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On the extension of positive definite kernels to topological algebras

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We define an extension of operator-valued positive definite functions from the real or complex setting to topological algebras, and describe their associated reproducing kernel spaces. The case of entire functions is of special interest, and we give a precise meaning to some power series expansions of analytic functions that appears in many algebras.

I. INTRODUCTION

It is often of interest in some areas of mathematics to consider the extension of the domain (and, in general, the range) of analytic functions from the field of complex or real numbers, here denoted by $\mathbb{K}$, to an algebra $\mathcal{A}$ over $\mathbb{K}$. More precisely, if f is analytic in a neighborhood $\Omega_{\mathbb{K}}$ of z, with Taylor expansion

$$f(z+h)=\sum_{n=0}^{\infty} h^n \frac{f^{(n)}(z)}{n!}, \tag{1.1}$$

where h is a complex number in the open disk of convergence centered at z, one formally defines

$$f(z+A)=\sum_{n=0}^{\infty} A^n \frac{f^{(n)}(z)}{n!}, \tag{1.2}$$

where $A \in \mathcal{A}\setminus\mathbb{K}$. Note that the restriction of $A \in \mathcal{A}\setminus\mathbb{K}$ is made just to keep Eq. (1.2) familiar. In fact, one could consider the change $A \mapsto A+h$, where $h \in \mathbb{K}$ is such that $z+h \in \Omega_{\mathbb{K}}$.

This type of extension is used many areas, like in supermathematics, and even in the theory of linear stochastic systems[1,2] and in the associated theory of strong algebras[3,4]. Moreover, extensions given by Eq. (1.2) contrasts with the one done, for instance, in the study of white noise space, where, with exception of[5], the complex coefficients of the power series (and not the variable) in Eq. (1.1) are replaced by elements that take value in the space of stochastic distributions S_{-1}.

In some cases, if the functions of interest in Eq. (1.1) form a reproducing kernel Hilbert space, the extension of their kernel can be straightforward. Let, for instance, $\mathcal{A}$ be the Grassmann algebra with a finite number, say N, of generators. In this case, a number $z \in \mathcal{A}$ can be written as

$$z = z_B + z_S,$$

where $z_B \in \mathbb{C}$ is called the body of z, and $z_S \in \mathcal{A}\setminus\mathbb{C}$ is such that $z_S^{N+1}=0$ and is called the soul of z. Then, one has

$$\mathscr{K}_N(z_B+z_S, w_B+w_S)=\begin{pmatrix}1 & z_S & \frac{z_S^2}{2!} \cdots & \frac{z_S^N}{N!}\end{pmatrix} K_N(z_B,w_B)\begin{pmatrix}1\\ \overline{w_S}\\ \frac{\overline{w_S}^2}{2!}\\ \vdots\\ \frac{\overline{w_S}^N}{N!}\end{pmatrix}, \tag{1.3}$$

where K_N is the $(N+1)\times(N+1)$ matrix function with (n,m) entry equals to

$$\frac{1}{n!m!}\frac{\partial^{n+m}K_N(z,w)}{\partial^n\partial^m z^n\overline{w}^m}. \tag{1.4}$$

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Eq. (1.3) can be written in this simple form because of the nilpotence of the soul of z, i.e., because $z_S^{N+1} = 0$.

In general, for a Grassmann algebra with an infinite number of generators or for other arbitrary algebras, an expression similar to Eq. (1.3) is desirable. However, this requires a more careful analysis — for instance, with the study of convergence.

The present work focus on functions given by Eq. (1.2) whenever $\mathcal{A}$ is a topological algebra, i.e., $\mathcal{A}$ is a locally convex topological vector space and the product ab is separately continuous in each of the variables — see Ref.[6] for more details. Because of that, the convergence of Eq. (1.2) is assumed to be in the topology of $\mathcal{A}$.

Our objective is, in case the original analytic functions f presented in Eq. (1.1) form a reproducing kernel Hilbert space, to introduce the structure of the corresponding space of extended functions given by Eq. (1.2) and the extension of the underlying operators in this scenario.

In a sense, our approach can be seeing as a reduction to the complex (or real) case. The reason is that, even though they are desirable, expressions like Eq. (1.3) do not seem to always exist. In general, we do not obtain a closed form for the kernel of functions f presented in Eq. (1.2). Then, we study objects of the following type instead

$$F(a, z, A) = \langle a, f(z + A)\rangle = \sum_{n=0}^{\infty} \langle a, (z + A)^n\rangle \frac{f^{(n)}(z)}{n!}, \tag{1.5}$$

where a belongs to the topological dual $\mathcal{A}'$. In other words, we replace the powers $(z+A)^n$ by $\langle a, (z+A)^n\rangle$ for every n. The exchange of order between the sum and the duality operation is justified because convergence in the topological algebra implies weak convergence.

