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Pseudo-contractions, Rigidity, Fixed Points and Related Questions

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Pseudo-contractions, rigidity, fixed points and related questions

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Abstract The class of holomorphic self-mappings of the open unit disk (which are contractions with respect to the Poincaré metric) admits a natural extension to the class of holomorphic pseudo-contractions. In this paper, we study various inequalities involving the values of derivatives of holomorphic pseudo-contractions at fixed points (particularly, at the Denjoy-Wolff fixed point).

Keywords Pseudo-contraction · Denjoy-Wolff point · angular derivative

Mathematics Subject Classification (2010) 32A10 · 47H10 · 30C80

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. By $\mathrm{Hol}(\mathbb{D}, \Omega)$ we denote the class of all holomorphic functions mapping the open unit disk $\mathbb{D}$ into a domain $\Omega \subseteq \mathbb{C}$. The class $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$ of analytic self-mappings of $\mathbb{D}$ is of particular interest and importance for geometric function theory and complex dynamics. By the classical Schwarz-Pick theorem, these functions are contractions with respect to the pseudo-hyperbolic (and also hyperbolic) metric in $\mathbb{D}$. If f is a strict contraction, then it has a unique fixed point $\zeta_f \in \mathbb{D}$. Furthermore, $|f'(\zeta_f)| < 1$, and iterations of f converge

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to ζ_f uniformly on compact subsets of $\mathbb{D}$. If f is not a strict contraction, then it may have no fixed point in $\mathbb{D}$; in this case the Denjoy-Wolff theorem guarantees the existence of a unique boundary fixed point $\zeta_f \in \mathbb{T} := \partial\mathbb{D}$ at which the angular derivative $f'(\zeta_f) \in (0, 1]$ exists (by the Julia lemma) and such that the iterations of f converge to ζ_f uniformly on compact subsets of $\mathbb{D}$. The point ζ_f is called the Denjoy-Wolff point of f and can be defined as a unique fixed point of f with the derivative not exceeding one in modulus.

It follows from Schwarz-Pick theorem and Julia lemma that given $\zeta \in \overline{\mathbb{D}}$ and $\lambda \in \mathbb{D}\backslash\{\zeta\}$, the set $\{f(\lambda) : f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ and $\zeta_f = \zeta\}$ is a closed disk and moreover, if $f(\lambda)$ is a boundary point of this disk, then $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ is defined by this condition uniquely. The latter observations admit far-reaching generalizations including multi-point versions, involving other (boundary) fixed points different from the Denjoy-Wolff point, incorporating higher order derivatives etc. The main objective of the present paper is to consider similar questions in the context of the class $\mathcal{PC}$ of holomorphic pseudo-contractions with respect to the Poincaré hyperbolic metric on $\mathbb{D}$ which is an analog of pseudo-contractions with respect to the underlined norm introduced and studied by Kato [17], Browder [9,10], Kirk [18], Morales [20,21] and others with connections to various problems in nonlinear functional analysis.

Alternatively, holomorphic pseudo-contractions can be defined as follows. Let us observe that if $F \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$, then for each $t \in [0, 1)$ and any point $w \in \mathbb{D}$, the function

$$G(z) = tF(z) + (1 - t)w$$

is analytic on $\mathbb{D}$ with boundary angular limits (whenever the latter exist) less then one in modulus. Hence, G has no boundary fixed points and therefore, its unique (Denjoy-Wolff) fixed point $z := J_t(w)$ belongs to $\mathbb{D}$. Note that for each $t \in [0, 1)$ the function J_t belongs to $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$ as a function of $w \in \mathbb{D}$.

Definition 1.1 A function F analytic on $\mathbb{D}$ is called *pseudo-contractive* if for each $t \in [0, 1)$ and each $w \in \mathbb{D}$, the equation

$$z = tF(z) + (1 - t)w$$

has a unique solution $z = J_t(w) \in \mathbb{D}$ which is analytic in w.

It follows from the latter definition and the preceding discussion that the class $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$ is contained in the class $\mathcal{PC}$ of holomorphic pseudo-contractions, and simple examples justify that this containment is proper. Although unimodular constant functions belong neither to $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$ nor to $\mathcal{PC}$, it is convenient to extend the class $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$ by these unimodular constants to the *Schur class* $\mathcal{S}$ of all analytic functions mapping the open unit disk into its closure $\overline{\mathbb{D}}$.

The outline of the paper is as follows. In Section 2, we survey various properties of Schur functions pertaining to boundary behavior near fixed points, and certain remarkable inequalities relating boundary derivatives at these points. The latter inequalities will be extended to the class $\mathcal{PC}$ in Sections 4 and 5. Basic properties of pseudo-contractions as well as their distinguished Denjoy-Wolff point are discussed in Section 3.

2 Background

In this section we recall several results concerning Schur-class functions which will be extended in subsequent sections to the functions of the class $\mathcal{PC}$.

We will write $z \overset{\mathbf{nt}}{\to} \zeta_0$ if z tends to a boundary point $\zeta_0 \in \mathbb{T} = \{z : |z| = 1\}$ non-tangentially; if the **nt**-limit of $f(z)$ exists (finitely) as $z \overset{\mathbf{nt}}{\to} \zeta_0$ for a function $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$, we will denote this limit by $f(\zeta_0)$. A point $z_0 \in \overline{\mathbb{D}}$ is called *a fixed point* of $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ if $f(z_0) = z_0$.

By the Denjoy-Wolff theorem, any function $f \in \mathcal{S}\backslash\{\mathrm{id}\}$ (id denotes the identity map $z \mapsto z$) which is not an elliptic automorphism of $\mathbb{D}$, has a unique attractive fixed point in $\overline{\mathbb{D}}$ (the *Denjoy-Wolff point*). More precisely, by Schwarz's lemma, any $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})\backslash\{\mathrm{id}\}$ has at most one fixed point $z_0 \in \mathbb{D}$, and then necessarily, $|f'(z_0)| \le 1$. Due to the Schwarz-Pick theorem, all (elliptic) self-mappings of $\mathbb{D}$ with the fixed point $z_0 \in \mathbb{D}$ are given by the formula

$$f(z) = \frac{z_0(1 - z\overline{z}_0) + (z - z_0)\mathcal{E}(z)}{1 - z\overline{z}_0 + \overline{z}_0(z - z_0)\mathcal{E}(z)}, \qquad \mathcal{E} \in \mathcal{S}\backslash\{1\}, \tag{2.1}$$

where $\mathcal{E} \not\equiv 1$ is a Schur-class function (the parameter $\mathcal{E} \equiv 1$ is excluded as it leads to the identity map). Moreover, unimodular constant parameters $\mathcal{E} \neq 1$ produce via formula (2.1) all elliptic automorphisms of $\mathbb{D}$ with $f'(z_0) = \mathcal{E}$. One can see from (2.1) that $f'(z_0) = \mathcal{E}(z_0) \in \mathbb{D}$.

If $f \in \mathcal{S}$ does not have a fixed point inside $\mathbb{D}$, then (and only then) it has a unique boundary fixed point $t_0 \in \mathbb{T}$ such that $0 \le f'(t_0) \le 1$. The existence of the boundary derivative $f'(t_0)$ of a function $f \in \mathcal{S}$ at a fixed boundary point t_0 is guaranteed by the Carathéodory-Julia-Wolff theorem. Moreover, for $f \in \mathcal{S}$, the following limits (finite or infinite) are equal:

$$f'(t_0) := \lim_{z \overset{\mathbf{nt}}{\to} t_0} f'(z) = \lim_{z \overset{\mathbf{nt}}{\to} t_0} \frac{1 - |f(z)|}{1 - |z|}. \tag{2.2}$$

Making use of Julia's lemma, it is not hard to show that all analytic mappings $f : \mathbb{D} \to \mathbb{D}$ with the boundary Denjoy-Wolff point $t_0 \in \mathbb{T}$ are given by the formula

$$f(z) = \frac{t_0 + (t_0 - 2z)\mathcal{E}(z)}{2 - z\bar{t}_0 - z\bar{t}_0\mathcal{E}(z)}, \qquad \mathcal{E} \in \mathcal{S}\backslash\{\pm 1\}. \tag{2.3}$$

Constant parameters $\mathcal{E} \equiv \pm 1$ are excluded in (2.1) as they give rise to the identity map or to $f \equiv t_0$ which is not a self-mapping of $\mathbb{D}$. Several further observations were done in [8] in the context of more general interpolation problems.

Remark 2.1 *The analytic map $f : \mathbb{D} \to \mathbb{D}$ of the form (2.3) is*
(a) hyperbolic (i.e., $f'(t_0) < 1$) if and only if $\mathcal{E}(t_0) = 1$ and $\mathcal{E}'(t_0) < \infty$;
(b) a hyperbolic automorphism if and only if $\mathcal{E}$ is an automorphism with $\mathcal{E}(t_0) = 1$;
(c) parabolic (i.e., $f'(t_0) = 1$) if and only if $\mathcal{E}(t_0) \neq 1$ or $\mathcal{E}'(t_0) = \infty$;
(d) a parabolic automorphism if and only if $\mathcal{E}$ is a unimodular constant.

The boundary fixed point with multiplicity greater than one (that is, $f(z) - z = o(z - t_0)^2$ as $z \overset{\mathbf{nt}}{\to} t_0$) may occur only in the parabolic case. However, the options are quite limited: if $f''(t_0) = 0$, then necessarily, $-t_0^2 f'''(t_0) \ge 0$, and in the case $f'''(t_0) = 0$, the rigidity result of Burns and Krantz [11] guarantees that $f = \mathrm{id}$. Details are recalled in the next remark.

Remark 2.2 *Let $f \in \mathcal{S}$, $t_0 \in \mathbb{T}$ and let $f(t_0) = t_0$, $f'(t_0) = 1$.*

1. *If the* **nt***-limit $f''(t_0)$ exists, then $\Re(t_0 f''(t_0)) \geq 0$.*
2. *If $\Re(t_0 f''(t_0)) = 0$ and the limit $f'''(t_0)$ exists (finitely), then $t_0^2 f'''(t_0) \leq 0$. Moreover, if $f'''(t_0) = 0$, then $f''(t_0) = 0$ and $f(z) \equiv z$.*

Theorem 2.3 below partly recaptures the results just recalled and singles out the Denjoy-Wolff point of a function $f \in \mathcal{S}$ as the only attractive fixed point. On the other hand, f may have (infinitely) many fixed points $\zeta \in \mathbb{T}$ with $f'(\zeta) > 1$. Various inequalities involving derivatives at fixed points of $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$ were established in [12]. Inequality (2.4) below was established in [6] separately for two generic cases: $t_0 = 0$ and $t_0 = 1$. The present formulation is more unified.

