6-27-2019

Composition of Rational Functions: State-Space Realization and Applications

Daniel Alpay
Chapman University, alpay@chapman.edu

Izchak Lewkowicz
Ben-Gurion University of the Negev

Composition of Rational Functions: State-Space Realization and Applications

Comments

This is a preprint draft of an article that was accepted for publication in *Linear Algebra and its Applications*, volume 580, in 2019 following peer review, and may not exactly replicate the final published version. The definitive publisher-authenticated version is available online at DOI: 10.1016/j.laa.2019.06.014.

Copyright

Elsevier

COMPOSITION OF RATIONAL FUNCTIONS: STATE-SPACE REALIZATION AND APPLICATIONS

DANIEL ALPAY AND IZCHAK LEWKOWICZ

ABSTRACT. We define two versions of compositions of matrix-valued rational functions of appropriate sizes and whenever analytic at infinity, offer a set of formulas for the corresponding state-space realization, in terms of the realizations of the original functions. Focusing on positive real functions, the first composition is applied to electrical circuits theory along with introducing a connection to networks of feedback loops. The second composition is applied to Stieltjes functions.

CONTENTS

1. INTRODUCTION

This work is focused on composition $F_L(F_R)$ of rational functions $F_L(z)$, $F_R(z)$, where the subscript stands for "left" and "right". In general composition of functions is classical, e.g. [22]. Although composition of rational functions plays an important role in the theory of dynamical systems (see e.g. [5], [12]), a few associated questions are yet unsolved. We here touch upon three aspects.

- Families of functions which are closed under composition.
- Applications of composition of functions.
- State realization of $F_L(F_R)$ in terms of the realizations of $F_L(z)$ and $F_R(z)$.

In principle there exist results on each of these items: For example, applications of compositions to electrical circuits theory was studied in [36]. Little is known in the setting of realization theory (besides the case where one of the function is a Moebius map; see for instance [10, Theorem 1.9, p. 35]). Realization of composition of, not necessarily rational, Q-functions was already addressed in [29].

In the present work we define and study two compositions of matrix-valued rational functions and whenever analytic at infinity, provide formulas for the respective state-space realization.

For the first composition we offer an application to electrical circuits. In turn, we introduce a connection with feedback-loop networks, see the various figures below. Implications of this novel idea go well beyond the scope of this work.

Stieltjes functions, were explored in [31] in the setting of moment problems; the Nevanlinna-Pick interpolation problem in this class was studied in that reference in the scalar case, and for the matrix-valued case see [2], [14, 15] and [24]. We here characterize their state-space realization and then show that both composition schemes, leave the family of Stieltjes functions invariant. Our motivation to consider Stieltjes functions stemmed in part from the following possible link with statistical physics. Positive measures σ on $(0, \infty)$ play an important role in statistical physics, as functions of repartition of energy levels of a particle (or, more generally, of a system). The associated Laplace transform

$$Z(\beta) = \int_0^\infty e^{-\beta e} d\sigma(e),$$

assumed convergent in $\operatorname{Re} \beta > 0$, is called the partition function. See for instance [32, p. 138], [39, p. 66]. When σ is discrete, with unit jumps at $E_1, E_2, \ldots$ we have

$$(1.1) \qquad\qquad\qquad Z(\beta) = \sum_j e^{-\beta E_j}.$$

One can associate with such a measure another object, namely the function φ defined by

$$(1.2) \qquad\qquad\qquad \varphi_\sigma(z) = \int_0^\infty \frac{z}{t - iz} d\sigma(t),$$

provided σ satisfies

$$(1.3) \qquad\qquad\qquad \int_0^\infty \frac{d\sigma(t)}{1 + t} < \infty$$

The function φ_σ is a Stieltjes function and the study of compositions of such functions, associating to two measures σ_1 and σ_2 on $(0, \infty)$ a third measure corresponding to the composition $\varphi_{\sigma_1}(\varphi_{\sigma_2})$ (together with possibly an imaginary constant; see formula (6.3)) should have some physical interpretation, in particular in the case of discrete finite measures.

2. Realization of rational functions analytic at infinity

We first recall that a $p \times m$-valued function $F(z)$, analytic at infinity, can be written in the form

$$(2.1) \qquad\qquad\qquad F(z) = D + C(zI_n - A)^{-1}B,$$

where $D = F(\infty)$ and where A, B, C are matrices of appropriate sizes. Expression (2.1) is called a realization. Sometimes we shall find it convenient to use, the same A, B, C, D, the engineering shorthand notation, (introduced by H.H. Rosenbrock, see e.g. [37, Chapter 1, Section 2]) of a $(n + p) \times (n + m)$ realization array[1] R_F,

$$(2.2) \qquad R_F = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

Whenever for a given F, analytic at infinity, the dimension of A in (2.1), (2.2) is the smallest possible, the realization is called *minimal* and the dimension of A is *the McMillan degree of $F(z)$*. In this case, the realization is unique up to a change of coordinates meaning that for a $n \times n$ non-singular matrix S,

$$(2.3) \qquad \left(\begin{smallmatrix} S & 0 \\ 0 & I_p \end{smallmatrix} \right)^{-1} \underbrace{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)}_{R_F} \left(\begin{smallmatrix} S & 0 \\ 0 & I_m \end{smallmatrix} \right),$$

is another minimal realization of $F(z)$ similar to R_F. In particular, the spectrum of the A part, is preserved.

Up to this point, the above realization description is a classical textbook material and we refer the reader to [6, Section 3.4], [10], [38, Section 6.4 and Remark 6.7.4].

For future reference we cite additional known results

Proposition 2.1. *Let*

$$F(z) = C(zI - A)^{-1}B + D$$

be a $p \times p$-valued rational function, where D is non-singular. Then, $(F(z))^{-1}$ is well defined and a realization array associated with it, i.e. with

$$(2.4) \qquad (F(z))^{-1} = \left(C(sI - A)^{-1}B + D \right)^{-1} := C_{\text{inv}}(sI - A_{\text{inv}})^{-1}B_{\text{inv}} + D_{\text{inv}},$$

can be written as

$$(2.5) \qquad \left(\begin{array}{c|c} A_{\text{inv}} & B_{\text{inv}} \\ \hline C_{\text{inv}} & D_{\text{inv}} \end{array} \right) = \left(\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array} \right).$$

For proof see e.g. [11, Theorem 2.4].

Remark 2.2. One can re-write Eq. (2.5) as

$$\left(\begin{smallmatrix} A_{\text{inv}} & B_{\text{inv}} \\ C_{\text{inv}} & D_{\text{inv}} \end{smallmatrix} \right) = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix} \right) + \left(\begin{smallmatrix} -B \\ I_p \end{smallmatrix} \right) D^{-1} \left(\begin{smallmatrix} C & I_p \end{smallmatrix} \right).$$

The following known result, see e.g. [11, Section 2.5], will also be useful.

