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Shifting the quantum-classical boundary: theory and experiment for statistically classical optical fields

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The growing recognition that entanglement is not exclusively a quantum property, and does not even originate with Schrödinger's famous remark about it [Proc. Cambridge Philos. Soc. 31, 555 (1935)], prompts the examination of its role in marking the quantum-classical boundary. We have done this by subjecting correlations of classical optical fields to new Bell-analysis experiments and report here values of the Bell parameter greater than $\mathcal{B} = 2.54$. This is many standard deviations outside the limit $\mathcal{B} = 2$ established by the Clauser–Horne–Shimony–Holt Bell inequality [Phys. Rev. Lett. 23, 880 (1969)], in agreement with our theoretical classical prediction, and not far from the Tsirelson limit $\mathcal{B} = 2.828\ldots$. These results cast a new light on the standard quantum-classical boundary description, and suggest a reinterpretation of it.

OCIS codes: (030.0030) Coherence and statistical optics; (260.5430) Polarization.

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1. INTRODUCTION

For many decades the term "entanglement" has been attached to the world of quantum mechanics [1]. However, it is true that nonquantum optical entanglement can exist (realized very early by Spreeuw [2]) and its applications have concrete consequences. These are based on entanglements between two, or more than two, degrees of freedom (DOFs), which are easily available classically [2–6]. Multientanglements of the same kind are also being explored quantum mechanically [7]. Applications in the classical domain have included, for example, the resolution of a long-standing issue concerning Mueller matrices [8], an alternative interpretation of the degree of polarization (DOP) [9], introduction of the Bell measure as a new index of coherence in optics [10], and innovations in polarization metrology [11]. Here we present theoretical and experimental results extending these results by showing that probabilistic classical optical fields can exhibit violations of the Clauser–Horne–Shimony–Holt (CHSH) Bell inequality [12] of quantum strength. This is evidence of a new kind that asks for reconsideration of the common understanding that Bell violation signals quantum physics. We emphasize that our discussion focuses on the nonquantum entanglement of nondeterministic classical optical fields and does not engage issues such as nonlocality that are important for some applications in quantum information.

The observations and applications of nonquantum wave entanglement noted above [2–6,8–11] exploited nonseparable correlations among two or more modes or DOFs of optical wave fields. Nonseparable correlations among modes are an example of entanglement, but are not enough for our present purpose. In addition, we want to conform to three criteria that Shimony has identified for Bell tests [13], facts of quantum nature that must be satisfied when examining possible tests of the quantum-classical border. Fortuitously, the ergodic stochastic optical fields of the classical theory of partial coherence and partial polarization (see Wolf [14]) satisfy these criteria fully (see Supplement 1), and we have used such fields as our test bed.

2. BACKGROUND THEORY

We will deal here only with the simplest suitable example, the theory of completely unpolarized classical light, and have explained elsewhere (see [15] and Supplement 1) the generalizations needed to treat partially polarized fields, which lead to the same conclusions. In all cases there are only two DOFs to deal with, namely, the direction of polarization and the temporal amplitude of the optical field. In both classical and quantum theories, these are fundamentally independent attributes. An electric field, for a beam traveling in z direction, is written as

$$\vec{E}(t) = \hat{x}E_x(t) + \hat{y}E_y(t). \tag{1}$$

In the classical theory of unpolarized light [16], an optical field's two amplitudes E_x and E_y are statistically completely uncorrelated and are treated as vectors in a stochastic function space. A scalar product of the vectors in this space corresponds physically to observable correlation functions such as $\langle E_x E_y \rangle$. For unpolarized light we have $\langle E_x E_x \rangle = \langle E_y E_y \rangle$ and $\langle E_x E_y \rangle = 0$.

