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Unary-Determined Distributive ℓ-magmas and Bunched Implication Algebras

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Abstract. A distributive lattice-ordered magma ($d\ell$-magma) $(A, \wedge, \vee, \cdot)$ is a distributive lattice with a binary operation $\cdot$ that preserves joins in both arguments, and when $\cdot$ is associative then $(A, \vee, \cdot)$ is an idempotent semiring. A $d\ell$-magma with a top $\top$ is *unary-determined* if $x{\cdot}y = (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y)$. These algebras are term-equivalent to a subvariety of distributive lattices with $\top$ and two join-preserving unary operations p, q. We obtain simple conditions on p, q such that $x{\cdot}y = (px \wedge y) \vee (x \wedge qy)$ is associative, commutative, idempotent and/or has an identity element. This generalizes previous results on the structure of doubly idempotent semirings and, in the case when the distributive lattice is a Heyting algebra, it provides structural insight into unary-determined algebraic models of bunched implication logic. We also provide Kripke semantics for the algebras under consideration, which leads to more efficient algorithms for constructing finite models.

Keywords: distributive lattice-ordered magmas, bunched implication algebras, idempotent semirings, enumerating finite models

1 Introduction

Idempotent semirings $(A, \vee, \cdot)$ play an important role in several areas of computer science, such as network optimization, formal languages, Kleene algebras and program semantics. In this setting they are often assumed to have constants $0, 1$ that are the additive and multiplicative identity respectively, with 0 also being an absorbing element. However semirings are usually only assumed to have two binary operations $+, \cdot$ that are associative such that $+$ is also commutative and $\cdot$ distributes over $+$ from the left and right [9]. They are (additively) idempotent if $x + x = x$, hence $+$ is a (join) semilattice, and *doubly idempotent* if $x \cdot x = x$ as well. If $\cdot$ is also commutative then it defines a meet semilattice. The special case when these two semilattices coincide corresponds exactly to the variety of distributive lattices, which have a well understood structure theory.

In [1] a complete structural description was given for finite commutative doubly idempotent semirings where either the multiplicative semilattice is a chain, or the additive semilattice is a Boolean algebra. Here we show that the second description can be significantly generalized to the setting where the additive semilattice is a distributive lattice, dropping the assumptions of finiteness, multiplicative commutativity and idempotence in favor of the algebraic condition

$x{\cdot}y = (px \wedge y) \vee (x \wedge qy)$ for two unary join-preserving operations p, q. While this property is quite restrictive in general, it does hold in all idempotent Boolean magmas and expresses a binary operation in terms of two simpler unary operations. A full structural description of all (finite) idempotent semirings is unlikely, but in the setting of unary-determined idempotent semirings progress is possible.

In Section 2 we provide the needed background and prove a term-equivalence between a subvariety of top-bounded $d\ell$-magmas and a subvariety of top-bounded distributive lattices with two unary operators. This is then specialized to cases where $\cdot$ is associative, commutative, idempotent or has an identity element. In the next section we show that when the distributive lattice is a Brouwerian algebra or Heyting algebra, then $\cdot$ is residuated if and only if both p and q are residuated. This establishes a connection with bunched implication algebras (BI-algebras) that are the algebraic semantics of bunched implication logic[14], used in the setting of separation logic for program verification, including reasoning about pointers [16] and concurrent processes [13]. Section 4 contains Kripke semantics for $d\ell$-magmas, called Birkhoff frames, and for the two unary operators p, q. This establishes the connection to the previous results in [1] and leads to the main result (Thm. 15) that preorder forest P-frames capture a larger class of multiplicatively idempotent BI-algebras and doubly idempotent semirings. Although the heap models of BI-algebras used in applications are not (multiplicatively) idempotent, they contain idempotent subalgebras and homomorphic images, hence a characterization of unary-determined idempotent BI-algebras does provide insight into the general case. In Section 5, as an application, we count the number of such algebras up to isomorphism if their partial order is an antichain and also if it is a chain.

2 A term-equivalence between distributive lattices with operators

A *distributive lattice-ordered magma*, or *$d\ell$-magma*, is an algebra $(A, \wedge, \vee, \cdot)$ such that $(A, \wedge, \vee)$ is a distributive lattice and $\cdot$ distributes over $\vee$, i. e., $x(y \vee z) = xy \vee xz$ and $(x \vee y)z = xz \vee yz$ for all $x, y, z \in A$. If the distributive lattice has a top element $\top$ or a bottom element $\bot$ then it is called $\top$-*bounded* or $\bot$-*bounded*, or simply *bounded* if both exist. A $d\ell$-magma A is *normal* and $\cdot$ is a *normal* operation if A is $\bot$-bounded and satisfies $x{\cdot}\bot = \bot = \bot{\cdot}x$. Similarly, a unary operation f on A is an *operator* if it satisfies $f(x \vee y) = fx \vee fy$, and it is *normal* if $f\bot = \bot$. For brevity and to reduce the number of nested parentheses, we write function application as fx rather than $f(x)$, with the convention that it has priority over $\cdot$ hence, e.g., $fxy = (f(x)){\cdot}y$ (this convention ensures unique readability). Note that since operators distribute over $\vee$ in each argument, they are order-preserving in each argument. The operation f is said to be *inflationary* if $x \leq fx$ for all $x \in A$.

A binary operation $\cdot$ is said to be *idempotent* if $xx = x$ for all $x \in A$, *commutative* if $xy = yx$ and *associative* if $(xy)z = x(yz)$. A *semigroup* is a set with an associative operation, a *band* is a semigroup that is also idempotent, and a *semilattice* is a commutative band. As usual, a semilattice is partially ordered by $x \sqsubseteq y \iff xy = x$, and in this case xy is the meet operation with respect to $\sqsubseteq$. We also use this terminology with the prefix $d\ell$, in which case the magma operation satisfies the corresponding identities.

A $d\ell$-magma is called *unary-determined* if it is $\top$-bounded and satisfies the identity

$$x{\cdot}y = (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y).$$

As examples, we mention that all doubly-idempotent semirings with a Boolean join-semilattice are unary-determined (see Lemma 3). Complete and atomic versions of such semirings are studied in [1], and the results from that paper are generalized here to unary-determined $d\ell$-magmas with point-free algebraic proofs. This is an improvement since the algebraic results apply to all members of the variety, while the previous results applied only to complete and atomic algebras.

A *$d\ell pq$-algebra* is a $\top$-bounded distributive lattice with two unary operators p, q that satisfy

$$x \wedge p\top \le qx, \quad x \wedge q\top \le px.$$

These two equational axioms are needed for our first result which shows that unary-determined $d\ell$-magmas and $d\ell pq$-algebras are term-equivalent. This means that although the two varieties are based on different sets of fundamental operations (called the signature of each class), each fundamental operation of an algebra in one variety is identical to a term-operation constructed from fundamental operations of an algebra in the other variety (and vice versa). From the point of view of category theory, term-equivalent varieties are model categories of the same Lawvere theory.

Although unary-determined $d\ell$-magmas and $d\ell pq$-algebras seem rather special, they are simpler than general $d\ell$-magmas, yet include interesting idempotent semirings (as reducts).

