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Nikolaos Galatos

Peter Jipsen

Michael Kinyon

Adam Přenosil

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Nick Galatos, Peter Jipsen, Michael Kinyon and Adam Přenosil*

Abstract. We prove that the lattice reduct of every lattice-ordered pregroup is semidistributive. This is a consequence of a certain weak form of the distributive law which holds in lattice-ordered pregroups.

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1. Introduction

Lattice-ordered pregroups, or ℓ-pregroups for short, were introduced by Lambek [8], who called them lattice-ordered monoids with adjoints. Their partially ordered counterparts were studied in more detail by Lambek [9, 10] and Buszkowski [1, 2, 3] with linguistic motivations (type grammar) in mind. An ℓ-pregroup is an algebra $\langle G, \wedge, \vee, \cdot, 1, ^{\ell}, ^{r}\rangle$ where $\langle G, \wedge, \vee\rangle$ is a lattice, $\langle G, \cdot, 1\rangle$ is a monoid such that multiplication is order-preserving in both arguments, and the unary maps $x \mapsto x^{\ell}$ and $x \mapsto x^{r}$ satisfy the inequalities

$$x^{\ell}x \leq 1 \leq xx^{\ell} \qquad \text{and} \qquad xx^{r} \leq 1 \leq x^{r}x.$$

Alternatively, they are involutive residuated lattices satisfying $x \cdot y \approx x + y$, where addition is the De Morgan dual of multiplication (see [6]). Imposing the equation $x^{\ell} \approx x^{r}$ on ℓ-pregroups yields the variety of ℓ-groups.

The major open question concerning these algebras is whether their lattice reducts are distributive, like the lattice reducts of ℓ-groups. We leave this question open, however, we describe some positive properties of lattice reducts of ℓ-pregroups. These follow from the fact that the distributive law for ℓ-pregroups holds at least up to certain idempotents.

The variety of ℓ-pregroups exhibits an order duality as well as a left–right duality: if $\langle G, \wedge, \vee, \cdot, 1, ^{\ell}, ^{r}\rangle$ is an ℓ-pregroup, then so are $\langle G, \vee, \wedge, \cdot, 1, ^{r}, ^{\ell}\rangle$ and $\langle G, \wedge, \vee, \odot, 1, ^{r}, ^{\ell}\rangle$, where $x \odot y := y \cdot x$. These symmetries imply that if a

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(quasi)equation holds in all ℓ-pregroups, then so does its order dual, obtained by switching $\vee$ and $\wedge$ as well as ${}^\ell$ and ^{r}, as well as its left–right dual, obtained by switching ${}^\ell$ and ^{r} and reversing the order of multiplication.

We recall that ℓ-pregroups satisfy the following equations:

$$x(y \wedge z) \approx xy \wedge xz, \quad xx^\ell x \approx x, \quad (x \wedge y)^\ell \approx x^\ell \vee y^\ell, \quad (x \vee y)^\ell \approx x^\ell \wedge y^\ell,$$
$$(x \wedge y)z \approx xz \wedge yz, \quad xx^r x \approx x, \quad (x \wedge y)^r \approx x^r \vee y^r, \quad (x \vee y)^r \approx x^r \wedge y^r.$$

Moreover, they also satisfy the equations $x^{\ell r} \approx x \approx x^{r \ell}$.

Let us now recall the definition of semidistributivity. A lattice is called *meet semidistributive* if it satisfies the quasiequation

$$x \wedge y \approx x \wedge z \implies x \wedge (y \vee z) \approx x \wedge y.$$

It is called *join semidistributive* if it satisfies the dual quasiequation, namely

$$x \vee y \approx x \vee z \implies x \vee (y \wedge z) \approx x \vee z.$$

It is called *semidistributive* if it is both meet and join semidistributive. We call an ℓ-pregroup modular or (semi)distributive if its lattice reduct is modular or (semi)distributive.

2. Main results

We now prove an analogue of the distributive law for ℓ-pregroups. The proof given below is the ℓ-pregroup analogue of the proof of distributivity for GBL-algebras due to Galatos & Tsinakis [7, Lemma 2.9].

