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3-9-2010

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### Recommended Citation

J. C. Howell, D. J. Starling, P. B. Dixon, P. K. Vudyasetu, and A. N. Jordan, *Interferometric Weak Value Deflections: Quantum and Classical Treatments*, *Phys. Rev. A* **81**(3), 033813. <https://doi.org/10.1103/PhysRevA.81.033813>

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## Interferometric Weak Value Deflections: Quantum and Classical Treatments

### Comments

This article was originally published in *Physical Review A*, volume 81, issue 3, in 2010. <https://doi.org/10.1103/PhysRevA.81.033813>

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# Interferometric weak value deflections: Quantum and classical treatments

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(Received 10 November 2009; published 9 March 2010)

We derive the weak value deflection given in an article by Dixon *et al.* [P. B. Dixon *et al.* Phys. Rev. Lett. **102** 173601 (2009)] both quantum mechanically and classically, including diffraction effects. This article is meant to cover some of the mathematical details omitted in that article owing to space constraints.

DOI: [10.1103/PhysRevA.81.033813](https://doi.org/10.1103/PhysRevA.81.033813)

PACS number(s): 42.50.Xa, 03.65.Ta, 06.30.Bp, 07.60.Ly

## I. INTRODUCTION

Weak values [1] have presented or inspired intriguing possibilities for precision measurement. A recent example is by Hosten and Kwiat [2], where they were able to amplify the deflections arising from the spin Hall effect of light. The light fields used in their experiment, as well as in Ref. [3], were coherent quasiclassical fields and no apparent quantum mechanical system was employed in either experiment. The classical behavior of these weak value deflection measurements has been known for some time [4]. Shortly after the Hosten and Kwiat article, Aiello and Woerdman [5] published the classical description to allow greater accessibility to the metrology community. In the present article, we derive both the quantum weak value amplification for a Sagnac interferometer [3,6] and its classical counterpart under the corresponding limits using the standard classical description of the electromagnetic field. We end with a detailed calculation of the diffraction effects, which are summarized in Ref. [3].

Consider the interferometric weak value experiment in Ref. [3]. We point out that all two-dimensional quantum systems are isomorphic to spin-1/2 particles. In the Hosten-Kwiat experiment, the two-dimensional system was the transverse polarization states of the light. For the weak value description in this article, we use the which-path states of a photon in a Sagnac interferometer as the two-state system (see Fig. 1).

We first derive the quantum mechanical weak value description for a single photon in Sec. II before proceeding to the derivation using classical fields in Sec. III. Finally, we consider the case of a diverging beam in Sec. IV.

## II. QUANTUM TREATMENT

The which-path (system) variable of a photon is coupled to its transverse momentum (meter) variable. The system eigenstates are  $\{|a_i\rangle\}$  and the meter eigenstates are  $\{|k_x\rangle\}$ . The preselected total state of the photon is the tensor product of the system and meter states, written as

$$|\Psi\rangle = \int dk_x \psi(k_x) a_{k_x}^\dagger |0\rangle |\varphi_1\rangle, \quad (1)$$

where  $a_{k_x}^\dagger$  is the usual creation operator for the transverse mode  $k_x$ ,  $|\varphi_1\rangle = \sum_i c_i |a_i\rangle$  is the input state of the photon, where  $\{c_i\}$  are the probability amplitudes for the system states, and  $\psi(k_x)$  is the transverse wave function. We assume that  $\psi(k_x)$  is a Gaussian in order to obtain an analytic solution.

We now describe the weak-measurement procedure. First, the photon undergoes a small unitary evolution, which couples one of the propagation directions in the interferometer to one momentum shift given by  $k$ , and the other direction to another momentum shift, given by  $-k$  (for a symmetric interferometer). In essence, the momentum shift  $k$ , upon detection, gives a small amount of which-path information about the path of the photon. The unitary evolution is given by  $U = e^{-ik\hat{A}x} \approx 1 - ik\hat{A}x$ , where  $\hat{A}$  is the which-path observable with eigenvalues given by  $\hat{A}|a_i\rangle = a_i|a_i\rangle$ , and where  $x$  is the transverse position variable of the photon. Measuring this small momentum shift constitutes the weak measurement. In this scheme, the weak measurement and post-selection measurement happen simultaneously at the output port of the beam splitter (measuring the transverse momentum and post selecting on the output port).