Theorem 1. (1) *Let $(A, \wedge, \vee, \top, p, q)$ be a $d\ell pq$-algebra and define $x{\cdot}y = (px \wedge y) \vee (x \wedge qy)$. Then $(A, \wedge, \vee, \top, \cdot)$ is a unary-determined $d\ell$-magma and p, q are given by $px = x{\cdot}\top$ and $qx = \top{\cdot}x$.*

(2) *Let $(A, \wedge, \vee, \top, \cdot)$ be a unary-determined $d\ell$-magma and define $px = x{\cdot}\top$, $qx = \top{\cdot}x$. Then $(A, \wedge, \vee, \top, p, q)$ is a $d\ell pq$-algebra and $\cdot$ is definable from p, q via $x{\cdot}y = (px \wedge y) \vee (x \wedge qy)$.*

Proof. (1) Assume p, q are unary operators on a $\top$-bounded distributive lattice $(A, \wedge, \vee, \top)$, and $xy = (px \wedge y) \vee (x \wedge qy)$. Then

$$\begin{aligned} x(y \vee z) &= (px \wedge (y \vee z)) \vee (x \wedge q(y \vee z)) \\ &= (px \wedge y) \vee (px \wedge z) \vee (x \wedge qy) \vee (x \wedge qz) \\ &= (px \wedge y) \vee (x \wedge qy) \vee (px \wedge z) \vee (x \wedge qz) \\ &= xy \vee xz. \end{aligned}$$

A similar calculation shows that $(x \vee y)z = xz \vee yz$, hence $\cdot$ is an operator.

Since p, q satisfy $x \wedge q{\cdot}\top \le px$, it follows that $x{\cdot}\top = (px \wedge \top) \vee (x \wedge q{\cdot}\top) = px \vee (x \wedge q\top) = px$, and similarly $\top{\cdot}x = qx$ is implied by $x \wedge p\top \le qx$. Now the identity $xy = (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y)$ holds by definition.

(2) Assume $(A, \wedge, \vee, \top, \cdot)$ is a unary-determined $d\ell$-magma, and define $px = x{\cdot}\top$, $qx = \top{\cdot}y$. Then p, q are unary operators and $px = x{\cdot}\top = (x{\cdot}\top \wedge \top) \vee (x \wedge \top\top) = px \vee (x \wedge q\top)$, hence $x \wedge q\top \leq px$. The inequation $x \wedge p\top \leq qx$ is proved similarly. The operation $\cdot$ can be recovered from p, q since $xy = (px \wedge y) \vee (x \wedge qy)$ follows from the identity we assumed. □

The preceding theorem shows that unary-determined $d\ell$-magmas and $d\ell pq$-algebras are "essentially the same", and we can choose to work with the signature that is preferred in a given situation. The unary operators of $d\ell pq$-algebras are simpler to handle, while the binary operator $\cdot$ is familiar in the semiring setting. Next we examine how standard properties of $\cdot$ are captured by identities in the language of $d\ell pq$-algebras.

Lemma 2. *Let $(A, \wedge, \vee, \top, p, q)$ be a $d\ell pq$-algebra and define $x{\cdot}y = (px \wedge y) \vee (x \wedge qy)$.*

(1) *The operator $\cdot$ is commutative if and only if $p = q$.*
(2) *If $p = q$ then $\cdot$ is associative if and only if $p((px \wedge y) \vee (x \wedge py)) = (px \wedge py) \vee (x \wedge ppy)$.*
(3) *The operator $\cdot$ is idempotent if and only if p and q are inflationary, if and only if $p\top = \top = q\top$.*
(4) *If $\cdot$ is idempotent then it is associative if and only if*

$$p((px \wedge y) \vee (x \wedge qy)) = (px \wedge py) \vee (x \wedge qy) \text{ and}$$
$$q((px \wedge y) \vee (x \wedge qy)) = (px \wedge y) \vee (qx \wedge qy).$$

(5) *The operator $\cdot$ has an identity 1 if and only if $p1{=}\top{=}q1$ and $(px \vee qx) \wedge 1 \leq x$.*
(6) *If $\cdot$ has an identity then $\cdot$ is idempotent.*

Proof. (1) Assuming $xy = yx$, we clearly have $x{\cdot}\top = \top{\cdot}x$, hence $px = qx$. The converse makes use of commutativity of $\wedge$ and $\vee$: $xy = (px \wedge y) \vee (x \wedge py) = (py \wedge x) \vee (y \wedge px) = yx$.

(2) Assume $p = q$. If $\cdot$ is associative then $(xy)\top = x(y\top)$, so by the previous theorem, $p(xy) = xpy$, which translates to

$$p((px \wedge y) \vee (x \wedge py)) = (px \wedge py) \vee (x \wedge ppy) \quad (*).$$

Conversely, suppose $(*)$ holds, and note that $p(xy) = p(yx)$ by (1), hence

$$p((px{\wedge}y){\vee}(x{\wedge}py)) = (px{\wedge}py){\vee}(ppx{\wedge}y) = (px{\wedge}py){\vee}(x{\wedge}ppy){\vee}(ppx{\wedge}y) \quad (**).$$

It suffices to prove $(xy)z \leq x(yz)$ since then $z(yx) \leq (zy)x$ follows by commutativity. Now

$$\begin{aligned}
(xy)z &= [p((px \wedge y) \vee (x \wedge py)) \wedge z] \vee [((px \wedge y) \vee (x \wedge py)) \wedge pz] \\
&= [((px \wedge py) \vee (x \wedge ppy)) \wedge z] \vee [px \wedge y \wedge pz] \vee [x \wedge py \wedge pz] \text{ using } (*) \\
&= [px \wedge py \wedge z] \vee [x \wedge ppy \wedge z] \vee [px \wedge y \wedge pz] \vee [x \wedge py \wedge pz] \\
&\leq [px \wedge py \wedge z] \vee [px \wedge y \wedge pz] \vee [x \wedge py \wedge pz] \vee [x \wedge y \wedge ppz] \vee [x{\wedge}ppy{\wedge}z] \\
&= [px \wedge py \wedge z] \vee [px \wedge y \wedge pz] \vee [x \wedge ((py \wedge pz) \vee (y \wedge ppz) \vee (ppy \wedge z))] \\
&= [px \wedge ((py \wedge z) \vee (y \wedge pz))] \vee [x \wedge p((py \wedge z) \vee (y \wedge pz))] \text{ using } (**) \\
&= x(yz).
\end{aligned}$$

(3) If $\cdot$ is idempotent, then $x = xx \le x{\cdot}\top = px$ and $x \le \top{\cdot}x = qx$. Conversely, if p, q are inflationary then $xx = (px \wedge x) \vee (x \wedge qy) = x \vee x = x$, hence $\cdot$ is idempotent. For the second equivalence, if $p\top = \top = q\top$ then p, q are inflationary since they satisfy $x \wedge p\top \le qx, x \wedge q\top \le px$. The reverse implication holds because $x \le px, qx$ implies $\top \le p\top, q\top$.

(4) Assume $\cdot$ is idempotent and associative. Then $(\top{\cdot}x)\top = \top(x{\cdot}\top)$, hence $qpx = pqx$. Furthermore, $pqx = \top{\cdot}x{\cdot}\top = \top xx\top = (qx)(px) = (pqx \wedge px) \vee (qx \wedge qpx)$. By (3) p, q are inflationary, so $px \le pqx$ and $qx \le qpx$. Therefore $pqx = px \vee qx$. Now we translate $(xy)\top = x(y\top)$ to obtain $p(xy) = x(py)$, hence

$$\begin{aligned} &p((px \wedge y) \vee (x \wedge qy)) = (px \wedge py) \vee (x \wedge qpy) = (px \wedge py) \vee (x \wedge (py \vee qy)) \\ &\;= (px \wedge py) \vee (x \wedge py) \vee (x \wedge qy) = (px \wedge py) \vee (x \wedge qy) \text{ since } x \le px \text{ by (3)}. \end{aligned}$$

The identity $q((px \wedge y) \vee (x \wedge qy)) = (px \wedge y) \vee (qx \wedge qy)$ has a similar proof.