Proposition 2.3. *Let*

$$F_1(z) = C_1(sI - A_1)^{-1}B_1 + D_1 \quad \text{and} \quad F_2(z) = C_2(sI - A_2)^{-1}B_2 + D_2$$

be $p \times k$ and $k \times m$-valued rational functions, respectively. Then, $F_1 F_2$ is a $p \times m$-valued rational function whose state-space realization may be given by

$$(2.6) \qquad R_{F_1 F_2} = \left(\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array} \right).$$

---

[1]R stands for "realization" or "Rosenbrock".

We next recall in the tensor (a.k.a. Kronecker) product $M \otimes N$ of a pair of matrices $M \in \mathbb{C}^{p \times l}$ and $N \in \mathbb{C}^{m \times q}$ so that $M \otimes N$ is of dimensions $mp \times lq$ and takes the form[2]

$$M \otimes N := \begin{pmatrix} m_{11}N & m_{12}N & \cdots & m_{1,l}N \\ m_{21}N & m_{22}N & \cdots & m_{2,l}N \\ \vdots & \vdots & \cdots & \vdots \\ m_{p1}N & m_{p2}N & \cdots & m_{p,l}N \end{pmatrix}.$$

For more information, see e.g. [27, Chapter 4].

We next formulate the focal problem of this work.

Problem formulation

Let $F_L(z)$ be[3] a $p \times p$-valued rational function of McMillan degree n and let $F_R(z)$ be a $q \times q$-valued rational function of McMillan degree m. Their minimal realization arrays are $(n+p) \times (n+p)$ and $(m+q) \times (m+q)$, respectively

$$(2.7) \qquad R_L = \left(\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array} \right) \qquad R_R = \left(\begin{array}{c|c} A_R & B_R \\ \hline C_R & D_R \end{array} \right),$$

i.e.

$$(2.8) \quad F_L(z) = D_L + C_L \left(zI_n - A_L \right)^{-1} B_L, \qquad F_R(z) = D_R + C_R(zI_m - A_R)^{-1}B_R.$$

Assuming that $F_L(F_R)$ a composition of these functions, is well defined, we seek a formula for a state-space realization of this composition[4],

$$(2.9) \qquad F_L(F_R(z)) = D_{\text{comp}} + C_{\text{comp}}(zI_k - A_{\text{comp}})^{-1}B_{\text{comp}},$$

in terms of the realization of the original systems (2.7), for some natural k, i.e. a corresponding realization array is,

$$(2.10) \qquad R_{F_L(F_R)} = \left(\begin{array}{c|c} \overbrace{A_{\text{comp}}}^{k} & B_{\text{comp}} \\ \hline C_{\text{comp}} & D_{\text{comp}} \end{array} \right).$$

In particular, find both: k, see (2.9), (2.10), the dimension of the realization of the composed system and the corresponding McMillan degree.

Remark 2.4. a. Even when one starts with minimal realizations of F_L and F_R, of McMillan degrees n and m respectively, k the dimension of the realization, see (2.9), (2.10) is not necessarily minimal, i.e. k may be bigger than the McMillan degree of the composition.

b. In Section 3 one obtains that in the realization, see (2.9), (2.10)

$$k = mn,$$

and in fact, this is the McMillan degree of the composed function, see Remark 3.8.
In contrast, in Section 5

$$k \leq n.$$

---

[2] $M = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1,l} \\ m_{21} & m_{22} & \cdots & m_{2,l} \\ \vdots & \vdots & \cdots & \vdots \\ m_{p1} & m_{p2} & \cdots & m_{p,l} \end{pmatrix}$

[3] Recall, the subscript stands for "Left" and "Right".

[4] The subscript stands for "composition".

3. Composition of functions - first version

Presentation of the first version of composition of functions, is split into three subcases.

3.1. The case where $f_R(z)$ is scalar.

Proposition 3.1. *Let* $f_R(z)$ *be a scalar-valued[5] rational function, of the form,*

$$f_R(z) = d_R + c_R \left(z I_m - A_R \right)^{-1} b_R \ ,$$

then

$$F_l \left(f_R(z) \right) = D_L + C_L \left(f_R(z) I_n - A_L \right)^{-1} B_L$$

admits a state space realization of the form

$$F_l \left(f_R(z) \right) = D_{\text{comp}} + C_{\text{comp}} (z I_k - A_{\text{comp}})^{-1} B_{\text{comp}} \ ,$$

of state dimension

$$k = mn,$$

and a realization array of the form

$$\left(\begin{array}{c|c} A_{\text{comp}} & B_{\text{comp}} \\ \hline C_{\text{comp}} & D_{\text{comp}} \end{array} \right) = \left(\begin{array}{c|c} I_n \otimes A_R - (I_n \otimes b_R)(d_R I_n - A_L)^{-1}(I_n \otimes c_R) & -(I_n \otimes b_R)(d_R I_n - A_L)^{-1} B_L \\ \hline C_L (d_R I_n - A_L)^{-1}(I_n \otimes c_R) & D_L + C_L (d_R I_n - A_L)^{-1} B_L \end{array} \right) .$$

Proof : By construction,

$$F_L(f_R) \quad = \quad D_L + C_L \left(f_R(z) I_n - A_L \right)^{-1} B_L$$

$$= \quad D_L + C_L \left(\begin{pmatrix} f_R(z) & & \\ & \ddots & \\ & & f_R(z) \end{pmatrix} - A_L \right)^{-1} B_L$$

$$= \quad D_L + C_L \left(\begin{pmatrix} d_R + c_R(z I_m - A_R)^{-1} b_R & & \\ & \ddots & \\ & & d_R + c_R(z I_m - A_R)^{-1} b_R \end{pmatrix} - A_L \right)^{-1} B_L$$

$$= \quad D_L + C_L \left(d_R I_n - A_L + \begin{pmatrix} c_R(z I_m - A_R)^{-1} b_R & & \\ & \ddots & \\ & & c_R(z I_m - A_R)^{-1} b_R \end{pmatrix} \right)^{-1} B_L$$

$$= \quad D_L + C_L \left((d_R I_n - A_L) + I_n \otimes \left(c_R \left(z I_m - A_R \right)^{-1} b_R \right) \right)^{-1} B_L$$

$$= \quad D_L + C_L \left(d_R I_n - A_L + (I_n \otimes c_R)(I_n \otimes (z I_m - A_R)^{-1})(I_n \otimes b_R) \right)^{-1} B_L$$

$$= \quad D_L + C_L \left(\underbrace{d_R I_n - A_L}_{\hat{D}} + \underbrace{(I_n \otimes c_R)}_{\hat{C}} \left(z I_{nm} - \underbrace{I_n \otimes A_R}_{\hat{A}} \right)^{-1} \underbrace{(I_n \otimes b_R)}_{\hat{B}} \right)^{-1} B_L$$

$$= \quad D_L + C_L \left(\hat{D}^{-1} + \hat{D}^{-1}\hat{C} \left(z I_{nm} - (\hat{A} + \hat{B}\hat{D}^{-1}\hat{C}) \right)^{-1} (-\hat{B}\hat{D}^{-1}) \right) B_L$$