To introduce the underlying reproducing kernel Hilbert space, more definitions are required. However, we already remark that, for the Fock space, with reproducing kernel given by

$$e^{z\overline{w}} = \sum_{n=0}^{\infty} \frac{z^n \overline{w}^n}{n!}, \tag{1.6}$$

the reproducing kernel of the space of extended functions is

$$\sum_{n=0}^{\infty} \frac{\langle a, (z + A)^n\rangle \overline{\langle b, (w + B)^n\rangle}}{n!}. \tag{1.7}$$

In the case of $\mathbb{K} = \mathbb{C}$ and $\mathcal{A} = \mathbb{C}^{n\times n}$, the kernel in Eq. (1.7) becomes

$$K(a, A, z, b, B, w) = \operatorname{Tr}\left((a^* \otimes b)e^{(zI_n+A)\otimes(\overline{w}I_n+B^*)}\right), \quad a, A, b, B \in \mathbb{C}^{n\times n}. \tag{1.8}$$

For yet another example, we take $\mathcal{A}$ to be the quaternions $\mathbb{H}$ and $\mathbb{K}$ to be the real numbers. Then, Eq. (1.7) becomes

$$\sum_{n=0}^{\infty} \frac{(\operatorname{Re} \overline{a}(t + p)^n)\,(\operatorname{Re}\,(\overline{q} + s)^n b)}{n!}. \tag{1.9}$$

Even though the above expression is similar, it is not equivalent to the reproducing kernel of the Fock space of slice hyperholomorphic functions[7], which is

$$\sum_{n=0}^{\infty} \frac{(t + p)^n(\overline{q} + s)^n}{n!}.$$

Although the real part is taken in Eq. (1.9), observe that non-real parts of the variables also play a role in the kernel. This can be seeing by varying the parameters a and b.

Besides this introduction, this paper contains four sections. In Section II, we formalize our definition of reproducing kernel Hilbert spaces associated to extensions of functions of the type given by Eq. (1.2). After that, the case of entire functions is considered in Section 3. Then, the definitions and results are generalized to arbitrary analytic functions in Section 4. Finally, extensions of operators are considered in the last section.

II. EXTENSION OF KERNELS TO TOPOLOGICAL ALGEBRAS

In this work, we assume that the algebra and its topological dual are endowed with involutions (for simplicity denoted by the same symbol) $A \mapsto A^*$ and $a \mapsto a^*$, which extend the complex conjugation, keep the algebraic structure, and satisfy

$$\overline{\langle a, A \rangle} = \langle a^*, A^* \rangle, \quad A \in \mathcal{A} \quad \text{and} \quad a \in \mathcal{A}'. \tag{2.1}$$

In particular, choosing $A = cI$, where $c \in \mathbb{K}$, we have

$$\langle a, A^* \rangle = \overline{c} \cdot \langle a, I \rangle. \tag{2.2}$$

Before focusing on the case of entire functions, consider the case of a positive definite $\mathbf{B}(\ell_2(\mathbb{N}_0))$-valued kernel and $\Omega_{\mathbb{K}} \subset \mathbb{K}$. By definition of an operator-valued positive definite function, the $\mathbb{K}$-valued function

$$\mathcal{K}((z, f), (w, e)) = \langle K(z, w)e, f \rangle_{\ell_2(\mathbb{N}_0)} \tag{2.3}$$

is positive definite on $\Omega_{\mathbb{K}} \times \ell_2(\mathbb{N}_0)$. Note that Eq. (2.3) can be rewritten as

$$\mathcal{K}((z, f), (w, e)) = \sum_{n,m=0}^{\infty} \overline{e_n} k_{nm}(z, w) f_m, \tag{2.4}$$

where $(k_{nm}(z, w))_{n,m=0}^{\infty}$ is the matrix representation of $K(z, w)$ with respect to the standard basis of $\ell_2(\mathbb{N}_0)$, and where the elements of $\ell_2(\mathbb{N}_0)$ are written as semi-infinite column vectors.

