathbb{R})} &= \gamma\sqrt{\frac{\pi}{2}}\,\exp\left(-\frac{2}{\gamma^2}(\bar{z}^2 + w^2)\right)\exp\left(\frac{(\bar{z}+w)^2}{\gamma^2}\right) \\
&= \gamma\sqrt{\frac{\pi}{2}}\,\exp\left(\frac{1}{\gamma^2}(-\bar{z}^2 - w^2 + 2\bar{z}w)\right) \\
&= \gamma\sqrt{\frac{\pi}{2}}\,\exp\left(-\frac{(w-\bar{z})^2}{\gamma^2}\right) \\
&= \gamma\sqrt{\frac{\pi}{2}}K_\gamma(w,z).
\end{aligned}$$

In order to justify the second part of the statement, we note that

$$\|\Phi(z)\|^2_{L^2(\mathbb{R})} = \langle \Phi(z), \Phi(z) \rangle_{L^2(\mathbb{R})}.$$

Thus, we apply Proposition 4.5 and get

$$\|\Phi(z)\|^2_{L^2(\mathbb{R})} = \gamma\sqrt{\frac{\pi}{2}}K_\gamma(z,z).$$

However, we have

$$\sqrt{K_\gamma(z,z)} = \exp\left(-\frac{(z-\bar{z})^2}{2\gamma^2}\right).$$

Hence, we conclude that

$$\|\Phi(z)\|_{L^2(\mathbb{R})} = \sqrt{\gamma\sqrt{\frac{\pi}{2}}}\exp\left(-\frac{(z-\bar{z})^2}{2\gamma^2}\right).$$

□

Remark 4.6. We note that in the case of the RBF kernel K_γ, the feature space and the feature map, which were discussed in Definition 2.1, are obtained by setting $H = L^2(\mathbb{R})$ and $\Psi = \Phi$.

Proposition 4.7. Let $\gamma > 0$ and $n \geq 0$, and set $\alpha = \frac{2}{\gamma^2}$. Then, it holds that

$$\mathfrak{B}^\gamma_{RBF}[\psi^\alpha_n](z) = e^\gamma_n(z), \quad \forall z \in \mathbb{C}, \tag{4.31}$$

where ψ^α_n denote the α-weighted normalized Hermite functions and e^γ_n is the orthonormal basis of the RBF space $\mathcal{H}_\gamma$ given by (3.21). Moreover, we also have

$$\|\mathfrak{B}^\gamma_{RBF}[\psi^\alpha_n]\|_{\mathcal{H}_\gamma} = \|\psi^\alpha_n\|_{L^2(\mathbb{R})} = 1. \tag{4.32}$$

Proof. We observe that using Corollary 4.2 combined with the properties of the Bargmann transform B_α, we have

$$\begin{aligned}\mathfrak{B}^\gamma_{RBF}[\psi^\alpha_n](z) &= \mathcal{M}^{-\gamma^2}_{RBF}[B_\alpha(\psi^\alpha_n)](z)\\ &= \exp\left(-\frac{z^2}{\gamma^2}\right)B_\alpha(\psi^\alpha_n)](z)\\ &= \exp\left(-\frac{z^2}{\gamma^2}\right)\sqrt{\frac{\alpha^n}{n!}}z^n\\ &= \exp\left(-\frac{z^2}{\gamma^2}\right)\sqrt{\frac{2^n}{\gamma^n n!}}z^n\\ &= e^\gamma_n(z).\end{aligned}$$

The second part of the statement comes from the fact that both e^γ_n and ψ^α_n are the orthonormal basis of $\mathcal{H}_\gamma$ and $L^2(\mathbb{R})$, respectively. □

Theorem 4.8. *The RBF Segal–Bargmann transform of form (I) defined by*

$$\mathfrak{B}^\gamma_{RBF}[\psi](z) := \langle\Phi(\bar{z}),\psi\rangle = \int_{\mathbb{R}}\mathcal{A}^\gamma_{RBF}(z,x)\psi(x)dx, \quad \psi \in L^2(\mathbb{R}), \tag{4.33}$$

is an isometric isomorphism mapping the standard Schrödinger Hilbert space $L^2(\mathbb{R})$ onto the RBF space $\mathcal{H}_\gamma$.

Proof. Let $\gamma > 0$, and set $\alpha = \frac{2}{\gamma^2}$. Then, for any $\psi \in L^2(\mathbb{R})$, we have

$$\begin{aligned}\|\mathfrak{B}^\gamma_{RBF}[\psi]\|^2_{\mathcal{H}_\gamma} &= \left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}}|\mathfrak{B}^\gamma_{RBF}[\psi](z)|^2\exp\left(\frac{(z-\bar{z})^2}{\gamma^2}\right)dA(z)\\ &= \left(\frac{2}{\pi\gamma^2}\right)\int_{\mathbb{C}}\left|\mathcal{M}^{-\gamma^2}_{RBF}[B_\alpha(\psi)]\right|^2\exp\left(\frac{(z-\bar{z})^2}{\gamma^2}\right)dA(z).\end{aligned}$$

