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REGULAR PAPER

Schrödinger evolution of superoscillations with δ- and δ'-potentials

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Abstract In this paper, we study the time persistence of superoscillations as the initial data of the time-dependent Schrödinger equation with δ- and δ'-potentials. It is shown that the sequence of solutions converges uniformly on compact sets, whenever the initial data converge in the topology of the entire function space $A_1(\mathbb{C})$. Convolution operators acting in this space are our main tool. In particular, a general result about the existence of such operators is proven. Moreover, we provide an explicit formula as well as the large time asymptotics for the time evolution of a plane wave under δ- and δ'-potentials.

Keywords Superoscillating functions · Convolution operators · Schrödinger equation · Singular potential · Entire functions with growth conditions

Mathematics Subject Classification 32A15 · 32A10 · 47B38

1 Introduction

Superoscillating functions have an oscillatory behaviour which is locally faster than their fastest Fourier component. This paradoxical property was discovered by the first author and his collaborators in their work about weak measurements [1] and afterwards investigated from a mathematical and quantitative point of view by Berry [18]. In antenna theory, this phenomenon was discovered by Toraldo di Francia [47] as pointed out also in [19]. Several authors have contributed to this field and without claiming completeness we mention [10,12,22,23] and also [34–37,39,45,46]. More recently, a special emphasis was given to the mathematical aspect of superoscillations, see, e.g., [2–8,27].

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The special topic which we want to investigate in this paper is the Schrödinger time evolution of superoscillating functions F, that is, we consider

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(t,x) &= \left(-\frac{\partial^2}{\partial x^2}+V(x)\right)\Psi(t,x), \quad t>0,\ x\in\mathbb{R}, \\ \Psi(0,x) &= F(x), \quad x\in\mathbb{R}, \end{aligned} \tag{1}$$

where V is the potential and F is the initial datum, which is assumed to be superoscillating. Already a number of different potentials were investigated, see the survey papers [9,11,14,21] and references therein. Our aim is to add the one-dimensional δ-potential and δ'-potential to this list. We mention that the δ-potential was already treated in [15]; however, a (technical) condition on the strength of the δ-interaction was imposed there, which we are able to avoid in the present paper. A detailed discussion of Schrödinger operators with δ and δ'-point interactions can be found in the standard monograph [13]. For further reading on Schrödinger operators with singular potentials, we refer the reader to, e.g., [16,17,24–26,29–33,38,40–44] and the references therein.

We briefly illustrate the concept of superoscillations. Consider for some fixed $k\in\mathbb{R}$ with $|k|>1$ the sequence of functions

$$F_n(z,k)=\sum_{j=0}^{n} C_j(n,k)\mathrm{e}^{i(1-\frac{2j}{n})z}, \quad z\in\mathbb{C}, \tag{2}$$

with coefficients

$$C_j(n,k)=\binom{n}{j}\left(\frac{1+k}{2}\right)^{n-j}\left(\frac{1-k}{2}\right)^{j}.$$

The notion superoscillatory now comes from the fact that, although all the Fourier coefficients $k_j(n)=1-\frac{2j}{n}$ are contained in the bounded interval $[-1,1]$, the whole sequence converges to

$$\lim_{n\to\infty} F_n(z,k)=\mathrm{e}^{ikz}, \tag{3}$$

a plane wave with wave vector $|k|>1$; cf. [28, Theorem 2.1] for more details. Besides the convergence (3), the important feature of the functions F_n is that also for finite n, they oscillate with frequencies close to k in certain intervals; the lengths of these intervals grow when n increases. Nevertheless, outside this interval, one obtains an exponential growth of the amplitude, which conversely means that the amplitude inside the superoscillatory region is exponentially small. Different types of functions, in the form of a square-integrable sinc function, which are band-limited and, in some intervals, oscillate faster than its highest Fourier component, can be found in [20].

Inspired by (2), we define the notion of superoscillations as follows:

Definition 1.1 A sequence of functions of the form

$$F_n(z)=\sum_{j=0}^{n} C_j(n)\mathrm{e}^{ik_j(n)z}, \quad z\in\mathbb{C}, \tag{4}$$

with coefficients $C_j(n)\in\mathbb{C}$ and $k_j(n)\in\mathbb{R}$, is said to be superoscillating, if there exists some $k\in\mathbb{R}$, such that

(i) $\sup_{n\in\mathbb{N}_0,\,j\in\{0,\dots,n\}}|k_j(n)|<k$, and
(ii) for some $B\geq 0$, one has $\lim_{n\to\infty}\left\|(F_n-\mathrm{e}^{ik\,\cdot})\mathrm{e}^{-B|\cdot|}\right\|_\infty=0$.

Note that (ii) is exactly the convergence in the space $A_1(\mathbb{C})$ introduced in Definition 2.1 below. Due to the above description of the exponential growth of the amplitude outside the superoscillatory region, it is reasonable to use the exponential weight $\mathrm{e}^{-B|\cdot|}$ as a damping factor in the uniform convergence.

The purpose of this paper is now to consider a superoscillating sequence $(F_n)_n$ as the initial datum of the time-dependent Schrödinger equation (1) with either a δ-potential or a δ'-potential and investigate the corresponding sequence of solutions $(\Psi_n)_n$. The main result of this paper is the following:

Theorem 1.2 *Let $(F_n)_n$ be a superoscillatory sequence with limit function*

$$F_n \xrightarrow{A_1} \mathrm{e}^{ik\,\cdot}, \tag{5}$$

for some $k \in \mathbb{R}$. Then, the solutions Ψ and Ψ_n of (1) with either $V = 2\alpha\delta$ or $V = \frac{2}{\beta}\delta'$ for $\alpha, \beta \in \mathbb{R}\backslash\{0\}$, and initial data $\mathrm{e}^{ik\,\cdot}$ and F_n, respectively, satisfy

$$\lim_{n\to\infty} \Psi_n(t,x) = \Psi(t,x)$$

uniformly on every compact subset of $(0,\infty) \times \mathbb{R}$.