Proposition 2.1. *The following inequalities hold in all ℓ-pregroups:*

$$x \wedge (y \vee z) \leq yy^\ell(x \wedge y) \vee zz^\ell(x \wedge z),$$
$$x \wedge (y \vee z) \leq (x \wedge y)y^r y \vee (x \wedge z)z^r z.$$

Proof. We only prove the first inequality:

$$\begin{aligned}
x \wedge (y \vee z) &\leq (y \vee z)(y \vee z)^\ell x \wedge (y \vee z) \\
&= (y \vee z)((y^\ell \wedge z^\ell)x \wedge 1) \\
&= y((y^\ell \wedge z^\ell)x \wedge 1) \vee z((y^\ell \wedge z^\ell)x \wedge 1) \\
&\leq y(y^\ell x \wedge 1) \vee z(z^\ell x \wedge 1) \\
&= (yy^\ell x \wedge y) \vee (zz^\ell x \wedge z) \\
&= (yy^\ell x \wedge yy^\ell y) \vee (zz^\ell x \wedge zz^\ell z) \\
&= yy^\ell(x \wedge y) \vee zz^\ell(x \wedge z).
\end{aligned}$$

The second inequality follows by left–right duality. □

The only difference between these inequalities and the usual distributive law is the presence of the idempotents yy^ℓ and zz^ℓ, or $y^r y$ and $z^r z$. For some special instances of x, y, z we obtain the full distributive law.

Corollary 2.2. *Suppose that either $ya = x = zb$ or $ay = x = bz$ holds in an ℓ-pregroup for some a and b. Then $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.*

Proof. In the former case we have $yy^\ell(x \wedge y) = (yy^\ell ya \wedge yy^\ell y) = ya \wedge y = x \wedge y$ and likewise $zz^\ell(x \wedge z) = x \wedge z$. The latter case follows by left–right duality. □

Another form of distributivity will in fact be more useful in our proofs.

Proposition 2.3. *The following inequalities hold in all ℓ-pregroups:*

$$x \wedge (y \vee z) \leq yy^\ell(x \wedge y) \vee z,$$
$$x \wedge (y \vee z) \leq (x \wedge y)y^r y \vee z.$$

Proof. In the former case it suffices to observe that $zz^\ell(x \wedge z) \leq zz^\ell x \wedge zz^\ell z \leq zz^\ell x \wedge z \leq z$. The latter case follows by left–right duality. □

Corollary 2.4. *Suppose that either $ya = x$ or $ay = x$ holds in an ℓ-pregroup for some a. Then $x \wedge (y \vee z) \leq (x \wedge y) \vee z$.*

Proof. In the former case $x \wedge (y \vee z) \leq yy^\ell(x \wedge y) \vee z = (yy^\ell ya \wedge yy^\ell y) \vee z = (ya \wedge y) \vee z = (x \wedge y) \vee z$. The latter case follows by left–right duality. □

We now use this limited form of distributivity to prove that ℓ-pregroups are semidistributive.

Lemma 2.5. *The inequality $x' \wedge (y' \vee z') \leq (x' \wedge y') \vee z'$ holds whenever there are x and y such that one of the following four cases obtains:*

$$\begin{array}{llll} x' = y^\ell x, & x' = xy^\ell, & x' = y^r x, & x' = xy^r, \\ y' = y^\ell y, & y' = yy^\ell, & y' = y^r y, & y' = yy^r. \end{array}$$

Proof. This follows from Corollary 2.4, since $y^\ell yy^\ell = y^\ell$ and $y^r yy^r = y^r$. □

Theorem 2.6. *Each ℓ-pregroup is semidistributive.*

Proof. By order duality it suffices to prove meet semidistributivity, i.e. that $x \wedge y = x \wedge z$ implies $x \wedge (y \vee z) \leq y$. Suppose therefore that $x \wedge y = x \wedge z$ and let $x' = y^\ell x$, $y' = y^\ell y$, and $z' = y^\ell z$. It follows that $x' \wedge y' = x' \wedge z'$.