Theorem 2.3 *Let $\zeta_0 \in \overline{\mathbb{D}}$ be the Denjoy-Wolff point of an $f \in \mathcal{S}\backslash\{\mathrm{id}\}$ and let $\zeta_1, \ldots, \zeta_n \in \mathbb{T}$ be other fixed points of f. Then for each $z \in \mathbb{D}\backslash\{\zeta_0\}$,*

$$\sum_{i=1}^{n} \frac{|\zeta_i - \zeta_0|^2}{|\zeta_i - z|^2} \frac{1}{f'(\zeta_i) - 1} \leq \frac{1 - |f(z)|^2}{1 - |z|^2} \cdot \frac{|z - \zeta_0|^2}{|f(z) - z|^2} - \frac{|\zeta_0 - f(z)|^2}{|f(z) - z|^2}. \tag{2.4}$$

In particular, if $f(0) \neq 0$ (i.e., $\zeta_0 \neq 0$), then letting $z = 0$ in (2.4) gives

$$\sum_{i=1}^{n} \frac{|\zeta_i - \zeta_0|^2}{f'(\zeta_i) - 1} \leq 2\Re\left(\frac{\zeta_0}{f(0)}\right) - 1 - |\zeta_0|^2. \tag{2.5}$$

Moreover, if equality holds in (2.4) for some $z \in \mathbb{D}\backslash\{\zeta_0\}$, then it holds for all $z \in \mathbb{D}$ which is the case if and only if f is a Blaschke product of degree $n + 1$.

We now point out several corollaries and particular cases of Theorem 2.3. We first note that although the inequality (2.4) is formulated for $z \in \mathbb{D}\backslash\{\zeta_0\}$, it can be extended to all of $\overline{\mathbb{D}}$ by taking the angular boundary limits. For example, if $a, b \in \mathbb{T}$ and $f(a) = b$, then we let $z = ra$ in (2.4) and pass to the limit as $r \to 1^-$ to get (see [13, Theorem 3])

$$\sum_{i=1}^{n} \frac{|\zeta_i - \zeta_0|^2}{|\zeta_i - a|^2} \frac{1}{f'(\zeta_i) - 1} \leq \frac{|f'(a)||a - \zeta_0|^2 - |\zeta_0 - b|^2}{|b - a|^2}.$$

To get the extension of (2.4) at the Denjoy-Wolff point ζ_0 of f, we first let

$$\mathbf{p} := \lim_{r \to 1^-} \frac{1 - |f(r\zeta_0)|^2}{1 - r^2|\zeta_0|^2} = \begin{cases} 1, & \text{if } \zeta_0 \in \mathbb{D}, \\ f'(\zeta_0), & \text{if } \zeta_0 \in \mathbb{T} \end{cases} \tag{2.6}$$

(see (2.2) for the case $\zeta_0 \in \mathbb{T}$). We next let $z = r\zeta_0$ in (2.4) and pass to the limit as $r \to 1^-$, making use of relations

$$\begin{aligned} f(r\zeta_0) - r\zeta_0 &= (f'(\zeta_0) - 1)\zeta_0(r - 1) + o(1 - r), \\ f(r\zeta_0) - \zeta_0 &= f'(\zeta_0)\zeta_0(r - 1) + o(1 - r) \end{aligned}$$

for the inequality below and using formula (2.6) for the subsequent equality:

$$\sum_{i=1}^{n} \frac{1}{f'(\zeta_i) - 1} \leq \frac{\mathbf{p} - |f'(\zeta_0)|^2}{|1 - f'(\zeta_0)|^2} = \begin{cases} \dfrac{1 - |f'(\zeta_0)|^2}{|1 - f'(\zeta_0)|^2}, & \text{if } \zeta_0 \in \mathbb{D}, \\ \dfrac{f'(\zeta_0)}{1 - f'(\zeta_0)}, & \text{if } \zeta_0 \in \mathbb{T}. \end{cases} \tag{2.7}$$

Inequalities (2.7) (as well as (2.5) for $\zeta_0 = 0$) were established in [12].

In the parabolic case (i.e., $|\zeta_0| = 1$ and $f'(\zeta_0) = 1$) the estimate (2.7) is not informative; however some meaningful estimates can be given in terms of higher order angular derivatives of f at the Denjoy-Wolff point ζ_0.

It is natural to start with the assumption that at least the radial limit $f''(\zeta_0)$ exists (and satisfies $\Re(\zeta_0 f''(\zeta_0)) \geq 0$, by Remark 2.2). However, this assumption imposes no extra restrictions on the derivatives at the repelling fixed points. It is known (see [7, Theorem 3.6]) that given any $\gamma_i > 1$ $(i = 1, \ldots, n)$ and β_0 with $\Re(\beta_0) \geq 0$, there exist functions $f \in \mathcal{S}$ with fixed points at ζ_i, such that

$$f'(\zeta_0) = 1, \;\; \zeta_0 f''(\zeta_0) = \beta_0 \quad \text{and} \quad f'(\zeta_i) = \gamma_i \;\; (i = 1, \ldots, n). \tag{2.8}$$

Moreover, if $\Re(\beta_0) > 0$, then there are functions $f \in \mathcal{S}$ subject to (2.8) with *any* prescribed radial limits $f^{(j)}(\zeta_0)$ for all $j \geq 3$. For this reason, we ought to assume, in addition to $f'(\zeta_0) = 1$, that the boundary limits $f''(\zeta_0)$ and $f'''(\zeta_0)$ exist finitely and that $\Re(\zeta_0 f''(\zeta_0)) = 0$. In this case, one can pass to the limit in (2.4) $z = r\zeta_0 \to \zeta_0$ (we refer to [6] for details) to show that

$$\sum_{i=1}^{n} \frac{1}{f''(\zeta_i) - 1} \leq \frac{2 \cdot |f'''(\zeta_0)|}{3 \cdot |f''(\zeta_0)|^2} - 1. \tag{2.9}$$

If $f(\zeta_0) = 0$ and $|f'''(\zeta_0)| < \infty$ and $f \neq \mathrm{id}$, we have

$$\zeta_0^2 f_3(\zeta_0) < 0, \tag{2.10}$$

by Remark 2.2. In this case, the inequality (2.4) is not sharp – as was shown in [6], the expression on the right side of (2.4) can be sharpened to

$$\begin{aligned} &\sum_{i=1}^{n} \frac{|\zeta_i - \zeta_0|^2}{|\zeta_i - z|^2} \frac{1}{f'(\zeta_i) - 1} \\ &\leq \frac{1 - |f(z)|^2}{1 - |z|^2} \cdot \frac{|z - \zeta_0|^2}{|f(z) - z|^2} - \frac{|\zeta_0 - f(z)|^2}{|f(z) - z|^2} + \frac{6}{\zeta_0^2 f'''(\zeta_0)|\zeta_0 - z|^2}, \end{aligned} \tag{2.11}$$

and letting $z = 0$ in this modified inequality leads us to

$$\sum_{i=1}^{n} \frac{|\zeta_i - \zeta_0|^2}{f'(\zeta_i) - 1} \leq 2\Re\left(\frac{\zeta_0}{f(0)}\right) - 2 + \frac{6}{\zeta_0^2 f'''(\zeta_0)}, \tag{2.12}$$

with equality if and only if f is a Blaschke product of degree $n + 2$. To get an analog of (2.9), we ought to assume at least the existence of the boundary limit $f^{(4)}(\zeta_0)$, which necessarily is subject to condition

$$\mathrm{Re}(\zeta_0^3 f^{(4)}(\zeta_0)) \leq -2\zeta_0^2 f'''(\zeta_0)). \tag{2.13}$$

Moreover, if (2.13) holds with equality then the boundary limit $f^{(5)}(\zeta_0)$ exists and

$$5\zeta_0^3 f^{(4)}(\zeta_0) + \zeta_0^4 f^{(5)}(\zeta_0) \geq 0. \tag{2.14}$$

Relations (2.13), (2.14) can be justified as in Remark 2.2 or derived directly from [4, Theorem 2.1] (see also [5, Theorem 2.3]). Again, it follows from [7, Theorem 3.6], that strict inequality in (2.14) imposes no extra restrictions on the derivatives at the repelling fixed points. In order to get an estimate similar to (2.9) we have to assume the existence of finite boundary limits $f^{(j)}(\zeta_0)$ for $j = 3, 4, 5$ subject to conditions (2.10), (2.14) and $\mathrm{Re}(\zeta_0^3 f^{(4)}(\zeta_0)) = -2\zeta_0^2 f'''(\zeta_0))$. Under these assumptions, one can pass to the limit in (2.11) as $z = r\zeta_0 \to \zeta_0$ (see [6] for details) to get

$$\sum_{i=1}^{n} \frac{1}{f'(\zeta_i) - 1} \leq \frac{5\zeta_0^3 f^{(4)}(\zeta_0) + \zeta_0^4 f^{(5)}(\zeta_0)}{20 \cdot |f'''(\zeta_0)|^2} + \frac{3 \cdot |f^{(4)}(\zeta_0)|^2}{8 \cdot |f'''(\zeta_0)|^3} - 1. \qquad (2.15)$$

3 Pseudo-contractions: basic properties and the generalized Denjoy-Wolff point

By Definition 1.1, with any pseudo-contraction F, one can associate a family of analytic functions $\{J_t\}_{t\in[0,1)} \subset \mathcal{S}$ uniquely defined from the equality

$$J_t = tF \circ J_t + (1-t)I, \qquad (3.1)$$

and, since any contraction $J_t \in \mathcal{S}$ is a pseudo-contraction on $\mathbb{D}$, for any fixed $t \in [0,1)$, we further can define the family $\{G_s\}_{s\in[0,1)} \subset \mathcal{S}$ as the unique solutions of the equation

$$G_s = sJ_t \circ G_s + (1-s)I. \qquad (3.2)$$

Lemma 3.1 *(a resolvent identity). Let $F : \mathbb{D} \to \mathbb{C}$ be a pseudo-contractive mapping on $\mathbb{D}$, and let J_t and G_s satisfy the equations* (3.1) *and* (3.2) *for some fixed $t, s \in [0,1)$. Then*

$$J_t \circ G_s = J_k, \quad \text{where} \quad k = \frac{t}{(1-t)(1-s)+t} \in [0,1). \qquad (3.3)$$

Proof: Write (3.1) and (3.2) equivalently in terms of I_F and $I - J_t$ as

$$I - J_t = p(I-F) \circ J_t, \quad \text{where} \quad p = \frac{t}{1-t} \qquad (3.4)$$

and

$$(I + r(I - J_t)) \circ G_s = I, \quad \text{where} \quad r = \frac{s}{1-s}. \qquad (3.5)$$

Observe that

$$\frac{pr}{s+pr} = k$$

and that for a fixed $t \in (0,1)$, the numbers k and s tend to 1 simultaneously.