Note that the mathematical rigorous implementation of a singular δ-potential or a δ'-potential in the Schrödinger equation (1) is via jump conditions at the location $x = 0$ of the interaction; cf. (14) and (15).

The proof of Theorem 1.2 is postponed to the end of Sect. 4. Before, in Sect. 2, we introduce the space $A_1(\mathbb{C})$ which is used in the convergence (5). We also construct continuous operators acting in this space, which will play a crucial role in the proof of Theorem 1.2. Moreover, in Sect. 3, we explicitly calculate the solution with a plane wave $F(x) = \mathrm{e}^{ikx}$ as initial condition.

2 Convolution operators in $A_1(\mathbb{C})$

In this section, we recall the definition of the space $A_1(\mathbb{C})$ of entire functions with exponential growth, already mentioned below Definition 1.1. For the analysis of superoscillations, this space (or slight modifications of it) is a convenient choice; cf. [7,11,14]. Note also that $A_1(\mathbb{C})$ is one particular space in the theory of analytically uniform spaces, see, e.g., [7, Chapter 4].

Definition 2.1 Let $\mathcal{H}(\mathbb{C})$ be the space of entire functions and define

$$A_1(\mathbb{C}) := \left\{ F \in \mathcal{H}(\mathbb{C}) \,\middle|\, \exists A, B \geq 0 \text{ such that } |F(z)| \leq A\mathrm{e}^{B|z|} \text{ for all } z \in \mathbb{C} \right\}.$$

A sequence $(F_n)_n \in A_1(\mathbb{C})$ is said to be A_1-convergent to $F \in A_1(\mathbb{C})$ if

$$\lim_{n\to\infty} \left\| (F_n - F)\mathrm{e}^{-B|\cdot|} \right\|_\infty = 0 \quad \text{for some } B \geq 0. \tag{6}$$

This type of convergence will be denoted by $F_n \xrightarrow{A_1} F$.

The following lemma shows that also the derivatives of functions in $A_1(\mathbb{C})$ are exponentially bounded.

Lemma 2.2 *Let $F \in A_1(\mathbb{C})$ admit the estimate*

$$|F(z)| \leq A\mathrm{e}^{B|z|}, \quad z \in \mathbb{C}, \tag{7}$$

for some $A, B \geq 0$. Then, the derivatives $F^{(n)}$, $n \in \mathbb{N}$, of F admit the estimate

$$|F^{(n)}(z)| \leq A(\mathrm{e}B)^n \mathrm{e}^{B|z|}, \quad z \in \mathbb{C}, \tag{8}$$

and, in particular, $F^{(n)} \in A_1(\mathbb{C})$ for all $n \in \mathbb{N}$.

Proof If $B = 0$ the statement holds, since in that case, the entire function F is bounded and hence constant, so that all its derivatives vanish identically. In the following, let $B \neq 0$. By Cauchy's integral formula, we have

$$F^{(n)}(z) = \frac{n!}{2\pi i} \int_{|\xi - z| = r} \frac{F(\xi)}{(\xi - z)^{n+1}} \mathrm{d}\xi = \frac{n!}{2\pi r^n} \int_0^{2\pi} \frac{F(z + r\mathrm{e}^{i\varphi})}{\mathrm{e}^{in\varphi}} \mathrm{d}\varphi, \quad z \in \mathbb{C},$$

for the nth derivative, where $r > 0$ is not yet specified. The exponential boundedness (7) of the integrand gives the estimate

$$\left|F^{(n)}(z)\right| \leq \frac{An!}{2\pi r^n}\int_0^{2\pi} \mathrm{e}^{B|z+re^{i\varphi}|}\mathrm{d}\varphi \leq \frac{An!}{r^n}\mathrm{e}^{B(|z|+r)}, \quad z \in \mathbb{C}.$$

It is easy to see that the right-hand side can be minimized choosing $r = \frac{n}{B}$, which gives

$$\left|F^{(n)}(z)\right| \leq \frac{AB^n n!}{n^n}\mathrm{e}^{B|z|+n}, \quad z \in \mathbb{C}.$$

Using $\frac{n!}{n^n} \leq 1$, this gives the estimate (8). $\square$

After this preparatory lemma, we consider an operator that can be used to reconstruct a given function from plane waves. Such operators will play the crucial role of time evolution operators in Sect. 4, acting on the initial datum of the Schrödinger equation and pointwise giving its solution.

Proposition 2.3 *Let $\Psi : \mathbb{R} \to \mathbb{C}$ be a function, which can be written as the absolute convergent power series*

$$\Psi(k) = \sum_{m=0}^{\infty} c_m k^m, \quad k \in \mathbb{R}, \tag{9}$$

with coefficients $(c_m)_m \in \mathbb{C}$. Then, the operator $U : A_1(\mathbb{C}) \to A_1(\mathbb{C})$, defined by

$$UF(\xi) := \sum_{m=0}^{\infty}(-i)^m c_m \frac{\mathrm{d}^m}{\mathrm{d}\xi^m}F(\xi), \quad F \in A_1(\mathbb{C}),\ \xi \in \mathbb{C}, \tag{10}$$

is continuous in $A_1(\mathbb{C})$ and satisfies

$$\Psi(k) = Ue^{ik\xi}\big|_{\xi=0}, \quad k \in \mathbb{R}. \tag{11}$$

Moreover, for $F \in A_1(\mathbb{C})$, such that $|F(\xi)| \leq A\mathrm{e}^{B|\xi|}$ and $S_B := \sum_{m=0}^{\infty}|c_m|B^m$, the estimate

$$|UF(\xi)| \leq AS_{\mathrm{e}B}\mathrm{e}^{\mathrm{e}B|\xi|}, \quad \xi \in \mathbb{C}, \tag{12}$$

is valid.