However, we know by Theorem 3.2 that $\mathcal{M}^{-\gamma^2}_{RBF}$ is an isometric operator from the Fock space $\mathcal{F}_\alpha(\mathbb{C})$ onto the RBF space $\mathcal{H}_\gamma$. Thus, we also use the classical result by Bargmann on the transform B_α and obtain

$$\begin{aligned}\|\mathfrak{B}^{\gamma}_{RBF}[\psi]\|^2_{\mathcal{H}_\gamma} &= \|\mathcal{M}^{-\gamma^2}_{RBF}[B_\alpha(\psi)]\|_{\mathcal{H}_\gamma}\\ &= \|B_\alpha(\psi)\|_{\mathcal{F}_\alpha(\mathbb{C})}\\ &= \|\psi\|_{L^2(\mathbb{R})}.\end{aligned}$$

□

B. The RBF diagram approach and inverse transform

We consider the composition of the classical Segal–Bargmann transform with the RBF multiplication operator that was introduced in Theorem 3.2. Namely, this RBF Segal–Bargmann transform of form (II) is well posed thanks to the following commutative diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathfrak{S}^\gamma} & \mathcal{H}_\gamma \\ {\scriptstyle B_{\frac{2}{\gamma^2}}}\downarrow & \nearrow {\scriptstyle \mathcal{M}^{-\gamma^2}_{RBF}} & \\ \mathcal{F}_{\frac{2}{\gamma^2}} & & \end{array}$$

In particular, we consider the following definition:

$$\mathfrak{S}^\gamma := \mathcal{M}^{-\gamma^2}_{RBF}\circ B_{\frac{2}{\gamma^2}}. \tag{4.34}$$

This second approach will help a lot to translate several results involving the Segal–Bargmann transform from the Fock to the RBF kernels. In particular, we prove the following result.

Proposition 4.9. The RBF Bargmann transform of form (I) coincides with the RBF Bargmann transform of form (II). In particular, for any $\varphi \in L^2(\mathbb{R})$, we have

$$\mathfrak{B}^{\gamma}_{RBF}[\varphi](z) = \mathfrak{S}^\gamma[\varphi](z) \quad \textit{for any } z \in \mathbb{C}.$$

Proof. With some calculations, we obtain that for any $\varphi \in L^2(\mathbb{R})$ and $z \in \mathbb{C}$, we have

$$\mathfrak{S}^\gamma[\varphi](z) = \left(\frac{2}{\pi\gamma^2}\right)^{\frac{1}{4}} \int_{\mathbb{R}} \exp\left(-\frac{(\sqrt{2}z - x)^2}{\gamma^2}\right)\varphi(x)dx. \tag{4.35}$$

Thus, we apply Proposition 4.3 and conclude that

$$\mathfrak{B}^{\gamma}_{RBF}[\varphi](z) = \mathfrak{S}^\gamma[\varphi](z) \quad \textit{for any } z \in \mathbb{C}.$$

□

We observe that for $z = 0$, we have

$$\mathfrak{S}^\gamma[\varphi](0) = \left(\frac{2}{\pi\gamma^2}\right)^{\frac{1}{4}} \int_{\mathbb{R}} e^{-\frac{x^2}{\gamma^2}}\varphi(x)dx. \tag{4.36}$$

We note that in order to calculate the inverse of the RBF Segal–Bargmann transform, it is enough to use expression (4.34) and Proposition 4.9, which leads to the following result.

Theorem 4.10. *For every $\gamma > 0$, we note that the RBF Segal–Bargmann transform inverse $(\mathfrak{B}^{\gamma}_{RBF})^{-1} : \mathcal{H}_\gamma \longrightarrow L^2(\mathbb{R})$ is given by*

$$(\mathfrak{B}^{\gamma}_{RBF})^{-1} = \left(B_{\frac{2}{\gamma^2}}\right)^{-1} \circ \mathcal{M}^{\gamma^2}_{RBF}. \tag{4.37}$$

More precisely, for any $f \in \mathcal{H}_\gamma$, we have the explicit expression

$$(\mathfrak{B}^{\gamma}_{RBF})^{-1}[f](x) = \left(\frac{2}{\pi\gamma^2}\right)^{\frac{1}{4}} \int_{\mathbb{C}} \overline{\mathcal{A}^{\gamma}_{RBF}(z,x)} f(z) \exp\left(\frac{(z-\bar{z})^2}{\gamma^2}\right)dA(z). \tag{4.38}$$

Proof. First, we observe that using formula (4.34) and Proposition 4.9, we have

$$(\mathfrak{B}^{\gamma}_{RBF})^{-1} = \left(\mathcal{M}^{-\gamma^2}_{RBF} \circ B_{\frac{2}{\gamma^2}}\right)^{-1} = \left(B_{\frac{2}{\gamma^2}}\right)^{-1} \circ \mathcal{M}^{\gamma^2}_{RBF}.$$

