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Reducing the Complexity of Linear Optics Quantum Circuits

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Integrated optical elements can simplify the linear optics used to simulate quantum circuits. These linear optical simulations of quantum circuits have been developed primarily in terms of the free space optics associated with single-photon interferometry. For an L-bit simulation the number of required free-space optical elements is $\propto 2^L$ if 50/50 beam splitters are used. The implementation (construction and alignment) of these circuits with these free-space elements is nontrivial. On the other hand, for the cases presented in this paper in which linear integrated optics (e.g., $2^L \times 2^L$ fiber couplers) are used, the number of optical devices *does not* grow exponentially with L. The problem is changed from having an exponential growth of the number of devices to having devices with an exponential growth in the number of ports. In addition to simplifying the construction, the association of an $N \times N$ fiber coupler with the discrete Fourier transform suggests alternative formulations for the circuits. Several examples of circuit reductions are given.

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I. INTRODUCTION

All-optical implementations of quantum logic hold a unique position in quantum computing. Typical quantum computers employ many-particle entanglements to realize logic gates. Linear optics setups employ the interference of the degrees of freedom of a single photon [1] to realize logic gates. The unitary evolution [2] of a single photon in a linear optics environment provides an excellent framework in which to model quantum algorithms.

There are distinct advantages and disadvantages of using linear optics to model quantum networks. The primary disadvantage of using linear optics is that the size of the apparatus grows with the Hilbert space [3,4] and not with the number of bits as in a typical quantum computer. This is a consequence of having each path being labeled as an eigenstate of the Hilbert space. For L bits, where each bit has two states, the Hilbert space grows as 2^L, and hence the number of paths is also 2^L. On the other hand, this affords tremendous manipulative power. One consequence of the enhanced manipulative power, is that it is easy to obtain several many-bit gates [5]. For example, in order to change the phase of one of the eigenstates, a simple phase delay in one of the paths is sufficient. For a typical quantum computer, it is difficult to change the phase of a single state without affecting the phases of the other states of the system. Another advantage is that the decoherence associated with linear optics is small compared with typical quantum computing systems.

Most of the proposed all-optical circuits [4,6,7] employ only 1- and 2-path free-space optical elements (e.g., 50/50 beam splitters, phase delays, polarization rotators, etc.). Using elements that are limited to only 1 and 2 paths implies that as the number of bits increases the number of optical elements increases exponentially. Then, for even two-bit circuits, the experimental optical realization is very difficult. An obvious experimental difficulty arises from the alignment of these elements.

The techniques presented in this paper are based upon the use of multiport devices. It is possible to construct multiport devices from free-space 50/50 beam splitters [8]. However, for an $N \times N$ device, $N-1$ beam splitters are needed. On the other hand, linear integrated optics has developed such a multiport device, which is commonly known as an $N \times N$ fiber coupler [9]. Thus, a single integrated device may replace many individual elements. The alignment difficulty of the circuit is also significantly reduced. Of course such integrated devices are not perfect and can affect the polarization of the light and introduce unwanted phase delays. Fortunately, polarization correction elements for use with fiber systems are well developed as are phase shifting elements. Note that such corrections have been experimentally achieved in Ref. [8]. Also, note that polarization preserving $N \times N$ fiber couplers are commercially available. The use of correcting elements does not affect the circuit simplification provided by the use of integrated optics, i.e., polarization rotators and phase delays were already commonly required in the free-space implementation of these circuits. In addition to the simplification provided by the replacement of many beam splitters by a single device, the association of the $N \times N$ fiber coupler with the discrete Fourier transform offers further opportunities for circuit reduction.

II. $N \times N$ FIBER COUPLER AS DISCRETE FOURIER TRANSFORM

A symmetric $N \times N$ fiber coupler performs the discrete Fourier transform (DFT) on the spatial modes [9,13]. The DFT has matrix elements given by

$$F_{mn} = \langle m|n \rangle = \frac{1}{\sqrt{N}} e^{i2\pi mn/N}, \tag{2.1}$$

where N is the number of eigenvectors in the Hilbert space (number of paths in a linear optics setup).