---

[5]To ease reading, scalar functions are denoted by small letters.

$$= \underbrace{D_L + C_L \hat{D}^{-1} B_L}_{D_{\text{comp}}} + \underbrace{C_L \hat{D}^{-1} \hat{C}}_{C_{\text{comp}}} \left(zI_{nm} - \underbrace{(\hat{A} + \hat{B} \hat{D}^{-1} \hat{C})}_{A_{\text{comp}}} \right)^{-1} \underbrace{(-\hat{B} \hat{D}^{-1} B_L)}_{B_{\text{comp}}},$$

where we have used Proposition 2.1 with

$$\hat{A} := I_n \otimes A_R \qquad \hat{B} = I_n \otimes b_R \qquad \hat{C} := I_n \otimes c_R \qquad \hat{D} := d_R I_n - A_L,$$

and thus the construction is complete.   □

Remark 3.2. One can re-write the last result as,

$$\left(\begin{array}{c|c} A_{\text{comp}} & B_{\text{comp}} \\ \hline C_{\text{comp}} & D_{\text{comp}} \end{array} \right) = \begin{pmatrix} I_n \otimes A_R & 0 \\ 0 & D_L \end{pmatrix} + \begin{pmatrix} -I_n \otimes b_R \\ C_L \end{pmatrix} \left(d_R I_n - A_L \right)^{-1} \left(I_n \otimes c_R \;\; B_L \right).$$

3.2. **The case where A_L is diagonalizable.** Here diagonalizability assumption of A_L, the state matrix associated with $F_L(z)$ essentially reduces the problem to a composition by a sum of *degree one* rational functions. The details are as follows.

We start by diverting a little, and exploit diagonalizability of A to obtain a result whose applicability is well beyond the scope of this work.

Proposition 3.3. *Let $A \in \mathbb{C}^{n \times n}$ be a diagonalizable matrix and let $a_1, \ldots, a_\nu \in \mathbb{C}$, for some $\nu \in [1, n]$, be its distinct eigenvalues. Denote by $n_1, \ldots, n_\nu$ the corresponding algebraic multiplicity, i.e. for some (non-unique) non-singular $V \in \mathbb{C}^{n \times n}$,*

$$(3.1) \qquad A = V^{-1} \begin{pmatrix} a_1 I_{n_1} & & & \\ & a_2 I_{n_2} & & \\ & & \ddots & \\ & & & a_\nu I_{n_\nu} \end{pmatrix} V \qquad \sum_{j=1}^{\nu} n_j = n.$$

(I) Let B a $n \times m$ matrix. Then, the pair A, B is controllable if and only if, with V from (3.1) one can write

$$(3.2) \qquad V^{-1} B = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_\nu \end{pmatrix} \begin{array}{l} \} \, n_1 \\ \} \, n_2 \\ \\ \} \, n_\nu \end{array}$$

where each of the matrices $\beta_1, \ldots, \beta_\nu$ is of a full rank.

In particular, $m \geq \max(n_1, n_2, \ldots, n_\nu)$.

(II) Let C be a $p \times n$ matrix. Then, the pair A, C is observable if and only if, with V from (3.1) one can write

$$(3.3) \qquad CV = (\underbrace{\gamma_1}_{n_1} \;\; \underbrace{\gamma_2}_{n_2} \;\; \cdots \;\; \underbrace{\gamma_\nu}_{n_\nu})$$

where each of the matrices $\gamma_1, \cdots, \gamma_\nu$ is of a full rank.

In particular, $p \geq \max(n_1, n_2, \ldots, n_\nu)$.

(III) Let

$$F(z) = C(zI_n - A)^{-1} B + D,$$

be a $p \times m$-valued rational function where A, B, C are as above. R_F, an $(n+p) \times (n+m)$ realization array of $F(z)$,

$$R_F = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right),$$

is minimal, if and only if each of the above 2ν matrices, β_1 , ... , β_ν and γ_1 , ... , γ_ν in (3.2) and (3.3) respectively, is of a full-rank.

Proof : (I) By the P-B-H eigenvector's test, see e.g. [28, Theorem 6.2-5], a pair A, B is *uncontrollable*, if and only if (up to relabeling the eigenvalues of A) there exists $0 \neq v \in \mathbb{C}^n$ so that

(3.4) $$v^*A = a_1 v^* \qquad \text{and} \qquad v^*B = 0.$$

Using (3.1) one can write

$$u^* := v^*V^{-1} \qquad \text{with} \qquad u = \begin{pmatrix} u_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad 0 \neq u_1 \in \mathbb{C}^{n_1}.$$

Substituting in (3.2), controllability means that

$$u_1^* \beta_1 \neq 0,$$

and since u_1 is arbitrary, one may conclude that the rank of the $n_1 \times m$ matrix β_1 is at least n_1. Since similar reasoning can be applied with $j = 2, \dots . \nu$, this part of the claim is established.

Item (II) follows from item (I) by controllability-observability duality.

Item (III) follows from the first two items by recalling that a realization is minimal if and only if it is both controllable and observable. $\square$

For a diagonalizable matrix A, the eigenvalues-eigenspaces description of in (3.1) is the best known. However, it is inherently non-unique, i.e. one can also write,

$$A = (WV)^{-1} \begin{pmatrix} a_1 I_{n_1} & & & \\ & a_2 I_{n_2} & & \\ & & \ddots & \\ & & & a_\nu I_{n_\nu} \end{pmatrix} WV \qquad W := \begin{pmatrix} W_1 & & & \\ & W_2 & & \\ & & \ddots & \\ & & & W_\nu \end{pmatrix} \qquad \begin{array}{c} \sum\limits_{j=1}^{\nu} n_j = n \\ W_j \in \mathbb{C}^{n_j \times n_j} \quad \text{non-singular.} \end{array}$$

We next introduce a *unique* eigenvalues-eigenspaces description of a diagonalizable matrix A, to be used in the sequel. This is an extended version of a classical result, see e.g. [25, Ch. 6, Thms. 8 & 9]

Lemma 3.4. *Let $A \in \mathbb{C}^{n \times n}$ be a diagonalizable matrix and let $a_1, \dots, a_\nu \in \mathbb{C}$, (with $\nu \in [1, n]$) be its distinct eigenvalues, i.e. ν is the degree of the minimal polynomial associated with A.*
Then, there exist (oblique) projections[6], $\Pi_1, \dots, \Pi_\nu$ satisfying,

$$\Pi_j \Pi_k = \begin{cases} \Pi_j & j=k \\ 0_n & j \neq k \end{cases} \qquad \sum_{j=1}^{\nu} \Pi_j = I_n ,$$

---

[6]For $j = 1, \dots , \nu$ the rank of Π_j is equal to the algebraic multiplicity of the corresponding a_j.

so that one can write,

$$A = \sum_{j=1}^{\nu} \Pi_j a_j .$$

Furthermore, this presentation is unique.