Conversely, assume the two identities hold. Then using distributivity

$$\begin{aligned} (xy)z &= [p((px \wedge y) \vee (x \wedge qy)) \wedge z] \vee [((px \wedge y) \vee (x \wedge qy)) \wedge qz] \\ &= [px \wedge py \wedge z] \vee [x \wedge qy \wedge z] \vee [px \wedge y \wedge qz] \vee [x \wedge qy \wedge qz] \\ &= [px \wedge py \wedge z] \vee [px \wedge y \wedge qz] \vee [x \wedge qy \wedge qz] \quad \text{since } x{\wedge}qy{\wedge}z \le x{\wedge}qy{\wedge}qz \\ &= [px \wedge py \wedge z] \vee [px \wedge y \wedge qz] \vee [x \wedge py \wedge z] \vee [x \wedge qy \wedge qz] \\ &= [px \wedge ((py \wedge z) \vee (y \wedge qz))] \vee [x \wedge q((py \wedge z) \vee (y \wedge qz))] = x(yz). \end{aligned}$$

(5) Assume x has an identity 1. Then $p1 = 1\top = \top = \top 1 = q1$ and $x = x1 = (px \wedge 1) \vee (x \wedge q1) = (px \wedge 1) \vee x$, so $px \wedge 1 \le x$ and similarly $qx \wedge 1 \le x$. Therefore $(px \vee qx) \wedge 1 = (px \wedge 1) \vee (qx \wedge 1) \le x$.

Conversely, suppose $p1 = \top = q1$ and $(px \vee qx) \wedge 1 \le x$. Then $x1 = (px \wedge 1) \vee (x \wedge q1) = (px \wedge 1) \vee x = x$ since $px \wedge 1 \le x$. Likewise $1x = x$.

(6) This follows from (3) since $x = x1 \le x{\cdot}\top = px$ and $x = 1x \le qx$. $\square$

Note that if A also has a bottom bound $\perp$ then p, q are normal if and only if $\cdot$ is normal, hence the term-equivalence preserves normality.

This term-equivalence is useful since distributive lattices with unary operators are considerably simpler than distributive lattices with binary operators. In particular, (2) and (4) show that associativity can be replaced by one or two 2-variable identities in this variety. This provides more efficient ways to construct associative operators from a (pair of) unary operator(s) on a distributive lattice. The variety of $\top$-bounded distributive lattices is obtained as a subvariety of $d\ell pq$-algebras that satisfy $px = x = qx$, or a subvariety of unary determined $d\ell$-magmas that satisfy $x \cdot y = x \wedge y$.

For small cardinalities, Table 1 shows the number of algebras that are unary-determined (shown in the even numbered rows) for several subvarieties of normal $d\ell$-magmas. As seen from rows 5-8, under the assumption of associativity, commutativity and idempotence of $\cdot$, the property of being unary-determined is a relatively mild restriction compared to the general case of normal $d\ell$-magmas.

A *Boolean magma* is a Boolean algebra with a binary operator. The next lemma shows that if the operator is idempotent, then it is always unary-deter-

	Cardinality $n =$	2	3	4	5	6	7	8
1	normal $d\ell$-magmas	2	20	1116				
2	normal $d\ell pq$-algebras	2	6	46	3435			
3	normal comm. $d\ell$-semigroups	2	8	57	392	3212		
4	normal assoc. $d\ell p$-algebras	2	4	13	35	109	315	998
5	normal comm. idem. $d\ell$-semigroups	1	2	8	25	97	366	
6	normal assoc. idem. $d\ell p$-algebras	1	2	7	18	57	163	521
7	normal comm. idem. $d\ell$-monoids	1	2	6	15	44	115	326
8	normal assoc. idem. $d\ell p1$-algebras	1	2	5	10	24	47	108
9	distributive lattices	1	1	2	3	5	8	15

Table 1. The number of algebras of cardinality n up to isomorphism.

mined, hence the results in the current paper generalize the theorems about idempotent Boolean nonassociative quantales in [1].

Lemma 3. *Every idempotent Boolean magma* $(A, \wedge, \vee, \neg, \bot, \top, \cdot)$ *is unary-determined, i.e., satisfies* $xy = (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y)$.

Proof. Idempotence is equivalent to $x \wedge y \le xy \le x \vee y$ since $(x \wedge y)^2 \le xy \le (x \vee y)^2$ holds in all partially ordered algebras where $\cdot$ is an order-preserving binary operation. The following calculation

$$\begin{aligned} x{\cdot}\top \wedge y &= x(y \vee \neg y) \wedge y = (xy \wedge y) \vee (x(\neg y)) \wedge y) \\ &\le xy \vee ((x \vee \neg y) \wedge y) = xy \vee (x \wedge y) \vee (\neg y \wedge y) = xy \end{aligned}$$

and a similar one for $x \wedge \top{\cdot}y \le xy$ prove that $xy \ge (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y)$.

Using Boolean negation, the opposite inequation is equivalent to

$$xy \wedge \neg(x{\cdot}\top \wedge y) \le x \wedge \top{\cdot}y.$$

By De Morgan's law it suffices to show $(xy \wedge \neg(x{\cdot}\top)) \vee (xy \wedge \neg y) \le x \wedge \top{\cdot}y$. Since $xy \le x{\cdot}\top$, the first meet disappears. Next, by idempotence, $xy \wedge \neg y \le (x \vee y) \wedge \neg y = (x \wedge \neg y) \vee (y \wedge \neg y) \le x$ and finally $xy \wedge \neg y \le xy \le \top{\cdot}y$. □

3 BI-algebras from Heyting algebras and residuated unary operations

We now recall some basic definitions about residuated operations, adjoints and residuated lattices. For an overview and additional details we refer to [6]. A *Brouwerian algebra* $(A, \wedge, \vee, \to, \top)$ is a $\top$-bounded lattice such that $\to$ is the *residual* of $\wedge$, i.e.,

$$x \wedge y \le z \iff y \le x \to z.$$

Since $\to$ is the residual of $\wedge$, we have that $\wedge$ is join-preserving, so the lattice is distributive [6, Lem. 4.1]. The $\top$-bound is included as a constant since it always exists when a meet-operation has a residual: $x \wedge y \leq x$ always holds, hence $y \leq (x \to x) = \top$. A *Heyting algebra* is a bounded Brouwerian algebra with a constant $\bot$ denoting the bottom element.

A dual operator is an n-ary operation on a lattice that preserves meet in each argument. A *residual* or *upper adjoint* of a unary operation p on a poset A is a unary operation p^* such that

$$px \leq y \iff x \leq p^* y$$

for all $x, y \in A$. If A is a lattice, then the existence of a residual guarantees that p is an operator and p^* is a dual operator [6, Lem. 3.5]. Moreover, if A is bounded, then $p\bot = \bot$ and $p^*\top = \top$.