Let $f \in \mathcal{H}_\gamma$, and set $\alpha = \frac{2}{\gamma^2}$. Then, thanks to Proposition 4.3 and the Segal–Bargmann transform inverse expression, we obtain

$$\begin{aligned}(\mathfrak{B}^{\gamma}_{RBF})^{-1}[f](x) &= \left(B_{\frac{2}{\gamma^2}}\right)^{-1} \circ \mathcal{M}^{\gamma^2}_{RBF}[f](x)\\ &= \left(B_{\frac{2}{\gamma^2}}\right)^{-1}\left[\exp\left(\frac{z^2}{\gamma^2}\right)f\right](x)\\ &= \left(\frac{2}{\pi\gamma^2}\right)^{\frac{1}{4}}\int_{\mathbb{C}}\overline{\mathcal{A}^{\alpha}_{SB}(z,x)}\,\exp\left(\frac{z^2}{\gamma^2}\right)f(z)\exp(-\alpha|z|^2)dA(z)\\ &= \left(\frac{2}{\pi\gamma^2}\right)^{\frac{1}{4}}\int_{\mathbb{C}}\overline{\exp\left(\frac{z^2}{\gamma^2}\right)\mathcal{A}^{\gamma}_{RBF}(z,x)}\,\exp\left(\frac{z^2}{\gamma^2}\right)f(z)\exp\left(-\frac{2}{\gamma^2}|z|^2\right)dA(z)\\ &= \left(\frac{2}{\pi\gamma^2}\right)^{\frac{1}{4}}\int_{\mathbb{C}}\overline{\mathcal{A}^{\gamma}_{RBF}(z,x)}f(z)\exp\left(\frac{(z-\bar{z})^2}{\gamma^2}\right)dA(z).\end{aligned}$$

□

Remark 4.11. We note that the RBF Segal–Bargmann transform is a unitary operator so that its adjoint coincides with its inverse. In particular, for every $f \in \mathcal{H}_\gamma$, we have

$$(\mathfrak{B}^{\gamma}_{RBF})^{*}[f](x) = \left(\frac{2}{\pi\gamma^2}\right)^{\frac{1}{4}}\int_{\mathbb{C}}\overline{\mathcal{A}^{\gamma}_{RBF}(z,x)}f(z)\exp\left(\frac{(z-\bar{z})^2}{\gamma^2}\right)dA(z). \tag{4.39}$$

V. CREATION, ANNIHILATION, AND WEYL OPERATORS ON RBF SPACES

Let X be the position operator on $L^2(\mathbb{R})$, which is defined by $X(\varphi)(x) = x\varphi(x)$ for any φ that belongs to the domain of X and $x \in \mathbb{R}$. We denote by $\mathcal{D}(X)$ the domain of X, which is given by

$$\mathcal{D}(X) := \{\varphi \in L^2(\mathbb{R}),\ X(\varphi) \in L^2(\mathbb{R})\}.$$

We denote by P the momentum operator on $L^2(\mathbb{R})$ defined by $P(\varphi) = \frac{d}{dx}\varphi$ for any φ that belongs to the domain of P, which is given by

$$\mathcal{D}(P) := \{\varphi \in L^2(\mathbb{R}),\ P(\varphi) \in L^2(\mathbb{R})\}.$$

Then, we can prove the following.

Proposition 5.1. It holds that

$$\frac{d}{dz}\mathfrak{B}^{\gamma}_{RBF} = -\frac{4}{\gamma^2}M_z\mathfrak{B}^{\gamma}_{RBF} + \frac{2\sqrt{2}}{\gamma^2}\mathfrak{B}^{\gamma}_{RBF}X \quad \text{on } \mathcal{D}(X). \tag{5.40}$$

Proof. Let $\varphi \in \mathcal{D}(X)$; we have

$$\mathfrak{B}^{\gamma}_{RBF}[\varphi](z) = \int_{\mathbb{R}}\mathcal{A}^{\gamma}_{RBF}(z,x)\varphi(x)dx.$$

Hence,

$$\frac{d}{dz}\mathfrak{B}^{\gamma}_{RBF}[\varphi](z) = \int_{\mathbb{R}}\frac{d}{dz}\mathcal{A}^{\gamma}_{RBF}(z,x)\varphi(x)dx. \tag{5.41}$$

However, we know by Proposition 4.3 that

$$\mathcal{A}^{\gamma}_{RBF}(z,x) = \exp\left(-\frac{(x-\sqrt{2}z)^2}{\gamma^2}\right) \quad \text{for any } z \in \mathbb{C}, x \in \mathbb{R}.$$

Thus, developing direct calculations, we obtain

$$\frac{d}{dz}\mathcal{A}^{\gamma}_{RBF}(z,x) = \frac{2\sqrt{2}}{\gamma^2}(x-\sqrt{2}z)\mathcal{A}^{\gamma}_{RBF}(z,x). \tag{5.42}$$