It now follows that the pencil associated with A can be written as,

$$(I_n z - A)^{-1} = \sum_{j=1}^{\nu} \Pi_j (z - a_j)^{-1} \Pi_j .$$

Note that using (3.1) the projections in Lemma 3.4 are actually given by,

$$\Pi_1 = V^{-1} \begin{pmatrix} I_{n_1} & & & \\ & 0 \cdot I_{n_2} & & \\ & & \ddots & \\ & & & 0 \cdot I_{n_\nu} \end{pmatrix} V \quad \cdots \quad \Pi_\nu = V^{-1} \begin{pmatrix} 0 \cdot I_{n_1} & & & \\ & 0 \cdot I_{n_2} & & \\ & & \ddots & \\ & & & I_{n_\nu} \end{pmatrix} V.$$

We can now use Lemma 3.4 to obtain a convenient state space realization of a rational function.

Lemma 3.5. *Let $F(z)$ be a $p \times m$-valued rational function and assume that the associated $n \times n$ state-matrix A, is diagonalizable.*
(I)   Denote by $a_1, \ldots, a_n$ the eigenvalues (including multiplicity) of A.
Then, there exist rank-one (oblique) projections $\Pi_1, \ldots \Pi_n$ satisfying,

$$\Pi_j \Pi_k = \begin{cases} \Pi_j & j=k \\ 0_n & j \neq k \end{cases} \qquad \sum_{j=1}^{n} \Pi_j = I_n ,$$

so that $F(z)$ admits a unique minimal realization of the form,

$$
\begin{aligned}
F(z) \quad &= \quad D + C \sum_{j=1}^{n} \Pi_j (z - a_j)^{-1} \Pi_j B \\
&= \quad D + \sum_{j=1}^{n} C_j (z - a_j)^{-1} B_j \\
&= \quad D + \sum_{j=1}^{n} (z - a_j)^{-1} C_j B_j \\
&= \quad D + \sum_{j=1}^{n} C_j B_j (z - a_j)^{-1}
\end{aligned}
$$

(3.5)

$$
\begin{aligned}
&j = 1, \ldots, n, \\
&B_j := \Pi_j B \quad n \times m \\
&\sum_{j=1}^{n} B_j = B \\
&C_j := C\Pi_j \quad p \times n, \\
&\sum_{j=1}^{n} C_j = C.
\end{aligned}
$$

(II)   For some $\nu \in [1, n]$, denote by $\hat{a}_1, \ldots, \hat{a}_\nu$, the distinct eigenvalues of A.
Then, there exist (oblique) projections[7] $\hat{\Pi}_1, \ldots \hat{\Pi}_\nu$ satisfying,

$$\hat{\Pi}_j \hat{\Pi}_k = \begin{cases} \hat{\Pi}_j & j=k \\ 0_n & j \neq k \end{cases} \qquad \sum_{j=1}^{\nu} \hat{\Pi}_j = I_n ,$$

---

[7]For $j = 1, \ldots, \nu$ the degree of the projection $\hat{\Pi}_j$ is equal to the algebraic multiplicity of $\hat{a}_j$.

so that $F(z)$ admits a minimal realization of the form,

$$F(z) = D + C \sum_{j=1}^{\nu} \hat{\Pi}_j (z - \hat{a}_j)^{-1} \hat{\Pi}_j B$$

$$= D + \sum_{j=1}^{\nu} C_j (z - \hat{a}_j)^{-1} B_j$$

(3.6)

$$\begin{array}{ll} & j=1, \ldots, \nu, \\ & B_j := \hat{\Pi}_j B \quad n \times m \\ & \sum_{j=1}^{\nu} B_j = B \end{array}$$

$$= D + \sum_{j=1}^{\nu} (z - \hat{a}_j)^{-1} C_j B_j$$

$$\begin{array}{l} C_j := C\hat{\Pi}_j \quad p \times n, \\ \sum_{j=1}^{\nu} C_j = C. \end{array}$$

$$= D + \sum_{j=1}^{\nu} C_j B_j (z - \hat{a}_j)^{-1}$$

Furthermore, if $\nu = n$ or when $m = p = 1$, this presentation is unique.

Recalling that minimality of realization is preserved under change of coordinates, see (2.3), we next exploit Lemma 3.4 to specify a minimal realization of a rational function to be used in the sequel.

One can now apply part (I) of Lemma 3.5 to $F_L(z)$ and consider a composition $F_L(F_R)$.

Proposition 3.6. *Consider the system in the problem formulation assuming that:*
(i) A_L, the $n \times n$ state-matrix associated with $F_L(z)$ is diagonalizable,
(ii) The eigenvalues of A_L (including multiplicity) denoted by $a_1, \ldots, a_n \in \mathbb{C}$, are so that the $q \times q$ matrices

(3.7) $$\Delta_j := D_R - a_j I_q \qquad j = 1, \ldots, n,$$

are all non-singular.

In each of the three following cases of composition $F_L(F_R)$, one obtains, a realization of as in Eqs. (2.9), and (2.10) i.e.

$$F_L(F_R(z)) = C_{\text{comp}} (z I_k - A_{\text{comp}})^{-1} B_{\text{comp}} + D_{\text{comp}},$$

where $k = mn$ and

(3.8) $$A_{\text{comp}} = \begin{pmatrix} A_R - B_R \Delta_1^{-1} C_R & & \\ & \ddots & \\ & & A_R - B_R \Delta_n^{-1} C_R \end{pmatrix} \quad \text{and} \quad D_{\text{comp}} = D_L + \sum_{j=1}^{n} C_j \Delta_j^{-1} B_j .$$

(I) If $q = n$, namely $F_R(z)$ is $n \times n$-valued, then in (2.9) and (2.10)

(3.9) $$B_{\text{comp}} = - \begin{pmatrix} B_R \Delta_1^{-1} B_1 \\ \vdots \\ B_R \Delta_1^{-1} B_n \end{pmatrix} \qquad C_{\text{comp}} = \begin{pmatrix} C_1 \Delta_1^{-1} C_R & \cdots & C_n \Delta_n^{-1} C_R \end{pmatrix} .$$

(II) If $q = p$, namely both $F_L(z)$ and $F_R(z)$ are $p \times p$-valued, then in (2.9) and (2.10),