Proof To see that for $F \in A_1(\mathbb{C})$, the image UF is again an element in $A_1(\mathbb{C})$, we have to show that UF is entire and exponentially bounded. Since $F \in A_1(\mathbb{C})$ is entire, we can use the power series representation

$$F(\xi) = \sum_{n=0}^{\infty}\frac{F^{(n)}(0)}{n!}\xi^n, \quad \xi \in \mathbb{C},$$

to write (10) as the power series

$$UF(\xi) = \sum_{m=0}^{\infty}(-i)^m c_m \sum_{n=m}^{\infty}\frac{F^{(n)}(0)}{(n-m)!}\xi^{n-m} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-i)^m c_m \frac{F^{(n+m)}(0)}{n!}\xi^n, \quad \xi \in \mathbb{C}.$$

In the last step, we interchanged the order of summation, which is allowed, since (8) ensures the estimate

$$\sum_{m=0}^{\infty}|c_m|\frac{|F^{(n+m)}(0)|}{n!} \leq \sum_{m=0}^{\infty}|c_m|\frac{A(\mathrm{e}B)^{n+m}}{n!} = AS_{\mathrm{e}B}\frac{(\mathrm{e}B)^n}{n!}, \tag{13}$$

and, hence, the absolute convergence of the double sum. This shows that UF can be represented as an everywhere convergent power series and, hence, is entire. Moreover, the estimate (13) gives the exponential boundedness

$$|UF(\xi)| \leq AS_{\mathrm{e}B}\sum_{n=0}^{\infty}\frac{(\mathrm{e}B)^n}{n!}|\xi|^n = AS_{\mathrm{e}B}\mathrm{e}^{\mathrm{e}B|\xi|}, \quad \xi \in \mathbb{C},$$

so that $UF \in A_1(\mathbb{C})$ and the estimate (12) is satisfied. Finally, (11) follows from

$$U\mathrm{e}^{ik\xi}\big|_{\xi=0} = \sum_{m=0}^{\infty}(-i)^m c_m \frac{\mathrm{d}^m}{\mathrm{d}\xi^m}\mathrm{e}^{ik\xi}\Big|_{\xi=0} = \sum_{m=0}^{\infty}(-i)^m c_m (ik)^m = \Psi(k).$$

$\square$

3 Plane wave under a δ- or δ'-potential

In quantum mechanics, a particle or wave interacting with a δ-potential or δ'-potential is described by the Schrödinger equation (1), formally putting $V = 2\alpha\delta$ or $V = \frac{2}{\beta}\delta'$, where $\alpha, \beta \in \mathbb{R}\setminus\{0\}$ model the strength of the respective peaks. The rigorous mathematical description of such singular potentials is via jump conditions at the location $x = 0$ of the potential. More precisely, for the δ-potential, we have to consider the system

$$i\frac{\partial}{\partial t}\Psi_\delta(t,x) = -\frac{\partial^2}{\partial x^2}\Psi_\delta(t,x), \quad t > 0,\ x \in \mathbb{R}\setminus\{0\}, \tag{14a}$$

$$\Psi_\delta(t,0^+) = \Psi_\delta(t,0^-), \qquad t > 0, \tag{14b}$$

$$\frac{\partial}{\partial x}\Psi_\delta(t,0^+) - \frac{\partial}{\partial x}\Psi_\delta(t,0^-) = 2\alpha\,\Psi_\delta(t,0), \qquad t > 0, \tag{14c}$$

$$\Psi_\delta(0^+,x) = F(x), \qquad x \in \mathbb{R}\setminus\{0\}, \tag{14d}$$

and for the δ'-potential, we have to consider the system

$$i\frac{\partial}{\partial t}\Psi_{\delta'}(t,x) = -\frac{\partial^2}{\partial x^2}\Psi_{\delta'}(t,x), \quad t > 0,\ x \in \mathbb{R}\setminus\{0\}, \tag{15a}$$

$$\frac{\partial}{\partial x}\Psi_{\delta'}(t,0^+) = \frac{\partial}{\partial x}\Psi_{\delta'}(t,0^-), \qquad t > 0, \tag{15b}$$

$$\Psi_{\delta'}(t,0^+) - \Psi_{\delta'}(t,0^-) = \frac{2}{\beta}\frac{\partial}{\partial x}\Psi_{\delta'}(t,0), \qquad t > 0, \tag{15c}$$

$$\Psi_{\delta'}(0^+,x) = F(x), \qquad x \in \mathbb{R}\setminus\{0\}, \tag{15d}$$

where F is an appropriate initial condition. In the case that F is a plane wave, we find an explicit representation of the solutions Ψ_δ and $\Psi_{\delta'}$ in Theorem 3.2 below. To write down these solutions efficiently, we will use a certain modified error function Λ and its properties. For the convenience of the reader, we recall the following lemma from [15].