Hence, we insert (5.42) in (5.41) and get

$$\begin{aligned}\frac{d}{dz}\mathfrak{B}^{\gamma}_{RBF}[\varphi](z) &= \frac{2\sqrt{2}}{\gamma^2}\int_{\mathbb{R}}(x-\sqrt{2}z)\mathcal{A}^{\gamma}_{RBF}(z,x)\varphi(x)dx\\ &= -\frac{4}{\gamma^2}M_z\mathfrak{B}^{\gamma}_{RBF}[\varphi](z) + \frac{2\sqrt{2}}{\gamma^2}\mathfrak{B}^{\gamma}_{RBF}[X(\varphi)](z).\end{aligned}$$

This ends the proof. □

Corollary 5.2. We have

$$(\mathfrak{B}^{\gamma}_{RBF})^{-1}\left(\frac{\gamma^2}{2\sqrt{2}}\frac{d}{dz} + \sqrt{2}M_z\right)\mathfrak{B}^{\gamma}_{RBF} = X \quad \textit{on } \mathcal{D}(X). \tag{5.43}$$

Proof. On $\mathcal{D}(X)$, we have by Proposition 5.1 that

$$\frac{d}{dz}\mathfrak{B}^{\gamma}_{RBF} + \frac{4}{\gamma^2}M_z\mathfrak{B}^{\gamma}_{RBF} = \frac{2\sqrt{2}}{\gamma^2}\mathfrak{B}^{\gamma}_{RBF}X.$$

Thus, we obtain

$$X = (\mathfrak{B}^{\gamma}_{RBF})^{-1}\left(\frac{\gamma^2}{2\sqrt{2}}\frac{d}{dz} + \sqrt{2}M_z\right)\mathfrak{B}^{\gamma}_{RBF} \quad \text{on } \mathcal{D}(X).$$

□

Let $\gamma > 0$ and $a \in \mathbb{C}$. Then, we denote by $\mathcal{W}^{\gamma,a}_{RBF}$ the RBF-Weyl operators on the spaces $\mathcal{H}_\gamma$ that are obtained using the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H}_\gamma & \xrightarrow{\mathcal{W}^{\gamma,a}_{RBF}} & \mathcal{H}_\gamma \\ {\scriptstyle \mathcal{M}^{\gamma^2}_{RBF}}\downarrow & & \uparrow{\scriptstyle \mathcal{M}^{-\gamma^2}_{RBF}} \\ \mathcal{F}_{\frac{2}{\gamma^2}} & \xrightarrow[\mathcal{W}_a^{\frac{2}{\gamma^2}}]{} & \mathcal{F}_{\frac{2}{\gamma^2}} \end{array}$$

Thus, we define the RBF-Weyl operator by

$$\mathcal{W}^{\gamma,a}_{RBF} := \mathcal{M}^{-\gamma^2}_{RBF}\circ\mathcal{W}_a^{\frac{2}{\gamma^2}}\circ\mathcal{M}^{\gamma^2}_{RBF}. \tag{5.44}$$

Theorem 5.3. *Let $\gamma > 0$ and $a \in \mathbb{C}$. Then, the RBF-Weyl operator $\mathcal{W}^{\gamma,a}_{RBF}$ is an isometric operator from the RBF space $\mathcal{H}_\gamma$ onto itself. Moreover, its adjoint and inverse are given by*

$$\left(\mathcal{W}^{\gamma,a}_{RBF}\right)^* = \left(\mathcal{W}^{\gamma,a}_{RBF}\right)^{-1} = \mathcal{W}^{\gamma,-a}_{RBF}.$$

Proof. We have

$$\begin{aligned}\left(\mathcal{W}^{\gamma,a}_{RBF}\right)^{-1} &= \left(\mathcal{M}^{-\gamma^2}_{RBF}\circ\mathcal{W}_a^{\frac{2}{\gamma^2}}\circ\mathcal{M}^{\gamma^2}_{RBF}\right)^{-1}\\ &= \left(\mathcal{M}^{\gamma^2}_{RBF}\right)^{-1}\circ\left(\mathcal{W}_a^{\frac{2}{\gamma^2}}\right)^{-1}\circ\left(\mathcal{M}^{-\gamma^2}_{RBF}\right)^{-1}\\ &= \mathcal{M}^{-\gamma^2}_{RBF}\circ\mathcal{W}_{-a}^{\frac{2}{\gamma^2}}\circ\mathcal{M}^{\gamma^2}_{RBF}.\end{aligned}$$

We know by classical results on the Weyl operators (see Ref. 12) that

$$\left(\mathcal{W}_a^{\frac{2}{\gamma^2}}\right)^{-1} = \mathcal{W}_{-a}^{\frac{2}{\gamma^2}}.$$

Moreover, thanks to Theorem 3.3, we also have

$$\left(\mathcal{M}_{RBF}^{-\gamma^2}\right)^{-1} = \mathcal{M}_{RBF}^{\gamma^2} \quad \text{and} \quad \left(\mathcal{M}_{RBF}^{\gamma^2}\right)^{-1} = \mathcal{M}_{RBF}^{-\gamma^2}.$$

Thus, we obtain

$$\begin{aligned}\left(\mathcal{W}_{RBF}^{\gamma,a}\right)^{-1} &= \mathcal{M}_{RBF}^{-\gamma^2} \circ \mathcal{W}_{-a}^{\frac{2}{\gamma^2}} \circ \mathcal{M}_{RBF}^{\gamma^2} \\ &= \mathcal{W}_{RBF}^{\gamma,-a}.\end{aligned}$$

□

We note that the semi-group property related to the RBF-Weyl operator is given by the following result.