Now it is possible to talk of entanglement of the classical field. This is because entangled states are superpositions of products of vectors from different vector spaces, whenever the superpositions cannot be rearranged into a single product that separates the two spaces [1]. Looking again at Eq. (1) we see that this is the case because we have taken $\vec{E}$ to be unpolarized. That is, by the definition of unpolarized light, there is no direction $\hat{u}$ of polarization that captures the total intensity, so $\vec{E}(t)$ cannot, for any direction $\hat{u}$, be written in the form $\vec{E}(t) = \hat{u}F(t)$, which would factorably separate the polarization and amplitude DOFs [17].

Beyond its probabilistic indeterminacy, the $\vec{E}$ in Eq. (1) has other quantum-like attributes—it has the same form as a quantum state superposition and can be called a pure state in the same sense. More precisely, it is a two-party state living in two vector spaces at once, a polarization space for $\hat{x}$ and $\hat{y}$, and an infinitely continuous stochastic function space for E_x and E_y.

The Bell inequality most commonly used for correlation tests is due to Clauser *et al.* [12]. It deals with correlations between two different DOFs when each is two dimensional. The Schmidt theorem of analytic function theory [18] ensures two-dimensionality by guaranteeing that among the infinitely many dimensions available to the amplitudes in Eq. (1), only two dimensions are active. This is a consequence arising just from the fact that the partner polarization vectors $\hat{x}$ and $\hat{y}$ live in a two-dimensional space.

For convenience, we introduce $\vec{e}$, the field normalized to the intensity $I = \langle E_x E_x + E_y E_y \rangle$,

$$\vec{e}(t) \equiv \vec{E}(t)/\sqrt{I} = \{\hat{x}e_x(t) + \hat{y}e_y(t)\}, \tag{2}$$

where now $\langle \vec{e} \cdot \vec{e} \rangle = \langle e_x e_x + e_y e_y \rangle = 1$.

For some simplification in writing, we will use Dirac notation for the vectors without, of course, imparting any quantum character to the fields. The unit polarization vectors $\hat{x}$ and $\hat{y}$ will be renamed as $\hat{x} \to |u_1\rangle$ and $\hat{y} \to |u_2\rangle$ and the unit amplitudes will be rewritten as $e_x \to |f_1\rangle$ and $e_y \to |f_2\rangle$. If desired, the Dirac notation can be discarded at any point and the vector signs and hats re-installed. For the case of unpolarized light, we have $\langle u_1|u_2\rangle = 0$ and $\langle f_1|f_2\rangle = 0$. Unit projectors in the two spaces take the form $1 = |u_1\rangle\langle u_1| + |u_2\rangle\langle u_2|$ and $1 = |f_1\rangle\langle f_1| + |f_2\rangle\langle f_2|$. In this notation, and in the original notation for comparison, the field takes the form

$$\vec{E}/\sqrt{I} = \hat{x}e_x + \hat{y}e_y = |\mathbf{e}\rangle = (|u_1\rangle|f_1\rangle + |u_2\rangle|f_2\rangle)/\sqrt{2}. \tag{3}$$

In this notation, the field actually looks like what it is, a two-party superposition of products in independent vector spaces, i.e., an entangled two-party state (actually a Bell state). Here the two parties are the independent polarization and amplitude DOFs.

The notation for a CHSH correlation coefficient $C(a, b)$ implies that arbitrary rotations of the unit vectors $|u_j\rangle$ and $|f_k\rangle (j, k = 1, 2)$ through angles a and b can be managed independently in the two spaces. An arbitrary rotation through angle a of the polarization vectors $|u_1\rangle$ and $|u_2\rangle$ takes the forms

$$|u_1^a\rangle = \cos a|u_1\rangle - \sin a|u_2\rangle \quad \text{and}$$
$$|u_2^a\rangle = \sin a|u_1\rangle + \cos a|u_2\rangle. \tag{4}$$

For function space rotations, we have $|f_1^b\rangle$ and $|f_2^b\rangle$ defined similarly:

$$|f_1^b\rangle = \cos b|f_1\rangle - \sin b|f_2\rangle \quad \text{and}$$
$$|f_2^b\rangle = \sin b|f_1\rangle + \cos b|f_2\rangle, \tag{5}$$

where the rotation angles a and b are unrelated.