We, then, extend $\mathcal{K}$ — or, more precisely, Eq. (2.4) — to the domain Ω in the following way:

$$\begin{aligned} \mathcal{K}((z, (A_n)_{n=0}^{\infty}, a), (w, (B_m)_{n=0}^{\infty}, b)) &= \sum_{n,m=0}^{\infty} \overline{\langle a, A_n^* \rangle} k_{nm}(z, w) \langle b, B_m^* \rangle \\ &= \sum_{n,m=0}^{\infty} \langle a^*, A_n \rangle k_{nm}(z, w) \overline{\langle b^*, B_m \rangle}. \end{aligned} \tag{2.5}$$

In this expression, a and b belong to the topological dual $\mathcal{A}'$ of the algebra $\mathcal{A}$ and the brackets denote the duality between $\mathcal{A}$ and $\mathcal{A}'$. Moreover, $(A_n)_{n=0}^{\infty}$ and $(B_m)_{m=0}^{\infty}$ are sequences of elements indexed by $\mathbb{N}_0$. If the entries k_{nm} are matrix-valued, say in $\mathbb{C}^{p \times p}$, we take the A_n and the B_m to be in $\mathcal{A}^{1 \times p}$. The duality expressions $\langle a, A_n^* \rangle$ and $\langle b, B_m^* \rangle$ are, then, in $\mathbb{C}^{1 \times p}$.

The function given by Eq. (2.5) is well-defined and positive definite on the set $\Omega \equiv \Omega_{\mathbb{K}} \times \Omega_{\mathcal{A}}$, with

$$\Omega_{\mathcal{A}} = \{(a, (A_n^*)_{n=0}^{\infty}) \text{ such that } (\langle a, A_n^* \rangle)_{n=0}^{\infty} \in \ell_2(\mathbb{N}_0)\}$$

and is the starting point of our study.

Remark 2.1. In principle, one can replace in Eq. (2.4) the $\mathbb{K}$ numbers e_n and f_m by elements in this algebra, and the complex conjugation by the conjugation in $\mathcal{A}$ to get the expression

$$M((z, (A_n)_{n=0}^{\infty}), (w, (B_m)_{m=0}^{\infty})) = \sum_{n,m=0}^{\infty} A_n k_{nm}(z, w) B_m^*, \tag{2.6}$$

where the sequences $\mathbf{A} = (A_n)_{n=0}^{\infty}$ and $\mathbf{B} = (B_m)_{m=0}^{\infty}$ are chosen such that Eq. (2.6) converges. When $\mathcal{A}$ is an algebra of Hilbert operators, or a C^*-algebra, one can define positivity for Eq. (2.6), but, in general, it is not so clear in which sense (2.6) defines a positive definite function, when convergent.

For general topological algebras, we define positivity using the $\mathbb{K}$-valued function in Eq. (2.5). For $(a, (A_u)) \in \Omega_{\mathcal{A}}$, we denote by $X(a, \mathbf{A})$ the $\ell_2(\mathbb{N}_0)$ element with u-component $\langle a, A_u^* \rangle$

$$X(a, \mathbf{A}) = \begin{pmatrix} \langle a, A_0^* \rangle \\ \langle a, A_1^* \rangle \\ \langle a, A_2^* \rangle \\ \vdots \end{pmatrix}. \tag{2.7}$$

Moreover, we let

$$K(z,w)=\Gamma(z)\Gamma(w)^* \tag{2.8}$$

be a minimal factorization of the $\mathbf{B}(\ell_2(\mathbb{N}_0))$-valued kernel $K(z,w)$ via a Hilbert space $\mathcal{G}$, meaning that $\Gamma(z)\in\mathbf{B}(\mathcal{G},\ell_2(\mathbb{N}_0))$ for every $z\in\Omega_{\mathbb{K}}$ and that the linear span of the range of the operators $\Gamma(w)^*$ is dense in $\mathcal{G}$, as w runs through $\Omega_{\mathbb{K}}$. One can, for instance, choose for $\mathcal{G}$ the reproducing kernel Hilbert space $\mathcal{H}(K)$ of $\ell_2(\mathbb{N}_0)$-valued functions with reproducing kernel $K(z,w)$ and

$$(\Gamma(w))(f)=f(w),\quad f\in\mathcal{H}(K).$$

Then,

$$\Gamma^*(w)\xi=K(\cdot,w)\xi,\quad \xi\in\ell_2(\mathbb{N}_0)$$

and the next proposition is immediate:

Proposition 2.2. *The factorization*

$$\langle K(z,w)X(b,\mathbf{B}),X(a,\mathbf{A})\rangle_{\ell_2(\mathbb{N}_0)}=\langle\Gamma(w)^*X(b,\mathbf{B}),\Gamma(z)^*X(a,\mathbf{A})\rangle_{\mathcal{G}},$$

holds and the reproducing kernel Hilbert space associated to Eq. (2.5) *consists of functions of the form*

$$F(z,\mathbf{A},a)=\langle f,\Gamma(z)^*X(a,\mathbf{A})\rangle_{\mathcal{G}},\quad f\in\mathcal{G}, \tag{2.9}$$

with inner product and norm induced from the inner product and the norm of $\mathcal{G}$.