Proposition 5.4. Let $\gamma > 0$ and $a, b \in \mathbb{C}$. Then, we have

$$\mathcal{W}_{RBF}^{\gamma,a}\mathcal{W}_{RBF}^{\gamma,b} = \exp(\varphi_\gamma(a,b))\mathcal{W}_{RBF}^{\gamma,a+b}$$

with $\varphi_\gamma(a,b) := -\frac{2i}{\gamma^2}\operatorname{Im}(a\bar{b})$. In particular, if a and b are real numbers, we have

$$\mathcal{W}_{RBF}^{\gamma,a}\mathcal{W}_{RBF}^{\gamma,b} = \mathcal{W}_{RBF}^{\gamma,a+b}.$$

Proof. We observe that $\left(\mathcal{M}_{RBF}^{\gamma^2}\right)^{-1} = \mathcal{M}_{RBF}^{-\gamma^2}$; thus, the following calculations hold:

$$\begin{aligned}\mathcal{W}_{RBF}^{\gamma,a} \circ \mathcal{W}_{RBF}^{\gamma,b} &= \left(\mathcal{M}_{RBF}^{-\gamma^2} \circ \mathcal{W}_a^{\frac{2}{\gamma^2}} \circ \mathcal{M}_{RBF}^{\gamma^2}\right) \circ \left(\mathcal{M}_{RBF}^{-\gamma^2} \circ \mathcal{W}_b^{\frac{2}{\gamma^2}} \circ \mathcal{M}_{RBF}^{\gamma^2}\right) \\ &= \mathcal{M}_{RBF}^{-\gamma^2} \circ \left(\mathcal{W}_a^{\frac{2}{\gamma^2}} \circ \mathcal{W}_b^{\frac{2}{\gamma^2}}\right) \circ \mathcal{M}_{RBF}^{\gamma^2}.\end{aligned}$$

However, we know by formula (2.7) that the Weyl operator satisfies the semi-group property given by

$$\mathcal{W}_a^{\frac{2}{\gamma^2}} \circ \mathcal{W}_b^{\frac{2}{\gamma^2}} = \exp(\varphi_\gamma(a,b))\mathcal{W}_{a+b}^{\frac{2}{\gamma^2}}$$

with $\varphi_\gamma(a,b) := -\frac{2i}{\gamma^2}\operatorname{Im}(a\bar{b})$. Therefore, we obtain

$$\begin{aligned}\mathcal{W}_{RBF}^{\gamma,a} \circ \mathcal{W}_{RBF}^{\gamma,b} &= \exp\left(-\frac{2i}{\gamma^2}\operatorname{Im}(a\bar{b})\right)\mathcal{M}_{RBF}^{-\gamma^2} \circ \mathcal{W}_{a+b}^{\frac{2}{\gamma^2}} \circ \mathcal{M}_{RBF}^{\gamma^2} \\ &= \exp\left(-\frac{2i}{\gamma^2}\operatorname{Im}(a\bar{b})\right)\mathcal{W}_{RBF}^{\gamma,a+b}.\end{aligned}$$

This ends the proof. □

We can compute an explicit expression of the RBF-Weyl operator, which is given in the next result.

Theorem 5.5. *Let $\gamma > 0$, $a \in \mathbb{C}$, and $f \in \mathcal{H}_\gamma$. Then, we have*

$$\mathcal{W}_{RBF}^{\gamma,a} f(z) = \exp\left(\frac{a^2 - |a|^2}{\gamma^2} + 2z\frac{(\bar{a} - a)}{\gamma^2}\right) f(z-a), \quad z \in \mathbb{C}.$$

Moreover, the RBF-Weyl operator reduces to the standard translation operator defined by

$$T_a[f](z) := f(z-a), \quad z \in \mathbb{C},$$

if and only if $a \in \mathbb{R}$.

Proof. Let $f \in \mathcal{H}_\gamma$; then, using expression (5.44), we have

$$\begin{aligned}
\mathcal{W}^{\gamma,a}_{RBF}[f](z) &= \left(\mathcal{M}^{-\gamma^2}_{RBF} \circ \mathcal{W}^{\frac{2}{\gamma^2}}_{a}\right)\left[\exp\left(\frac{z^2}{\gamma^2}\right)f\right](z) \\
&= \exp\left(-\frac{z^2}{\gamma^2}\right)\mathcal{W}^{\gamma,a}_{RBF}\left[\exp\left(\frac{z^2}{\gamma^2}\right)f\right](z) \\
&= \exp\left(-\frac{z^2}{\gamma^2}\right)\exp\left(\frac{(z-a)^2}{\gamma^2}\right)f(z-a)\exp\left(\frac{2}{\gamma^2}\left(z\bar{a} - \frac{|a|^2}{2}\right)\right) \\
&= \exp\left(\frac{a^2}{\gamma^2} - 2\frac{za}{\gamma^2} + 2z\bar{a} - \frac{|a|^2}{\gamma^2}\right)f(z-a) \\
&= \exp\left(\frac{a^2 - |a|^2}{\gamma^2}\right)\exp\left(2z\frac{(\bar{a}-a)}{\gamma^2}\right)f(z-a), \quad z \in \mathbb{C}.
\end{aligned}$$

