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A Panorama on Superoscillations

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Abstract. Purpose of this note is to give an overview on superoscillating sequences and some of their properties. We discuss their persistence in time under Schrödinger equation, we propose various classes of superoscillating functions and we also briefly mention how they can be used to approximate some generalized functions.

INTRODUCTION

Since their introduction, see e.g. [1, 9], superoscillating functions have attracted a lot of attention. They are band-limited functions that show an apparently paradoxical behavior, in fact they can oscillate arbitrarily faster than their fastest Fourier component. Their study originated in quantum theory, but there were anticipations in theory of radar and in optics. Modern applications are wide and range from optical vortices to sub-wavelength microscopy and related areas of nanoscience and information theory. In this note we will not tackle the applications, for which we refer the reader to [8], but we illustrate some of the main mathematical properties of this important class of functions. For more information we refer the interested reader to [4] and to the recent survey papers [6, 5, 7]. The mathematical study of superoscillations has nowadays various directions of investigation, but in this overview we shall concentrate on three main problems: the evolution of superoscillating sequences when evolved using Schrödinger equation, the search for larger classes of superoscillatory functions, and the use of superoscillating sequences to approximate important sets of (generalized) functions.

SUPEROSCILLATING SEQUENCES AND THEIR EVOLUTION

The archetype superoscillatory function can be constructed as follows: let a be a real number such that $a > 1$ and consider the sequence of complex valued functions $F_n(x, a)$ defined on $\mathbb{R}$ by

$$F_n(x,a) = \left(\cos\left(\frac{x}{n}\right) + ia\sin\left(\frac{x}{n}\right)\right)^n = \sum_{k=0}^{n} C_k(n,a)\exp(i(1-2k/n)x) \tag{1}$$

where $C_k(n,a) := \binom{n}{k}\left(\frac{1+a}{2}\right)^{n-k}\left(\frac{1-a}{2}\right)^k$, and $\binom{n}{k}$ denotes the binomial coefficients. If $x \in \mathbb{R}$ is fixed, and we let n go to infinity, we immediately obtain that $\lim_{n\to\infty} F_n(x,a) = e^{iax}$. The convergence is uniform on all compact sets in $\mathbb{R}$ but it is not uniform on $\mathbb{R}$, see [2]. The representation in terms of $\exp(i(1-2k/n)x)$, together with the calculation of the limit of $F_n(x,a)$ when n goes to infinity, explains why such a sequence is called superoscillating (or superoscillatory).

Since superoscillations arise from weak measurements in quantum mechanics it is important to study the persistence in time of superoscillations. In other words, it is important to study the evolution of superoscillating functions under Schrödinger equation (with different potentials). The simplest Cauchy problem one can consider is the case of the free particle $i\partial_t\psi(x,t) = -\partial_{xx}\psi(x,t)$, $\psi(x,0) = F_n(x,a)$. An interesting fact is that the solution of the evolution problem involves refined techniques in complex analysis. One should note that for any finite $n \in \mathbb{N}$ the functions $F_n(x,a)$ rise to values $O(n!)$ outside the range $|x| < O(\sqrt{n})$, and are ultimately destroyed for values of the time greater than $O(n)$, see [9]. However, we have proved that for the free particle when $n \to \infty$ the superoscillations persist for any

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time t. The study of the evolution of superoscillations under different potentials requires to determine the continuity of operators such as

$$P_\lambda(t, \partial_x) = \sum_{n=0}^{\infty} \frac{\lambda(t)^n}{n!} \partial_x^{pn}$$

where $\lambda(t)$ is a given bounded function for the parameter $t \in [0, T]$, and $p \in \mathbb{N}$. For some type of potentials we have studied the continuity of these kind of operators acting on the analytic extensions of $F_n(z, a)$ to $\mathbb{C}$ of the functions $F_n(x, a)$ and we proved that the solution $\psi_n(x, a)$ of the associated Cauchy problem $i\partial_t \psi(x, t) = H\psi(x, t)$, $\psi(x, 0) = F_n(x, a)$ can be written as $\psi_n(x, a) = P_\lambda(t, \partial_x)F_n(x, a)$. One then extends Schrödinger equation to the complex plane and demonstrates that the operator $P_\lambda(t, d/dz)$ acts continuously on an appropriate space of entire functions containing $F_n(z, t)$. That this is indeed possible is non trivial and relies on some subtle estimates in spaces of entire functions with growth. The desired result is then obtained by restricting back to the real axis. In order to explain the general ideas behind our method, we introduce the following:

Definition 1 *We define the class $\mathcal{A}_1$ to be the set of entire functions such that there exists $C > 0$ and $B > 0$ for which $|f(z)| \leq C \exp(B|z|)$, $\forall z \in \mathbb{C}$.*

The crucial tool to obtain our result is next result proved in [7]:

Theorem 2 *Let $\lambda(t)$ be a bounded function for $t \in [0, T]$ for some $T \in (0, \infty)$ and let $f \in \mathcal{A}_1$. Then, for $p \in \mathbb{N}$, we have $P_\lambda(t, \partial_z)f \in \mathcal{A}_1$ and $P_\lambda(t, \partial_z)$ is continuous on $\mathcal{A}_1$, that is $P_\lambda(t, \partial_z)f \to 0$ as $f \to 0$.*

The theory allows to treat also more general situations like the one of the harmonic oscillator which depends on the following corollary:

Corollary 3 *Let $p = 2$ and $\lambda(t) = \frac{i}{2} \sin t \cos t$. The operator $P(t, \partial_z) := \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{i}{2} \sin t \cos t\Big)^n \partial_z^{2n}$, is continuous on $\mathcal{A}_1$.*

This results allows to treat the Cauchy problem for the quantum harmonic oscillator

$$i\frac{\partial \psi(t, x)}{\partial t} = \frac{1}{2}\Big(-\frac{\partial^2}{\partial x^2} + x^2\Big)\psi(t, x), \quad \psi(0, x) = F_n(x, a).$$

Using the continuity proven in the Corollary, one shows that it admits the solution

$$\psi_n(t, x) = (\cos t)^{-1/2} e^{-(i/2)x^2 \tan t} \sum_{k=0}^{n} C_k(n, a) \exp(ix(1 - 2k/n)/\cos t - (i/2)(1 - 2k/n)^2 \tan t).$$

By taking the limit for $n \to \infty$ we then obtain

$$\lim_{n\to\infty} \psi_n(t, x) = (\cos t)^{-1/2} \exp(-(i/2)(x^2 + a^2) \tan t + iax/\cos t).$$

This fact shows that the superoscillations are amplified by the potential and that the solution blows up in $t = \pi/2$. We note that, even when $a \in (0, 1)$, for the harmonic oscillator there is always a superoscillatory phenomenon since in the solution there is the term $e^{-(i/2)(x^2+a^2)\tan t + iax/\cos t}$, and it does not matter how small is a.

NEW CLASSES OF SUPEROSCILLATING SEQUENCES

An interesting byproduct of the construction in the previous section is the fact that from the sequence $F_n(x, a)$ one can construct several other superoscillating sequences. Another possibility to enlarge the class is to consider superoscillating sequences as described below:

Definition 4 *Let $a \in \mathbb{R}$ and let $C_j(n, a)$ and $k_j(n)$ be real valued functions of the variables n, a and n, respectively. The function*

$$Y_n(x, a) = \sum_{j=0}^{n} C_j(n, a) e^{ik_j(n)x},$$

(called generalized Fourier sequence) is said to be a superoscillating sequence *if: $|k_j(n)| \leq 1$; and if there exists a compact subset of $\mathbb{R}$, which will be called* a superoscillation set, *on which Y_n converges uniformly to $e^{ig(a)x}$ where g is a continuous real value function such that $|g(a)| > 1$.*

We note that, obviously, the classical Fourier expansion does not fulfill the previous definition, since its frequencies are not, in general, bounded. However, this definition is enough general to include various possibilities.
Another direction to enlarge the class of superoscillating functions is to consider the case of several variables, see [3]. We will consider n-tuples of complex numbers $(u_1, \ldots, u_n)$. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be a multi-index of length $|\alpha| = \sum_{j=1}^n \alpha_j$.

Definition 5 (Generalized Fourier sequence in several variables) *Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ be a multi-index and let*

$$P(u_1, u_2, \ldots, u_m) = \sum_{|\alpha| \leq h} a_\alpha u_1^{\alpha_1} \ldots u_r^{\alpha_m}, \quad for \quad a_\alpha \in \mathbb{C}$$

be a polynomial of degree h. Set

$$Z_{k_j}(x_j) := e^{i x_j k_j(n)}, \quad j = 1, \ldots, m,$$

where $k_j(n)$, $j = 1, \ldots, m$, are real valued sequences. We call generalized Fourier sequence in several variables *a sequence of the form*

$$f_n(x_1, \ldots, x_m) = \sum_{k=0}^{n} \mathcal{K}_k(n, a) P(Z_{k_1}(x_1), \ldots, Z_{k_m}(x_m)), \tag{2}$$

where $a \in \mathbb{R}$ and $\mathcal{K}_k(n, a)$, $k = 0, \ldots, m$, $m \in \mathbb{N}$ is a real valued sequence.