(3.10) $$B_{\text{comp}} = - \begin{pmatrix} B_R \Delta_1^{-1} C_1 B_1 \\ \vdots \\ B_R \Delta_n^{-1} C_n B_n \end{pmatrix} \qquad C_{\text{comp}} = \begin{pmatrix} \Delta_1^{-1} C_R & \cdots & \Delta_n^{-1} C_R \end{pmatrix} ,$$

or

$$(3.11) \qquad B_{\text{comp}} = - \begin{pmatrix} B_R \Delta_1^{-1} \\ \vdots \\ B_R \Delta_n^{-1} \end{pmatrix} \qquad C_{\text{comp}} = \begin{pmatrix} C_1 B_1 \Delta_1^{-1} C_R & \cdots & C_n B_n \Delta_n^{-1} C_R \end{pmatrix}.$$

(III)   Assume that $p = 1$ i.e. $f_L(z)$ is scalar-valued and $F_R(s)$ is $q \times q$-valued where q is a parameter. Here we define n scalars

$$(3.12) \qquad \eta_j := C_j B_j \qquad j = 1, \ \ldots \ , \ n,$$

and then in (2.9) and (2.10),

$$(3.13) \qquad B_{\text{comp}} = - \begin{pmatrix} B_R \Delta_1^{-1} \\ \vdots \\ B_R \Delta_n^{-1} \end{pmatrix} \qquad C_{\text{comp}} = \begin{pmatrix} \eta_1 \Delta_1^{-1} C_R & \cdots & \eta_n \Delta_n^{-1} C_R \end{pmatrix}.$$

Proof of Proposition 3.6   We here find it convenient to introduce an auxiliary function $\tilde{F}_L$ and to apply (3.5) to it, i.e.

$$\tilde{F}_L(z) = D_L + \sum_{j=1}^{n} \gamma_j \, (z - a_j)^{-1} \, \beta_j \ ,$$

where the parameters $\beta_1 \ \ldots \ , \ \beta_n$ and $\gamma_1 \ , \ \ldots \ , \ \gamma_n$ will be defined in the sequel.

We now consider realization of composition of these functions namely, using Eq. (3.7),

$$\tilde{F}_L(F_R) \quad = \quad D_L + \sum_{j=1}^{n} \gamma_j \, (F_R - a_j I_q)^{-1} \, \beta_j$$

$$(3.14) \qquad = \quad D_L + \sum_{j=1}^{n} \gamma_j \underbrace{\left(C_R (z I_m - A_R)^{-1} B_R + D_R - a_j I_q \right)^{-1}}_{(F_R(z) - a_j I_q)^{-1}} \beta_j$$

$$= \quad D_L + \sum_{j=1}^{n} \gamma_j \left(C_R (z I_m - A_R)^{-1} B_R + \Delta_j \right)^{-1} \beta_j \ .$$

Since by assumption the $q \times q$ matrices $\Delta_1, \ \ldots \ , \ \Delta_n$ are all non-singular, using Proposition 2.1, one can equivalently write Eq. (3.14) as

$$\tilde{F}_L(F_R) \quad = \quad D_L + \sum_{j=1}^{n} \gamma_j \left((\Delta_j^{-1} C_R) \left((z I_m - (A_R - B_R \Delta_j^{-1} C_R))^{-1} (-B_R \Delta_j^{-1}) + \Delta_j^{-1} \right) \beta_j \right.$$

$$= \quad D_L + \sum_{j=1}^{n} \gamma_j \Delta_j^{-1} \beta_j + \sum_{j=1}^{n} \left(\gamma_j \Delta_j^{-1} C_R \right) \left(z I_m - (A_R - B_R \Delta_j^{-1} C_R) \right)^{-1} \left(B_R \Delta_j^{-1} \beta_j \right) \ .$$

This can be compactly written as

$$(3.15) \quad R_{\tilde{F}_L(F_R)} = \left(\begin{array}{c|c} A_{\text{comp}} & B_{\text{comp}} \\ \hline C_{\text{comp}} & D_{\text{comp}} \end{array} \right) = \left(\begin{array}{ccc|c} A_R - B_R \Delta_1^{-1} C_R & & & -B_R \Delta_1^{-1} \beta_1 \\ & \ddots & & \vdots \\ & & A_R - B_R \Delta_n^{-1} C_R & -B_R \Delta_n^{-1} \beta_n \\ \hline \gamma_1 \Delta_1^{-1} C_R & \cdots & \gamma_n \Delta_n^{-1} C_R & D_L + \sum_{j=1}^{n} \gamma_j \Delta_j^{-1} \beta_j \end{array} \right).$$

(I) To obtain (3.9) substitute

$$\beta_j = B_j \qquad \gamma_j = C_j \qquad j = 1, \ \ldots \ , \ n.$$

(II)(a) To obtain (3.10) substitute

$$\beta_j = C_j B_j \qquad \gamma_j \equiv 1 \qquad j = 1, \ \ldots \ , \ n.$$

(II)(b) To obtain (3.11) substitute

$$\beta_j \equiv 1 \qquad \gamma_j = C_j B_j \qquad j = 1, \ \ldots \ , \ n.$$

(III) To obtain (3.13) substitute

$$\beta_j \equiv 1 \qquad \gamma_j = C_j B_j = \eta_j \qquad j = 1, \ \ldots \ , \ n,$$

so the construction is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3.3. The case where $f_L(z)$ is scalar with A_L is non-diagonalizable.

Here, one can still obtain realization of composition of functions. However, the technical details are not as elegant as the diagonalizable case. For simplicity of exposition this is illustrated through an example.

Example 3.7. Consider the case where

$$f_L(z) = d_L + \tfrac{1}{(s+a)^2} \qquad\quad a > 0 \quad d_L \in \mathbb{R} \quad \text{parameters}[8].$$

To see that here, A_L is not diagonalizable, recall that a minimal realization of f_L may be given by,

$$R_{f_L} = \left(\begin{array}{cc|c} -a & 1 & 0 \\ 0 & -a & 1 \\ \hline 1 & 0 & d_L \end{array} \right).$$

Let now $F_R(z)$ be a $q \times q$-valued rational function. Then, the composition $f_l(F_R)$ is given by

$$f_l(F_R) = d_L I_q + (aI_q + F_R)^{-2}$$

and we compute, in stages, a corresponding state-space realization.