From the previous expression, it is easy to see that if $a \in \mathbb{R}$, we will have $a^2 = |a|^2$ and $\bar{a} = a$. Thus, it follows that for every $a \in \mathbb{R}$, we have

$$\mathcal{W}^{\gamma,a}_{RBF}[f](z) = f(z-a) = T_a[f](z). \tag{5.45}$$

For the converse, if we assume that (5.45) holds, we will get that

$$\exp\left(\frac{a^2 - |a|^2}{\gamma^2}\right)\exp\left(2z\frac{(\bar{a}-a)}{\gamma^2}\right) = 1 \quad \textit{for any } z \in \mathbb{C}.$$

In particular, we obtain that $a = \bar{a}$, which shows that $a \in \mathbb{R}$; this ends the proof. □

VI. THE FOURIER TRANSFORM ON RBF SPACES

Let $\alpha > 0$; we denote by F_α the Fourier transform on $L^2(\mathbb{R})$ defined by

$$\mathsf{F}_\alpha(\varphi)(\lambda) := \sqrt{\frac{\alpha}{2\pi}}\int_{\mathbb{R}} \exp(-\alpha i\lambda x)\varphi(x)dx.$$

It is possible to use a commutative diagram in order to consider the composition

$$\mathcal{Z}_\alpha = B_\alpha \circ \mathsf{F}_\alpha \circ B_\alpha^{-1} : \mathcal{F}_\alpha(\mathbb{C}) \longrightarrow \mathcal{F}_\alpha(\mathbb{C}), \tag{6.46}$$

where B_α is the classical Segal–Bargmann transform associated with the Fock space $\mathcal{F}_\alpha(\mathbb{C})$.

It turns out that $\mathcal{Z}_\alpha$ reduces to a simple composition operator $\mathcal{C}_\phi$ with the symbol given by the function $\phi(z) = -iz$ (see Ref. 15), which means

$$\mathcal{C}_\phi f(z) := f \circ \phi(z) = f(-iz), \quad z \in \mathbb{C}, \ f \in \mathcal{F}_\alpha(\mathbb{C}). \tag{6.47}$$

Let us fix $\gamma > 0$. Then, using the RBF-Bargmann transform given by (4.34), we can introduce the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{H}_\gamma & \xrightarrow{S_\gamma} & \mathcal{H}_\gamma \\
{\scriptstyle (\mathfrak{B}^\gamma_{RBF})^{-1}}\downarrow & & \uparrow{\scriptstyle \mathfrak{B}^\gamma_{RBF}} \\
L^2(\mathbb{R}) & \xrightarrow[\mathsf{F}_{\frac{2}{\gamma^2}}]{} & L^2(\mathbb{R})
\end{array}$$

Thus, we have

$$S_\gamma := \mathfrak{B}^\gamma_{RBF} \circ \mathsf{F}_{\frac{2}{\gamma^2}} \circ (\mathfrak{B}^\gamma_{RBF})^{-1}.$$

We first observe that we have

$$\mathfrak{B}^\gamma_{RBF} := \mathcal{M}^{-\gamma^2}_{RBF} \circ B_{\frac{2}{\gamma^2}} \quad \text{and} \quad (\mathfrak{B}^\gamma_{RBF})^{-1} = B^{-1}_{\frac{2}{\gamma^2}} \circ \mathcal{M}^{\gamma^2}_{RBF}.$$

Then, we can prove the following result.

Proposition 6.1. Let $\gamma > 0$; it holds that

$$S_\gamma = \mathcal{M}_{RBF}^{-\gamma^2} \circ \mathcal{C}_\phi \circ \mathcal{M}_{RBF}^{\gamma^2},$$

where $\phi(z) = -iz$ for any $z \in \mathbb{C}$.

Proof. We note that thanks to Proposition 4.9, we know that $\mathfrak{B}_{RBF}^{\gamma} = \mathfrak{S}^{\gamma}$. Thus, we can provide the following calculations:

$$\begin{aligned} S_\gamma &= \mathfrak{B}_{RBF}^{\gamma} \circ \mathsf{F}_{\frac{2}{\gamma^2}} \circ \left(\mathfrak{B}_{RBF}^{\gamma}\right)^{-1} \\ &= \mathfrak{S}_{RBF}^{\gamma} \circ \mathsf{F}_{\frac{2}{\gamma^2}} \circ \left(\mathfrak{S}_{RBF}^{\gamma}\right)^{-1} \\ &= \left(\mathcal{M}_{RBF}^{-\gamma^2} \circ B_{\frac{2}{\gamma^2}}\right) \circ \mathsf{F}_{\frac{2}{\gamma^2}} \circ \left(B_{\frac{2}{\gamma^2}}^{-1} \circ \mathcal{M}_{RBF}^{\gamma^2}\right). \end{aligned}$$