Lemma 3.1 *For the function*

$$\Lambda(z) := e^{z^2}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2}d\xi\right), \quad z \in \mathbb{C},$$

the following statements hold:

(i) $\Lambda(-z) = 2e^{z^2} - \Lambda(z), \quad z \in \mathbb{C}$;

(ii) $\Lambda'(z) = 2z\Lambda(z) - \frac{2}{\sqrt{\pi}}, \quad z \in \mathbb{C}$;

(iii) *for* $|z| \to \infty$ *one has*

$$\Lambda(z) = \begin{cases} \frac{1}{\sqrt{\pi}z} + \mathcal{O}\left(\frac{1}{|z|^2}\right), & \text{if } \operatorname{Re}(z) \geq 0, \\ 2e^{z^2} + \frac{1}{\sqrt{\pi}z} + \mathcal{O}\left(\frac{1}{|z|^2}\right), & \text{if } \operatorname{Re}(z) \leq 0; \end{cases} \tag{16}$$

(iv) Λ *admits the power series representation*

$$\Lambda(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\frac{n}{2}+1)} z^n, \quad z \in \mathbb{C}. \tag{17}$$

The next theorem is the main result in this section.

Theorem 3.2 *The Schrödinger equations (14) and (15) with initial datum $F(x) = e^{ikx}$, $k \in \mathbb{R}$, admit the explicit solutions*

$$\Psi_\delta(t,x;k) = \Psi_{\text{free}}(t,x;k) + \varphi_\delta(t,x;k) + \varphi_\delta(t,-x;-k), \tag{18a}$$

$$\Psi_{\delta'}(t,x;k) = \Psi_{\text{free}}(t,x;k) + \varphi_{\delta'}(t,x;k) + \varphi_{\delta'}(t,-x;-k), \tag{18b}$$

where $\Psi_{\mathrm{free}}(t,x;k)=\mathrm{e}^{ikx-ik^2t}$ *is the solution of the unperturbed system and*

$$\varphi_\delta(t,x;k)=\frac{\alpha}{2(\alpha+ik)}\mathrm{e}^{-\frac{x^2}{4it}}\left(\Lambda\left(\frac{|x|}{2\sqrt{it}}+\alpha\sqrt{it}\right)-\Lambda\left(\frac{|x|}{2\sqrt{it}}-ik\sqrt{it}\right)\right), \tag{19a}$$

$$\varphi_{\delta'}(t,x;k)=\frac{\operatorname{sgn}(x)}{2(\beta+ik)}\mathrm{e}^{-\frac{x^2}{4it}}\left(\beta\Lambda\left(\frac{|x|}{2\sqrt{it}}+\beta\sqrt{it}\right)+ik\Lambda\left(\frac{|x|}{2\sqrt{it}}-ik\sqrt{it}\right)\right). \tag{19b}$$

Moreover, for $t\to\infty$*, the wave functions (18) satisfy*

$$\Psi_\delta(t,x;k)=\mathrm{e}^{-ik^2t}\left(\mathrm{e}^{ikx}-\frac{\alpha}{\alpha-i|k|}\mathrm{e}^{i|kx|}\right)+1_{\mathbb{R}^-}(\alpha)\frac{2\alpha^2}{\alpha^2+k^2}\mathrm{e}^{\alpha|x|+i\alpha^2t}+\mathcal{O}\left(\frac{1}{t}\right), \tag{20a}$$

$$\Psi_{\delta'}(t,x;k)=\mathrm{e}^{-ik^2t}\left(\mathrm{e}^{ikx}+\frac{ik\operatorname{sgn}(x)}{\beta-i|k|}\mathrm{e}^{i|kx|}\right)-1_{\mathbb{R}^-}(\beta)\frac{2i\beta k\operatorname{sgn}(x)}{\beta^2+k^2}\mathrm{e}^{\beta|x|+i\beta^2t}+\mathcal{O}\left(\frac{1}{t}\right), \tag{20b}$$

where $1_{\mathbb{R}^-}$ *denotes the characteristic function of the negative half line.*

Remark 3.3 Note that the second terms on the right-hand sides in the asymptotics (20) only appear for the attractive cases $\alpha,\beta<0$. These terms are connected to the negative bound states which are only present in the attractive cases. More precisely, these terms can be seen as the time evolution of the eigenfunctions $\mathrm{e}^{\alpha|x|}$ and $\operatorname{sgn}(x)\mathrm{e}^{\beta|x|}$ corresponding to the eigenvalues $-\alpha^2$ and $-\beta^2$, respectively.

Proof of Theorem 3.2 Since the strategy of the proof is the same for Ψ_δ and $\Psi_{\delta'}$, and since the δ-potential is already treated in [15], we restrict our considerations to the δ'-case. The verification of (15) and (20b) is splitted into four steps.

Step 1: We check that (18b) satisfies the differential equation (15a). Since this is obvious for the free part Ψ_{free}, it suffices to check this for (19b). In fact, it suffices to verify that

$$\phi(t,x;\omega):=\mathrm{e}^{-\frac{x^2}{4it}}\Lambda\left(\frac{|x|}{2\sqrt{it}}+\omega\sqrt{it}\right),\quad t>0,\ x\in\mathbb{R}\setminus\{0\}, \tag{21}$$

is a solution of (15a) for every $\omega\in\mathbb{C}$. With the help of Lemma 3.1 (ii), we first compute

$$\begin{aligned}
\frac{\partial}{\partial t}\phi(t,x;\omega)&=i\mathrm{e}^{-\frac{x^2}{4it}}\left(\omega^2\Lambda\left(\frac{|x|}{2\sqrt{it}}+\omega\sqrt{it}\right)+\frac{1}{it\sqrt{\pi}}\left(\frac{|x|}{2\sqrt{it}}-\omega\sqrt{it}\right)\right),\\
\frac{\partial}{\partial x}\phi(t,x;\omega)&=\operatorname{sgn}(x)\mathrm{e}^{-\frac{x^2}{4it}}\left(\omega\Lambda\left(\frac{|x|}{2\sqrt{it}}+\omega\sqrt{it}\right)-\frac{1}{\sqrt{i\pi t}}\right),\\
\frac{\partial^2}{\partial x^2}\phi(t,x;\omega)&=\mathrm{e}^{-\frac{x^2}{4it}}\left(\omega^2\Lambda\left(\frac{|x|}{2\sqrt{it}}+\omega\sqrt{it}\right)+\frac{1}{it\sqrt{\pi}}\left(\frac{|x|}{2\sqrt{it}}-\omega\sqrt{it}\right)\right).
\end{aligned} \tag{22}$$

Using this with $\omega=\beta$ and $\omega=-ik$ immediately shows that $\varphi_{\delta'}$ and hence $\Psi_{\delta'}$ solve (15a).