Lemma 2.5 now implies that $x' \wedge (y' \vee z') \leq (x' \wedge y') \vee z' = (x' \wedge z') \vee z' = z'$, therefore $x' \wedge (y' \vee z') \leq x' \wedge z' = x' \wedge y' \leq y'$. But multiplying the inequality $x' \wedge (y' \vee z') \leq y'$ by y on the left yields that $x \wedge (y \vee z) \leq yy^\ell(x \wedge (y \vee z)) = y(x' \wedge (y' \vee z')) \leq yy' = yy^\ell y = y$. □

Each modular join semidistributive (or meet semidistributive) lattice is in fact distributive: modularity implies that it does not contain the pentagon $\mathbf{N_5}$ as a sublattice, while semidistributivity implies that it does not contain the diamond $\mathbf{M_3}$ as a sublattice.

Corollary 2.7. *Each modular ℓ-pregroup is distributive.*

The problem of determining whether ℓ-pregroups are distributive is therefore equivalent to the problem of determining whether they are modular, i.e. whether some ℓ-pregroup contains the pentagon $\mathbf{N_5}$ as a sublattice.

We can in fact obtain more information about the lattice reducts of ℓ-pregroups with the help of Lemma 2.5, namely that certain non-distributive lattices cannot occur as sublattices of ℓ-pregroups.

Recall that the *monolith* of a subdirectly irreducible algebra is its smallest congruence other than the identity relation.

Definition 2.8. Let $\mathbf{L}$ be a subdirectly irreducible lattice and μ be its monolith. We shall say that μ *involves* a if $\langle a, b\rangle \in \mu$ for some b distinct from a, i.e. if the μ-equivalence class of a is not a singleton. A triple of elements $\langle a, b, c\rangle$ of $\mathbf{L}$ will be called *forbidden* if $a \wedge (b \vee c) \nleq (a \wedge b) \vee c$ and moreover μ involves b. The lattice $\mathbf{L}$ will be called *forbidden* if it contains a forbidden triple.

Theorem 2.9. *Forbidden lattices are not sublattices of any ℓ-pregroup.*

Proof. Let $\mathbf{L}$ be a subdirectly irreducible sublattice of an ℓ-pregroup $\mathbf{G}$ with monolith μ and a forbidden triple $\langle a, b, c\rangle$. Then $\langle b, d\rangle \in \mu$ for some $d \in \mathbf{L}$ distinct from b. We may assume without loss of generality that either $d > b$ or $d < b$. Suppose first that $d > b$.

We use $\lambda_y \colon \mathbf{L} \to \mathbf{G}$ to denote the left multiplication map $\lambda_y \colon x \mapsto yx$ and $\rho_y \colon \mathbf{L} \to \mathbf{G}$ to denote the right multiplication map $\rho_y \colon x \mapsto xy$. Recall that these maps are lattice homomorphisms.

Firstly, observe that $\lambda_{bb^\ell} \colon \mathbf{L} \to \mathbf{G}$ is a lattice embedding: if it were not, then $b = \lambda_{bb^\ell} b = \lambda_{bb^\ell} d \geq d$, since $\langle b, d\rangle \in \mu$. It follows that the map $\lambda_{b^\ell} \colon \mathbf{L} \to \mathbf{G}$ is also a lattice embedding, since $\lambda_{bb^\ell} = \lambda_b \circ \lambda_{b^\ell}$.

Lemma 2.5 states that $\lambda_{b^\ell} a \wedge (\lambda_{b^\ell} b \vee \lambda_{b^\ell} c) \leq (\lambda_{b^\ell} a \wedge \lambda_{b^\ell} b) \vee \lambda_{b^\ell} c$. Since λ_{b^ℓ} is a lattice embedding, it follows that $a \wedge (b \vee c) \leq (a \wedge b) \vee c$, contrary to the hypothesis that $\langle a, b, c\rangle$ is a forbidden triple.