Next, the correlation between the polarization (u) and function (f) DOFs is given by the standard average

$$C(a, b) = \langle \mathbf{e}|Z^u(a) \otimes Z^f(b)|\mathbf{e}\rangle, \tag{6}$$

where Z is shorthand for the difference projection: $Z^u(a) \equiv |u_1^a\rangle\langle u_1^a| - |u_2^a\rangle\langle u_2^a|$, analogous to a σ_z spin operation. $C(a, b)$ is thus a combination of four joint projections such as

$$P_{11}(a, b) = \langle \mathbf{e}|(|u_1^a\rangle|f_1^b\rangle\langle f_1^b|\langle u_1^a|)|\mathbf{e}\rangle = |\langle f_1^b|\langle u_1^a|\mathbf{e}\rangle|^2. \tag{7}$$

This is all classical and all of the correlation projections $P_{jk}(a, b)$, with $j, k = 1, 2$, have familiar roles in classical optical polarization theory [16].

Gisin [19] observed that any quantum state entangled in the same way as the classical pure state Eq. (2) will lead to a violation of the CHSH inequality, which takes the form $\mathcal{B} \le 2$, where

$$\mathcal{B} = |C(a, b) - C(a', b) + C(a, b') + C(a', b')|. \tag{8}$$

The same result will be found here, as one uses only DOF independence and properties of positive functions and normed vectors to arrive at it (see details in Supplement 1). We note again that the issue of entanglement itself is pertinent to the discussion, but the usefulness of entanglement as a resource for particular applications is not. Thus we have reached the main goal of our theoretical background sketch. This was to demonstrate the existence of a purely classical field theory that can exhibit a violation of the CHSH Bell inequality.

3. EXPERIMENTAL TESTING

The remaining task is to show that experimental observation confirms this theoretical prediction, in effect shifting one's interpretation of tests of the quantum-classical border by showing that, along with quantum fields, classical fields conforming to the Shimony Bell-test criteria are capable of Bell violation. In order to make such a demonstration, a classical field source must be used. This means a source producing a field that is quantum mechanical (since we believe all light fields are intrinsically quantum), but a field whose quantum statistics are not distinguishable from classical statistics. This is only necessary up to second order in the field because the CHSH procedure engages no higher order statistics. Such sources are easily available. Since the earliest testing of laser light it has been known that a laser operated below threshold has a statistical character not distinguishable from classical thermal statistics. So in our experiments we have used a broadband laser diode operated below threshold.

Our experiment repeatedly records the correlation function $C(a, b)$ defined in Eq. (6) for four different angles in order to construct the value of the Bell parameter $\mathcal{B}$. This is done through measurements of the joint projections $P_{jk}(a, b)$. We will describe explicitly only the recording of $P_{11}(a, b)$, identified in Eq. (7), but the others are done similarly in an obvious way. In the classical context that we are examining, the optical field is macroscopic and correlation detection is essentially calorimetric (i.e., using a power meter, not requiring or employing individual photon recognition).

4. POLARIZATION TOMOGRAPHY

The first step is to tomographically determine the polarization state of the test field. A polarization tomography setup is shown in Fig. 1. Using a polarizing beam splitter (PBS) and half-wave plates (HWP) and a quarter-wave plate (QWP) to project onto circular and diagonal bases, the Stokes parameters (S_1, S_2, S_3), relative to $S_0 = 1$, are found to be (−0.0827, −0.0920, −0.0158), providing a small nonzero DOP equal to 0.125. This departure from 0 requires a slight modification of the theory presented above (see Supplement 1) and reduces the maximum possible value of $\mathcal{B}$ able to be achieved for our specific experimental field to $\mathcal{B} = 2.817$, below but close to $\mathcal{B} = 2\sqrt{2} = 2.828\ldots$, the theoretical maximum for completely unpolarized light.