For simplicity, the calculation here will assume a collimated beam with no divergence. As seen in Fig. 1, a piezo-driven mirror imparts a weak transverse momentum shift  $k$  in opposite directions relative to the optical axis at the exit face of the beam splitter. As noted earlier, this deflection gives partial information about which way the photon went in the interferometer because of the transverse position shift of the photon in the detection plane. For very small momentum shifts and short distances, the deflection is very small compared to the transverse diameter of the beam, and thus the system eigenstates are only weakly discriminated. As the photon passes through the beam splitter for the second time, there is a post-selection on the state  $|\varphi_2\rangle$ , which is nearly orthogonal to the input state. This yields a post-selected meter state

$$\begin{aligned} \langle\varphi_2|U|\Psi\rangle &\approx \int dk_x \psi(k_x) a_{k_x}^\dagger |0\rangle \langle\varphi_2|\varphi_1\rangle \\ &- i \int dk_x \psi(k_x) kx a_{k_x}^\dagger |0\rangle \langle\varphi_2|\hat{A}|\varphi_1\rangle. \end{aligned} \quad (2)$$

As can be seen, if the pre- and the post-selected system states are nearly orthogonal, the probability for the photon to pass through the post-selecting device (i.e., the beam splitter) is small. However, for the photons that do pass through, we must renormalize the single photon meter state. We define the renormalized state as

$$\begin{aligned} |\Psi'\rangle &= \int dk_x \psi(k_x) a_{k_x}^\dagger |0\rangle \\ &- i \int dk_x \psi(k_x) kx a_{k_x}^\dagger |0\rangle \frac{\langle\varphi_2|\hat{A}|\varphi_1\rangle}{\langle\varphi_2|\varphi_1\rangle}, \end{aligned} \quad (3)$$

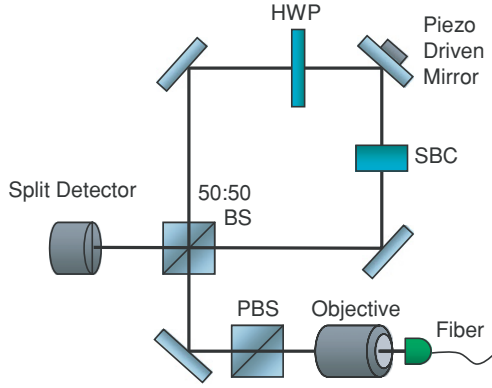


FIG. 1. (Color online) Experimental setup. A fiber-coupled laser is launched into free space and collimated. The beam passes through a polarizing beam splitter (PBS) and then enters a Sagnac interferometer that consists of three mirrors: a 50:50 beam splitter (BS), a half wave plate (HWP), and a Soleil-Babinet compensator (SBC). The SBC and HWP in the interferometer allow the output intensity of the interferometer to be tuned. The output port is monitored with a split detector. The piezo-driven mirror gives a small beam deflection.

where the term  $\frac{\langle \varphi_2 | \hat{A} | \varphi_1 \rangle}{\langle \varphi_2 | \varphi_1 \rangle}$  is the standard weak value term. A quick example shows why this term is imaginary. Suppose the preselected spin state is given by  $|\varphi_1\rangle = \frac{1}{\sqrt{2}}(e^{-i\phi/2}|+\rangle + e^{i\phi/2}|-\rangle)$  and the post-selected state is  $|\varphi_2\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$ . We then see that  $\langle \varphi_2 | \varphi_1 \rangle = i \sin(\phi/2)$ . If the observable  $\hat{A}$  is the Pauli operator  $\sigma_z$ , we find  $\langle \varphi_2 | \hat{A} | \varphi_1 \rangle = \cos(\phi/2)$ . Thus,  $\frac{\langle \varphi_2 | \hat{A} | \varphi_1 \rangle}{\langle \varphi_2 | \varphi_1 \rangle}$  is purely imaginary. Noting this, we let  $A_w = \frac{|\langle \varphi_2 | \hat{A} | \varphi_1 \rangle|}{|\langle \varphi_2 | \varphi_1 \rangle|}$ . In this example, small  $\phi$  produces a large  $A_w$ . We will see that this corresponds to a standard weak value enhancement.