Step 2: We verify that the jump conditions (15b) and (15c) are satisfied. For this, we calculate the limits $x\to 0^\pm$ of (21) and (22) and insert them into (19b), to get

$$\begin{aligned}
\varphi_{\delta'}(t,0^\pm;k)&=\pm\frac{\beta\Lambda\left(\beta\sqrt{it}\right)+ik\Lambda\left(-ik\sqrt{it}\right)}{2(\beta+ik)},\\
\frac{\partial}{\partial x}\varphi_{\delta'}(t,0^\pm;k)&=\frac{\beta^2\Lambda\left(\beta\sqrt{it}\right)+k^2\Lambda\left(-ik\sqrt{it}\right)}{2(\beta+ik)}-\frac{1}{2\sqrt{i\pi t}}.
\end{aligned}$$

Plugging this into (18b) gives

$$\Psi_{\delta'}(t,x;k) = \mathrm{e}^{-ik^2t} \mp \frac{ik\beta}{\beta^2+k^2}\Lambda\left(\beta\sqrt{it}\right) \pm \frac{ik}{2}\left(\frac{\Lambda\left(-ik\sqrt{it}\right)}{\beta+ik} + \frac{\Lambda\left(ik\sqrt{it}\right)}{\beta-ik}\right),$$

$$\frac{\partial}{\partial x}\Psi_{\delta'}(t,x;k) = ik\mathrm{e}^{-ik^2t} - \frac{ik\beta^2}{\beta^2+k^2}\Lambda\left(\beta\sqrt{it}\right) + \frac{k^2}{2}\left(\frac{\Lambda\left(-ik\sqrt{it}\right)}{\beta+ik} - \frac{\Lambda\left(ik\sqrt{it}\right)}{\beta-ik}\right).$$

Finally, using property (i) in Lemma 3.1 of $\Lambda(-z)$ leads to

$$\Psi_{\delta'}(t,0^\pm;k) = \mathrm{e}^{-ik^2t}\left(1 \pm \frac{ik}{\beta+ik}\right) \mp \frac{ik\beta}{\beta^2+k^2}\Lambda\left(\beta\sqrt{it}\right) \mp \frac{k^2}{\beta^2+k^2}\Lambda\left(ik\sqrt{it}\right),$$

$$\frac{\partial}{\partial x}\Psi_{\delta'}(t,0^\pm;k) = \beta\left(\frac{ik}{\beta+ik}\mathrm{e}^{-ik^2t} - \frac{ik\beta}{\beta^2+k^2}\Lambda\left(\beta\sqrt{it}\right) - \frac{k^2}{\beta^2+k^2}\Lambda\left(ik\sqrt{it}\right)\right),$$

from which it is clear that (15b) and (15c) are both satisfied.

Step 3: We check the initial condition (15d). In fact, due to the asymptotic behaviour (16), we have $\varphi_{\delta'}(t,x;k) \to 0$ for $t \to 0^+$, and hence

$$\Psi_{\delta'}(0^+,x;k) = \Psi_{\mathrm{free}}(0^+,x;k) = \mathrm{e}^{ikx}.$$

Step 4: In this step, we show the large time asymptotics (20b) of the solution. Note first that for large t, we have

$$\mathrm{Re}\left(\frac{|x|}{2\sqrt{it}} + \beta\sqrt{it}\right) \geq 0, \quad \text{if } \beta > 0,$$
$$\mathrm{Re}\left(\frac{|x|}{2\sqrt{it}} + \beta\sqrt{it}\right) \leq 0, \quad \text{if } \beta < 0.$$

Hence, from (16), we get for every fixed $x \in \mathbb{R}\setminus\{0\}$ the asymptotic behaviour

$$\begin{aligned}\Lambda\left(\frac{|x|}{2\sqrt{it}} + \beta\sqrt{it}\right) &= \mathbb{1}_{\mathbb{R}^-}(\beta)2\mathrm{e}^{\left(\frac{|x|}{2\sqrt{it}}+\beta\sqrt{it}\right)^2} + \frac{1}{\sqrt{\pi}\left(\frac{|x|}{2\sqrt{it}} + \beta\sqrt{it}\right)} + \mathcal{O}\left(\frac{1}{\left|\frac{|x|}{2\sqrt{it}} + \beta\sqrt{it}\right|^2}\right)\\ &= \mathbb{1}_{\mathbb{R}^-}(\beta)2\mathrm{e}^{\left(\frac{|x|}{2\sqrt{it}}+\beta\sqrt{it}\right)^2} + \frac{2\sqrt{it}}{\sqrt{\pi}\,(|x|+2\beta it)} + \mathcal{O}\left(\frac{1}{t}\right).\end{aligned}$$