First, a realization of $F_R + aI_q$ is trivially given by

$$R_{F_R + aI_q} = \left(\begin{array}{c|c} A_R & B_R \\ \hline C_R & D_R + aI_q \end{array} \right)$$

and following Proposition 2.1 one has that,

$$R_{(F_R + I_q)^{-1}} = \left(\begin{array}{c|c} A_R - B_R(D_R + aI_q)^{-1}C_R & -B_R(D_R + aI_q)^{-1} \\ \hline (D_R + aI_q)^{-1}C_R & (D_R + aI_q)^{-1} \end{array} \right).$$

Next, following Proposition 2.3

$$R_{(F_R + aI_q)^{-2}} = \left(\begin{array}{cc|c} A_R - B_R(D_R + aI_q)^{-1}C_R & -B_R(D_R + aI_q)^{-2}C_R & -B_R(D_R + aI_q)^{-2} \\ 0 & A_R - B_R(D_R + aI_q)^{-1}C_R & -B_R(D_R + aI_q)^{-1} \\ \hline (D_R + aI_q)^{-1}C_R & (D_R + aI_q)^{-2}C_R & (D_R + aI_q)^{-2} \end{array} \right)$$

and finally,

$$R_{f_L(F_R)} = \left(\begin{array}{cc|c} A_R - B_R(D_R + aI_q)^{-1}C_R & -B_R(D_R + aI_q)^{-2}C_R & -B_R(D_R + aI_q)^{-2} \\ 0 & A_R - B_R(D_R + aI_q)^{-1}C_R & -B_R(D_R + aI_q)^{-1} \\ \hline (D_R + aI_q)^{-1}C_R & (D_R + aI_q)^{-2}C_R & d_L I_q + (D_R + aI_q)^{-2} \end{array} \right).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

---

[8]In Section 4 we shall use the fact that $f_L(z)$ is positive real whenever $d_L \geq \frac{1}{8a^2}$.

Remark 3.8. In Propositions 3.1 and 3.6 and in Example 3.7, the degree of the realization of $F_L(F_R)$, the composition of functions, is (up to minimality) equal to the product of the McMillan degrees of the original realizations.

4. APPLICATIONS TO ELECTRICAL CIRCUITS AND TO FEEDBACK-LOOP NETWORKS

In the sequel we shall denote by[9] $\mathbb{C}_L$, $\mathbb{C}_R$ the open left, right, halves of the complex plane (and by $\overline{\mathbb{C}}_R$ the closed right half of the complex plane).

We shall denote by $(\overline{\mathbf{P}}_k)$ $\mathbf{P}_k$ the set of $k \times k$ positive (semi) definite matrices.

Recall that a $p \times p$-valued functions $F(z)$ is said to be *positive* if

(4.1) $$\forall z \in \mathbb{C}_r \qquad \begin{array}{c} F(z) \quad \text{analytic} \\[6pt] (F(z) + (F(z))^*) \in \mathbf{P}_p \ . \end{array}$$

In engineering it is further restricted so that

$$F(z)_{|z \in \mathbb{R}} \in \mathbb{R}^{p \times p},$$

and then called *positive real*. For details see e.g. [6], [13], [16], [17], [18], [19], [20], [40].

We first establish a connection with the previous section.

Observation 4.1. *Whenever both $F_L(z)$ and $F_R(z)$ are positive real, then in each of the three above cases, i.e. Propositions 3.1, 3.6 and Example 3.7, the resulting composed function $F_L(F_R)$, is positive real.*

Indeed, in terminology of scalar functions, a positive real function maps the right half plane to itself.

Duality between rational positive real functions and the driving point immittance of $R - L - C$ electrical circuits, has already been recognized for about ninety years, e.g. [16], [17], [18], [19]. This has lead to rich and well-established theory, see e.g. [6], [13], [20], [40].

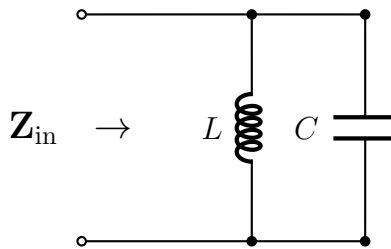This duality is illustrated through two simple examples in Figures 1 and 2.

FIGURE 1. $\mathbf{Z}_{\text{in}}(z) = ((zL)^{-1} + zC)^{-1}$.

One can next address a higher level of this duality between positive real rational functions and the driving point immittance of $R - L - C$ electrical circuits: Composition of rational functions is translated, in circuits language, to substituting elements by sub-networks,

---

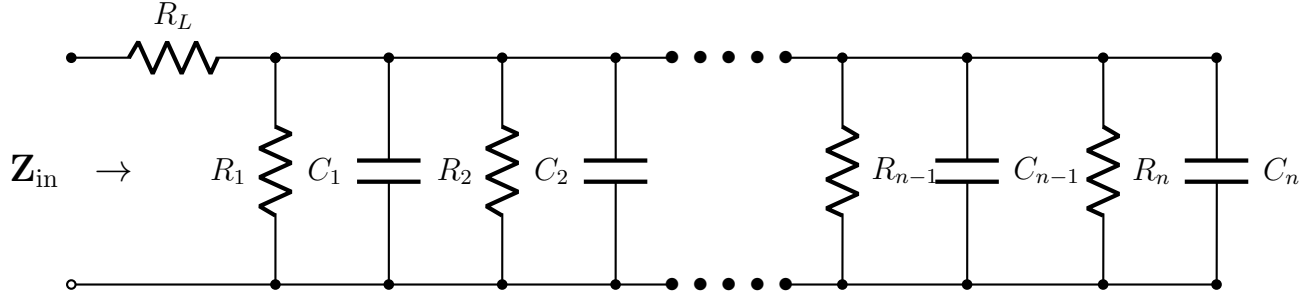[9]As before, the subscript stands for "left" or "right"

FIGURE 2. $Z_{\text{in}}(z) = R_L + \sum_{j=1}^{n} \left(R_j^{-1} + zC_j\right)^{-1}\Big|_{R_L=d_L \quad R_j=\frac{\gamma_j}{a_j} \quad C_j=\frac{1}{\gamma_j}} = d_L + \sum_{j=1}^{n} \gamma_j (z + a_j)^{-1}.$

while preserving the original configuration. For instance, zL and zC in Figure 1 are substituted in Figure 3 by the *impedance network* Z_G and the *admittance network* Y_F, respectively. This suggests constructing an elaborate network when the basic building blocks are $p \times p$-valued positive real functions, see Section 7.
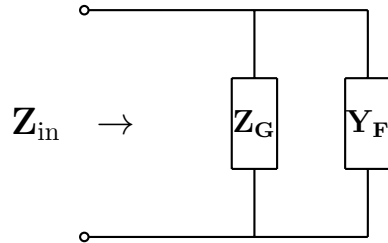

FIGURE 3. $Z_{\text{in}}(z) = \left(Y_F + Z_G^{-1}\right)^{-1}.$

To the above mentioned duality we now add a third aspect, namely *interconnection of feedback loops*. This is best illustrated by an example. Let G and F be square matrix-valued (possibly scalar) rational functions so that

$$\det(G) \not\equiv 0 \qquad \text{and} \qquad \det(G^{-1} + F) \not\equiv 0.$$

Then, the input-output relation of the feedback loop in Figure 4 is well defined and is given by

$$\text{Out} = \left(F + G^{-1}\right)^{-1} \cdot \text{In}.$$
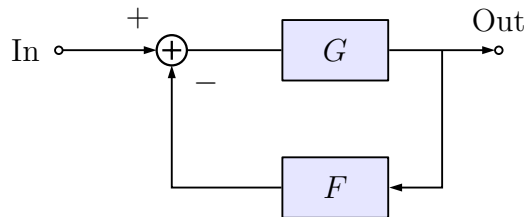

FIGURE 4. $\text{Out} = (F + G^{-1})^{-1} \cdot \text{In}$

Now, one can identify F and G in Figure 4, with Y_F and Z_G respectively, from Figure 3.