A binary operation $\cdot$ on a poset is *residuated* if there exist a *left residual* $\backslash$ and a *right residual* $/$ such that

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z/y.$$

A *residuated ℓ-magma* $(A, \wedge, \vee, \cdot, \backslash, /)$ is a lattice with a residuated binary operation. In this case $\cdot$ is an operator and $\backslash, /$ are dual operators in the "numerator" argument. In the "denominator" $\backslash, /$ map joins to meets, hence they are order reversing. A *residuated Brouwerian-magma* is a residuated ℓ-magma expanded with $\to, \top$ such that $(A, \wedge, \vee, \to, \top)$ is a Brouwerian algebra.

A *residuated lattice* is a residuated ℓ-magma with $\cdot$ associative and a constant 1 that is an identity element, i.e., $(A, \cdot, 1)$ is a monoid. A *generalized bunched implication algebra*, or GBI-algebra, $(A, \wedge, \vee, \to, \top, \cdot, 1, \backslash, /)$ is a $\top$-bounded residuated lattice with a residual $\to$ for the meet operation, i.e., $(A, \wedge, \vee, \to, \top)$ is a Brouwerian algebra. A GBI-algebra is called a *bunched implication algebra* (BI-algebra) if $\cdot$ is commutative and A also has a bottom element, denoted by the constant $\bot$, hence a BI-algebra has a Heyting algebra reduct. These algebras are the algebraic semantics for bunched implication logic, which is the propositional part of separation logic, a Hoare logic used for reasoning about memory references in computer programs. In this setting the operation $\cdot$ is usually denoted by $*$, the left residual $\backslash$ is denoted $-\!\!*$, and $/$ can be omitted since $x/y = y -\!\!* x$.

Note that the property of being a residual can be expressed by inequalities (p^* is a residual of p if and only if $p(p^*x) \leq x \leq p^*(px)$ for all x, and p, p^* are order preserving), hence the classes of all Brouwerian algebras, Heyting algebras, residuated ℓ-magmas, residuated Brouwerian-magmas, residuated lattices, (G)BI-algebras, and pairs of residuated unary maps on a lattice are varieties (see e.g. [6, Thm 2.7, Lem. 3.2.]). Recall also that a $\top$-bounded magma is unary-determined if it satisfies the identity $xy = (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y)$.

We are now ready to prove a result that upgrades the term-equivalence of Theorem 1 to Brouwerian algebras with two pairs of residuated maps and unary-determined residuated Brouwerian-magmas.

Theorem 4. (1) *Let $(A, \wedge, \vee, \to, \top, p, p^*, q, q^*)$ be a Brouwerian algebra with unary operators p, q and their residuals p^*, q^* such that $x \wedge p\top \leq qx$, $x \wedge q\top \leq px$. If we define $x{\cdot}y = (px \wedge y) \vee (x \wedge qy)$,*

$$x \backslash y = (px \to y) \wedge q^*(x \to y) \quad \textit{and} \quad x/y = p^*(y \to x) \wedge (qy \to x)$$

then $(A, \wedge, \vee, \top, \cdot, \backslash, /)$ *is a unary-determined residuated Brouwerian-magma and the unary operations are recovered by* $px = x{\cdot}\top$*,* $p^*x = x/\top$*,* $qx = \top{\cdot}x$ *and* $q^*x = \top\backslash x$*.*

(2) *Let* $(A, \wedge, \vee, \to, \top, \cdot, \backslash, /)$ *be a unary-determined residuated Brouwerian-magma and define* $px = x{\cdot}\top$*,* $p^*x = x/\top$*,* $qx = \top{\cdot}x$ *and* $q^*x = \top\backslash x$*. Then* $(A, \wedge, \vee, \to, \top, p, p^*, q, q^*)$ *is a Brouwerian algebra with a unary operators* p, q *and dual operators* p^*, q^* *that satisfies* $x \wedge p\top \leq qx$*,* $x \wedge q\top \leq px$*.*

Proof. (1) The following calculation shows that $\cdot$ is residuated.

$$\begin{aligned} x \cdot y \leq z &\iff (px \wedge y) \vee (x \wedge qy) \leq z &&\iff px \wedge y \leq z \text{ and } x \wedge qy \leq z \\ &\iff y \leq px \to z \text{ and } y \leq q^*(x \to z) &&\iff y \leq (px \to z) \wedge q^*(x \to z) \end{aligned}$$

hence $x\backslash z = (px \to z) \wedge q^*(x \to z)$ and similarly $z/y = p^*(y \to z) \wedge (qy \to z)$. By Theorem 1 it follows that $px = x{\cdot}\top, qx = \top{\cdot}x$ and $xy = (x{\cdot}\top \wedge y) \vee (x \wedge \top{\cdot}y)$. Since $x{\cdot}\top \leq y \iff x \leq y/\top$ we obtain $p^*(x) = x/\top$, and similarly $q^*(x) = \top\backslash x$.

(2) Since $\cdot$ is residuated it follows that p^* and q^* are the unary residuals of p, q respectively. The remaining parts hold by Theorem 1. ☐

Recall that a closure operator p is an order-preserving unary function on a poset such that $x \leq px = ppx$. A $d\ell p$-algebra where p is a closure operator is called a $d\ell p$-closure algebra. If $\cdot$ is idempotent and associative then $x{\cdot}\top = x(\top\top) = (x\top)\top$, so $px = x{\cdot}\top$ is a closure operator.

Lemma 5. *Assume* $\mathbf{A}$ *is a* $d\ell p$*-closure algebra and let* $x{\cdot}y = (px \wedge y) \vee (x \wedge py)$*. Then* $\cdot$ *is associative if and only if* $px \wedge py \leq p((px \wedge y) \vee (x \vee py))$*.*

Proof. By Lemma 2 $\cdot$ is associative if and only if the identity $p((px \wedge y) \vee (x \vee py)) = (px \wedge py) \vee (x \wedge py)$ holds. This is equivalent to $px \wedge py \leq p((px \wedge y) \vee (x \vee py))$ since $x \wedge py \leq px \wedge py$, $p(px \wedge y) \leq ppx \wedge py = px \wedge py$ and similarly $p(x \wedge py) \leq px \wedge py$. ☐

Hence the preceding theorems specialize to a term-equivalence for a subvariety of unary-determined BI-algebras as follows:

Corollary 6. *1. Let* $(A, \wedge, \vee, \to, \top, \bot, p, p^*, 1)$ *be a Heyting algebra with a closure operator* p*, residual* p^* *and constant* 1 *such that* $px \wedge py \leq p((px \wedge y) \vee (x \wedge py))$*,* $p1 = \top$ *and* $px \wedge 1 \leq x$*. If we define* $x{*}y = (px \wedge y) \vee (x \wedge py)$ *and* $x{-\!*}y = (px \to y) \wedge p^*(x \wedge y)$ *then* $(A, \wedge, \vee, \top, \to, *, -\!*, 1)$ *is a unary-determined BI-algebra and* $x{*}\top \wedge y{*}\top \leq ((x{*}\top \wedge y) \vee (x \wedge y{*}\top)){*}\top$ *holds.*

2. Let $(A, \wedge, \vee, \to, \top, \bot, *, -\!*, 1)$ *be a unary-determined BI-algebra, and define* $px = x{*}\top$ *and* $p^*x = \top{-\!*}x$*. Then* $(A, \wedge, \vee, \to, \top, \bot, p, p^*, 1)$ *is a Heyting algebra with a closure operator* p *that has* p^* *as residual and satisfies* $px \wedge py \leq p((px \wedge y) \vee (x \wedge py))$*,* $p1 = \top$ *and* $px \wedge 1 \leq x$*.*

By Lemma 2(6) unary-determined BI-algebras satisfy $x{*}x = x$, which does not hold in BI-algebras that model applications (e.g., heap storage). However, as mentioned in the introduction, they are members of the variety of BI-algebras, and understanding their properties via this term-equivalence is useful for the general theory. E.g., structural results about algebraic object (such as rings) often start by investigating the idempotent algebras, followed by sets of idempotent elements in more general algebras. Line 8 in Table 1 also shows that finite unary-determined BI-algebras are not rare (normal join-preserving operators are automatically residuated in the finite case, hence the algebras counted in Line 8 are indeed term-equivalent to unary-determined BI-algebras).