Substituting (3.4) into (3.5) gives

$$(I + rp(I-F) \circ J_t) \circ G_s = I. \qquad (3.6)$$

We next introduce the function $\Psi_s = J_t \circ G_s \in \mathcal{S}$ and write (3.6) and (3.2) as

$$G_s + rp(I-F) \circ \Psi_s = I \quad \text{and} \quad G_s = s\Psi_s + (1-s)I, \qquad (3.7)$$

respectively. Eliminating G_s in the latter system gives

$$s\Psi_s + pr(I-F) \circ \Psi_s = sI,$$

or, which is the same,

$$(s+pr)\Psi_s = prF\circ\Psi_s + sI.$$

Taking $s\in(0,1)$, we can cancel $s+pr = s(1+\frac{t}{(1-t)(1-s)})$ in the latter equation getting

$$\Psi_s = \frac{pr}{s+pr}F\circ\Psi_s + \frac{s}{s+pr}I = kF\circ\Psi_s + (1-k)I.$$

Since the latter equation has a unique solution Ψ_s which by definition is J_k, we conclude that $J_k = \Psi_s := J_t\circ G_s$ for k defined as in (3.3). □

Lemma 3.2 *Let $F:\mathbb{D}\to\mathbb{C}$ be a pseudo-contractive mapping on $\mathbb{D}$, and let J_t satisfy the equation* (3.1)*. Then for each $w\in\mathbb{D}$ the function $J_t(w)$ is differentiable at $t=0^+$ and moreover,*

$$\frac{\partial J_t(w)}{\partial t}\left(t=0^+\right) = \lim_{t\to 0^+}\frac{J_t(w)-w}{t} = F(w)-w. \tag{3.8}$$

Proof: Fix any $w\in\mathbb{D}$. Since for sufficiently small $t\geq 0$, the net $F(J_t(w))$ is bounded it follows by the definition of J_t that

$$|J_t(w)-w| = t\,|F(J_t(w))-w| \to 0 \quad \text{as } t\to 0.$$

In turn, this implies (3.8) because of continuity of F at the point w. □

The Denjoy-Wolff point of a pseudo-contraction. The Denjoy-Wolff point of a function $F\in\mathrm{Hol}(\mathbb{D},\mathbb{D})$ can be defined as the only fixed point $\zeta_0\in\overline{D}$ with $|F'(\zeta_0)|\leq 1$. Alternatively, this point can be defined as the point to which the iterates of F converge uniformly on compact subsets of $\mathbb{D}$. The existence of such point for any $F\in\mathrm{Hol}(\mathbb{D},\mathbb{D})$ (not an elliptic automorphism) and its uniqueness are guaranteed by the classical Denjoy-Wolff theorem [14,26]. It turns out that such a distinguished fixed point exists for any $F\in\mathcal{PC}$.

Theorem 3.3 *Suppose that $F\in\mathrm{Hol}(\mathbb{D},\mathbb{C})$ is pseudo-contractive, i.e., for each $t\in[0,1)$, the equation*

$$J_t = tF\circ J_t + (1-t)I, \tag{3.9}$$

has a holomorphic solution $J_t\in\mathcal{S}$. Then there is a unique fixed point $\tau\in\overline{\mathbb{D}}$ of F such that for each $z\in\mathbb{D}$ the curve $J_t(z)$ converges to this point . Moreover,

1. *If $\tau\in\mathbb{D}$, then $\Re(F'(\tau))<1$;*
2. *If $\tau\in\mathbb{T}$, then the angular derivative $F'(\tau)$ is a real number and $F'(\tau)\leq 1$.*

Proof: For a fixed $t\in(0,1)$, define G_s via the equation (3.2) and let k be defined as in (3.3). We recall that k and s tend to 1 simultaneously make use of (3.3) and (3.2) to estimate and write

$$\begin{aligned}|G_s(z)-J_k(z)| &= |G_s(z)-(J_t\circ G_s)(z)|\\ &= |(1-s)z+(s-1)J_t\circ G_s(z)|\\ &\leq (1-s)\left[|J_tG_s(z)|+|z|\right]\leq 2(1-s)\to 0 \quad \text{as } s\to 1^-.\end{aligned}$$

On the other hand, since J_t is a self-mapping of $\mathbb{D}$, it is known (see, e.g., [16]) that for each $z\in\mathbb{D}$ the approximating curve $\{G_s(z)\}$ converges to a point $\tau\in\overline{\mathbb{D}}$

(which is the Denjoy-Wolff point of J_t) as s tends to 1^-. Since s and k related as in (3.3) tend to 1 simultaneously, we get the first statement of the theorem.

To complete the proof, we write the equation (3.9) in the form

$$J_t\left(w\right) = tF\left(J_t\left(w\right)\right) + (1-t)w$$

and differentiate it with respect to $w \in \mathbb{D}$ for a fixed $t \in [0,1)$. We have

$$J_t'(w) = tF'(J_t(w))J_t'(w) + (1-t).$$

Thus,

$$J_t'(w) = \frac{1-t}{1-tF'(J_t(w))}. \tag{3.10}$$

If $\tau \in \mathbb{D}$ is a fixed point of F, it also must be the fixed point of J_t for each $t \geq 0$, so we get

$$J_t'(\tau) = \frac{1-t}{1-tF'(\tau)}. \tag{3.11}$$

Then we have from the Schwarz-Pick Lemma

$$1 \geq \left|J_t'(\tau)\right|^2 = \frac{(1-t)^2}{\left|1-tF'\left(\tau\right)\right|^2}.$$

Therefore,

$$\Re(F'(\tau) \leq 1 + \frac{1}{2}\left(\left|F'\left(\tau\right)\right|^2 - 1\right)t.$$

Letting $t \to 0$ in the latter inequality gives $\Re(F'(\tau) \leq 1$.

If F is fixed point-free in $\mathbb{D}$, then so is J_t for all $t \in [0,1)$. Fix any $t \in (0,1)$ and let $\tau \in \mathbb{T}$ be the boundary Denjoy-Wolff point for J_t. Using the formal notation $J_t'(\tau)$ and $F'(\tau)$ for the angular derivatives at $\tau \in \mathbb{T}$ we still derive formula (3.11) from (3.10). It follows from the Julia-Wolff-Carathéodory theorem that $0 < J_t'(\tau) \leq 1$. But this happens if and only if $F'\left(\tau\right) \leq 1$. □

It can be also shown that for each $t > 0$ the iterates of J_t also converge to the same point τ uniformly on compacta. Therefore, it is natural to call this point the *generalized Denjoy-Wolff* fixed point of F.

Remark 3.4 Combining Lemma 3.2 with the results in [22] (see also [19] and [23]) we conclude: *a mapping F is pseudo-contractive if and only if $I-F$ is a semi-complete vector field on $\mathbb{D}$.*

Therefore, F is a pseudo-contraction if and only if $\mathrm{id}-F$ is a generator. Combining this observation with the Berkson-Porta formula for generators [3], we arrive at the following conclusion:

Proposition 3.5 *Any pseudo-contraction F admits a unique representation*

$$F(z) = z - (z-\tau)(1-z\overline{\tau})p(z), \quad \tau \in \overline{\mathbb{D}} \tag{3.12}$$

where p is a Carathéodory-class function (i.e., analytic and with non-negative real part in $\mathbb{D}$).

The proposition below collects several well known facts concerning the functions from the Carathéodory class which will be denoted by $\mathcal{C}$ in what follows.

Proposition 3.6 *Let p be a Carathéodory-class function. Then*

1. *There is an essentially unique positive measure $d\sigma$ on $[0, 2\pi)$ such that*

$$p(z) = i\Im p(0) + \int_0^{2\pi} \frac{e^{i\phi} + z}{e^{i\phi} - z} d\sigma(\phi) \quad \text{for all} \quad z \in \mathbb{D}. \tag{3.13}$$

2. *For any $\zeta = e^{i\psi} \in \mathbb{T}$, the following* **nt**-*limits exist:*

$$\lim_{z \overset{\mathbf{nt}}{\to} \zeta} (z - \zeta)p(z) = -2\zeta\sigma(\{\psi\}), \quad \lim_{z \overset{\mathbf{nt}}{\to} \zeta} (z - \zeta)^2 p'(z) = 2\zeta\sigma(\{\psi\}). \tag{3.14}$$

3. *If $p(\zeta) = \lim_{z \overset{\mathbf{nt}}{\to} \zeta} p(z) = 0$, then the limit $p'(\zeta) = \lim_{z \overset{\mathbf{nt}}{\to} \zeta} p'(z)$ exits and*

$$\lim_{z \overset{\mathbf{nt}}{\to} \zeta} \frac{\Re(p(z))}{1 - |z|} = -\zeta p'(\zeta) \geq 0. \tag{3.15}$$

   Furthermore, if $p'(\zeta)$, then $p \equiv 0$.

4. *If $p \not\equiv 0$, then $\frac{1}{p(z)}$ also belongs to the Carathéodory class.*

In the next two sections we will discuss various questions concerning pseudo-contractions versus their Schur-class counterparts. Here we start with a question of this sort: for which Carathéodory functions p, the formula (3.12) produces a Schur-class function?