5. EXPERIMENTAL BELL TEST

The experimental test has two major components, as shown in Fig. 1: a source of light to be measured and a Mach–Zehnder (MZ) interferometer. The source utilizes a 780 nm laser diode, operated in the multimode region below threshold, giving it a short coherence length of the order of 1 mm. The beam is assumed to be statistically ergodic, stable and stationary, as commonly delivered from such a multimode below-threshold diode. It is incident on a 50:50 beam splitter and recombined on a PBS after adequate delay so that the light to be studied is an incoherent mix of horizontal and vertical polarizations before being sent to the measurement area via a single-mode fiber. A HWP in one arm controls the relative power, and thus, the DOP. QWP and HWP help correct polarization changes introduced by the fiber.

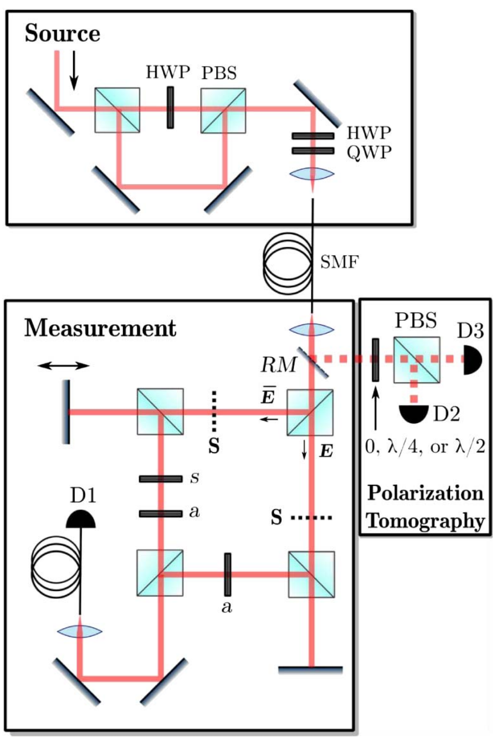

Fig. 1. Experimental setup consists of a source of unpolarized light and a measurement using a modified MZ interferometer. HWP and a QWP control the polarization of the source. All beam splitters are 50:50 unless marked as a PBS. Intensities needed for obtaining the required joint projections are measured at detector D1. Shutters S independently block the arms of the interferometer in order to measure light through the arms separately. A removable mirror (RM) directs the light to a polarization tomography setup where the orthogonal components of the polarization in the basis determined by the wave plate are measured at detectors D2 and D3.

In Fig. 1 the partially polarized beam entering the MZ is separated by a 50:50 beam splitter into a primary test beam $|\mathbf{E}\rangle$ and an auxiliary beam $|\bar{\mathbf{E}}\rangle$. The two beams inherit the same statistical properties from their mother beam, and thus, both can be expressed as in Eq. (3), with intensities I and $\bar{I}$. The phase of the auxiliary beam $|\bar{\mathbf{E}}\rangle$ is shifted by an unimportant factor i at the beam splitter.

To determine the joint projection $P_{11}(a, b)$ of the test beam $|\mathbf{E}\rangle$, the first step is to project the field to obtain $|\mathbf{E}_1^a\rangle \equiv |u_1^a\rangle\langle u_1^a|\mathbf{E}\rangle$. This can be realized by the polarizer labeled a on the bottom arm of the MZ. The transmitted beam retains both $|f\rangle$ components in function space:

$$|\mathbf{E}_1^a\rangle = \sqrt{I_1^a}|u_1^a\rangle(c_{11}|f_1^b\rangle + c_{12}|f_2^b\rangle), \quad (9)$$

where I_1^a is the intensity and c_{11} and c_{12} are normalized amplitude coefficients with $|c_{11}|^2 + |c_{12}|^2 = 1$. Here c_{11} relates to P_{11} in an obvious way: $P_{11}(a, b) = I_1^a|c_{11}|^2/I$. One sees that the intensities I and I_1^a can be measured directly, but not the coefficient c_{11}.