4 Relational semantics for $d\ell$-magmas

We now briefly recall relational semantics for bounded distributive lattices with operators and then apply correspondence theory to derive first-order conditions for the equational properties of the preceding sections.

An element in a lattice is *completely join-irreducible* if it is not the supremum of all the elements strictly below it. The set of all completely join-irreducible elements of a lattice A is denoted by $J(A)$, and it is partially ordered by restricting the order of A to $J(A)$. For example, if A is a Boolean lattice, then $J(A) = At(A)$ is the antichain of *atoms*, i.e., all elements immediately above the bottom element. The set $M(A)$ of completely meet-irreducible elements is defined dually. A lattice is *perfect* if it is complete (i.e., all joins and meets exist) and every element is a join of completely join-irreducibles and a meet of completely meet-irreducibles. For a Boolean algebra, the notion of perfect is equivalent to being *complete* (i.e., joins and meets of all subsets exist) and *atomic* (i.e., every non-bottom element has an atom below it).

Recall that for a poset $\mathbf{W} = (W, \leq)$, a *downset* is a subset X such that $y \leq x \in X$ implies $y \in X$. As in modal logic, W is considered a set of "worlds" or states. We let $D(\mathbf{W})$ be the set of all downsets of $\mathbf{W}$, and $(D(\mathbf{W}), \cap, \cup)$ the *lattice of downsets*. The collection $D(\mathbf{W})$ is a perfect distributive lattice with infinitary meet and join given by (arbitrary) intersections and unions. The following result, due to Birkhoff [2] for lattices of finite height, shows that up to isomorphism all perfect distributive lattices arise in this way. The poset $J(D(\mathbf{W}))$ contains exactly the principal downsets ${\downarrow}x = \{y \in W \mid y \leq x\}$.

Theorem 7 ([3, 10.29]). *For a lattice A the following are equivalent:*

1. *A is distributive and perfect.*
2. *A is isomorphic to the lattice of downsets of a partial order.*

Note that the set of upsets of a poset is also a perfect distributive lattice, and if it is ordered by reverse inclusion then this lattice is isomorphic to the downset lattice described above. It is also well known that the maps J and D are functors for a categorial duality between the category of posets with order-preserving maps and the category of perfect distributive lattices with complete lattice homomorphisms (i.e., maps that preserve arbitrary joins and meets).

A *complete* operator on a complete lattice is an operation that is either completely join-preserving, completely meet-preserving, maps all arbitrary meets to joins or all arbitrary joins to meets in each argument. A lattice-ordered algebra is called *perfect* if its lattice reduct is perfect and every fundamental operation on it is a complete operator. The duality between the category of perfect distributive lattices and posets extends to the category of perfect distributive lattices with (a fixed signature of) complete operators. The corresponding poset category has additional relations of arity $n+1$ for each operator of arity n, and the relations have to be upward or downward closed in each argument. For example, a binary relation $Q \subseteq W^2$ is upward closed in the second argument if $xQy \leq z \implies xQz$. Here $xQy \leq z$ is an abbreviation for xQy and $y \leq z$.

Perfect distributive lattices with operators are algebraic models for many logics, including relevance logic, intuitionistic logic, Hajek's basic logic, Łukasiewicz logic and bunched implication logic [7,6]. In such an algebra $\mathbf{A}$, a join-preserving binary operation is determined by a ternary relation R on $J(\mathbf{A})$ given by

$$xRyz \iff x \leq yz.$$

The notation $xRyz$ is shorthand for $(x, y, z) \in R$. For $b, c \in A$ the product bc is recovered as $\bigvee\{x \in J(\mathbf{A}) \mid xRyz \text{ for some } y \leq b \text{ and } z \leq c\}$.

The relational structure $(J(\mathbf{A}), \leq, R)$ is an example of a Birkhoff frame. In general, a *Birkhoff frame* [5] is a triple $\mathbf{W} = (W, \leq, R)$ where $(W, \leq)$ is a poset, and $R \subseteq W^3$ satisfies the following three properties (downward closure in the 1st, and upward closure in the 2nd and 3rd argument):

$$\begin{array}{l} \text{(R1) } u \leq xRyz \implies uRyz \\ \text{(R2) } xRyz \;\&\; y \leq v \implies xRvz \\ \text{(R3) } xRyz \;\&\; z \leq w \implies xRyw. \end{array}$$

A Birkhoff frame $\mathbf{W}$ defines the downset algebra $\mathbf{D}(\mathbf{W}) = (D(\mathbf{W}), \bigcap, \bigcup, \cdot)$ by

$$Y \cdot Z = \{x \in W \mid xRyz \text{ for some } y \in Y \text{ and } z \in Z\}.$$

The property (R1) ensures that $Y \cdot Z \in D(\mathbf{W})$.

In relevance logic [4] similar ternary frames are known as Routley-Meyer frames. In that setting upsets are used to recover the distributive lattice-ordered relevance algebra, and this choice implies that $J(A)$ with the induced order from A is dually isomorphic to $(W, \leq)$. Another difference is that Routley-Meyer frames have a unary relation and axioms to ensure it is a left identity element of the $\cdot$ operation.

The duality between perfect $d\ell$-magmas and Birkhoff frames is recalled below. Here we assume that the binary operation on a complete $d\ell$-magma is a complete operator, i.e., distributes over arbitrary joins in each argument. Such algebras are also known as *nonassociative quantales* or *prequantales*.

Theorem 8 ([5]).

1. *If $\mathbf{A}$ is a perfect $d\ell$-magma and $R \subseteq J(A)^3$ is defined by $xRyz \Leftrightarrow x \leq yz$ then $J(\mathbf{A}) = (J(A), \leq, R)$ is a Birkhoff frame, and $\mathbf{A} \cong \mathbf{D}(J(\mathbf{A}))$.*
2. *If $\mathbf{W}$ is a Birkhoff frame then $\mathbf{D}(\mathbf{W})$ is a perfect $d\ell$-magma, and $\mathbf{W} \cong (J(D(\mathbf{W})), \subseteq, R_\downarrow)$, where $(\downarrow x, \downarrow y, \downarrow z) \in R_\downarrow \Leftrightarrow xRyz$.*

A ternary relation R is called *commutative* if $xRyz \implies xRzy$ for all x, y, z. The justification for this terminology is provided by the following result.