Lemma 3.7 *Let F be of the form (3.12). Then F belongs to the Schur class $\mathcal{S}$ if and only if*

$$2\Re\left(\frac{1}{p(z)}\right) \geq 1 - |z|^2 + |z - \tau|^2. \tag{3.16}$$

Proof: Let $|\tau| < 1$. By the Schwarz-Pick theorem, F belongs to $\mathcal{S}$ if and only if the function

$$\frac{F(z) - \tau}{1 - F(z)\overline{\tau}} \cdot \frac{1 - z\overline{\tau}}{z - \tau} = \frac{1 - (1 - z\overline{\tau})p(z)}{1 + (z - \tau)\overline{\tau}p(z)} \tag{3.17}$$

belongs to $\mathcal{S}$ (observe that the equality in (3.17) holds by (3.12)). The latter is equivalent to

$$\begin{aligned} 0 &\leq |1 + (z - \tau)\overline{\tau}p(z)|^2 - |1 - (1 - z\overline{\tau})p(z)|^2 \\ &= (1 - |\tau|^2)|p(z)|^2 \left(\frac{1}{p(z)} + \frac{1}{\overline{p(z)}} - 1 + z\overline{\tau} + \overline{z}\tau - |\tau|^2 \right), \end{aligned}$$

which in turn, is equivalent to (3.16) if $p \not\equiv 0$.

Let $|\tau| = 1$. In this case, by the Julia's lemma, F belongs to $\mathcal{S}$ if and only if the function

$$\frac{(2 - z\overline{\tau})F(z) - \tau}{2z - \tau - z\overline{\tau}F(z)} = \frac{1 - (2 - z\overline{\tau})p(z)}{1 + z\overline{\tau}p(z)} \tag{3.18}$$

belongs to $\mathcal{S}$ (the equality in (3.18) follows from (3.12)). The latter is equivalent to

$$\begin{aligned} 0 &\le |1+z\overline{\tau}p(z)|^2 - |1-(2-z\overline{\tau})p(z)|^2 \\ &= 2|p(z)|^2\left(\frac{1}{p(z)} + \frac{1}{\overline{p(z)}} - 2 + z\overline{\tau} + \overline{z}\tau\right), \end{aligned}$$

which is equivalent to (3.16). □

Observe that equality holds in (3.16) (for one or, equivalently, for all $z \in \mathbb{D}$) if and only if the Schur-class function on the right side of (3.17) (or (3.18)) is a unimodular constant (say γ), that is if and only if

$$p(z) = \begin{cases} \dfrac{1-\gamma}{1-\gamma|\tau|^2-(1-\gamma)\overline{\tau}z}, & \text{if} \quad |\tau|<1, \\ \dfrac{1-\gamma}{2-(1-\gamma)\overline{\tau}z}, & \text{if} \quad |\tau|=1, \end{cases} \qquad |\gamma|=1.$$

We next remark that condition (3.16) means that $p \in \mathcal{C}$ is necessarily of the form

$$\frac{1}{p(z)} = \frac{1}{\widetilde{p}(z)} - z\overline{\tau} + \frac{1+|\tau|^2}{2} \quad \text{for some} \quad \widetilde{p} \in \mathcal{C}.$$

Substituting the latter formula in (3.16) we conclude: *a function $f \in \mathcal{S}$ with the Denjoy-Wolff point τ admits a unique representation*

$$f(z) = z - \frac{(z-\tau)(1-z\overline{\tau})\widetilde{p}(z)}{1+\left(\frac{1+|\tau|^2}{2} - z\overline{\tau}\right)\widetilde{p}(z)}, \qquad \widetilde{p} \in \mathcal{C}. \tag{3.19}$$

If $|\tau|=1$ (respectively, $|\tau|<1$), the latter formula is equivalent to formula (2.3) (resp., (2.1)) via the respective Cayley transforms

$$\widetilde{p} = \frac{1+\mathcal{E}}{1-\mathcal{E}} \quad \text{and} \quad \widetilde{p} = \frac{2}{1-|\tau|^2}\cdot\frac{1-\mathcal{E}}{1+\mathcal{E}}.$$

We now take another look at the Denjoy-Wolff point in the light of the Berkson-Porta formula (3.12). It is clear from (3.12) that $z=\tau$ is a fixed point of F. Other fixed points (if any) occur at any point $\zeta \in \mathbb{T}$ at which $p(\zeta)=0$.

The interior case: If $|\tau|<1$, then τ is a unique fixed point of F inside $\mathbb{D}$, as $p \not\equiv 0$ cannot have zeros in $\mathbb{D}$. It is readily seen from (3.12) that

$$F'(\tau) = 1-(1-|\tau|^2)p(\tau), \tag{3.20}$$

and hence, $\Re(F'(\tau))<1$. Also, it follows from (3.20) that

$$|F'(\tau)| \le 1 \iff 2\Re\left(\frac{1}{p(\tau)}\right) \ge 1-|\tau|^2 \tag{3.21}$$

telling us that the interior fixed point τ of F is non-repelling if and only if the corresponding function $p \in \mathcal{C}$ satisfies the inequality in (3.21). Letting $z=\tau$ in (3.16) we get the same inequality confirming that the interior fixed point of a Schur-class function is always non-repelling. Also, $F'(\tau)=1$ if and only if $p(\tau)=0$ which holds only if $p \equiv 0$.

The boundary case: If $|\tau| = 1$ and $\phi = \arg\tau$, then we make use of formulas (3.14) (with $\zeta = \tau = e^{i\phi}$) along with the Berkson-Porta representation (3.12) to conclude

$$F'(\tau) = \lim_{z\to\tau}(1 + 2\overline{\tau}(z-\tau)p(z) + \overline{\tau}(z-\tau)^2 p'(z) = 1 - 2\sigma(\{\phi\}) \le 1.$$

Uniqueness: If $\zeta \neq \tau$ is another (boundary) fixed point of F, then necessarily $p(\zeta) = 0$. Then there exists the limit $\zeta p'(\zeta) \le 0$. Then

$$\begin{aligned} F'(\zeta) &= \lim_{z\to\zeta}(1 + 2\overline{\tau}(z-\tau)p(z) + \overline{\tau}(z-\tau)^2 p'(z) \\ &= 1 - |\zeta-\tau|^2 \zeta p'(\zeta) \ge 1, \end{aligned} \tag{3.22}$$

with equality if and only if $p \equiv 0$. Thus, $\tau \in \overline{\mathbb{D}}$ is the unique boundary fixed point of F with $\Re F' \le 1$.

4 Various inequalities: pseudo-contractions versus contractions

In this section we establish several inequalities concerning pseudo-contractions and compare them with their Schur-class counterparts.

Proposition 4.1 *Let F be a pseudo-contraction with a boundary fixed point $\zeta_0 \in \mathbb{T}$. Then $F'(\zeta_0)$ is a real number which satisfies*

$$F'(\zeta_0) \ge 1 - 2\Re(\overline{\zeta}_0 F(0)). \tag{4.1}$$

Moreover, if the equality holds, then F is a polynomial of the form

$$F(z) = a + bz - \overline{a}z^2, \quad \text{where} \quad a = F(0),\ b = 1 + \zeta_0\overline{F(0)} - \overline{\zeta}_0 F(0). \tag{4.2}$$

Proof: Let us take F in the form (3.12) and verify (4.1) case by case.

Case 1: $\tau = 0$. In this case, $F(z) = z(1 - p(z))$ and $F(0) = 0$. Since $F(\zeta_0) = \zeta_0$, we have $p(\zeta_0) = 0$ and then $\zeta_0 p'(\zeta_0) \le 0$, by (3.15). Then

$$F'(\zeta_0) = 1 - \zeta_0 p'(\zeta_0) \ge 1 = 1 - 2\Re(\overline{\zeta}_0 F(0))$$

which proves (4.1) in this case. If $F'(\zeta_0) = 1$, then $p'(\zeta_0) = 0$ and hence, $p \equiv 0$. Therefore, $F = \mathrm{id}$, which proves (4.2) with $a = 0$ and $b = 1$.

Case 2: $\tau = \zeta_0 = e^{i\psi_0}$. In this case, $F'(\zeta_0) = 1 - 2\sigma(\{\psi_0\})$ (by (3.14)), whereas

$$\Re(\overline{\zeta}_0 F(0)) = \Re(p(0)) = \int_0^{2\pi} d\sigma(\phi) \ge \sigma(\{\psi_0\}). \tag{4.3}$$

Inequality (4.1) is verified as follows:

$$F'(\zeta_0) = 1 - 2\sigma(\{\psi_0\}) \ge 1 - 2\Re(p(0)) = 1 - 2\Re(\overline{\zeta}_0 F(0)). \tag{4.4}$$

By (4.3), equality holds in (4.4) if and only if the measure σ is supported by the single point ψ_0 and hence,

$$p(z) = p(0) + \frac{z}{\tau - z}\big(p(0) + \overline{p(0)}\big) = \frac{\tau p(0) + z\overline{p(0)}}{\tau - z} = \frac{F(0) + z\tau\overline{F(0)}}{\tau - z}.$$

Substituting the latter formula into (3.12) gives (4.2).