For $P_{11}(a, b)$ our aim is to produce a field that combines a projection onto $|f_1^b\rangle$ in function space with the $|u_1^a\rangle$ projection in polarization space. The challenge of overcoming the lack of polarizers for the projection of a nondeterministic field in an arbitrary direction in its independent infinite-dimensional function space is managed by a "stripping" technique [Supplement 1] applied to the auxiliary $\bar{\mathbf{E}}$ field in the left arm. We pass $\bar{\mathbf{E}}$ through a polarizer rotated from the initial $|u_1\rangle - |u_2\rangle$ basis by a specially chosen angle s so that the statistical component $|f_2^b\rangle$ is stripped off. The transmitted beam $|\bar{E}_1^s\rangle$ then has only the $|f_1^b\rangle$ component, as desired: $|\bar{\mathbf{E}}_1^s\rangle = i\sqrt{\bar{I}_1^s}|u_1^s\rangle|f_1^b\rangle$. Here $\bar{I}_1^s$ is the corresponding intensity and the special stripping angle s is given by $\tan s = (\kappa_1/\kappa_2) \tan b$ (see [15] and Supplement 1).

The function-space-oriented beam $|\bar{\mathbf{E}}_1^s\rangle$ is then sent through another polarizer a to become $|\bar{\mathbf{E}}_1^a\rangle = |u_1^a\rangle\langle u_1^a|\bar{\mathbf{E}}_1^a\rangle = i\sqrt{\bar{I}_1^a}|u_1^a\rangle|f_1^b\rangle$, where $\bar{I}_1^a$ is the corresponding intensity. Finally, the beams $|\mathbf{E}_1^a\rangle$ and $|\bar{\mathbf{E}}_1^a\rangle$ are combined by a 50:50 beam splitter, which yields the outcome beam $|\mathbf{E}_1^T\rangle = (|\bar{\mathbf{E}}_1^a\rangle + i|\mathbf{E}_1^a\rangle)/\sqrt{2}$. The total intensity I_1^T of this outcome beam can be easily expressed in terms of the needed coefficient c_{11}.

Some simple arithmetic will immediately provide the joint projection $P_{11}(a, b)$ in terms of various measurable intensities:

$$P_{11}(a, b) = (2I_1^T - \bar{I}_1^a - I_1^a)^2/4I\bar{I}_1^a. \quad (10)$$

Other $P_{jk}(a, b)$ values can be obtained similarly by rotations of polarizers a and s. To make our measurements, polarizers a were simultaneously rotated using motorized mounts, whereas the third polarizer s was fixed at different values in a sequence of runs.

6. RESULTS

For each angle, measurements were made at detector D1 for the total intensity I^T and for the separate intensities from each arm I^a and $\bar{I}^a$ by closing the shutters S alternately. In this way, the measurements of the polarization space and statistical amplitude space are carried out separately. From these measurements, the needed correlations $C(a, b)$ were determined and Eq. (8) was used to evaluate the CHSH parameter B.

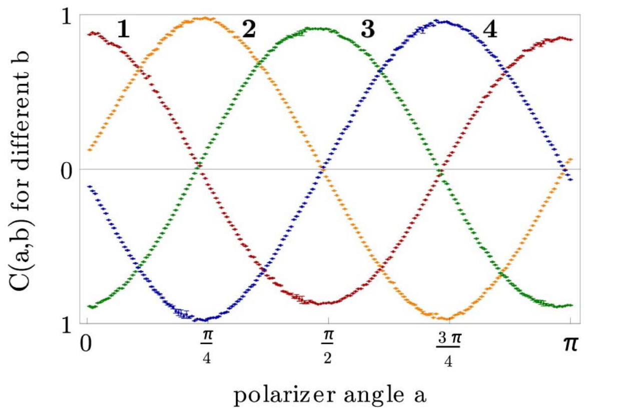

Fig. 2. Plots of the correlation functions $C(a, b)$ obtained by rotating polarizer a in the polarization space and keeping angle b in the function space constant. Curves 1–4 correspond to different fixed values of b separated by $\pi/4$. The invariant cosine function required to violate the Bell inequality is clearly present. Error bars are included but are scarcely visible.