Lemma 9. *For any Birkhoff frame* $\mathbf{W}$*,* $\mathbf{D}(\mathbf{W})$ *is commutative if and only if* R *is commutative.*

Lemma 10. *Let* $\mathbf{W}$ *be a Birkhoff frame. Then* $\mathbf{D}(\mathbf{W})$ *is idempotent if and only if* $xRxx$ *and* $(xRyz \implies x \le y$ *or* $x \le z)$ *for all* $x, y, z \in W$*.*

Proof. Assume $\mathbf{D}(\mathbf{W})$ is idempotent, and let $x \in W$. Then ${\downarrow}x \cdot {\downarrow}x = {\downarrow}x$ since ${\downarrow}x \in D(\mathbf{W})$. From $x \in {\downarrow}x$ we deduce $x \in {\downarrow}x \cdot {\downarrow}x$, whence it follows that $xRyz$ for some $y \in {\downarrow}x, z \in {\downarrow}x$. Therefore $xRyz$ for $y \le x, z \le x$, which implies $xRxx$ by (R2) and (R3).

Next assume $xRyz$ holds. Then $x \in {\downarrow}\{y,z\}{\cdot}{\downarrow}\{y,z\} = {\downarrow}\{y,z\}$ by idempotence. Hence for some $w \in \{y,z\}$ we have $x \le w$, and it follows that $x \le y$ or $x \le z$.

For the converse, assume $xRxx$ and $(xRyz \implies x \le y$ or $x \le z)$ for all $x, y, z \in W$ and let $X \in D(\mathbf{W})$. From $xRxx$ we obtain $X \subseteq X \cdot X$.

For the reverse inclusion, let $x \in X \cdot X$. Then $xRyz$ holds for some $y, z \in X$. By assumption $xRyz$ implies $x \le y$ or $x \le z$. Since X is a downset, $x \le y \implies x \in X$ and $x \le z \implies x \in X$. Hence $X \cdot X = X$. □

The previous two results are examples of correspondence theory, since they show that an equational property on a perfect $d\ell$-magma corresponds to a first-order condition on its Birkhoff frame.

The relational semantics of a perfect $d\ell pq$-magma is given by a *PQ-frame*, which is a partially-ordered relational structure $(W, \le, P, Q)$ such that P, Q are binary relations on W, $u \le xPy \le v \implies uPv$ and $u \le xQy \le v \implies uQv$. Relations with this property are called *weakening relations* [11,5], and this is what ensures that if we define $p(Y) = \{x \mid \exists y(xPy \ \&\ y \in Y)\}$ for a downset Y, then p is a complete normal join-preserving operator that produces a downset, and P is uniquely determined by $xPy \Leftrightarrow x \in p({\downarrow}y)$. Similarly, a normal operator q is defined from Q, and uniquely determines Q. The residual p' of p is a completely meet-preserving operator, defined by $p'(Y) = \{x \mid \forall y(yPx \Rightarrow y \in Y)\}$, and likewise for q'. If $P = Q$ then we omit Q and refer to $(W, \le, P)$ simply as a *P-frame*.

We now list some correspondence results for $d\ell pq$-magmas. We begin with a theorem that restates the term-equivalence of Theorem 1 as a definitional equivalence on frames. A direct proof of this result is straightforward, but it also follows from Theorem 1 by correspondence theory.

Theorem 11. (1) *Let* $(W, \le, P, Q)$ *be a PQ-frame such that* $x \le y \ \&\ xPz \Rightarrow xQy$ *and* $x \le y \ \&\ xQz \Rightarrow xPy$*. If we define* $xRyz \Leftrightarrow (xPy \ \&\ x \le z)$ *or* $(x \le y \ \&\ xQz)$ *then* $(W, \le, R)$ *is a Birkhoff frame, and* P, Q *are obtained from* R *via* $xPy \Leftrightarrow \exists z(xRyz)$ *and* $xQy \Leftrightarrow \exists z(xRzy)$*.*

(2) *Let* $(W, \le, R)$ *be a Birkhoff frame that satisfies* $xRyz \Leftrightarrow (\exists z(xRyz) \ \&\ x \le z)$ *or* $(x \le y \ \&\ \exists z(xRzy))$ *and define* $xPy \Leftrightarrow \exists z(xRyz),\ xQy \Leftrightarrow \exists z(xRzy)$*. Then* $(W, \le, P, Q)$ *is a PQ-frame in which* $x \le y \ \&\ xPz \Rightarrow xQy$ *and* $x \le y \ \&\ xQz \Rightarrow xPy$ *hold.*

Note that the universal formula $x \leq y \ \&\ xPz \implies xQy$ corresponds to the $d\ell pq$-magma axiom $Y \wedge p\top \leq qY$.

A significant advantage of PQ-frames over Birkhoff frames is that binary relations have a graphical representation in the form of directed graphs (whereas ternary relations are 3-ary hypergraphs that are more complicated to draw). Equational properties from Lemma 2, Cor. 6 correspond to the following first-order properties on PQ-frames.

Lemma 12. *Assume* $\mathbf{A}$ *is a perfect* $d\ell pq$*-algebra and* $\mathbf{W} = (W, \leq, P, Q)$ *is its corresponding* PQ*-frame. The constant* $1 \in A$ *(when present) is assumed to correspond to a downset* $E \subseteq W$*. Then*

(1) $a \leq pa$ *holds in* $\mathbf{A}$ *if and only if* P *is reflexive,*
(2) $ppa \leq pa$ *holds in* $\mathbf{A}$ *if and only if* P *is transitive,*
(3) $pa = qa$ *holds in* $\mathbf{A}$ *if and only if* $P = Q$*,*
(4) $p1 = \top$ *holds in* $\mathbf{A}$ *if and only if* $\forall x \exists y (y \in E \ \&\ xPy)$ *holds in* $\mathbf{W}$*,*
(5) $pa \wedge 1 \leq a$ *holds in* $\mathbf{A}$ *if and only if* $x \in E \ \&\ xPy \Rightarrow x \leq y$ *holds in* $\mathbf{W}$*,*
(6) $pa \wedge pb \leq p((pa \wedge b) \vee (a \wedge pb))$ *holds in* $\mathbf{A}$ *if and only if*

$$wPx \ \&\ wPy \Rightarrow \exists v (wPv \ \&\ (vPx \ \&\ v \leq y \text{ or } v \leq x \ \&\ vPy)) \quad \text{holds in } \mathbf{W}.$$

Proof. (1)-(3) These correspondences are well known from modal logic.

(4) For $x \in J(A)$ and $E = {\downarrow}1$ we have $x \leq p1$ if and only if there exists $y \in J(A)$ such that $y \leq 1$ and $x \leq py$, or equivalently, $y \in E$ and xPy.

(5) In the forward direction, let $a = {\downarrow}y$. Then it follows that $x \in p({\downarrow}y) \cap E$ implies $x \in {\downarrow}y$, and consequently $x \in E \ \&\ xPy \implies x \leq y$.

In the other direction, let Y be a downset of W and assume $x \in pY \cap E$. Then $x \in E$ and xPy for some $y \in Y$. Hence $x \leq y$, or equivalently $x \in {\downarrow}y \subseteq Y$. Thus, $pY \cap E \subseteq Y$, so the algebra $\mathbf{A}$ satisfies $pa \wedge 1 \leq a$ for all $a \in A$.