Case 3: $\tau \in \overline{\mathbb{D}}\backslash\{0,\zeta_0\}$. Since $F(\zeta_0)=\zeta_0$, we have from (3.12) $p(\zeta_0)=0$ and

$$F(0)=\tau p(0), \qquad F(\zeta_0)=\zeta_0-|\zeta_0-\tau|^2\zeta_0 p(\zeta_0),$$
$$F'(\zeta_0)=1-|\zeta_0-\tau|^2\zeta_0 p'(\zeta_0). \tag{4.5}$$

Since $p(\zeta_0)=0$, we have (as in Case 1) $\zeta_0 p'(\zeta_0)\le 0$, by (3.15). Moreover,

$$-\zeta_0 p'(\zeta_0)\ge \frac{|p(0)|^2}{p(0)+\overline{p(0)}}, \tag{4.6}$$

by the Julia lemma. Furthermore, equality holds in (4.6) if and only if p is of the form as below:

$$-\zeta_0 p'(\zeta_0)=\frac{|p(0)|^2}{p(0)+\overline{p(0)}} \iff p(z)=\frac{|p(0)|^2(1-z\overline{\zeta}_0)}{\overline{p(0)}+p(0)\overline{\zeta}_0 z}. \tag{4.7}$$

We derive from (4.5), (4.6) and (4.7):

$$\begin{aligned} F'(\zeta_0)&=1-|\zeta_0-\tau|^2\zeta_0 p'(\zeta_0)\\ &\ge 1+\frac{|\zeta_0-\tau|^2|p(0)|^2}{p(0)+\overline{p(0)}}=1+\frac{|\zeta_0-\tau|^2|F(0)|^2}{\overline{\tau}F(0)+\tau\overline{F(0)}}. \end{aligned} \tag{4.8}$$

Moreover, equality holds in (4.8) if and only if

$$F(z)=z-(z-\tau)(1-z\overline{\tau})\frac{|F(0)|^2(1-z\overline{\zeta}_0)}{\tau\overline{F(0)}+z\overline{\tau}\overline{\zeta}_0F(0)}. \tag{4.9}$$

We next observe that

$$\frac{|\zeta_0-\tau|^2|F(0)|^2}{\overline{\tau}F(0)+\tau\overline{F(0)}}=-(\overline{\zeta}_0F(0)+\zeta_0\overline{F(0)})+\frac{|F(0)+\zeta_0\tau\overline{F(0)}|^2}{\overline{\tau}F(0)+\tau\overline{F(0)}}. \tag{4.10}$$

Since $\Re(\tau F(0))=|\tau|^2\Re(p(0))>0$, we conclude from (4.10) that

$$\frac{|\zeta_0-\tau|^2|F(0)|^2}{\overline{\tau}F(0)+\tau\overline{F(0)}}\ge -(\overline{\zeta}_0F(0)+\zeta_0\overline{F(0)}), \tag{4.11}$$

and that, moreover, equality holds in (4.11) if and only if

$$F(0)=-\zeta_0\tau\overline{F(0)}. \tag{4.12}$$

In particular, $\tau=-\overline{\zeta}_0F(0)/\overline{F(0)}\in\mathbb{T}$. Now we complete the proof: combining (4.8) and (4.11) gives (4.1). Equality holds in (4.1) if and only if we have equalities in (4.8) and (4.11). In the latter case we combine (4.9) and (4.12) to arrive at

$$\begin{aligned} F(z)&=z-(z-\tau)(1-z\overline{\tau})\frac{|F(0)|^2(1-z\overline{\zeta}_0)}{-\overline{\zeta}_0F(0)+z\overline{\zeta}_0\overline{\tau}F(0)}\\ &=z+(z-\tau)(\zeta_0-z)\overline{F(0)}\\ &=z+(z\overline{F(0)}+\overline{\zeta}_0F(0))(\zeta_0-z)\\ &=-\overline{F(0)}z^2+(1+\zeta_0\overline{F(0)}-\overline{\zeta}_0F(0))z+F(0) \end{aligned}$$

which is the same as (4.2). □

The corresponding result for Schur-class functions follows immediately by the Julia lemma:

Remark 4.2 *If $F \in \mathcal{S}$, $\zeta_0 \in \mathbb{T}$ and $F(\zeta_0) = \zeta_0$, then*

$$F'(\zeta_0) \geq \frac{|\zeta_0 - F(0)|^2}{1 - |F(0)|^2}. \tag{4.13}$$

Equality holds in (4.13) *if F is an automorphism of $\mathbb{D}$:*

$$F(z) = \frac{z\gamma + F(0)}{1 + z\gamma\overline{F(0)}}, \quad \text{where} \quad \gamma = \frac{1 - \overline{\zeta}_0 F(0)}{1 - \zeta_0\overline{F(0)}} \in \mathbb{T}.$$

The inequality (4.13) can be equivalently written as

$$\left| F(0) - \frac{\zeta_0}{1 + F'(\zeta_0)} \right| \leq \frac{F'(\zeta_0)}{1 + F'(\zeta_0)} \tag{4.14}$$

showing that for a prescribed value $F'(\zeta_0)$, the value $F(0)$ has to be within the closed disk of radius $\frac{F'(\zeta_0)}{1+F'(\zeta_0)}$ centered at $\frac{\zeta_0}{1+F'(\zeta_0)}$. Remark 4.2 asserts that if $F(0)$ belongs to the boundary of the latter disk, the function F is determined uniquely, which can be viewed as a rigidity result. Observe that for pseudo-contractions, this result is slightly different: $F(0)$ must belong to the half-plane $\Re(\overline{\zeta}_0 z) \geq 1 - F'(\zeta_0)$, and rigidity occurs if and only if $F(0)$ belongs to the boundary of this half-plane (the line $\Re(\overline{\zeta}_0 z) = 1 - F'(\zeta_0)$).

If $0 < |F(0)| < 1$, then the inequality (4.13) is indeed sharper than (4.1) as

$$1 - 2\Re(\overline{\zeta}_0 F(0)) = |\zeta_0 - F(0)|^2 - |F(0)|^2 \leq \frac{|\zeta_0 - F(0)|^2}{1 - |F(0)|^2}.$$

It follows from (4.5) that $F'(\zeta_0) = 1$ only if $p'(\zeta_0) = 0$ (in which case $p(z) \equiv 0$) or $\tau = \zeta_0$. We will discuss the latter case in some more detail. We thus assume that

$$F(\tau) := \lim_{z \overset{\mathrm{nt}}{\to} \tau} F(z) = \tau \quad \text{and} \quad F'(\tau) := \lim_{z \overset{\mathrm{nt}}{\to} \tau} F'(z) = 1. \tag{4.15}$$

If the higher order boundary derivatives of F at τ also exist, they are subject to certain conditions clarified below.

Remark 4.3 *Let $F \in \mathcal{PC}$ satisfy conditions* (4.15)*. Then $\Re(\overline{\tau} F(0)) \geq 0$. Moreover,*

1. *If $\Re(\overline{\tau} F(0)) = 0$, then $F = \mathrm{id}$.*
2. *If the limit $F''(\tau)$ exists, then $\Re(\tau F''(\tau)) \geq 0$.*
3. *If $\Re(\tau F''(\tau)) = 0$ and the limit $F'''(\tau)$ exists, then $\tau^2 F'''(\tau) \leq 0$.*
4. *If $\Re(\tau F''(\tau)) = 0$ and $F'''(\tau) = 0$, then $F = \mathrm{id}$.*

Proof: If F satisfies conditions (4.15), then, according to (3.12),

$$F(z) = z + (z - \tau)^2 \overline{\tau} p(z), \qquad p \in \mathcal{C} \tag{4.16}$$

where τ is not a mass point for the Herglotz measure σ of the Carathéodory-class function p. By (4.16), $\overline{\tau} F(0) = p(0)$ and the first statement follows since $p \in \mathcal{C}$. It is also clear from (4.16) that the limit $F''(\tau)$ exists if and only if the limit $p(\tau)$ does, in which case, $F''(\tau) = 2\overline{\tau} p(\tau)$. Since $p \in \mathcal{C}$, we have $\Re(p(\tau)) \geq 0$, and the second statement follows.

If $\Re(\tau F''(\tau)) > 0$, nothing particular can be said about higher order boundary derivatives of F at τ (in case they exist). On the other hand, if $\Re(\tau F''(\tau)) = 0$ (and therefore, $\Re(p(\tau)) = 0$), then the limits $F'''(\tau)$ and $p'(\tau)$ exist; if they exist finitely, then it follows from (4.16) that $F'''(\tau) = 6\overline{\tau}p'(\tau)$, which being combined with (3.15) implies $\tau^2 F'''(\tau) \le 0$. If $F'''(\tau) = 0$, then $p'(\tau) = 0$ and then $p \equiv 0$ (and hence, $F = \mathrm{id}$) by the Julia lemma. □

The next theorem can be viewed as a higher order analog of Proposition 4.1. Its assumptions exclude trivial cases discussed in Remark 4.3.

Theorem 4.4 *Let $F \in \mathcal{PC}$ satisfy conditions (4.15) and $\Re(\overline{\tau}F(0)) > 0$. Let us assume that the* **nt**-*limits $F''(\tau)$ and $F'''(\tau)$ exist and satisfy conditions*

$$\Re(\tau F''(\tau)) = 0 \quad \text{and} \quad \tau^2 F'''(\tau) < 0.$$

Then necessarily,

$$|F(0) - \tau(\mu + \nu)| \le \mu \quad \text{or, equivalently,} \quad \mu \ge \frac{|\overline{\tau}F(0) - \nu|^2}{2\Re(\overline{\tau}F(0))}, \tag{4.17}$$

where we have set

$$\nu := \frac{\tau}{2}F''(\tau) \in i\mathbb{R} \quad \text{and} \quad \mu := -\frac{\tau^2}{6}F'''(\tau) > 0. \tag{4.18}$$

Furthermore, if $\mu = \dfrac{|F(0) - \tau\nu|^2}{2|\Re(\overline{\tau}F(0))|}$, then necessarily

$$F(z) = z + (z - \tau)^2 \frac{\overline{\tau}F(0)\big[\tau\overline{F(0)} - \overline{\nu}\big] - \overline{F(0)}\big[\overline{\tau}F(0) - \nu\big]z}{\tau\big[\tau\overline{F(0)} - \overline{\nu}\big] + z\big[\overline{\tau}F(0) - \nu\big]}. \tag{4.19}$$