There are several instances in which an $N \times N$ device can be used in place of free-space optics. The most obvious use for symmetric $N \times N$ fiber couplers is in generating the DFT. The DFT is at the heart of several important quantum algorithms [10–13]. In order to implement the DFT using free-space optics $(NL)/2$ 50/50 beam splitters are necessary. On

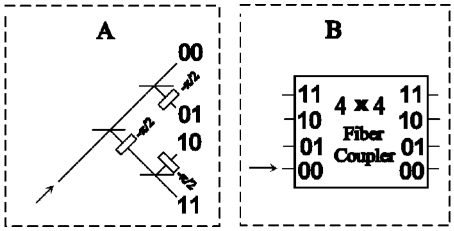

FIG. 1. Three cascading 50/50 beam splitters (in dashed box A) are replaced by a single 4×4 fiber coupler (in dashed box B) in order to generate equal probability amplitude in each spatial mode. This is a device used for preparing an input state.

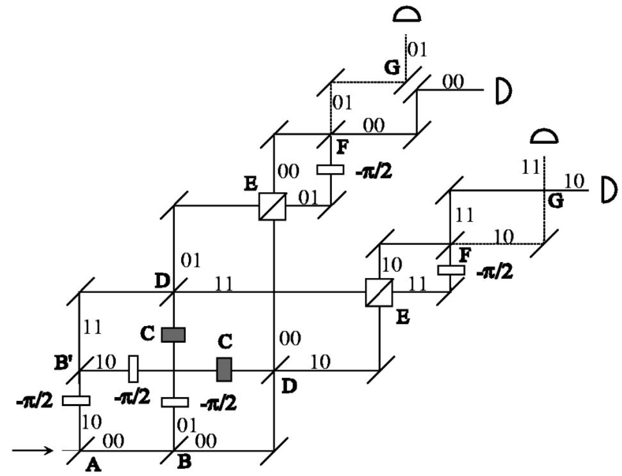

FIG. 2. Linear optical implementation of a teleportation circuit utilizing free space optical elements. This circuit is reproduced with permission from Ref. [4]. Copyright 1998 by the American Physical Society.

the other hand only a single $N \times N$ coupler is needed. For example, in Ref. [13] the DFT and its inverse were the only operations needed in order to realize the quantum baker's map (QBM). It was shown that two 2-bit DFTs and one 3-bit inverse DFT were needed. Hence, 20 beam splitters were required to realize the map. On the other hand, it was also shown that three multiport devices could implement the map. In addition, one gains a better visual physical intuition for the simple QBM. From a device point of view, as the number of bits increases the number of multiport devices remains constant as long as there exists a device with a sufficient number of ports. Hence, the problem is changed from having an exponential increase in the number of optical elements to having multiport devices with an exponential number of ports.

III. CIRCUIT REDUCTION

A. Input

A preliminary example of circuit reduction via integrated optics is in preparation of initial states. Often it is necessary to prepare a state vector such that the photon has an equal probability amplitude in each path with no relative phase delays. For example, in Ref. [4] (the teleportation scheme from this paper has been reproduced by permission and is shown in Fig. 2) and [7] (three-bit search algorithm) cascading 50/50 beam splitters were used to create equal photon probability amplitudes in each of the paths. For the four-path setups employed in those circuits three beam splitters were needed. On the other hand, a single 4×4 coupler would have performed the same operation. In general, for $N=2^L$ paths, $N-1$ 50/50 beam splitters are needed to create a state with equal amplitudes in each of the paths, whereas an $N\times N$ symmetric multiport device is sufficient to generate the input. Consider Fig. 1. Inside the dashed box labeled A are three 50/50 beam splitters. This configuration is similar to the device employed in the teleportation circuit in Ref. [4] and shown in Fig. 2. The substitute device shown inside the dashed box labeled B is realized by the 4×4 coupler. It performs the same transformation with only a single device. It should be noted that this is only true if the photon enters the 00 path. Allowing the photon to enter another path will cause different phase delays as governed by the DFT.