As an engineering application of item (III) of Proposition 3.6 see Figures 2, 5, 6.

Figure 7 presents an engineering application of Example 3.7.

Connection between positive real rational functions and feedback loops is further elaborated on in Section 7.
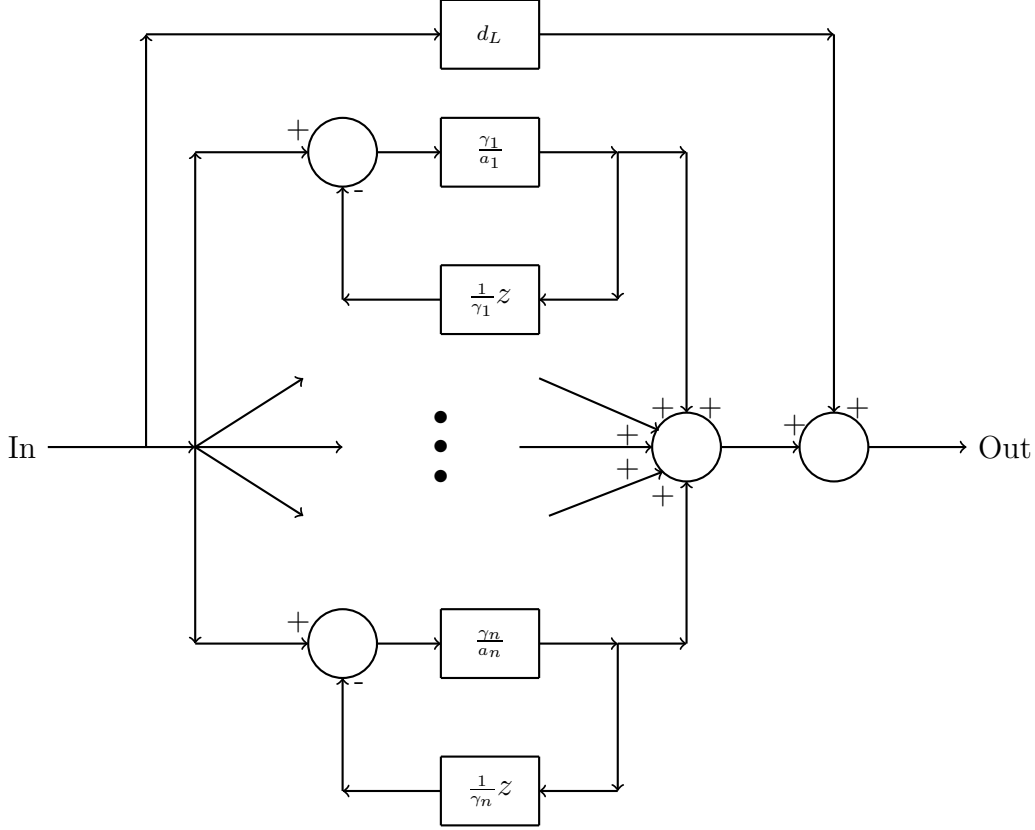
FIGURE 5. $\text{Out} = f_L(z) \cdot \text{In}$ with $f_L(z) = d_L + \sum_{j=1}^{n} \gamma_j (z + a_j)^{-1}$

5. COMPOSITION OF FUNCTIONS - SECOND VERSION

Here, we address a second version of composition of realizations. Specifically, using (2.8) we set

$$(5.1) \qquad F_L(F_R(z)) = D_L + C_L(F_R(z) - A_L)^{-1} B_L.$$

To this end, we assume that

n, the McMillan degree of $F_L(z)$, is equal to the dimension of $F_R(z)$.

Moreover assume that the $n \times n$ matrix $D_R - A_L$ is non-singular, i.e.