Figure 2 shows $C(a, b)$ obtained by measuring the joint projections $P_{jk}(a, b)$ for a complete rotation of polarizer a, with different curves corresponding to b (and thus s) fixed at different values. It is apparent from the near-identity of the curves that, to good approximation, the correlations are a function of the difference in angles, i.e., $C(a, b) = C(a - b)$. The maximum value for B can then be found straightforwardly from any one of the curves in Fig. 2. Among them, the smallest and largest values of B (obtained for curves 1 and 4) are 2.548 ± 0.004 and 2.679 ± 0.007, respectively.

To be careful, we note that in our experiments the field was almost but not quite completely unpolarized; thus, not quite the same field was sketched in the Background Theory section. Thus, we could not expect to get the maximum quantum result $\mathcal{B} = 2\sqrt{2} = 2.828\ldots$ for the Bell parameter, although the values achieved also present a strong violation. The background theory is mildly more complicated for partially polarized rather than unpolarized light, but when worked out for the DOP of our light beams (see [15] and Supplement 1) it supports the values we observed.

7. SUMMARY

In summary, we first sketched the purely classical theory of optical beam fields (1) that satisfy the Bell-test criteria of Shimony [13,15]. Their bipartite pure state form shows the entanglement of their two independent DOFs [20]. The classical theory defines them as dynamically probabilistic fields, meaning that individual field measurements yield values that cannot be predicted except in an average sense, which is another feature shared with quantum systems but also associated for more than 50 years with the well-understood and well-tested optical theory of partial coherence [16]. Our theoretical sketch for the simplest case, unpolarized light, indicated that such fields or states are predicted to possess a range of correlation strengths equal to that of two-party quantum systems, that is, outside the bound $\mathcal{B} \leq 2$ of the CHSH Bell inequality and potentially as great as $\mathcal{B} = 2\sqrt{2}$. In our experimental test, we used light whose statistical behavior (field second-order statistics) is indistinguishable from classical, viz., the light from a broadband laser diode operating below threshold. Our detections of whole-beam intensity are free of the heralding requirements familiar in paired-photon CHSH experiments. Repeated tests confirmed that such a field can strongly violate the CHSH Bell inequality and can attain Bell-violating levels of correlation similar to those found in tests of maximally entangled quantum systems.

One naturally asks, how are these results possible? We know that a field with classically random statistics is a local real field, and we also know that Bell inequalities prevent local physics from containing correlations as strong as what quantum states provide. But the experimental results directly contradict this. The resolution of the apparent contradiction is not complicated, but does mandate a shift in the conventional understanding of the role of Bell inequalities, particularly as markers of a classical-quantum border. Bell himself came close to addressing this point. He pointed out [21] that even adding classical indeterminism would still not be enough for any type of hidden variable system to overcome the restriction imposed by his inequalities. This is correct as far as it goes, but fails to engage the point that local fields can be statistically classical and exhibit entanglement at the same time. For the fields under study, the entanglement is a strong correlation that is intrinsically present between the amplitude and polarization DOFs, and it is embedded in the field from the start (as it also is embedded *ab initio* in any quantum states that violate a Bell inequality). The possibility of such pre-existing structural correlation is bypassed in a CHSH derivation. Thus one sees that Bell violation has less to do with quantum theory than previously thought, but everything to do with entanglement.

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See Supplement 1 for supporting content.

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