We, now, introduce the matrix representation

$$\Gamma(z)f=\begin{pmatrix}f_0(z)\\ f_1(z)\\ \vdots\end{pmatrix} \tag{2.10}$$

of $\Gamma(z)$, and we associate to Eq. (2.9) the $\mathcal{A}$-valued function

$$F(z,\mathbf{A})=\sum_{n=0}^{\infty}A_nf_n(z). \tag{2.11}$$

Proposition 2.3. *The series* (2.11) *is weakly convergent on the set of sequences $\mathbf{A}=(A_u)$ such that $X(a,\mathbf{A})\in\ell_2(\mathbb{N}_0)$ for all $a\in\mathcal{A}'$.*

Proof. Starting from (2.9), we have:

$$\begin{aligned}\langle f,\Gamma(z)^*X(a,\mathbf{A})\rangle_{\mathcal{G}}&=\langle\Gamma(z)f,X(a,\mathbf{A})\rangle_{\ell_2(\mathbb{N}_0)}\\&=\sum_{n=0}^{\infty}\langle a^*,A_n\rangle f_n(z)\\&=\lim_{N\to\infty}\langle a^*,\sum_{n=0}^{N}A_nf_n(z)\rangle.\end{aligned}$$

□

We are interested in the special case $f_n(z)=\dfrac{f^{(n)}(z)}{n!}$ and $A_n=(A^*)^n$. We, then, have the condition

$$\sum_{n=0}^{\infty}\Big|\langle a^*,A^n\rangle\Big|^2<\infty \tag{2.12}$$

to insure that $X(a,A)\in\ell_2(\mathbb{N})$.

In the next section, after introducing our approach for a general topological algebra, we explore two cases of special interest: strong algebras and Banach algebras.

III. ANALYTIC KERNELS FOR ENTIRE FUNCTIONS

Let $K(z,w)$ be a $\mathbb{K}^{p\times p}$-valued kernel, positive definite for $z,w\in\Omega_{\mathbb{K}}\subset\mathbb{K}$, and analytic in the variables z and $\overline{w}$. Also, let $\mathcal{H}(K)$ denote the associated reproducing kernel Hilbert space with reproducing kernel K. Recall, see Ref.[8], that the elements of $\mathcal{H}(K)$ are, then, analytic in $\Omega_{\mathbb{K}}$ and that, for every $n\in\mathbb{N}_0$, $w\in\Omega_{\mathbb{K}}$, and $\eta\in\mathbb{K}^p$, the function

$$D_{n,w}\eta\,:\,z\mapsto\frac{1}{n!}\frac{\partial^n K(z,w)\eta}{\partial\overline{w}^n}\;\in\;\mathcal{H}(K). \tag{3.1}$$

Furthermore,

$$\langle f,D_{n,w}\eta\rangle_{\mathcal{H}(K)}=\frac{\eta^* f^{(n)}(w)}{n!},\quad\forall f\in\mathcal{H}(K). \tag{3.2}$$

In particular,

$$\langle D_{m,w}\xi\,,\,D_{n,z}\eta\rangle_{\mathcal{H}(K)}=\frac{1}{m!n!}\eta^*\frac{\partial^{n+m}K(z,w)}{\partial z^n\partial\overline{w}^m}\xi=\eta^*\mathscr{K}_{n,m}(z,w)\xi, \tag{3.3}$$

where $z,w\in\Omega_{\mathbb{K}}$ and $\xi,\eta\in\mathbb{K}^p$, and $\mathscr{K}_{n,m}(z,w)$ has been defined in Eq. (1.4).
We denote as vectors

$$\begin{pmatrix}\xi_0\\ \xi_1\\ \xi_2\\ \vdots\end{pmatrix} \tag{3.4}$$

the elements of $\ell_2(\mathbb{N}_0,\mathbb{K}^p)$, i.e., the sequences of elements of $\mathbb{K}^p$ such that $\sum_{n=0}^\infty\|\xi_n\|^2<\infty$. Also, we let f be a $\mathbb{C}^p$-valued function analytic in $\Omega_{\mathbb{K}}\subset\mathbb{K}$. Then,

$$J_z(f)\stackrel{\text{def.}}{=}\begin{pmatrix}f(z)\\ f^{(1)}(z)\\ \frac{f^{(2)}(z)}{2!}\\ \frac{f^{(3)}(z)}{3!}\\ \vdots\end{pmatrix},\quad z\in\Omega_{\mathbb{K}}, \tag{3.5}$$

is called the jet function generated by f — see Ref.[9] (p. 222). We denote by $J(f)$ the function

$$z\mapsto J_z(f). \tag{3.6}$$

Now, we focus on the case of entire functions, i.e., $\Omega_{\mathbb{K}}=\mathbb{K}$.