Proof: Due to conditions (4.15), we can take F in the form (4.16) and then consider the Schwarz-Pick inequality for the corresponding $p \in \mathcal{C}$

$$\left[\frac{p(z_i) + \overline{p(z_j)}}{1 - z_i\overline{z}_j}\right] \ge 0$$

based on three points $\zeta_1 \overset{\mathbf{nt}}{\to} \tau$, $z_2 = 0$ and $z_3 = z$. On account of (3.15) and since $p(\tau) = \nu = -\overline{\nu}$ and $\tau p'(\tau) = -\mu$, the inequality above takes the form

$$\begin{bmatrix} \mu & \overline{p(0)} - \overline{\nu} & \dfrac{\overline{p(z)} - \overline{\nu}}{1 - \tau\overline{z}} \\ p(0) - \nu & p(0) + \overline{p(0)} & p(0) + \overline{p(z)} \\ \dfrac{p(z) - \nu}{1 - z\overline{\tau}} & p(z) + \overline{p(0)} & \dfrac{p(z) + \overline{p(z)}}{1 - |z|^2} \end{bmatrix} \ge 0 \tag{4.20}$$

and holds for all $z \in \mathbb{D}$. In particular,

$$\mu \ge \frac{|p(0) - \nu|^2}{p(0) + \overline{p(0)}} = \frac{|\overline{\tau}F(0) - \nu|^2}{2\Re(p(0))} = \frac{|F(0) - \tau\nu|^2}{2\Re(\overline{\tau}F(0))}.$$

Writing the latter inequality equivalently as

$$\begin{aligned}0 &\ge |F(0)-\tau\nu|^2-\mu(\overline{\tau}F(0))+\tau\overline{F(0)})\\ &=|F(0)|^2-\overline{\tau}(\overline{\nu}+\mu)F(0)-\tau(\nu+\mu)\overline{F(0)}+|\nu|^2\\ &=|F(0)-\tau(\mu+\nu)|^2+|\nu|^2-|\mu+\nu|^2\\ &=|F(0)-\tau(\mu+\nu)|^2-\mu^2,\end{aligned}$$

we confirm (4.17). If

$$\mu=\frac{|F(0)-\tau\nu|^2}{2|\Re(\overline{\tau}F(0))|}=\frac{|p(0)-\nu|^2}{2\Re(p(0))}, \tag{4.21}$$

we then have from (4.20),

$$\begin{aligned}0 &\le \begin{bmatrix}p(0)-\nu & -\mu & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mu & \overline{p(0)}-\overline{\nu} & \dfrac{\overline{p(z)}-\overline{\nu}}{1-\tau\overline{z}}\\ p(0)-\nu & p(0)+\overline{p(0)} & p(0)+\overline{p(z)}\\ \dfrac{p(z)-\nu}{1-z\overline{\tau}} & p(z)+\overline{p(0)} & \dfrac{p(z)+\overline{p(z)}}{1-|z|^2}\end{bmatrix}\begin{bmatrix}\overline{p(0)}-\overline{\nu} & 0\\ -\mu & 0\\ 0 & 1\end{bmatrix}\\ &=\begin{bmatrix}0 & \overline{\Upsilon(z)}\\ \Upsilon(z) & \dfrac{p(z)+\overline{p(z)}}{1-|z|^2}\end{bmatrix},\end{aligned} \tag{4.22}$$

where

$$\Upsilon(z)=\frac{p(z)-\nu}{1-z\overline{\tau}}\cdot(\overline{p(0)}-\overline{\nu})-\mu\big(p(z)+\overline{p(0)}\big).$$

It follows from (4.22) that $\Upsilon(z)\equiv 0$ from which we have

$$\frac{p(z)-\nu}{p(z)+\overline{p(0)}}=\frac{(1-z\overline{\tau})\mu}{\overline{p(0)}-\overline{\nu})}=\frac{(1-z\overline{\tau})(p(0)-\nu)}{2\Re(p(0))},$$

where the second equality follows from (4.21). Solving the latter equality for p gives

$$\begin{aligned}p(z)&=\frac{-\overline{\tau}(p(0)-\nu)\overline{p(0)}z+p(0)(\overline{p(0)}-\overline{\nu})}{\overline{\tau}(p(0)-\nu)z+\overline{p(0)}-\overline{\nu}}\\ &=\frac{p(0)\gamma+\overline{p(0)}z}{\gamma-z},\quad \text{where}\quad \gamma=-\tau\cdot\frac{\overline{p(0)}-\overline{\nu}}{p(0)-\nu}\in\mathbb{T}.\end{aligned} \tag{4.23}$$

Observe that (4.23) can be written in the form

$$p(z)=i\Im(p(0))+\Re(p(0))\cdot\frac{\gamma+z}{\gamma-z}.$$

Replacing $p(0)$ by $\overline{\tau}F(0)$ in (4.23) and substituting the latter into (4.16) we arrive at (4.19). □

If $\nu=0$ in Theorem 4.4, we get the following result:

Corollary 4.5 *Let F be pseudo-contractive and let us assume that*

$$\lim_{z \overset{\text{nt}}{\to} \tau} \frac{F(z)-z}{(z-\tau)} = -\overline{\tau}^2\mu \quad (\mu \geq 0). \tag{4.24}$$

If $\mu = 0$ or $\Re(F(0)) = 0$, then $F(z) \equiv z$. If $\mu > 0$ and $\Re(F(0)) > 0$, then necessarily

$$|F(0) - \tau\mu| \leq \mu \quad \text{or, equivalently,} \quad \mu \geq \frac{|F(0)|^2}{2\Re(\overline{\tau}F(0))}.$$

If $\mu = \dfrac{|F(0)|^2}{2|\Re(\overline{\tau}F(0))|}$ then necessarily

$$F(z) = z - \frac{(z-\tau)^3\overline{\tau}^2|F(0)|^2}{\tau\overline{F(0)} + z\overline{\tau}^2F(0)}. = z - \frac{2\mu\cos\frac{\theta}{2}(z-1)^3}{e^{-\frac{i\theta}{2}} + ze^{\frac{i\theta}{2}}}. \tag{4.25}$$

We verify that for $\nu = 0$, the formula (4.19) amounts to (4.25)

$$F(z) = z + (z-\tau)^2\frac{|F(0)|^2(1-z\overline{\tau})}{\tau^2\overline{F(0)} + z\overline{\tau}F(0)} = z - \frac{(z-\tau)^3\overline{\tau}^2|F(0)|^2}{\tau\overline{F(0)} + z\overline{\tau}^2F(0)}$$

and observe that all the statements in Corollary 4.5 follow from the corresponding statements in Theorem 4.4. Let us consider the more restrictive case where F is a Schur-class function.

Theorem 4.6 *Let $F \in \mathcal{S}$ satisfy conditions (4.15). Let us assume that the limits $F''(\tau)$ and $F'''(\tau)$ exist and satisfy conditions*

$$\Re(\tau F''(\tau)) = 0 \quad \text{and} \quad \tau^2 F'''(\tau) \leq 0. \tag{4.26}$$

Let $\mu \geq 0$ and $\nu \in i\mathbb{R}$ be defined as in (4.18). Then

1. $\Re(\overline{\tau}F(0)) \geq |F(0)|^2$. *Moreover, if $\Re(\overline{\tau}F(0)) = |F(0)|^2$, then necessarily,*

$$F(z) = \frac{z - \tau^2\overline{F(0)}}{1 - 2\tau\overline{F(0)} + \overline{F(0)}z}. \tag{4.27}$$

2. $\mu \geq |\nu|^2$. *Moreover, if $\mu = |\nu|^2$, then necessarily,*

$$F(z) = \frac{z(\tau - 2F(0)) + \tau F(0)}{\tau - z\overline{\tau}F(0)}. \tag{4.28}$$

3. *If $\mu > |\nu|^2$ and $\Re(\overline{\tau}F(0)) > |F(0)|^2$, then necessarily*

$$\mu \geq \frac{|\overline{\tau}F(0) - \nu|^2 - |\nu|^2|F(0)|^2}{\Re(\overline{\tau}F(0)) - |F(0)|^2}, \tag{4.29}$$

or equivalently,

$$\left|F(0) - \frac{\tau(\mu+\nu)}{2\mu+1-|\nu|^2}\right| \leq \frac{\mu}{2\mu+1-|\nu|^2}. \tag{4.30}$$

4. *If (4.29) holds with equality, then F is unique and is a Blaschke product of degree two.*

Proof: We start with a generalized Schwarz-Pick inequality

$$\begin{bmatrix} \frac{1-|F(z_1)|^2}{1-|z_1|^2} & \frac{\partial}{\partial \overline{z}_1}\frac{1-|F(z_1)|^2}{1-|z_1|^2} & \frac{1-F(z_1)\overline{F(z_2)}}{1-z_1\overline{z}_2} & \frac{1-F(z_1)\overline{F(z)}}{1-z_1\overline{z}} \\ \frac{\partial}{\partial z_1}\frac{1-|F(z_1)|^2}{1-|z_1|^2} & \frac{\partial^2}{\partial z_1\partial \overline{z}_1}\frac{1-|F(z_1)|^2}{1-|z_1|^2} & \frac{\partial}{\partial z_1}\frac{1-F(z_1)\overline{F(z_2)}}{1-z_1\overline{z}_2} & \frac{\partial}{\partial z_1}\frac{1-F(z_1)\overline{F(z)}}{1-z_1\overline{z}} \\ \frac{1-F(z_2)\overline{F(z_1)}}{1-z_2\overline{z}_1} & \frac{\partial}{\partial \overline{z}_1}\frac{1-F(z_2)\overline{F(z_1)}}{1-z_2\overline{z}_1} & \frac{1-|F(z_2)|^2}{1-|z_2|^2} & \frac{1-F(z_2)\overline{F(z)}}{1-z_2\overline{z}} \\ \frac{1-F(z)\overline{F(z_1)}}{1-z\overline{z}_1} & \frac{\partial}{\partial \overline{z}_1}\frac{1-F(z)\overline{F(z_1)}}{1-z\overline{z}_1} & \frac{1-F(z)\overline{F(z_2)}}{1-z\overline{z}_2} & \frac{1-|F(z)|^2}{1-|z|^2} \end{bmatrix} \geq 0. \tag{4.31}$$

Letting $z_2 = 0$ and $z_1 \overset{\mathbf{nt}}{\to} \tau$ in (4.31) and making use of conditions (4.15) and (4.26), we get the inequality

$$\begin{bmatrix} 1 & \tau\overline{\nu} & 1-\tau\overline{F(0)} & \frac{1-\tau\overline{F(z)}}{1-\tau\overline{z}} \\ \overline{\tau}\nu & \mu & -\overline{F(0)} & \frac{\overline{z}-\overline{F(z)}}{(1-\tau\overline{z})^2} \\ 1-\overline{\tau}F(0) & -F(0) & 1-|F(0)|^2 & 1-F(0)\overline{F(z)} \\ \frac{1-\overline{\tau}F(z)}{1-\overline{\tau}z} & \frac{z-F(z)}{(1-z\overline{\tau})^2} & 1-F(z)\overline{F(0)} & \frac{1-|F(z)|^2}{1-|z|^2} \end{bmatrix} \geq 0 \tag{4.32}$$

holding for all $z \in \mathbb{D}$. From the latter positivity condition we get

$$1-|F(0)|^2 \geq |1-\overline{\tau}F(0)|^2 = |\tau - F(0)|^2, \tag{4.33}$$

which is equivalent to $\Re(\overline{\tau}F(0)) \geq |F(0)|^2$. If (4.33) holds with equality, we multiply the matrix in (4.32) by

$$T = \begin{bmatrix} 1-\overline{\tau}F(0) & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

on the left and by A^* on the right to get the positive matrix

$$\begin{bmatrix} 0 & \overline{\Upsilon(z)} \\ \Upsilon(z) & \frac{1-|F(z)|^2}{1-|z|^2} \end{bmatrix} \geq 0, \quad \text{where} \quad \Upsilon(z) = (1-\tau\overline{F(0)})\frac{1-\overline{\tau}F(z)}{1-\overline{\tau}z} - 1 + F(z)\overline{F(0)}.$$

It follows from the latter positivity that $\Upsilon \equiv 0$, which implies that in this case F is uniquely defined by formula (4.27).