Similarly, we get the asymptotics

$$\Lambda\left(\frac{|x|}{2\sqrt{it}} - ik\sqrt{it}\right) = \mathbb{1}_{\mathbb{R}^-}(k)2\mathrm{e}^{\left(\frac{|x|}{2\sqrt{it}}-ik\sqrt{it}\right)^2} + \frac{2\sqrt{it}}{\sqrt{\pi}\,(|x|+2kt)} + \mathcal{O}\left(\frac{1}{t}\right)$$

for the second term in (19b). Substituting this into (19b) gives

$$\varphi_{\delta'}(t,x;k) = \mathrm{sgn}(x)\left(\frac{\mathbb{1}_{\mathbb{R}^-}(\beta)\beta}{\beta+ik}\mathrm{e}^{\beta|x|+i\beta^2t} + \frac{\mathbb{1}_{\mathbb{R}^-}(k)ik}{\beta+ik}\mathrm{e}^{-ik|x|-ik^2t} + \frac{|x|\sqrt{it}}{\sqrt{\pi}(|x|+2\beta it)(|x|+2kt)}\right) + \mathcal{O}\left(\frac{1}{t}\right).$$

Since the third summand in this expression is of order $t^{-3/2}$ and hence, in particular, $\mathcal{O}(\frac{1}{t})$, the expansion reduces to

$$\varphi_{\delta'}(t,x;k) = \operatorname{sgn}(x)\left(\frac{1_{\mathbb{R}^-}(\beta)\beta}{\beta+ik}\mathrm{e}^{\beta|x|+i\beta^2 t} + \frac{1_{\mathbb{R}^-}(k)ik}{\beta+ik}\mathrm{e}^{-ik|x|-ik^2 t}\right) + \mathcal{O}\left(\frac{1}{t}\right).$$

Using this in (18b), immediately gives the asymptotics (20b) of $\Psi_{\delta'}$. □

4 Proof of Theorem 1.2

For the proof of Theorem 1.2, some preparatory statements are needed. The following lemma provides a simple algebraic reformulation of the power series of the difference quotient of an entire function.

Lemma 4.1 *Let $F \in \mathcal{H}(\mathbb{C})$ be an entire function, that is, F admits the power series representation*

$$F(z) = \sum_{n=0}^{\infty} f_n z^n, \quad z \in \mathbb{C}, \tag{23}$$

with the coefficients $f_n = \frac{F^{(n)}(0)}{n!}$. Then, for every $a \in \mathbb{C}\backslash\{0\}$, the difference quotient admits the series representation

$$\frac{F(z+a)-F(z)}{a} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{n+m+1}\binom{n+m+1}{m+1} z^n a^m, \quad z \in \mathbb{C}.$$

Proof Inserting the power series (23) into the difference quotient gives

$$\frac{F(z+a)-F(z)}{a} = \sum_{n=1}^{\infty} f_n \frac{(z+a)^n - z^n}{a}. \tag{24}$$

We can now use the binomic formula

$$(z+a)^n = z^n + \sum_{m=1}^{n}\binom{n}{m} z^{n-m}a^m$$

to rewrite the series (24) as

$$\begin{aligned}\frac{F(z+a)-F(z)}{a} &= \sum_{n=1}^{\infty}\frac{f_n}{a}\sum_{m=1}^{n}\binom{n}{m} z^{n-m}a^m \\ &= \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\frac{f_n}{a}\binom{n}{m} z^{n-m}a^m \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_{n+m+1}\binom{n+m+1}{m+1} z^n a^m.\end{aligned}$$

□

The following Lemma 4.2 for the δ-potential and its counterpart Lemma 4.3 for the δ'-potential are further useful ingredients in the proof of Theorem 1.2. Here, Lemma 4.1 is used to construct time evolution operators, which take the initial value of the Schrödinger equation and pointwise give the solutions (18). In the case of a δ-potential, a similar operator was already provided in [15]; however, there the (technical) restriction $|k| < |\alpha|$ (wave vector smaller than the potential strength) appeared, which is avoided here.

Lemma 4.2 *For fixed $x \in \mathbb{R}\backslash\{0\}$ and $t > 0$, there exists a continuous linear operator $U_\delta(t,x) : A_1(\mathbb{C}) \to A_1(\mathbb{C})$, such that (18a) can be represented as*

$$\Psi_\delta(t,x;k) = U_\delta(t,x)\mathrm{e}^{ik\xi}\big|_{\xi=0}. \tag{25}$$

Furthermore, for every $B \geq 0$, there exists $\widetilde{S}_{B,\delta}(t,x)$, continuous in t and x, such that the estimate

$$|U_\delta(t,x)F(\xi)| \leq A\widetilde{S}_{\mathrm{e}B,\delta}(t,x)\mathrm{e}^{\mathrm{e}B|\xi|} \tag{26}$$

holds for every $F \in A_1(\mathbb{C})$ satisfying $|F(\xi)| \leq A\mathrm{e}^{B|\xi|}$.

Proof Since Ψ_δ decomposes into (18a), it is sufficient to ensure (25) and (26) for its components Ψ_{free} and φ_δ. Starting with Ψ_{free}, we can write it as the power series

$$\begin{aligned}\Psi_{\mathrm{free}}(t,x;k) &= \sum_{m=0}^{\infty}\frac{1}{m!}(ikx - ik^2t)^m \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{j=0}^{m}\binom{m}{j}(ikx)^{m-j}(-ik^2t)^j \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{j=m}^{2m}\binom{m}{j-m}(ix)^{2m-j}(-it)^{j-m}k^j \\ &= \sum_{j=0}^{\infty}\sum_{m=\lceil\frac{j}{2}\rceil}^{j}\frac{(ix)^{2m-j}(-it)^{j-m}}{m!}\binom{m}{j-m}k^j,\end{aligned} \tag{27}$$

which coincides with (9). Hence, by Proposition 2.3, there exists a continuous operator $U_{\mathrm{free}}(t,x) : A_1(\mathbb{C}) \to A_1(\mathbb{C})$, such that

$$\Psi_{\mathrm{free}}(t,x;k) = U_{\mathrm{free}}(t,x)\mathrm{e}^{ik\xi}\big|_{\xi=0}. \tag{28}$$