Lemma 3.1. *Let $K(z,w)$ be a $\mathbb{K}^{p\times p}$-valued positive definite kernel, entire in z and $\overline{w}$, with associated reproducing kernel Hilbert space $\mathcal{H}(K)$. Also, let $f\in\mathcal{H}(K)$. Then, for all $z\in\mathbb{K}$, the operator $J_z\in\mathbf{B}(\mathcal{H}(K),\ell_2(\mathbb{N}_0,\mathbb{K}^p))$, and its adjoint is given by*

$$J_z^*(u)=\sum_{n=0}^\infty\left(\frac{1}{n!}\frac{\partial^n K(\cdot,w)}{\partial\overline{w}^n}\Big|_{w=z}\right)u_n. \tag{3.7}$$

Proof. The elements of $\mathcal{H}(K)$ are entire, and so for every $z\in\mathbb{K}$, the series

$$f(z+1)=\sum_{n=0}^\infty 1^n\frac{f^{(n)}(z)}{n!}$$

converges in norm, which implies that

$$\sum_{n=0}^{\infty}\|\frac{f^{(n)}(z)}{n!}\|^2<\infty.$$

The computation of J_z^* goes as follows: with $u=(u_n)_{n=0}^{\infty}\in\ell_2(\mathbb{N}_0,\mathbb{K}^p)$ and $f\in\mathcal{H}(K)$,

$$\begin{aligned}\langle J_z(f),u\rangle_{\ell_2(\mathbb{N}_0)}&=\sum_{n=0}^{\infty}u_n^*\frac{f^{(n)}(z)}{n!}\\&=\sum_{n=0}^{\infty}\langle f,D_{n,z}u_n\rangle_{\mathcal{H}(K)}\\&=\langle f,J_z^*(u)\rangle_{\ell_2(\mathbb{N}_0)}.\end{aligned}$$

$\square$

Theorem 3.2. *Let $K(z,w)$ be a $\mathbb{K}^{p\times p}$-valued function, entire in z and $\overline{w}$. Then:*(1) *For every pair $(z,w)\in\mathbb{K}^2$, the semi-infinite block matrix*

$$\mathscr{K}_{n,m}(z,w)=\frac{1}{m!n!}\frac{\partial^{n+m}K(z,w)}{\partial z^n\partial\overline{w}^m},\quad n,m=0,1,\ldots \tag{3.8}$$

defines a bounded operator, which is denoted by $\mathscr{K}(z,w)$, from $\ell_2(\mathbb{N}_0,\mathbb{K}^p)$ into itself.
(2) *The operator $\mathscr{K}(w,w)$ is Hermitian if the kernel $K(z,w)$ is Hermitian.*
(3) *The $\mathbf{B}(\ell_2,\ell_2)$-valued function $\mathscr{K}(z,w)$ is positive definite in $\mathbb{K}$ if the kernel $K(z,w)$ is positive definite in $\mathbb{K}$.*
(4) *Assume the kernel $K(z,w)$ is positive definite in $\mathbb{K}$. Then,*

$$\mathscr{K}(z,w)=J_zJ_w^*. \tag{3.9}$$

Proof. Let $(z,w)\in\mathbb{K}^2$. The power series expansion

$$K(z+M,w+M)=\sum_{n,m=0}^{\infty}\mathscr{K}_{n,m}(z,w)M^{n+m} \tag{3.10}$$

converges for every $M>0$, and, in particular, there is a positive number $C=C(z,w,M)$ such that

$$|\mathscr{K}_{n,m}(z,w)M^{n+m}|\leq C<\infty,\quad\forall n,m=0,1,\ldots \tag{3.11}$$