The inequality $\mu \geq |\nu|^2$ follows from (4.32) since the leading 2×2 minor in the matrix on the left side is nonnegative. If $\mu = |\nu|^2$, we conclude from (4.32) that

$$\overline{\nu}\tau(1-\overline{\tau}F(0)) + F(0) = 0 \quad \text{and} \quad \overline{\nu}\tau \cdot \frac{1-\overline{\tau}F(z)}{1-\overline{\tau}z} = \frac{z-F(z)}{(1-z\overline{\tau})^2}.$$

Combining the latter relations leads us to (4.28).

Under the assumption $\Re(\overline{\tau}F(0)) > |F(0)|^2$, the positive semidefinitness of the 3×3 leading submatrix M of the matrix on the left side of (4.32) is equivalent to its determinant being nonnegative:

$$\det M = (\mu - |\nu|^2)(1-|F(0)|^2 - |\tau - F(0)|^2) - |F(0) + \overline{\nu}(\tau - F(0))|^2 \geq 0.$$

Routine computations (using in particular, the equality $\Re\nu = 0$) simplify the latter inequality to

$$|F(0)|^2(|\nu|^2 - 2\mu - 1) + \overline{\tau}F(0)(\mu + \overline{\nu}) + \tau\overline{F(0)}(\mu + \nu) - |\nu|^2 \geq 0,$$

which in turn, is equivalent to (4.29) and also to (4.30), as

$$\left|F(0)-\frac{\tau(\mu+\nu)}{2\mu+1-|\nu|^2}\right|^2\le\frac{|\tau(\mu+\nu)|^2-|\nu|^2}{(2\mu+1-|\nu|^2)^2}=\frac{\mu^2}{(2\mu+1-|\nu|^2)^2},\tag{4.34}$$

where the rightmost equality holds, since $\mu\in\mathbb{R}$, $\nu\in i\mathbb{R}$ and $|\tau|=1$. The last statement follows from general interpolation results: if (4.29) holds with equality, the leading submatrix M (considered above) is singular as its determinant equals zero. Since M has rank 2, the only Schur-class function satisfying conditions (4.15) and (4.18) is a Blaschke product of degree two. To construct it, one can take a column vector $\mathbf{c}\in\mathbb{C}^3\in\mathrm{KerM}$ (this vector is essentially unique as $\dim\mathrm{KerM}=1$, and then from (4.32) we see that

$$\begin{aligned}0&\equiv\begin{bmatrix}\frac{1-\overline{\tau}F(z)}{1-\overline{\tau}z} & \frac{z-F(z)}{(1-z\overline{\tau})^2} & 1-F(z)\overline{F(0)}\end{bmatrix}\mathbf{c}\\ &=c_1\frac{1-\overline{\tau}F(z)}{1-\overline{\tau}z}+c_2\frac{z-F(z)}{(1-z\overline{\tau})^2}+c_3(1-F(z)\overline{F(0)}).\end{aligned}$$

The function F is uniquely recovered from the latter identity. ◻

Remark 4.7 Let us note that under the assumptions of Theorem 4.6 (see (4.15), (4.18) and (4.26)),

$$\mu-|\nu|^2=-\frac{\tau^2}{6}F'''(\tau)+\frac{\tau^2}{4}F''(\tau)=-\frac{\tau^2}{6}S_F(\tau),$$

where S_F is the Schwarzian derivative of F. Part (2) in Theorem 4.6 is a particular case of a Tauraso-Vlacci result [25] stating that if $F\in\mathcal{S}$ admits boundary limits $F(\tau)\in\mathbb{T}$, $F'(\tau)$, $F''(\tau)$ and $F'''(\tau)$, such that $\Re(\tau F''(\tau)=|F'(\tau)|^2-|F'(\tau)|$ and if $S_F(\tau)=0$, then F is an automorphism of the unit disk.

Remark 4.8 We also note that the inequalities (4.17) and (4.30) define two disks – the set of possible values of $F(0)$ where F is respectively, in $\mathcal{PC}$ and $\mathcal{S}$. The disk (4.30) is properly included in the disk (4.17) as is well expected since $\mathcal{S}\subset\mathcal{PC}$.

Letting $\nu=0$ in Theorem 4.6, we get the following result.

Corollary 4.9 *Let $F\in\mathcal{S}$ satisfy the condition* (4.24) *and let us assume that $\mu>0$ and $\Re(\overline{\tau}F(0))>|F(0)|^2$. Then necessarily,*

$$\mu\ge\frac{|F(0)|^2}{\Re(\overline{\tau}F(0))-|F(0)|^2}=\frac{1}{2\Re\left(\frac{\tau-F(0)}{F(0)}\right)}$$

or equivalently,

$$\left|F(0)-\frac{\tau\mu}{2\mu+1}\right|\le\frac{\mu}{2\mu+1}\quad\textit{or, equivalently,}\quad\mu\ge\frac{1}{2\Re\left(\frac{1-F(0)}{F(0)}\right)}.$$

5 Cowen-Pommerenke inequalities

In this section we present the analog of Theorem 2.3 in the context of pseudo-contractions.

Theorem 5.1 *Let F be a pseudo-contraction with the Denjoy-Wolff point τ and boundary fixed points $\zeta_1,\ldots,\zeta_n\in\mathbb{T}$. Then for any $z\in\mathbb{D}\backslash\{\tau\}$,*

$$\sum_{i=1}^{n}\frac{|\zeta_i-\tau|^2}{|\zeta_i-z|^2(F'(\zeta_i)-1)}\le\frac{2}{1-|z|^2}\Re\biggl(\frac{(z-\tau)(1-z\overline{\tau})}{z-F(z)}\biggr). \tag{5.1}$$

In particular, if $\tau\neq 0$ (i.e., $F(0)\neq 0$), then

$$\sum_{i=1}^{n}\frac{|\zeta_i-\tau|^2}{F'(\zeta_i)-1}\le 2\Re\biggl(\frac{\tau}{F(0)}\biggr). \tag{5.2}$$

Proof: Let us take F in the form (3.12). Applying the multipoint Julia lemma to the Carathéodory function p at the fixed boundary points $\zeta_1,\ldots,\zeta_n$ and an arbitrary point $\zeta_{n+1}=z\in\mathbb{D}$, and taking into account that $p(\zeta_i)=0$ for $i=1,\ldots,n$, we conclude that the matrix $P=[p_{ij}]_{i,j=0}^{n+1}$ with the entries

$$\begin{aligned}
p_{ii}&=-\zeta_i p'(\zeta_i);\quad i=1,\ldots,n;\\
p_{ij}=\overline{p}_{ji}&=\frac{p(\zeta_i)+\overline{p(\zeta_j)}}{1-\zeta_i\overline{\zeta}_j}=0;\quad i\neq j;\ i,j=1,\ldots,n;\\
p_{i,n+1}=\overline{p}_{n+1,i}&=\frac{p(\zeta_i)+\overline{p(z)}}{1-\zeta_i\overline{z}}=\frac{\overline{p(z)}}{1-\zeta_i\overline{z}};\quad i=1,\ldots,n;\\
p_{n+1,n+1}&=\frac{p(z)+\overline{p(z)}}{1-|z|^2}
\end{aligned}$$

is positive semidefinite. Decomposing P as $P=\begin{bmatrix}A & B\\ B^* & p_{n+1,n+1}\end{bmatrix}$, we observe that condition $P\ge 0$ is equivalent to $p_{n+1,n+1}-B^*A^{-1}B\ge 0$, or, more explicitly, to

$$\frac{p(z)+\overline{p(z)}}{1-|z|^2}\ge\sum_{i=1}^{n}\frac{-|p(z)|^2}{|1-\zeta_i\overline{z}|^2\zeta_i p'(\zeta_i)}.$$

Write the latter inequality in the form

$$\frac{2\Re(\frac{1}{p(z)})}{1-|z|^2}\ge\sum_{i=1}^{n}\frac{1}{-|\zeta_i-z|^2\zeta_i p'(\zeta_i)}. \tag{5.3}$$

By (3.22) and (3.12),

$$\frac{1}{p(z)}=\frac{(z-\tau)(1-z\overline{\tau})}{z-F(z)}\quad\text{and}\quad-\zeta_i p'(\zeta_i)=\frac{F'(\zeta_i)-1}{|\zeta_i-\tau|^2}\quad(i=1,\ldots,n).$$

Substituting the latter equalities into (5.3) we get (5.1). Letting $z=0$ in (5.1) gives (5.2). □

If $z=\tau\in\mathbb{D}$, the inequality (5.1) still makes sense. Taking the limit as $z\to\tau$ and taking into account that

$$z-F(z)=(z-\tau)(1-F'(\tau))+O((z-\tau)^2),$$

we arrive at the following statement.