Then, we shall use the classical result given by formulas (6.46) and (6.47) for the parameter $\alpha = \frac{2}{\gamma^2}$ to get

$$\begin{aligned} S_\gamma &= \mathcal{M}_{RBF}^{-\gamma^2} \circ \left(B_{\frac{2}{\gamma^2}} \circ \mathsf{F}_{\frac{2}{\gamma^2}} \circ B_{\frac{2}{\gamma^2}}^{-1}\right) \circ \mathcal{M}_{RBF}^{\gamma^2} \\ &= \mathcal{M}_{RBF}^{-\gamma^2} \circ \mathcal{Z}_{\frac{2}{\gamma^2}} \circ \mathcal{M}_{RBF}^{\gamma^2} \\ &= \mathcal{M}_{RBF}^{-\gamma^2} \circ \mathcal{C}_\phi \circ \mathcal{M}_{RBF}^{\gamma^2}. \end{aligned}$$

This ends the proof. □

As a consequence, we can prove the next result.

Theorem 6.2. *Let $\gamma > 0$ and $f \in \mathcal{H}_\gamma$. Then, we have*

$$S_\gamma f(z) = \exp\left(-\frac{z^2}{2\gamma^2}\right) f(-iz), \quad z \in \mathbb{C}.$$

In particular, if we set $\phi(z) = -iz$, then we have

$$S_\gamma = \mathcal{M}_{RBF}^{-2\gamma^2} \circ \mathcal{C}_\phi.$$

Proof. Let $f \in \mathcal{H}_\gamma$; thanks to Proposition 6.1, we have

$$S_\gamma = \mathcal{M}_{RBF}^{-\gamma^2} \circ \mathcal{C}_\phi \circ \mathcal{M}_{RBF}^{\gamma^2}.$$

Thus, we set $\psi_f(z) = \mathcal{C}_{-iz} \circ \mathcal{M}_{RBF}^{\gamma^2}[f](z)$ and get

$$\begin{aligned} \psi_f(z) &= \mathcal{C}_\phi\left[\exp\left(\frac{z^2}{\gamma^2}\right) f\right](z) \\ &= \exp\left(-\frac{z^2}{\gamma^2}\right) f(-iz). \end{aligned}$$

As a consequence, we obtain

$$\begin{aligned} S_\gamma f(z) &= \mathcal{M}_{RBF}^{-\gamma^2}[\psi_f](z) \\ &= \exp\left(-\frac{z^2}{\gamma^2}\right)\psi_f(z) \\ &= \exp\left(-2\frac{z^2}{\gamma^2}\right) f(-iz). \end{aligned}$$

□

Let $a \in \mathbb{R}$; we consider on $L^2(\mathbb{R})$ the translation operator defined by $\tau_a : \varphi \longmapsto \tau_a\varphi(x) := \varphi(x-a)$. Then, using the RBF-Bargmann transform given by (4.34), we can introduce the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{H}_\gamma & \xrightarrow{L_\gamma^a} & \mathcal{H}_\gamma \\
{\scriptstyle (\mathfrak{B}^\gamma_{RBF})^{-1}}\downarrow & & \uparrow{\scriptstyle \mathfrak{B}^\gamma_{RBF}} \\
L^2(\mathbb{R}) & \xrightarrow[\tau_a]{} & L^2(\mathbb{R})
\end{array}
$$

Thus, we consider the factorization given by the operator

$$L_\gamma^a := \mathfrak{B}^\gamma_{RBF} \circ \tau_a \circ (\mathfrak{B}^\gamma_{RBF})^{-1}.$$

Proposition 6.3. Let $a \in \mathbb{R}$ and $\gamma > 0$. Then, it holds that

$$L_\gamma^a = \mathcal{W}^{\gamma,a}_{RBF}. \tag{6.48}$$

Proof. Some calculations and a classical result on the Weyl operators (see Ref. 15) show that

$$
\begin{aligned}
L_\gamma^a &= \left(\mathcal{M}^{-\gamma^2}_{RBF} \circ B_{\frac{2}{\gamma^2}}\right) \circ \tau_a \circ \left(B^{-1}_{\frac{2}{\gamma^2}} \circ \mathcal{M}^{\gamma^2}_{RBF}\right) \\
&= \mathcal{M}^{-\gamma^2}_{RBF} \circ \left(B_{\frac{2}{\gamma^2}} \circ \tau_a \circ B^{-1}_{\frac{2}{\gamma^2}}\right) \circ \mathcal{M}^{\gamma^2}_{RBF} \\
&= \mathcal{M}^{-\gamma^2}_{RBF} \circ \mathcal{W}_a^{\frac{2}{\gamma^2}} \circ \mathcal{M}^{\gamma^2}_{RBF} \\
&= \mathcal{W}^{\gamma,a}_{RBF}.
\end{aligned}
$$