Now, let $u=(u_n)_{n=0}^{\infty}$ and $v=(v_n)_{n=0}^{\infty}$ be two sequences in $\ell_2(\mathbb{N}_0,\mathbb{K}^p)$. Then, for $M>1$, and using the Cauchy-Schwartz inequality,

$$\|\sum_{n=0}^{\infty}\frac{u_n}{M^n}\|\leq\left(\sum_{n=0}^{\infty}\|u_n\|^2\right)^{1/2}\left(\sum_{n=0}^{\infty}M^{-2n}\right)^{1/2}=\frac{\|u\|}{\sqrt{1-\frac{1}{M^2}}}.$$

Also, a similar result holds for v. Then, we have

$$|u_n^*\mathscr{K}_{n,m}(z,w)v_m|\leq\frac{\|u_n\|}{M^n}C(z,w,M)\frac{\|v_m\|}{M^m}.$$

Finally,

$$|\sum_{n,m=0}^{\infty}u_n^*\mathscr{K}_{n,m}(z,w)v_m|\leq\frac{C}{1-\frac{1}{M^2}}\|u\|\cdot\|v\|.$$

$\square$

As a consequence of the previous theorem, we can write:

Corollary 3.3. *Let* $\eta=(\eta_m)_{m=0}^{\infty}\in\ell_2(\mathbb{N}_0,\mathbb{K}^p)$ *and* $w\in\mathbb{K}$. *Then,*

$$\sum_{m=0}^{\infty}D_{m,w}\eta_m\ \in\ \mathcal{H}(K) \tag{3.12}$$

and

$$\mathscr{K}(z,w)\eta=\begin{pmatrix}(\sum_{m=0}^{\infty}D_{m,w}\eta_m)(z)\\(\sum_{m=0}^{\infty}D_{m,w}\eta_m)^{(1)}(z)\\\frac{1}{2!}(\sum_{m=0}^{\infty}D_{m,w}\eta_m)^{(2)}(z)\\\vdots\end{pmatrix}. \tag{3.13}$$

Proof. The first claim comes from

$$\langle\eta,\mathscr{K}(w,w)\eta\rangle_{\ell_2(\mathbb{N}_0)}=\|\sum_{m=0}^{\infty}D_{m,w}\eta_m\|^2_{\mathcal{H}(K)}.$$

By Eq. (3.2),

$$\frac{1}{n!}\left(\sum_{m=0}^{\infty}D_{m,w}\eta_m\right)^{(n)}=\langle\sum_{m=0}^{\infty}D_{m,w}\eta_m\,,\,D_{n,z}\rangle_{\mathcal{H}(K)}.$$

However, this is the n-th entry of $\mathscr{K}(z,w)\eta$, as can be seen from Eq. (3.3). □

Theorem 3.4. *Assume* $K(z,w)$ *is a* $\mathbb{K}^{p\times p}$*-valued positive definite function in* $\mathbb{K}$ *and entire in the variables* z *and* $\overline{w}$. *The reproducing kernel Hilbert space associated to* $\mathscr{K}(z,w)$ *is the space of jet functions generated by the elements of* $\mathcal{H}(K)$, *with inner product*

$$\langle J(f),J(g)\rangle_{\mathcal{H}(\mathscr{K})}=\langle f,g\rangle_{\mathcal{H}(K)}. \tag{3.14}$$

Proof. Using Eq. (3.13), we have

$$\begin{aligned}\langle J(f)(\cdot),\mathscr{K}(\cdot,w)\eta\rangle_{\mathcal{H}(\mathscr{K})}&=\langle f,\sum_{m=0}^{\infty}D_{m,w}\eta_m\rangle_{\mathcal{H}(K)}\\&=\sum_{m=0}^{\infty}\eta_m^*\frac{f^{(m)}(w)}{m!}\\&=\langle J(f)(w),\eta\rangle_{\ell_2(\mathbb{N}_0,\mathbb{C}^p)}.\end{aligned} \tag{3.15}$$

□

We, now, present two special cases.

Strong algebras

The notion of strong algebra was introduced in Refs.[3,4]. It originated from a space of stochastic distributions defined by Y. Kondratiev[10,11] and a related inequality proved by Våge[12,13].