Arguing in the same way as in (27) the respective constant $S_{B,\mathrm{free}}(t,x)$ in the bound (12) can be estimated by

$$S_{B,\mathrm{free}}(t,x) \leq \sum_{j=0}^{\infty}\sum_{m=\lceil\frac{j}{2}\rceil}^{j}\frac{|x|^{2m-j}t^{j-m}}{m!}\binom{m}{j-m}B^j = \sum_{m=0}^{\infty}\frac{(B|x|+B^2t)^m}{m!} = \mathrm{e}^{B|x|+B^2t} =: \widetilde{S}_{B,\mathrm{free}}(t,x). \tag{29}$$

Next, we rearrange the terms of the function φ_δ in (19a) in the form

$$\varphi_\delta(t,x;k) = \frac{\alpha\sqrt{it}}{2}\mathrm{e}^{-\frac{x^2}{4it}}\frac{\Lambda\left(\frac{|x|}{2\sqrt{it}} + \alpha\sqrt{it}\right) - \Lambda\left(\frac{|x|}{2\sqrt{it}} - ik\sqrt{it}\right)}{(\alpha+ik)\sqrt{it}}. \tag{30}$$

It follows that this is a difference quotient of the form (23) with $z = \frac{|x|}{2\sqrt{it}} - ik\sqrt{it}$ and $a = (\alpha+ik)\sqrt{it}$. Hence, we obtain the series expansion

$$\varphi_\delta(t,x;k) = \frac{\alpha\mathrm{e}^{-\frac{x^2}{4it}}}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\sqrt{it})^{n+m+1}}{\Gamma(\frac{n+m+3}{2})}\binom{n+m+1}{m+1}\left(\frac{|x|}{2it} - ik\right)^n(\alpha+ik)^m,$$

where we used the coefficients in the power series representation (17) of Λ. In a similar way as in (27) and (29) for the free part, we can use the binomic formula to further expand this series into the form (9). Hence, we get an operator $U_{\delta,+}(t,x) : A_1(\mathbb{C}) \to A_1(\mathbb{C})$ satisfying

$$\varphi_\delta(t,x;k) = U_{\delta,+}(t,x)\mathrm{e}^{ik\xi}\big|_{\xi=0},$$

with a constant $S_{B,\delta,+}(t,x)$ in the corresponding bound (12). This constant can be estimated by

$$\begin{aligned} S_{B,\delta,+}(t,x) &\leq \frac{|\alpha|}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\sqrt{t})^{n+m+1}}{\Gamma(\frac{n+m+3}{2})}\binom{n+m+1}{m+1}\left(\frac{|x|}{2t}+B\right)^{n}(|\alpha|+B)^{m} \\ &= \frac{|\alpha|}{2}\frac{\Lambda\left(-\frac{|x|}{2\sqrt{t}}-(2B+|\alpha|)\sqrt{t}\right)-\Lambda\left(-\frac{|x|}{2\sqrt{t}}-B\sqrt{t}\right)}{(|\alpha|+B)\sqrt{t}} =: \widetilde{S}_{B,\delta,+}(t,x). \end{aligned}$$

The same reasoning for $\varphi_\delta(t,-x;-k)$ leads to an operator $U_{\delta,-}(t,x): A_1(\mathbb{C}) \to A_1(\mathbb{C})$ and a constant $\widetilde{S}_{B,\delta,-}(t,x)$. Finally, we add all three terms together to get

$$\begin{aligned} U_\delta(t,x) &:= U_{\text{free}}(t,x)+U_{\delta,+}(t,x)+U_{\delta,-}(t,x), \\ \widetilde{S}_{B,\delta}(t,x) &:= \widetilde{S}_{B,\text{free}}(t,x)+\widetilde{S}_{B,\delta,+}(t,x)+\widetilde{S}_{B,\delta,-}(t,x), \end{aligned}$$

which satisfy (25) and (26). □

Although the proof is slightly different, an analogous statement as in Lemma 4.2 holds for the δ'-potential.

Lemma 4.3 *For fixed $x \in \mathbb{R}\backslash\{0\}$ and $t > 0$, there exists a continuous linear operator $U_{\delta'}(t,x): A_1(\mathbb{C}) \to A_1(\mathbb{C})$, such that (18b) can be represented as*

$$\Psi_{\delta'}(t,x;k) = U_{\delta'}(t,x)e^{ik\xi}\big|_{\xi=0}. \tag{31}$$

Furthermore, for every $B \geq 0$, there exists $\widetilde{S}_{B,\delta'}(t,x)$, continuous in t and x, such that the estimate

$$|U_{\delta'}(t,x)F(\xi)| \leq A\widetilde{S}_{eB,\delta'}(t,x)e^{eB|\xi|} \tag{32}$$

holds for every $F \in A_1(\mathbb{C})$ satisfying $|F(\xi)| \leq Ae^{B|\xi|}$.

Proof Since $\Psi_{\delta'}$ decomposes into (18b), it is sufficient to ensure (31) and (32) for its components Ψ_{free} and $\varphi_{\delta'}$. Since Ψ_{free} is already treated in (28) and (29), we only consider $\varphi_{\delta'}$. For this, we first split (19b) into $\varphi_{\delta'} = \varphi_{\delta'}^{(0)} + \varphi_{\delta'}^{(1)}$, where

$$\varphi_{\delta'}^{(0)}(t,x;k) := \frac{\operatorname{sgn}(x)\beta\sqrt{it}}{2}e^{-\frac{x^2}{4it}}\frac{\Lambda\left(\frac{|x|}{2\sqrt{it}}+\beta\sqrt{it}\right)-\Lambda\left(\frac{|x|}{2\sqrt{it}}-ik\sqrt{it}\right)}{(\beta+ik)\sqrt{it}}, \tag{33a}$$

$$\varphi_{\delta'}^{(1)}(t,x;k) := \frac{\operatorname{sgn}(x)}{2}e^{-\frac{x^2}{4it}}\Lambda\left(\frac{|x|}{2\sqrt{it}}-ik\sqrt{it}\right). \tag{33b}$$

By comparing (33a) with (30), we see that, besides the presence of the prefactor $\operatorname{sgn}(x)$, both coincide if α is replaced by β. Hence, the same computations as in then proof of Lemma 4.2 lead to operators $U_{\delta',\pm}^{(0)}(t,x)$ and corresponding constants $\widetilde{S}_{B,\delta',\pm}^{(0)}(t,x)$ for $\varphi_{\delta'}^{(0)}(t,\pm x;\pm k)$.