(6) In the forward direction, let $a = {\downarrow}x$ and $b = {\downarrow}y$. Then it follows from the inequation that $w \in p{\downarrow}x \cap {\downarrow}y \implies w \in p((p{\downarrow}x \cap {\downarrow}y) \cup ({\downarrow}x \cap p{\downarrow}y))$ for all $w \in W$. This in turn implies $wPx \ \&\ wPy \implies \exists v (wPv \ \&\ v \in (p{\downarrow}x \cap {\downarrow}y) \cup ({\downarrow}x \cap p{\downarrow}y))$, which translates to the given first-order condition.

In the reverse direction, let X, Y be downsets of W and assume $w \in pX \cap pY$. Then wPx and wPy for some $x \in X$ and $y \in Y$. It follows that there exists a $v \in W$ such that $(wPv \ \&\ (vPx \ \&\ v \leq y \text{ or } v \leq x \ \&\ vPy))$, hence $v \in (pX \cap Y) \cup (X \cap pY)$. Therefore $w \in p(pX \cap Y) \cup (X \cap pY)$. □

Recall that a ternary relation R is commutative if $xRyz \Leftrightarrow xRzy$ for all x, y. From Theorem 11 we also obtain the following result.

Corollary 13. *Let* $(W, \leq, P, Q)$ *be a* PQ*-frame and define* R *as in Thm. 11(1). Then* R *is commutative if and only if* $xPy \Leftrightarrow xQy$ *for all* $x, y \in W$*.*

This corollary shows that in the commutative setting a PQ-frame only needs one of the two binary relations. Hence we define $\mathbf{W} = (W, \leq, P)$ to be a *P-frame* if P is a weakening relation, i.e., $u \leq xPy \leq v \implies uPv$.

We now turn to the problem of ensuring that the binary operation of a $d\ell$-magma is associative. For Birkhoff frames the following characterization of associativity is well known from relation algebras [10] (in the Boolean case) and from the Routley-Meyer semantics for relevance logic [4] in general.

Lemma 14. *Let $\mathbf{W} = (W, \leq, R)$ be a Birkhoff frame. Then $\mathbf{D}(\mathbf{W})$ is an associative ℓ-magma if and only if $\forall wxyz(\exists u(uRxy \,\&\, wRuz) \Leftrightarrow \exists v(vRyz \,\&\, wRxv))$. If R is commutative then the equivalence can be replaced by the implication $\forall uwxyz(uRxy \,\&\, wRuz \Rightarrow \exists v(vRyz \,\&\, wRxv))$.*

This lemma is another correspondence result that follows from translating $w \in (XY)Z \Leftrightarrow w \in X(YZ)$ for $X, Y, Z \in D(\mathbf{W})$. In the commutative case $(XY)Z \subseteq X(YZ)$ implies the reverse inclusion, hence only one of the implications is needed. We now show that for a large class of P-frames the 5-variable universal-existential formula for associativity can be replaced by simpler universal formulas with only three variables.

A *preorder forest P-frame* is a P-frame such that P is a preorder (i.e. reflexive and transitive) and satisfies the formula

$$\text{(Pforest)} \quad xPy \text{ and } xPz \implies x \leq y \text{ or } x \leq z \text{ or } yPz \text{ or } zPy.$$

Note that since P is a weakening relation, reflexivity of P implies that $\leq \subseteq P$ because xPx and $x \leq y$ implies xPy.

It is interesting to visualize the properties that define preorder forest P-frames by implications between Hasse diagrams with $\leq$-edges (solid) and P-edges (dotted) as in Figure 1. However, one needs to keep in mind that dotted lines could be horizontal (if xPy and yPx) and that any line could be a loop if two variables refer to the same element.

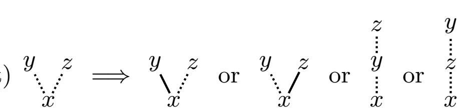
(Pforest) 

Fig. 1. The (Pforest) axiom. The partial order $\leq$ and the preorder P are denoted by solid lines and dotted lines respectively.

We are now ready to state the main result. We use the algebraic characterization of associativity in Lemma 2.

Theorem 15. *Let $\mathbf{W} = (W, \leq, P)$ be a preorder forest P-frame and $\mathbf{D}(\mathbf{W})$ its corresponding downset algebra. Then the operation $x{\cdot}y = (px \wedge y) \vee (x \wedge py)$ is associative in $\mathbf{D}(\mathbf{W})$.*

Proof. Let $\mathbf{W} = (W, \leq, P)$ be a preorder forest P-frame and $\mathbf{D}(\mathbf{W})$ its $d\ell p$-algebra of downsets with operator p. Since P is a preorder, $\mathbf{D}(\mathbf{W})$ is a $d\ell p$-closure algebra. By Lemma 5, a $d\ell p$-closure algebra is associative if and only if

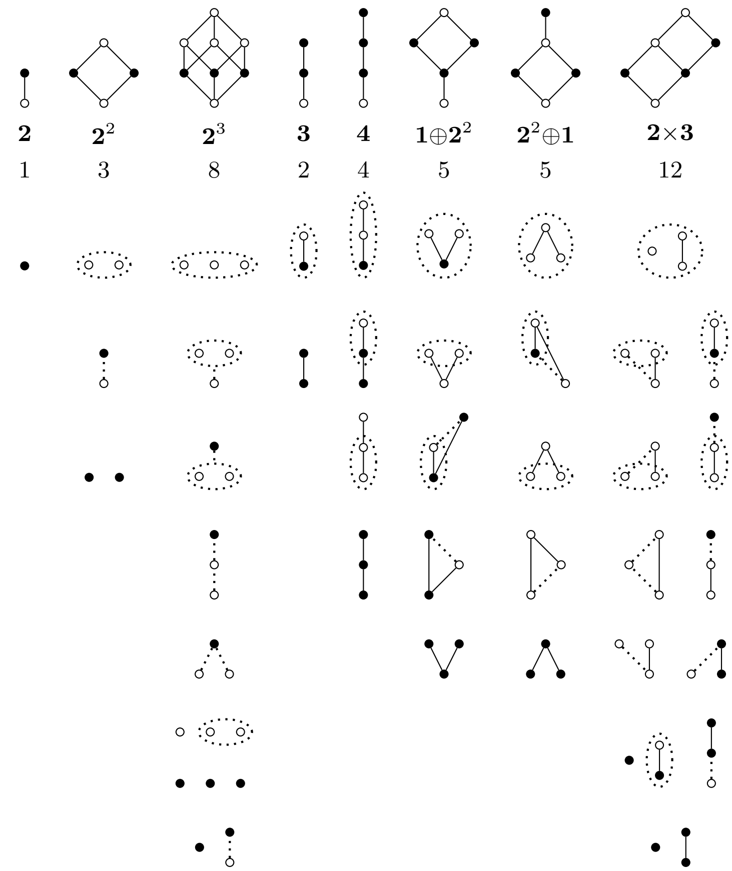


Fig. 2. All 40 preorder forest P-frames $(W, \leq, P)$ with up to 3 elements. Solid lines show $(W, \leq)$, dotted lines show the additional edges of P, and the identity (if it exists) is the set of black dots. The first row shows the lattice of downsets, and the Boolean quantales from [1] appear in the first three columns.

$p(x) \wedge p(y) \leq p(p(x) \wedge y) \vee (x \wedge p(y))$. By Lemma 12 this is equivalent to the frame property

$$(*) \qquad xPy \;\&\; xPz \Rightarrow \exists w(xPw \;\&\; (wPy \;\&\; w \leq z \text{ or } w \leq y \;\&\; wPz)).$$

We now show that this frame property holds in $\mathbf{W}$. We know that P is reflexive and (Pforest) holds.