Definition 3.5. *An algebra* $\mathcal{A}$ *which is an inductive limit of a family of Banach spaces* $\{\mathcal{B}_t\,:\,t\in T\}$ *directed under inclusion is called a strong algebra if, for every* $t\in T$, *there exists* $h(t)\in T$ *such that, for every* $s\geq h(t)$, *there exists a positive constant* $c_{s,t}$ *such that, for every* $A\in\mathcal{B}_t$ *and* $B\in\mathcal{B}_s$, *the products* AB *and* BA *belong to* $\mathcal{B}_s$ *and*

$$\|AB\|_s\leq c_{s,t}\|A\|_t\cdot\|B\|_s\quad\text{and}\quad\|BA\|_s\leq c_{s,t}\|A\|_t\cdot\|B\|_s. \tag{3.16}$$

Let $A \in \mathcal{B}_t \subset \mathcal{A}$, where $\mathcal{A}$ is a strong algebra. Also, let $s = h(t)$ and $d_t = c_{h(t),t}$. An easy induction shows that

$$\|A^n\|_{h(t)} \leq d_t^{n-1} \|A\|_t^n, \quad n = 1, 2, \ldots \tag{3.17}$$

Thus, for $a \in \mathcal{A}'$, we have

$$|\langle a, A^n \rangle| \leq \|a\|' \cdot d_t^{n-1} \|A\|_t^n, \quad n = 1, 2, \ldots \tag{3.18}$$

Hence, for $a \in \mathcal{A}'$,

$$|\langle a, A^n \rangle| \leq \frac{\|a\|'}{d_t} \left(c_t \|A\|_t\right)^n, \quad n = 1, 2, \ldots \tag{3.19}$$

Therefore, the following theorem holds:

Theorem 3.6. *In a strong algebra $\mathcal{A}$, the power series given by Eq.* (1.2)

$$f(z + A) = \sum_{n=0}^{\infty} A^n \frac{f^{(n)}(z)}{n!}$$

converges for every $A \in \mathcal{A}$. In particular, Eq. (2.12) *holds for all $a \in \mathcal{A}'$ and all $A \in \mathcal{A}$.*

Proof. This follows from Eq. (3.18). □

Banach algebras

The case of Banach algebras is much simpler than that of strong algebras. Indeed, when $\mathcal{A}$ is a Banach algebra, it and its dual are endowed with a norm, denoted by $\|\cdot\|$ and $\|\cdot\|'$, respectively, and we have

$$|\langle a, A^n \rangle| \leq \|a\|' \cdot \|A\|^n.$$

Then, a version of Theorem 3.6 also holds in this case.

IV. GENERAL ANALYTIC KERNELS

We, now, consider the case of kernels whose domain of analyticity in z and $\overline{w}$ is not necessarily the entire set $\mathbb{K}$.

Theorem 4.1. *Let $K(z, w)$ be a $\mathbb{K}^{p \times p}$-valued kernel analytic in z and $\overline{w}$ in the open set $\Omega_{\mathbb{K}}$. Then, for every $(z, w) \in \Omega_{\mathbb{K}}^2$, there exists M_0 (which depends on (z, w)) such that, for every $M \in (0, M_0)$, the infinite matrix*

$$\frac{1}{m!n!} \frac{\partial^{n+m} K(z, w)}{\partial z^n \partial \overline{w}^m} M^{n+m} \tag{4.1}$$

defines a bounded operator from $\ell_2(\mathbb{N}_0)$ into itself.

Proof. Let $z, w \in \Omega_{\mathbb{K}}$. Then, there exists $M_0 > 0$ (which depends on z and w) for which the power expansion

$$K(z + M_0, w + M_0) = \sum_{n,m=0}^{\infty} \mathscr{K}_{n,m}(z, w) M_0^{n+m}$$

converges. Let C be such that

$$|\mathscr{K}_{n,m}(z, w) M_0^{n+m}| \leq C.$$

For $M \in (0, M_0)$ we have

$$\begin{aligned}|\eta_n^* \mathscr{K}_{n,m}(z,w) M^{n+m} \xi_m| &= |\eta_n^* \frac{M_0^n}{M^n} \mathscr{K}_{n,m}(z,w) M_0^{n+m} \xi_m \frac{\rho^m}{\rho_0^m}| \\ &\le C \left(\|a_n\| \frac{M^n}{M_0^n} \right) \left(\|\xi_m\| \frac{M^m}{M_0^m} | \right).\end{aligned}$$

Hence,

$$|\sum_{n,m=0}^{\infty} \eta_n^* \mathscr{K}_{n,m}(z,w) M^{n+m} \xi_m| \le C\|\eta\| \cdot \|\xi\| \frac{1}{1 - \frac{M}{M_0}}.$$

□

As a consequence, we have the following proposition, where

$$e(z) \begin{pmatrix} I \\ \overline{z} I \\ \overline{z}^2 I \\ \overline{z}^3 I \\ \vdots \end{pmatrix}, \quad z \in \mathbb{K}. \tag{4.2}$$