For the treatment of (33b), we first use (17) to rewrite $\varphi_{\delta'}^{(1)}(t,x;k)$ as the series

$$\varphi_{\delta'}^{(1)}(t,x;k) = \frac{\operatorname{sgn}(x)e^{-\frac{x^2}{4it}}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\frac{n}{2}+1)}\left(\frac{|x|}{2\sqrt{it}}-ik\sqrt{it}\right)^{n}.$$

Proposition 2.3 then gives a continuous operator: $U_{\delta',+}^{(1)}(t,x): A_1(\mathbb{C}) \to A_1(\mathbb{C})$, such that

$$\varphi_{\delta'}^{(1)}(t,x;k) = U_{\delta',+}^{(1)}(t,x)e^{ik\xi}\big|_{\xi=0},$$

as well as a corresponding constant $S^{(1)}_{B,\delta',+}(t,x)$ in the bound (12). The constant can be estimated by

$$S^{(1)}_{B,\delta',+}(t,x) \leq \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{\Gamma(\frac{n}{2}+1)}\left(\frac{|x|}{2\sqrt{t}}+B\sqrt{t}\right)^n = \frac{1}{2}\Lambda\left(-\frac{|x|}{2\sqrt{t}}-B\sqrt{t}\right) =: \widetilde{S}^{(1)}_{B,\delta',+}(t,x).$$

In the same way, we get: $U^{(1)}_{\delta',-}(t,x) : A_1(\mathbb{C}) \to A_1(\mathbb{C})$ and $\widetilde{S}^{(1)}_{B,\delta',-}$ for $\varphi_{\delta'}(t,-x;-k)$. Now, we sum up all terms to end up with

$$\begin{aligned} U_{\delta'}(t,x) &:= U_{\text{free}}(t,x) + U^{(0)}_{\delta',+}(t,x) + U^{(0)}_{\delta',-}(t,x) + U^{(1)}_{\delta',+}(t,x) + U^{(1)}_{\delta',-}(t,x),\\ \widetilde{S}_{B,\delta'}(t,x) &:= \widetilde{S}_{B,\text{free}}(t,x) + \widetilde{S}^{(0)}_{B,\delta',+}(t,x) + \widetilde{S}^{(0)}_{B,\delta',-}(t,x) + \widetilde{S}^{(1)}_{B,\delta',+}(t,x) + \widetilde{S}^{(1)}_{B,\delta',-}(t,x), \end{aligned}$$

and it follows that (31) and (32) are satisfied. □

Now, we are ready to prove Theorem 1.2. Since the argument is the same for the δ-potential and the δ'-potential, we will only consider the δ'-case. From Lemma 4.3, we know that the solution $\Psi_{\delta'}$ of the Schrödinger equation (15) with the initial datum $F(x) = \mathrm{e}^{ikx}$ can be represented in the form (31). Since the operator $U_{\delta'}(t,x)$ and the Schrödinger equation are both linear, we also get

$$\Psi_n(t,x) = U_{\delta'}(t,x)F_n(\xi)\big|_{\xi=0}$$

for every solution Ψ_n of (15) with the initial datum F_n of the form (4).

Let, now, $(F_n)_n$ be the superoscillating sequence from (5). Then, from the definition of the A_1-convergence (6), we get the estimate

$$\left|F_n(\xi) - \mathrm{e}^{ik\xi}\right| \leq A_n \mathrm{e}^{B|\xi|}, \quad \xi \in \mathbb{C},$$

where $B \geq 0$ is as in (6) and $A_n = \sup_{\xi\in\mathbb{C}} |(F_n(\xi) - \mathrm{e}^{ik\xi})\mathrm{e}^{-B|\xi|}| \to 0$ for $n \to \infty$. Consequently, from the estimate (32), we get for any compact $K \subseteq (0,\infty)\times\mathbb{R}$ the uniform convergence

$$\begin{aligned} \sup_{(t,x)\in K}\left|\Psi_n(t,x) - \Psi(t,x)\right| &= \sup_{(t,x)\in K}\left|U_{\delta'}(t,x)\left(F_n(\xi) - \mathrm{e}^{ik\xi}\right)\big|_{\xi=0}\right| \\ &\leq \sup_{(t,x)\in K} A_n \widetilde{S}_{\mathrm{e}B,\delta'}(t,x)\mathrm{e}^{\mathrm{e}B|\xi|}\big|_{\xi=0} \\ &= A_n \sup_{(t,x)\in K}\widetilde{S}_{\mathrm{e}B,\delta'}(t,x) \overset{n\to\infty}{\longrightarrow} 0; \end{aligned}$$

note that $\sup_{(t,x)\in K}\widetilde{S}_{\mathrm{e}B,\delta'}(t,x)$ is finite, which follows from the continuity of $\widetilde{S}_{\mathrm{e}B,\delta'}$; cf. Lemma 4.3. This completes the proof of Theorem 1.2.

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Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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