If instead of $d > b$ we have $d < b$, we use the map $\rho_{b^\ell b}$ instead of λ_{bb^ℓ} to show that $\rho_{b^\ell} \colon \mathbf{L} \to \mathbf{G}$ is a lattice embedding. Then again $\rho_{b^\ell} a \wedge (\rho_{b^\ell} b \vee \rho_{b^\ell} c) \leq (\rho_{b^\ell} a \wedge \rho_{b^\ell} b) \vee \rho_{b^\ell} c$ by Lemma 2.5, hence $a \wedge (b \vee c) \leq (a \wedge b) \vee c$ using the fact that ρ_{b^ℓ} is a lattice embedding. $\square$

Corollary 2.10. *A simple non-distributive lattice cannot occur as a sublattice of an ℓ-pregroup.*

It is not immediately obvious that this corollary does not follow directly from semidistributivity by some lattice-theoretic argument. For example, the only simple semidistributive lattice with a greatest (or least) element is the two-element chain (see [4]), therefore the corollary does not provide any new information about which lattices with a greatest (or least) element occur as sublattices of ℓ-pregroups. Nevertheless, it is indeed not a direct consequence of semidistributivity: Freese & Nation [4] managed to construct a simple semidistributive lattice which is not distributive.

Finally, let us show that in ℓ-pregroups only powers of positive elements are positive, a fact which is well known in the case of ℓ-groups. The argument in fact applies to each lattice-ordered monoid satisfying $x \approx (1 \wedge x)(1 \vee x)$ where products distribute over joins and meets. The fact that each ℓ-pregroup satisfies this equation was proved in [5, Lemma 1].

Proposition 2.11. *In every ℓ-pregroup $1 \wedge x^n \leq x$ holds for each $n \geq 1$.*

Proof. We first observe that $1 \wedge y \leq x(1 \vee x)^m$ if and only if $1 \wedge y \leq x(1 \vee x)^{m+1}$ for all $m \geq 0$ (where $z^0 := 1$ for each z):

$$\begin{aligned} 1 \wedge y \leq x(1 \vee x)^m &\iff 1 \wedge y \leq (1 \wedge x)(1 \vee x)(1 \vee x)^m \\ &\iff 1 \wedge y \leq (1 \wedge x)(1 \vee x)^{m+1} \\ &\iff 1 \wedge y \leq (1 \vee x)^{m+1} \wedge x(1 \vee x)^{m+1} \\ &\iff 1 \wedge y \leq (1 \vee x)^{m+1} \text{ and } 1 \wedge y \leq x(1 \vee x)^{m+1} \\ &\iff 1 \wedge y \leq x(1 \vee x)^{m+1}. \end{aligned}$$

It follows that $1 \wedge x^n \leq x$ holds if and only if $1 \wedge x^n \leq x(1 \vee x)^{n-1}$. But $1 \wedge x^n \leq x^n \leq xx^{n-1} \leq x(1 \vee x)^{n-1}$. $\square$

Corollary 2.12. *Let $n \geq 1$. In every ℓ-pregroup $1 \leq x^n$ if and only if $1 \leq x$.*

This yields an alternative proof of the following known fact.

Corollary 2.13. *In every ℓ-pregroup $1 \leq x \vee x^\ell$.*

Proof. By the previous corollary it suffices to prove that $1 \leq (x \vee x^\ell)^2$: $1 \leq xx^\ell \leq xx \vee xx^\ell \vee x^\ell x \vee x^\ell x^\ell = (x \vee x^\ell)^2$. $\square$

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Nick Galatos
Department of Mathematics
University of Denver
Denver, CO 80208
USA
URL: `https://cs.du.edu/~ngalatos/`
e-mail: `ngalatos@du.edu`

Peter Jipsen
Mathematics
Chapman University
Orange, CA 92866
USA
URL: `https://www1.chapman.edu/~jipsen/`
e-mail: `jipsen@chapman.edu`

Michael Kinyon
Department of Mathematics
University of Denver
Denver, CO 80208
USA
URL: `https://cs.du.edu/~mkinyon/`
e-mail: `mkinyon@du.edu`

Adam Přenosil*
Department of Mathematics
Vanderbilt University
Nashville, TN 37240
USA
URL: `https://sites.google.com/site/adamprenosil`
e-mail: `adam.prenosil@vanderbilt.edu`