Corollary 5.2 *If $\tau\in\mathbb{D}$, then*

$$\sum_{i=1}^{n}\frac{1}{F'(\zeta_i)-1}\le 2\Re\left(\frac{1}{1-F'(\tau)}\right). \tag{5.4}$$

We next let $z=r\tau\in\mathbb{D}$ and take the limit on the right side of (5.1) as $r\to 1^-$. Since

$$z-F(z)=(z-\tau)(1-F'(\tau))+o((z-\tau)),\quad \frac{1-z\overline{\tau}}{1-|z|^2}\to\frac{1}{2},$$

and since in this case, $F'(\tau)$ is a real number, we get

Corollary 5.3 *If $\tau\in\mathbb{T}$ and $F'(\tau)<1$, then*

$$\sum_{i=1}^{n}\frac{1}{F'(\zeta_i)-1}\le\frac{1}{1-F'(\tau)}. \tag{5.5}$$

If $F'(\tau)=1$, the inequality (5.5) is not meaningful. To get a meaningful estimate for the sum on the left side of (5.5) we assume that the non-tangential limits

$$F''(\tau)=\lim_{z\to\tau}F''(z)\quad\text{and}\quad F'''(\tau)=\lim_{z\to\tau}F'''(z) \tag{5.6}$$

exist. Then it follows from the representation (3.12), that the limits $p(\tau)$ and $p'(\tau)$ also exist and satisfy

$$2p(\tau)=\tau F''(\tau),\quad 6p'(\tau)=\tau F'''(\tau).$$

Therefore, $\Re(\tau F''(\tau))\ge 0$ and in case $\Re(\tau F''(\tau))=0$, we have $\tau^2F'''(\tau)\le 0$, by the Julia lemma applied to the Carathéodory function p. If $\Re(\tau F''(\tau))\ge 0$, the expression on the left side of (5.6) cannot be estimated in terms of $F''(\tau)$; rigorous justification of the latter is beyond the scope of this paper.

Corollary 5.4 *If $\tau\in\mathbb{T}$ is the Denjoy-Wolff point of a pseudo-contraction F and if $F'(\tau)=1$, $\Re(\tau F''(\tau))=0$ and $\tau^2F'''(\tau)<0$, then*

$$\sum_{i=1}^{n}\frac{1}{F'(\zeta_i)-1}\le\frac{2}{3}\frac{|F'''(\tau)|}{|F''(\tau)|^2}. \tag{5.7}$$

Indeed, letting $z = r\tau$ and and taking into account the assumptions on $F''(\tau)$, $F'''(\tau)$, we get

$$\begin{aligned}\Re\left(\frac{(z-\tau)(1-z\overline{\tau})}{z-F(z)}\right) &= \Re\left(\frac{1}{\frac{\tau F''(\tau)}{2}+(r-1)\tau^2\frac{F'''(\tau)}{6}+o((1-r))}\right)\\ &= 2\frac{\Re(\tau F''(\tau))+\frac{r-1}{3}\Re(\tau^2F'''(\tau))+o((1-r))}{|F''(\tau)|^2+o((1))}\\ &= \frac{2(r-1)}{3}\cdot\frac{\tau^2F'''(\tau))+o((1-r))}{|F''(\tau)|^2+o((1))}.\end{aligned}$$

Therefore,

$$\lim_{z=\tau r\to\tau}\frac{2}{1-|z|^2}\Re\left(\frac{(z-\tau)(1-z\overline{\tau})}{z-F(z)}\right)=\frac{2}{3}\frac{|F'''(\tau)|}{|F''(\tau)|^2}.$$

Now we taking the limit as $z = r\tau \to \tau$ in (5.1) and get (5.7).

If $F''(\tau) = 0$, the inequality (5.7) is not informative. To handle this case, we assume that the non-tangential limits

$$F_j := \lim_{z\to\tau}\frac{F^{(j)}(z)}{j!}\quad(0\le j\le 5)\tag{5.8}$$

exist. We thus assume that

$$F_0=\tau,\quad F_1=1,\quad F_2=0,\quad \tau^2F_3<0.\tag{5.9}$$

Furthermore, if the limit F_4 exists for a pseudo-contraction F, then necessarily $2\Re(\tau^3F_4) \le -\tau^2F_3$. However, if $2\Re(\tau^3F_4) < -\tau^2F_3$, no estimates can be obtained in (5.13) in terms of F_4; justification of the latter is again beyond the scope of this paper. We assume that $2\Re(\tau^3F_4) = -\tau^2F_3$, in which case $\tau^3F_4+\tau^4F_5$ is necessarily non-negative.

Theorem 5.5 *Let $\tau \in \mathbb{T}$ be the Denjoy-Wolff point of a pseudo-contraction F such that the boundary limits* (5.9) *exist and let $\zeta_1, \dots, \zeta_n \in \mathbb{T}$ be other boundary fixed points of F. Then*

$$\sum_{i=1}^{n}\frac{|\zeta_i-\tau|^2}{|\zeta_i-z|^2(F'(\zeta_i)-1)}\le\frac{2}{1-|z|^2}\Re\left(\frac{(z-\tau)(1-z\overline{\tau})}{z-F(z)}\right)+\frac{1}{|\tau-z|^2\tau^2F_3}\tag{5.10}$$

for each $z \in \mathbb{D}$. In particular,

$$\sum_{i=1}^{n}\frac{|\zeta_i-\tau|^2}{F'(\zeta_i)-1}\le 2\Re\left(\frac{\tau}{F(0)}\right)+\frac{1}{\tau^2F_3}.\tag{5.11}$$

If in addition, the boundary limits F_4 and F_5 exist and are subject to relations

$$2\Re(\tau^3F_4)=-\tau^2F_3,\quad \tau^3F_4+\tau^4F_5\ge 0,\tag{5.12}$$

then

$$\sum_{i=1}^{n}\frac{1}{F'(\zeta_i)-1}\le\frac{\tau^3F_4+\tau^4F_5}{|F_3|^2}-\frac{|F_4|^2}{|F_3|^3}.\tag{5.13}$$

Proof: It follows from (3.12) and (5.9) that $p(\tau)=0$ and $\tau p'(\tau)<0$. Thus, the point τ is a boundary zero of p. By virtue of (5.3),

$$\frac{2\Re(\frac{1}{p(z)})}{1-|z|^2}\geq\sum_{i=1}^{n}\frac{1}{-|\zeta_i-z|^2\zeta_i p'(\zeta_i)}-\frac{1}{|\tau-z|^2\tau p'(\tau)},$$

which is equivalent to (5.10). Letting $z=0$ in (5.10) gives (5.11).

For $z=r\tau\in\mathbb{D}$, we have, on account of (5.9), (5.13),

$$\begin{aligned}
&(1-r)\cdot\left(2\Re\left(\frac{(z-\tau)(1-z\overline{\tau})}{z-F(z)}\right)+\frac{1-|z|^2}{|\tau-z|^2\tau^2F_3}\right)\\
&=-2\Re\left(\frac{1}{\tau^2F_3+(r-1)\tau^3F_4+(r-1)^2\tau^4F_5+o((1-r)^2)}\right)+\frac{1+r}{\tau^2F_3}\\
&=-\frac{2\tau^2F_3+2(r-1)\Re(\tau^3F_4)+2(r-1)^2\Re(\tau^4F_5)+o((1-r)^2)}{|F_3|^2+2(r-1)\Re(\overline{\tau}F_3\overline{F}_4)+(r-1)^2(2\Re(\overline{\tau}^2F_3\overline{F}_5)+|F_4|^2)+o((1-r)^2)}\\
&\quad+\frac{1+r}{\tau^2F_3}.
\end{aligned}$$

Making use of equalities

$$\frac{|F_3|^2}{\tau^2F_3}=\tau^2F_3,\qquad\frac{\Re(\overline{\tau}F_3\overline{F}_4)}{\tau^2F_3}=\Re(\tau^3F_4),\qquad\frac{\Re(\overline{\tau}^2F_3\overline{F}_5)}{\tau^2F_3}=\Re(\tau^4F_5),$$

and of equalities (5.12), we continue the above calculation as follows:

$$\begin{aligned}
&(1-r)\cdot\left(2\Re\left(\frac{(z-\tau)(1-z\overline{\tau})}{z-F(z)}\right)+\frac{1-|z|^2}{|\tau-z|^2\tau^2F_3}\right)\\
&=-\frac{(2r-6)\Re(\tau^3F_4)+2(r-1)^2\Re(\tau^4F_5)+o((1-r)^2)}{|F_3|^2+2(r-1)\Re(\overline{\tau}F_3\overline{F}_4)+(r-1)^2(2\Re(\overline{\tau}^2F_3\overline{F}_5)+|F_4|^2)+o((1-r)^2)}\\
&\quad+\frac{(1+r)\left[(2r-4)\Re(\tau^3F_4)+(r-1)^2\left(2\Re(\tau^4F_5)+\frac{|F_4|^2}{\tau^2F_3}\right)+o((1-r)^2)\right]}{|F_3|^2+2(r-1)\Re(\overline{\tau}F_3\overline{F}_4)+(r-1)^2(2\Re(\overline{\tau}^2F_3\overline{F}_5)+|F_4|^2)+o((1-r)^2)}\\
&=(1-r)^2\cdot\frac{2\Re(\tau^3F_4)+2r\Re(\tau^4F_5)+(1+r)\frac{|F_4|^2}{\tau^2F_3}}{|F_3|^2+o(1)}+o((1-r)^2)\\
&=2(1-r)^2\cdot\frac{\tau^3F_4+\tau^4F_5-\frac{|F_4|^2}{|F_3|}}{|F_3|^2}+o((1-r)^2).
\end{aligned}\tag{5.14}$$

Therefore,

$$\lim_{z=r\tau\to\tau}\left(\frac{2\Re\left(\frac{(z-\tau)(1-z\overline{\tau})}{z-F(z)}\right)}{1-|z|^2}+\frac{1}{|\tau-z|^2\tau^2F_3}\right)=\frac{\tau^3F_4+\tau^4F_5}{|F_3|^2}-\frac{|F_4|^2}{|F_3|^3}.$$

Taking the limit in (5.10) as $z=r\tau\to\tau$ we arrive at (5.13). □

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