As long as the second term on the right hand side is much smaller than the first of Eq. (4), we can reexponentiate to obtain

$$|\Psi'\rangle = \int dk_x \psi(k_x) e^{-k A_w x} a_{k_x}^\dagger |0\rangle. \quad (4)$$

To obtain the probability amplitude distribution in the transverse plane, we define a positive-frequency field operator

$$E^+(x) = \int dk_x E_0 e^{-ik_x x} a_{k_x}, \quad (5)$$

where  $E_0$  is the electric field amplitude. Incorporating this result and using the commutation relation  $[a_{k_x}, a_{k_x}^\dagger] = \delta(k_x' - k_x)$ , the state becomes

$$\langle 0 | E^+(x) | \Psi' \rangle = E_0 \int dk_x e^{-ik_x x} \psi(k_x) e^{-k A_w x}. \quad (6)$$

From this point, we will not worry about the normalization of the state and use the Gaussian wave function. Using the fact that  $A_w \approx 2/\phi$  for small  $\phi$ , we find

$$\begin{aligned} \langle 0 | E^+(x) | \Psi' \rangle &\propto e^{-2kx/\phi} \int dk_x e^{-ik_x x} e^{-k_x^2 \sigma^2} \\ &= \exp \left[ -\frac{x^2}{4\sigma^2} - \frac{2kx}{\phi} \right], \end{aligned} \quad (7)$$

where  $\sigma$  is the Gaussian beam radius. After completing the square,

$$\langle 0 | E^+(x) | \Psi' \rangle \propto \exp \left[ -\frac{1}{4\sigma^2} \left( x + \frac{4k\sigma^2}{\phi} \right)^2 \right]. \quad (8)$$

One can see that, at the detector, there will be a transverse position shift of the beam given by  $d_w = 4k\sigma^2/\phi$ , where  $d_w$  denotes the weak value transverse deflection.

### III. CLASSICAL TREATMENT

We now derive the same result classically using standard wave optics. This can be done by denoting the transverse two-port input field of the interferometer as

$$\mathbf{E}_{\text{in}} = \begin{pmatrix} E_0 \exp\left(\frac{-x^2}{4\sigma^2}\right) \\ 0 \end{pmatrix}, \quad (9)$$

where the second position in the column vector denotes the input port with no electric field. The field then passes through a 50:50 beam splitter with matrix representation

$$\mathbf{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \quad (10)$$

We now define a matrix that gives both an opposite-momentum shift  $k$  and a relative phase between the two paths:

$$\mathbf{M} = \begin{pmatrix} e^{i(-kx+\phi/2)} & 0 \\ 0 & e^{-i(-kx+\phi/2)} \end{pmatrix}. \quad (11)$$

We want to determine the field at the ‘‘dark’’ output port (i.e., the port with the lowest intensity of light coming out of it) of the interferometer. The evolution of the light is represented by the matrix combination

$$\mathbf{E}_{\text{out}} = (\mathbf{B}\mathbf{M}\mathbf{B})\mathbf{E}_{\text{in}}. \quad (12)$$

The output field at the dark port is renormalized by noting that the detector only measures the total flux falling on it to determine the deflection. For small  $k$ , the renormalized [by  $\sin(\phi/2)$ ], measured output signal at the dark port will be of the form

$$E_{\text{out}}^d = \frac{\sin(-kx + \phi/2)}{\sin(\phi/2)} \exp\left(\frac{-x^2}{4\sigma^2}\right). \quad (13)$$

For small angles, we obtain

$$E_{\text{out}}^d \approx \left( 1 - \frac{2kx}{\phi} \right) \exp\left(\frac{-x^2}{4\sigma^2}\right), \quad (14)$$

which we reexponentiate and, after completing the square in the exponent, find

$$E_{\text{out}}^d \propto \exp \left[ -\frac{1}{2\sigma^2} \left( x + \frac{4k\sigma^2}{\phi} \right)^2 \right]. \quad (15)$$

We see that we obtain the same deflection as the quantum mechanical weak value treatment.