Assume xPy and xPz. By (Pforest) there are four cases:

1. $x \leq y$: take $w = x$. Then xPx, $x \leq y$ and xPz, hence $(*)$ holds.
2. $x \leq z$: again take $w = x$. Then the other disjunct of $(*)$ holds.
3. yPz: take $w = y$. Then xPy, $y \leq y$ and yPz, hence $(*)$ holds.
4. zPy: take $w = z$. Then xPz, zPy and $y \leq y$, hence again $(*)$ holds. □

The universal class of preorder forest P-frames is strictly contained in the class of all P-frames in which $x{\cdot}y$ is associative. In fact the latter class is not closed under substructures, hence not a universal class: $W = \{0,1,2,3\}$, $\leq = id_W \cup \{(0,1),(0,2),(0,3)\}$, $P = \leq \cup \{(1,0),(1,2),(1,3)\}$ is a P-frame with associative $\cdot$ (use e.g. Lemma 5), but restricting $\leq, P$ to the subset $\{1,2,3\}$ gives a P-frame where $\cdot$ fails to be associative, hence (Pforest) also fails.

A *dℓ-semilattice* is an associative commutative idempotent distributive ℓ-magma. The point of the previous result is that it allows the construction of perfect associative commutative idempotent $d\ell$-magmas and idempotent bunched implication algebras from preorder forest P-frames. This is much simpler than constructing the ternary relation R of the Birkhoff frame of such algebras. For example the Hasse diagrams for all the preorder forest P-frames with up to 3 elements are shown in Figure 2, with the preorder P given by dotted lines and ovals. The corresponding ternary relations can be calculated from P, but would have been hard to include in each diagram.

We now examine when a preorder forest P-frame will have an identity element. For any P-frame $\mathbf{W}$ we define $E = \{x \in W \mid \forall y(xPy \Rightarrow x \leq y)\}$.

Lemma 16. *Let $\mathbf{W}$ be a P-frame. Then E is an identity element for $\cdot$ in the downset algebra $D(\mathbf{W})$ if and only if E is a downset and $pE = W$.*

Proof. In the forward direction, E is certainly a downset and it follows from Lemma 2(5) that $pE = W$ since W is the top element in $D(\mathbf{W})$.

Conversely, by the definition of E, if $x \in E$ then $xPy \Rightarrow x \leq y$ holds for all $y \in W$. Hence by Lemma 12(5) for all $X \in D(\mathbf{W})$ we have $pX \cap E \subseteq X$. Since $pE = W$ together with Lemma 2(5), it follows that E is an identity element in the downset algebra. □

5 Counting preorder forests and linear P-frames

In the case when the poset $(W, \leq)$ is an antichain, a preorder forest P is simply a preorder $P \subseteq W^2$ such that xPy and xPz implies yPz or zPy. A *preorder tree* is a connected component of a preorder forest. A *rooted* preorder forest is defined to have an equivalence class of P-maximal elements in each component. For finite preorder forests this is always the case. Let F_n denote the number of preorder forests and T_n the number of preorder trees with n elements (up to isomorphism). We also let $F_0 = 1$.

A preorder forest *has singleton roots* if the P-maximal equivalence class of each component is a singleton set. The number of preorder forests and trees with singleton roots is denoted by F_n^s and T_n^s respectively.

Note that every preorder forest gives rise to a unique preorder tree with a singleton root by adding one new element r such that for all $x \in W$ we have xPr. It follows that $T_n^s = F_{n-1}$.

Every preorder tree with a non-singleton root equivalence class and n elements is obtained from a preorder tree with $n-1$ elements by adding one more

cardinality $n =$	1	2	3	4	5	6	7
preorder trees $T_n =$	1	2	5	13	37	108	337
$c_n =$	1	5	16	57	186	668	
preorder forests $F_n =$	1	3	8	24	71	224	
preorder trees with singleton roots $T_n^s =$	1	1	3	8	24	71	224
$c_n^s =$	1	3	10	35	121	438	
preorder forests with singleton roots $F_n^s =$	1	2	5	14	41	127	

Table 2. Number of preorder trees and forests (up to isomorphism)

element to the root equivalence class. Hence for $n > 0$ we have $T_n = F_{n-1} + T_{n-1}$. The Euler transform of T_n is used to calculate the next value of F_n as follows:

$$c_n = \sum_{d|n} d \cdot T_n \qquad\qquad F_n = \frac{1}{n} \sum_{k=1}^{n} c_k \cdot F_{n-k}.$$

Since preorder forests with singleton roots are disjoint unions of preorder trees with singleton roots, F_n^s is calculated by an Euler transform from T_n^s.

Corollary 17. *The sequence F_n^s is the Euler transform of T_n^s.*

While it is difficult to count preorder forest P-frames in general, it is simple to count the linear ones. Let L_n be the number of linearly ordered preorder forest P-frames with n elements. Note that $(P3)$ is actually redundant for linearly ordered frames.

Theorem 18. *For linearly ordered forest P-frames $L_n = 2^{n-1}$. In the algebraic setting there are 2^{n-2} unary-determined commutative doubly idempotent linear semirings with n elements, and $n-1$ of them have an identity element.*

Proof. Let $\mathbf{W}$ be a linearly ordered P-frame with elements $W = \{1 < 2 < \cdots < n\}$ such that P is transitive and $(P0)$ holds. Then each possible relation P on W is determined by choosing a subset S of the edges $\{(2,1), (3,2), \ldots, (n, n-1)\}$ and defining P to be the transitive closure of $S \cup \leq$. Since there are $n-1$ such edges to choose from, the number of p-frames is 2^{n-1}.

Let $\mathbf{A}$ be a unary-determined commutative doubly idempotent linear semiring with n elements. Then the P-frame $\mathbf{W}$ associated with $\mathbf{A}$ has $n-1$ elements, is linearly ordered, and P is reflexive and transitive since $\cdot$ is idempotent and associative. Hence there are 2^{n-2} such algebras.

By Lemma 16, the subset $E = \{x \in W \mid \forall y(xPy \Rightarrow x \leq y)\}$ will be an identity of the downset algebra if and only if it is a downset of W and $p(E) = W$. This will only be the case if there exists an element $w \in W$ such that for all $y \in W$ we have $y \geq w$ if and only if wPy. Every choice of $w \in W$ determines one such P, hence there are $n-1$ algebras with an identity element. $\square$

6 Conclusion

We showed that unary-determined $d\ell$-magmas have a simple algebraic structure given by two unary operators and that their relational frames are definitionally equivalent to frames with two binary relations. The complex algebras of these frames are complete distributive lattices with completely distributive operators, hence they have residuals and can be considered Kripke semantics for unary-determined bunched implication algebras and bunched implication logic. Associativity of the binary operator for idempotent unary-determined algebras can be checked by an identity with 2 rather than 3 variables, and for the frames by a 3-variable universal formula rather than a 6-variable universal-existential formula. All idempotent Boolean magmas are unary-determined, hence these results significantly extend the structural characterization of idempotent atomic Boolean quantales in [1] and relate them to bunched implication logic. As an application we counted the number of preorder forest P-frames with n elements for which the partial order is an antichain, as well as the number of linearly ordered preorder P-frames.

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