B. Circuit sectioning and replacement

An attractive and general reduction scheme is to have replacement components. Suppose a circuit is generated from 1 and 2 path optical elements. In many cases, the circuit can be broken down into sections, which can be reproduced by finding the same transformation generated from products of the DFT and other matrices (e.g., phase matrix, relabelling of paths, etc.). For example, the teleportation circuit in Ref. [4], Fig. 2, can be broken down into four sections which we will call input, polarization, reduction, and measurement sections. For the moment, we will concern ourselves with the reduction section. The reduction section is transformations D, F, and G in their circuit. Transformation D(F) performs the Hadamard transformation on the most(least) significant path bit. Transformation G is a control-not gate where the most(least) significant path-bit is the control(target)-bit. Now consider the simplified circuit shown in Fig. 3. Transformations D and E commute. However, in a standard teleportation circuit, transformations C and D correspond to the Bell measurement. Hence, commuting operations D and E does lose the one-to-one correspondence with teleportation. Still, the reduced circuit does the same transformation as the teleportation circuit of Fig. 2. Thus, for the sake of demonstrating the reduction technique, we will commute operations D and E and transformation E will be placed in the polarization section of the circuit.

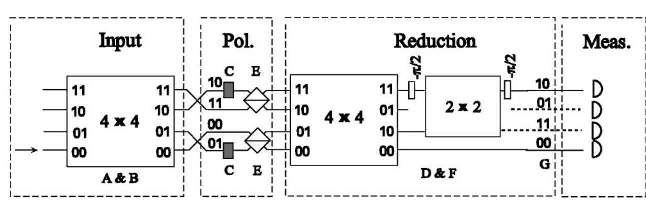

FIG. 3. Linear optical implementation of a teleportation circuit utilizing integrated optical elements. The circuit is broken down into four sections. The input and reduction sections use linear integrated optics to reduce the complexity of the circuit.

The reduction section requires four beam splitters and eight phase delays in the original scheme. It will be shown using linear integrated optics that the reduction section can be implemented with one 4×4 fiber coupler one 2×2 fiber coupler and two phase delays.

The key part of this reduction technique is finding a transformation that is equivalent to two successive Hadamard transformations. A 2-bit DFT can be obtained from

$$F=S_{01}A_0R_{10}A_1, \tag{3.1}$$

where

$$A_j=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{3.2}$$

is the Hadamard transformation, S_{01} is a swap gate which swaps the zeroth and first bit and R_{ij} for $(i<j)$ is defined to be

$$R_{ij}|k_{L-1},\ldots,k_0\rangle=e^{i\varphi_{ij}}|k_{L-1},\ldots,k_0\rangle, \tag{3.3}$$

where

$$\varphi_{ij}=\begin{cases}\pi/2^{j-i} & \text{if } k_i=k_j=1,\\ 0 & \text{otherwise,}\end{cases} \tag{3.4}$$

and $k_m\in\{0,1\}$. Here, L is the number of qubits and hence $N=2^L$. The matrix representation of R_{ij} is given by

$$R_{ij}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&e^{i\varphi_{ij}}\end{pmatrix}. \tag{3.5}$$

However, the problem is reversed. We want to find a substitution for successive Hadamard transformations using the DFT. We want to find a matrix B which is defined by

$$BF=\mathrm{CNOT}_{10}A_0A_1, \tag{3.6}$$

where CNOT_{10} is a control-not gate with the most(least) significant bit acting as the control(target) bit. This implies

$$B=\mathrm{CNOT}_{10}A_0A_1F^{-1}=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&\frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}&0&\frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}\\0&\frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}}&0&\frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\end{pmatrix}. \tag{3.7}$$