Definition 6 (Superoscillating sequence in several variables) *Let $a \in \mathbb{R}$. A generalized Fourier sequence $f_n(x_1, \ldots, x_m)$ of the form (2) is said to be* a superoscillating sequence *if*

$$\lim_{n \to \infty} f_n(x_1, \ldots, x_m) = Q(e^{i g_1(a) x_1}, \ldots, e^{i g_m(a) x_m})$$

where $Q(u_1, \ldots, u_m)$ is a polynomial, $|k_j(n)| \leq 1$ for $j = 1, \ldots, m$, and there exists a compact subset of $\mathbb{R}^m$, which will be called a superoscillation set*, on which f_n converges uniformly to $Q(e^{i g_1(a) x_1}, \ldots, e^{i g_m(a) x_m})$, where the functions g_j are continuous, real valued and satisfy $|g_j(a)| > 1$ for $j = 1, \ldots, m$.*

The following theorem extends a result originally proved in [3] in the case that the powers are even numbers. The statement is provided in the case of two variables, but it can be generalized to an arbitrary number of variables, see also [6].

Theorem 7 *Let $a > 1$. For m_1, $m_2 \in \mathbb{N}$ let us set*

$$z_{k,m_1}(x) := e^{ix(1-2k/n)^{m_1}}, \qquad z_{k,q}(y) := e^{iy(1-2k/n)^{m_2}},$$

and assume that there exists $r \in \mathbb{N}$ such that $m_1 = r m_2$. Consider the polynomial of degree h in the two variables u, v

$$P(v, u) = \sum_{|\alpha| \leq h} a_\alpha v^{\alpha_1} u^{\alpha_2},$$

where $a_\alpha \in \mathbb{C}$ and α is a multi-index of length $|\alpha|$. Define

$$f_n(x, y) = \sum_{k=0}^{n} C_k(n, a) P(z_{k,m_1}(x), z_{k,m_2}(y)).$$

Then $f_n(x, y)$ is superoscillating, that is

$$\lim_{n \to \infty} f_n(x, y) = P(e^{ixa^{m_1}}, e^{iya^{m_2}}).$$

APPROXIMATION OF (GENERALIZED) FUNCTIONS

Since superoscillating sequences can be used to approximate exponentials, which form a basis for several spaces of functions, one should be able to approximate functions (and generalized functions) in these spaces using superoscillating functions. This point of view was already adopted in [2], where it is discussed the approximation of the value of a band limited function φ in the Schwartz space $S(R)$ of rapidly decreasing functions at an arbitray point a, given the values of φ near the origin.

In [10] the idea is further developed to approximate Schwartz tempered functions as well as Schwartz tempered distributions by using superoscillations. In this paper, the authors use the Hermite orthonormal basis for the space $L^2(\mathbb{R})$, and replace, in it, the exponentials with their superoscillating approximants. A similar result can be proved for tempered distributions.

A more recent work, see [11], treats the approximation of hyperfunctions. The case of pointwise supported hyperfunctions can be easily addressed, at least in the compact case, by noticing that every such hyperfunction is a suitable sum of infinite derivatives of Dirac's deltas.

A hyperfunction $\mathfrak{f}$ is said to be *low frequency* (or *band limited*) if it admits as a representative a pair $[F^+, F^-]$ with $F^+ \in \mathcal{O}(\Pi^+)$ and $F^- \in \mathcal{O}(\Pi^-)$ which are both restrictions respectively to Π^+ and Π^- of entire functions f^+ and f^- of the form $\sum_{\kappa\in\Lambda} a_\kappa \exp(i\kappa z)$, where Λ must be contained in $K \cap \mathbb{Q}$, K being a compact in $\mathbb{R}$. We denote by $\mathcal{B}_{\mathrm{LF}}(\mathbb{R})$ this subset of the set of hyperfunctions $\mathcal{B}(\mathbb{R})$ whose elements are low frequency hyperfunctions for $K = [-1, 1]$. The main result (where the convergence $\xrightarrow{\text{rep}}$ is meant in the sense of representatives) is:

Theorem 8 *Any hyperfunction with compact support* $\mathfrak{f}$ *can be approximated on* $\mathbb{R}$ *as* $\mathfrak{f}_N \xrightarrow{\text{rep}} \mathfrak{f}$ *when* $N \in \mathbb{N}^*$ *tends to infinity, where each hyperfunction* $\mathfrak{f}_N$ $(N \geq 1)$ *belongs to* $\mathcal{B}_{\mathrm{LF}}(\mathbb{R})$.

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