### IV. PROPAGATION EFFECTS

To consider the case of a diverging beam, we insert a negative focal length lens before the interferometer and use

standard Fourier optics methods in the paraxial approximation outlined in Goodman [7]. In the case of the quantum treatment, phase factors and Fourier transforms are applied to the quantum state  $\Psi(x)$  or  $\tilde{\Psi}(k_x)$  by convention. Similarly, in the classical treatment, they are applied to the electric field  $E$ .

Passing through the lens, the wave function (electric field) acquires a multiplicative phase factor  $\exp[ik_0x^2/(2s_i)]$ , where  $k_0$  is the wave number of the light and  $s_i$  is the image distance behind the lens, resulting in a spreading beam. Propagation effects are accounted for by Fourier transforming the state (field) at the lens, and applying a multiplicative phase factor  $\exp[-ip^2l_{lm}/(2k_0)]$  to the momentum-space wave function (field), where  $l_{lm}$  is the distance between the lens and the mirror. The effect of the oscillating mirror is to shift the state (field) by a very small transverse momentum  $k$ ,  $\Phi(p) \rightarrow \Phi(p \pm k)$  [ $E(k_x) \rightarrow E(k_x \pm k)$ ], where the direction of the shift depends on which path the photon takes in the interferometer. Propagation from the mirror to the detector results in a final multiplicative phase factor  $\exp[-ip^2l_{md}/(2k_0)]$  on the momentum-space wave function (field), where  $l_{md}$  is the distance between the mirror and the detector. After applying an inverse Fourier transform, the individual amplitudes in both arms are given by

$$\Psi_{1,2}(x) \propto \exp\left[\frac{-ik_0x^2 \pm 2ilkx}{2(l + l_{md})}\right] \quad (16)$$

up to normalization, where  $l = l_{lm} - a^2s_i/[a^2 + is_i/(2k_0)]$  and  $a$  is the beam radius at the lens. These amplitudes (fields) now interfere with a relative phase  $\phi$ , and the position of the beam is monitored with a split detector at the dark port. Because the relative momentum shift  $k$  given by the movable mirror is so small, the post-selection probability is given only by the overlap of pre- and post-selected states,

$P_{ps} = \sin^2(\phi/2) \approx \phi^2/4$  for  $\phi \ll 1$  as before. If we consider the beam far from the focus, such that the wavelength  $\lambda \ll 2\pi a^2/s_i$ , we find that the beam deflection is given by

$$d'_w = \left(\frac{4ka^2}{\phi}\right) \left[\frac{l_{im}(l_{im} + l_{md})}{s_i^2}\right] = d_w F, \quad (17)$$

where  $l_{im}$  is the distance between the image and the mirror. We can rewrite this expression in terms of easily measurable quantities by noting that the beam radius of a Gaussian beam can be written as  $\sigma(z) = \sigma(0)[1 + (z/z_R)^2]^{1/2}$ . Here,  $z$  is the propagation distance, with  $z = 0$  defined as the location of the minimum beam radius, and where  $z_R$  is defined as the Rayleigh range. However, for slow divergence of the beam, we can simply write  $\sigma = a(l_{im} + l_{md})/s_i$ . We can then eliminate the image distances  $s_i$  and  $l_{im}$  and write

$$F = \frac{\sigma^2 l_{md} + \sigma a l_{md}}{a^2(l_{im} + l_{md})}. \quad (18)$$

This multiplicative factor  $F$  is the same as found in Ref. [3].

## V. CONCLUSION

We have derived the weak value deflection measurement results in the article by Dixon *et al.* [3] using both classical and quantum methods. The results for a diverging beam using classical Fourier techniques with quantum wave functions and classical electric fields were shown in more detail.

## ACKNOWLEDGMENTS

This work was supported by DARPA DSO Slow Light, ARO and DOD PECASE.

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