By observing the B matrix, one can get a general feel for what simple transformations could yield such a matrix. First of all, one notices that the upper-left most element is 1 on the diagonal. Secondly, the second row has a 1 in the third column. In addition, one can see there are elements in the third and fourth row of the second column. These elements imply using swap gate (S_{01}), which is defined above, as one of the transformations needed to realize the B operation. Another observation is that the third and fourth rows have equal amplitudes in both the second and fourth columns. One also notices that the phases change sign between either the rows or columns. The relative amplitudes and phases of the last two rows could be realized by a lossless symmetric beam splitter, which will be referred to as the (J) operation, which has the 10 and 11 paths entering the input ports [4] (up to a phase). In addition, the phases of the third and fourth rows implies the use of control-not (CNOT_{10}) with most significant bit being the control bit. Indeed, the transformations needed to realize the gate are

$$B=\mathrm{CNOT}_{10}J(S_{01}). \tag{3.8}$$

The linear integrated optics realization is shown inside the dashed box labeled reduction in Fig. 3. The swap gate is realized by relabeling the outputs of the 4×4 output coupler where $01\mapsto 10$ and vice versa. Since we are interested in working with linear integrated optics, it would be appropriate to use a 2×2 fiber coupler instead of a symmetric beam splitter. Since a 2×2 fiber coupler performs a one-bit DFT, then

$$J=\begin{pmatrix}1&0\\0&e^{i(\pi/2)}\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1&0\\0&e^{i(\pi/2)}\end{pmatrix}. \tag{3.9}$$

Hence, placing a $\pi/2$ phase shift in the input and output port of the of the 11 path entering and exiting a 2×2 fiber coupler will yield the same result as a symmetric beam splitter. It should be noted that path 01 goes around the 2×2 fiber coupler, i.e., it does not interact with it. The CNOT_{10} is implemented by simply relabeling the paths in which the most significant bit is high. In other words, path $11\mapsto 10$ and vice versa.

The circuit shown in Fig. 3 represents the simplified version of the teleportation circuit first modeled in Fig. 2. While the outputs of the two circuits are the same, the correspondence to a teleportation circuit has been lost.

C. Alternative approaches and Grover's search

Finally, the last reduction example makes explicit use of the fact that many circuits do not have a unique construction. All circuits can be considered to have some initial state, which after the transformation provided by the circuit, yields some desired output. The truth table defined by the input and output state makes no explicit reference to the transformation needed to obtain the output from the input. In this section, it will be shown that, in some cases, more than one transformation can be used to obtain the same truth table. This reduction technique differs from the technique explained in the previous subsection. In the previous subsection, the *same* transformation was obtained using higher-dimensional integrated optics instead of free-space optics. In this subsection *different* transformations are used to obtain the same truth table.

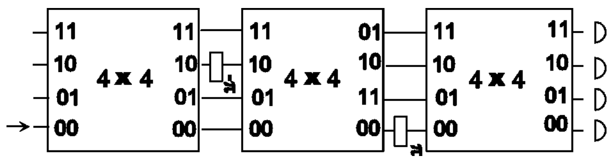

FIG. 4. A simple circuit to model Grover's search algorithm. One iteration employs only two 4×4 fiber couplers.

Grover showed that quantum computers can search rapidly using almost any unitary transformation [14]. He showed that it is possible to reach some target state denoted $|\tau\rangle$ from initial state $|\gamma\rangle$ in $O(1/|U_{\tau\gamma}|)$ steps, where $|U_{\tau\gamma}|$ is the magnitude of the matrix element connecting state $|\tau\rangle$ and $|\gamma\rangle$. Hence, as long as there is a non-zero matrix element connecting state $|\tau\rangle$ and $|\gamma\rangle$, then any unitary matrix, with this property, will be able to search for state $|\tau\rangle$.