□

VII. CONCLUDING REMARKS

In a forthcoming work, we plan to investigate further results on RBF spaces using Fock spaces in the several variables case. This is not the only way to extend the case of one complex variable case to a multi-dimensional case; in fact, one could consider the RBF kernels in the hypercomplex case. In the specific case of quaternions, one may consider the setting of slice hyperholomorphic functions; indeed, it is possible to introduce a quaternionic version of the RBF kernels thanks to the properties satisfied by quaternionic intrinsic regular functions. Inspired from the calculations in Proposition 3.1, we introduce the quaternionic slice regular RBF kernel, which is defined by

$$K_{\gamma,S}(q,p) := e^{-\frac{q^2}{\gamma^2}} F^S_{\frac{2}{\gamma^2}}(q,p) e^{-\frac{\bar{p}^2}{\gamma^2}}, \quad \forall (q,p) \in \mathbb{H} \times \mathbb{H}, \tag{7.49}$$

where $F^S_{\frac{2}{\gamma^2}}$ is the slice hyperholomorphic Fock space kernel (see Ref. 16), which is defined in terms of the $\star$-exponential function,

$$F^S_{\frac{2}{\gamma^2}}(q,p) := e_\star\left(\frac{2}{\gamma^2} q\bar{p}\right) = \sum_{n=0}^{\infty} \frac{2^n}{\gamma^{2n} n!} q^n \bar{p}^n.$$

We note that by restricting both the variables q and p to the real line $\mathbb{R}$, we get the classical Gaussian RBF kernel on $\mathbb{R}$.

ACKNOWLEDGMENTS

The authors thank Prof. P. Zunino for asking a stimulating question which lead to this paper. Daniel Alpay acknowledges the Foster G. and Mary McGaw Professorship in Mathematical Sciences, which supported this research. Kamal Diki acknowledges the Grand Challenges Initiative (GCI) at Chapman University for supporting his research. He also acknowledges Dipartimento di Matematica of Politecnico di Milano for the excellent work conditions he had in Milano while developing this research.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Daniel Alpay: Conceptualization (equal); Methodology (equal); Writing – original draft (equal). **Fabrizio Colombo**: Conceptualization (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **Kamal Diki**: Conceptualization (equal); Methodology (equal); Writing – original draft (equal). **Irene Sabadini**: Conceptualization (equal); Methodology (equal); Writing – original draft (equal).

DATA AVAILABILITY

There is no data to support the findings of this study.

REFERENCES

[1] D. Alpay, *An Advanced Complex Analysis Problem Book. Topological Vector Spaces, Functional Analysis, and Hilbert Spaces of Analytic Functions* (Birkhäuser, Basel, 2015).
[2] I. Steinwart and A. Christmann, *Support Vector Machines* (Springer-Verlag New York, 2008), ISBN: 978-0-387-77242-4.
[3] N. Aronszajn, "Theory of reproducing kernels," Trans. Am. Math. Soc. **68**, 337–404 (1950).
[4] M. A. Aizerman, "Theoretical foundations of the potential function method in pattern recognition learning," Autom. Remote Control **25**, 821–837 (1964).
[5] B. Schölkopf, K. Tsuda, and J. P. Vert, *Kernel Methods in Computational Biology* (MIT press, 2004).
[6] E. Parzen, "An approach to time series analysis," Ann. Math. Stat. **32**, 951–989 (1961).
[7] E. Parzen, "Statistical interference on time series by RKHS methods," Stanford University California Department of Statistics, 1970.
[8] T. M. Mitchell, *Machine Learning* (McGraw-Hill, 1997).
[9] J. Shawe-Taylor and N. Cristianini, *Kernel Methods for Pattern Analysis* (Cambridge University Press, 2004).
[10] E. J. Hartman, J. D. Keeler, and J. M. Kowalski, "Layered neural networks with Gaussian hidden units as universal approximations," Neural Comput. **2**(2), 210–215 (1990).
[11] I. Steinwart, D. Hush, and C. Scovel, "An explicit description of the reproducing kernel Hilbert spaces of Gaussian RBF kernels," IEEE Trans. Inf. Theory **52**, 4635–4643 (2006).
[12] K. Zhu, *Analysis on Fock Spaces* (Springer, New York, 2012).
[13] B. Hall, *Quantum Theory for Mathematicians* (Springer, 2013).
[14] V. Bargmann, "On a Hilbert space of analytic functions and an associated integral transform," Commun. Pure Appl. Math. **14**, 187–214 (1961).
[15] K. Zhu, "Towards a dictionary for the Bargmann transform," *Handbook of Analytic Operator Theory*, Handbook in Mathematics Series (CRC Press/Chapman Hall, CRC Press, Boca Raton, FL, 2019), pp. 319–349.
[16] D. Alpay, F. Colombo, I. Sabadini, and G. Salomon, "The Fock space in the slice hyperholomorphic setting," *Hypercomplex Analysis: New Perspectives and Applications* (Springer, 2014), pp. 43–59.