Proposition 4.2. *Let $D(M)$ denotes the diagonal operator with diagonal equals to $(I, M, M^2, M^3, \cdots)$. Thus,*

$$\langle D(M) \mathscr{K}(z+h, w+k) D(M) \xi, \eta \rangle = \langle \mathscr{K}(z,w) e(k) \xi, e(h) \eta \rangle. \tag{4.3}$$

The new positive definite kernel is, then,

$$\langle D(M) \mathscr{K}(z,w) D(M) X(b,B), X(a,A) \rangle_{\ell_2(\mathbb{N}_0)}. \tag{4.4}$$

V. OPERATORS

Let K be an analytic kernel, and T in $\mathbf{B}(\mathcal{H}(K))$. Then, T has a natural extension to an operator from $\mathcal{H}(\widetilde{K})$ into itself via the formula

$$\widetilde{T}(J(f)) = J(Tf) \tag{5.1}$$

By definition of the norm in $\mathbf{B}(\mathcal{H}(K))$, the operator $\widetilde{T}$ is bounded if and only if T is bounded. Furthermore we have the following result, the proof of which we omit.

Proposition 5.1. *Let T and S be possibly unbounded operators in $\mathcal{H}(K)$. Then,*

$$(\widetilde{TS})(J(f)) = \widetilde{T}(\widetilde{S} J(f)), \quad f \in \operatorname{Dom} S \text{ such that, } Sf \in \operatorname{Dom} T, \tag{5.2}$$

$$\widetilde{T^*}(J(f)) = \widetilde{T}^* J(f), \quad f \in \operatorname{Dom} T. \tag{5.3}$$

In general, the case of unbounded operators is of interest, as the Fock space example shows. Then, if

$$Z \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5.4}$$

and

$$S \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{5.5}$$

we have

$$\widetilde{M_z}(J(f)) = (zI + Z)J(f) \tag{5.6}$$

and

$$\widetilde{\partial_z}(J(f)) = SJ(f). \tag{5.7}$$

Next, letting $h \in \mathcal{G}$ as in Eq. (2.8) and $\Gamma_n(z)h = \frac{f^{(n)}(z)}{n!}$, we extend the operators to the space of $\mathbb{K}$-valued functions with reproducing kernel given by Eq. (2.5) as

$$T_{\mathcal{A}}\left(\sum_{n=0}^{\infty}\langle a^*, A^n\rangle \Gamma_n(z)h\right) = \sum_{n=0}^{\infty}\langle a^*, A^n\rangle \Gamma_n(z)Th. \tag{5.8}$$

When $\mathcal{G} = \mathcal{H}(K)$ in the factorization given by Eq. (2.8), we have

$$T_{\mathcal{A}}\left(\sum_{n=0}^{\infty}\langle a^*, A^n\rangle \Gamma_n(z)h\right) = \sum_{n=0}^{\infty}\langle a^*, A^n\rangle \frac{(Th)^{(n)}(z)}{n!} \tag{5.9}$$

or, equivalently,

$$\langle a^*, (Tf)(z + A)\rangle = \sum_{n=0}^{\infty}\langle a^*, A^n\rangle \frac{(Th)^{(n)}(z)}{n!}. \tag{5.10}$$

Proposition 5.2. *Let T, S be two possibly unbounded linear operators from $\mathcal{H}(K)$ into itself. Then,*

$$(TS)_{\mathcal{A}} = (T)_{\mathcal{A}}(S)_{\mathcal{A}}, \tag{5.11}$$
$$(T_{\mathcal{A}})^* = (T^*)_{\mathcal{A}}. \tag{5.12}$$

Finally, we note that the operators $T, \widetilde{T}$ and $T_{\mathcal{A}}$ are related by

$$T_{\mathcal{A}}\left(\sum_{n=0}^{\infty}\langle a^*, A^n\rangle \Gamma_n h\right) = \langle \widetilde{T}J(f), X(a, (A^n)_{n=0}^{\infty})\rangle_{\ell_2(\mathbb{N}_0)}. \tag{5.13}$$

Indeed, we have

$$\begin{aligned} T_{\mathcal{A}}\left(\sum_{n=0}^{\infty}\langle a^*, A^n\rangle \Gamma_n h\right) &= \langle J(Tf), X(a, (A^n)_{n=0}^{\infty})\rangle_{\ell_2(\mathbb{N}_0)} \\ &= \langle \widetilde{T}(f), X(a, (A^n)_{n=0}^{\infty})\rangle_{\ell_2(\mathbb{N}_0)}. \end{aligned}$$

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DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available within the article.

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