Grover showed that there exists a unitary transformation Q given by $-I_\gamma U^{-1} I_\tau U$, which after the appropriate number of iterations of Q will evolve from state $|\gamma\rangle$ to $U^{-1}|\tau\rangle$. Here, $I_\gamma(I_\tau)$ is a diagonal matrix with all of the diagonal elements equal to 1 except for the $\gamma\gamma(\tau\tau)$ element which is equal to -1.

In Ref. [7] several all-optical implementations of Grover's search algorithm are proposed and studied. However, it can be seen that the number of 1- and 2-path elements grows exponentially with the number of bits. Even employing a compiling technique and using polarization as one of the bits, a 3-bit search required 21 elements.

The method proposed here is to use the DFT instead of the Walsh-Hadamard transformation (WHT). The most important similarity between the DFT and WHT is that $|F_{ij}| = |W_{ij}| = 1/\sqrt{N}$, where F represents the DFT and W represents the WHT. Hence, either transformation can search equally well. Using the DFT instead of the WHT means that $N\times N$ fiber couplers can be used.

Consider the 2-bit 4-path search circuit proposed in Fig. 4. This figure shows one iteration of a transformation which searches for the $|10\rangle$ state starting from the $|00\rangle$ state. The first $N\times N$ fiber coupler performs the DFT. The second transformation is a phase operation, which simply retards the phase of the 10 path by π. This operations marks the state to be searched for. The third operation performs and inverse DFT F^{-1}. The inverse DFT is given by

$$(F_N)^{-1} = T(F_N), \tag{3.10}$$

where for two bits T, in matrix form, is given by

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{3.11}$$

The T operation simply swaps rows of the DFT. For example, the 11 path is relabled as the 01 path. The fourth operation is a π phase delay in 00 path. The first four operations represents the Q operation. Hence, the circuit has transformed the state from $|\gamma\rangle$ to $U^{-1}|\tau\rangle$. Hence, the fifth and last operation is to take another DFT. In this way, the target state $|\tau\rangle$ has been found from the input state.

The key result of employing the DFT and using $N\times N$ couplers is that for one iteration the number of optical elements remains the same regardless of the number of bits, just as in the quantum baker's map. It should be noted that this proposal has neglected counting the fibers connecting the couplers as optical elements.

IV. DISCUSSION

The purpose of this paper has been to draw upon the distinct advantages of linear integrated optics to reduce the number of optical devices in quantum circuits. While it has been shown that several aspects of circuits can be simplified, many of the same experimental difficulties still remain. For example, spurious phase shifts introduced in the inputs, circuit, or output can greatly affect the function of the circuit. Also, couplers have imperfections that cause loss, unequal amplitude splittings, and undesirable phase shifts. In addition, fiber couplers and fibers, many times, do not preserve the polarization. Hence, efforts need to be taken to restore the polarization or polarization preserving optical devices must be used. As a general rule, the difficulties in implementing the reduced circuits are the same as the unreduced circuits, it is just that there are fewer of them.

In summary, it has been shown that many circuits can be simplified by using linear integrated optics. The circuit reductions are implemented using higher order multiport devices from linear integrated optics, rather than the typical free-space 50/50 beam splitters. We have demonstrated that for many circuits the problem can be reduced from having a large number of optical elements to having a multiport device with a large number of ports. This promises a significant simplification in the physical realization of these circuits.

ACKNOWLEDGMENTS

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constructed by crossing the 110 and 101 paths and relabeling the paths. Any phase gate can be realized by placing phase shifters in the appropriate paths. In a scheme where two of the bits are represented by spatial degrees of freedom and one of the bits is represented by the polarization degrees of freedom, a Deutsch gate can be realized by placing a polarization rotator in the 11 path. Other combinations of the Toffoli and Fredkin gates exist and all of them only require a single unitary operation.

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