$$(5.2) \qquad \det(D_R - A_L) \neq 0.$$

We can now present the main result of this section.
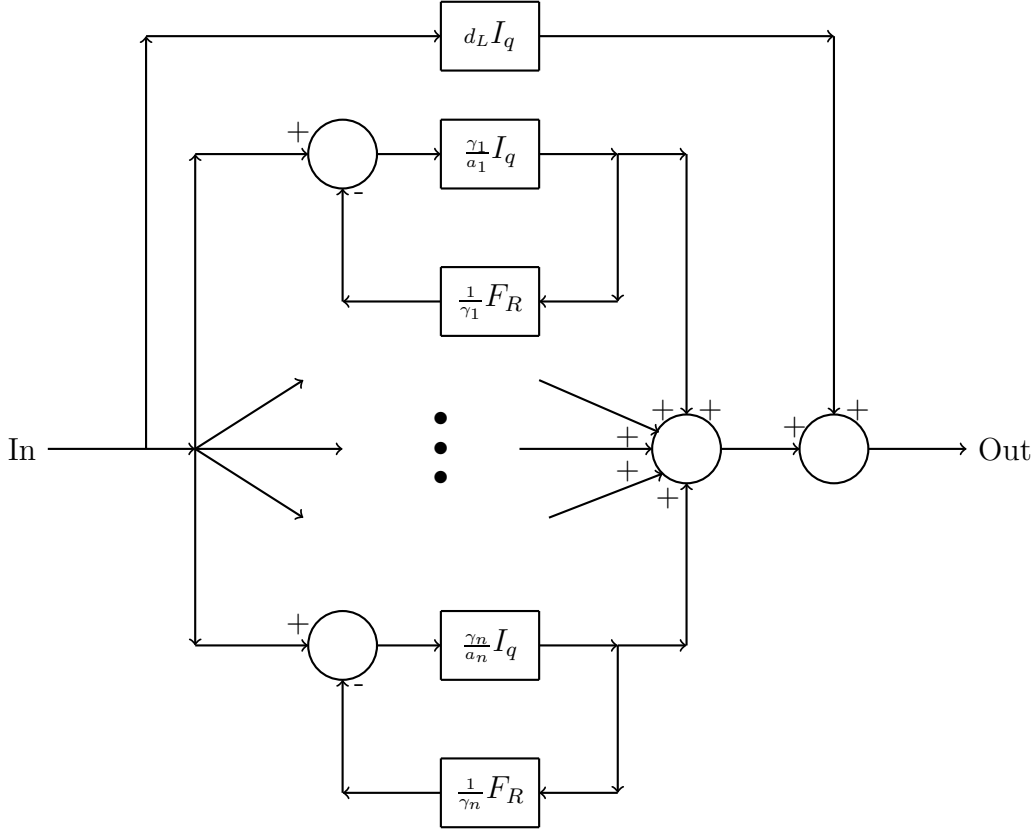
FIGURE 6. $\mathrm{Out} = f_L\left(F_R(z)\right)\cdot \mathrm{In}$ with $f_L(z) = d_L + \sum_{j=1}^{n}\gamma_j(z + a_j)^{-1}$
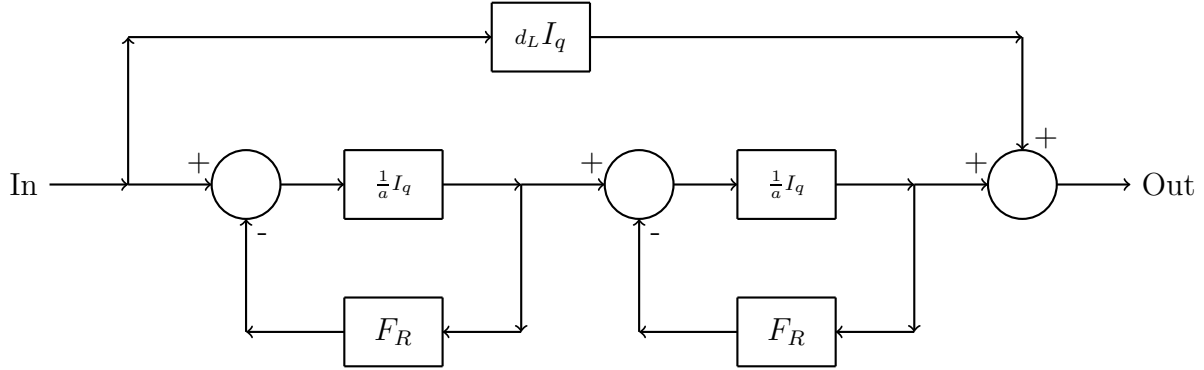
FIGURE 7. $\mathrm{Out} = f_L\left(F_R(z)\right)\cdot \mathrm{In}$ with $f_L(z) = d_L + \frac{1}{(s+a)^2}$ $a > 0$, $d_L \in \mathbb{R}$ parameters.

Proposition 5.1. *Under the above premises, the composed function, see (2.9), (2.10), (5.1) and (5.2), is $p \times p$-valued and of McMillan degree m. A corresponding realization array is given by*

$$\left(\begin{array}{c|c} A_{\mathrm{comp}} & B_{\mathrm{comp}} \\ \hline C_{\mathrm{comp}} & D_{\mathrm{comp}} \end{array}\right) = \left(\begin{array}{c|c} A_R - B_R(D_R - A_L)^{-1}C_R & B_R(D_R - A_L)^{-1}B_L \\ \hline -C_L(D_R - A_L)^{-1}C_R & D_L + C_L(D_R - A_L)^{-1}B_L \end{array}\right).$$

Remark 5.2. Sometimes we shall find it convenient to re-write the result of Proposition 5.1 as

$$\begin{pmatrix} A_{\text{comp}} & B_{\text{comp}} \\ C_{\text{comp}} & D_{\text{comp}} \end{pmatrix} = \begin{pmatrix} A_R & 0 \\ 0 & D_L \end{pmatrix} + \begin{pmatrix} B_R \\ C_L \end{pmatrix} \left(D_R - A_L \right)^{-1} \begin{pmatrix} -C_R & B_L \end{pmatrix}.$$

Remark 5.3. Note that by (2.3) the matrix A_L is coordinates-dependent. This in particular implies that almost always one can make condition (5.2) satisfied.

Remark 5.4. A simple example illustrating the difference between the two versions of composition dealt with in this work, is when $F_R(z) = f_R(z)I_n$, where $f_R(z)$ is scalar rational.

Proof of Proposition 5.1: In the sequel we shall use the identity,

$$
\begin{array}{c}
X \in \mathbb{C}^{n \times m} \\
Y \in \mathbb{C}^{m \times n} \\
-1 \notin \text{spect}(XY).
\end{array}
$$

$$(5.3) \qquad (I_n + XY)^{-1} = I_n - X(I_m + YX)^{-1}Y$$

We now have

$$F_L(F_R(z)) = D_L + C_L \left(\underbrace{C_R(zI_m - A_R)^{-1}B_R + D_R}_{F_R(z)} - A_L \right)^{-1} B_R$$

$$= D_L + C_L \left(C_R(zI_m - A_R)^{-1}B_R + (D_R - A_L) \right)^{-1} B_R$$

$$= D_L + C_L(D_R - A_L)^{-1} \left(I_n + \underbrace{C_R(zI_m - A_R)^{-1}}_{X} \underbrace{B_R(D_R - A_L)^{-1}}_{Y} \right)^{-1} B_L$$

$$= D_L + C_L(D_R - A_L)^{-1} \times$$

$$\times \left(I_n - \underbrace{C_R(zI_m - A_R)^{-1}}_{X} \left(I_m + \underbrace{B_R(D_R - A_L)^{-1}}_{Y} \underbrace{C_R(zI_m - A_R)^{-1}}_{X} \right)^{-1} \underbrace{B_R(D_R - A_L)^{-1}}_{Y} \right) B_L$$

$$= D_L + C_L(D_R - A_L)^{-1} \times$$

$$\times \left(I_n - C_R \left(zI_m - A_R + B_R(D_R - A_L)^{-1}C_R \right)^{-1} B_R(D_R - A_L)^{-1} \right) B_L$$

$$= \underbrace{D_L + C_L(D_R - A_L)^{-1}B_L}_{D_{\text{comp}}} -$$

$$\underbrace{-C_L(D_R - A_L)^{-1}C_R}_{C_{\text{comp}}} \left(zI \underbrace{-A_R + B_R(D_R - A_L)^{-1}C_R}_{-A_{\text{comp}}} \right)^{-1} \underbrace{B_R(D_R - A_L)^{-1}B_L}_{B_{\text{comp}}},$$

A critical part is when one substitutes in (5.3) the values, $X := C_R(zI_m - A_R)^{-1}$ and $Y := B_R(D_R - A_L)^{-1}$. $\square$

We conclude this section by examining the extent to which the main result is coordinates-dependent.

Remark 5.5. Assume the realizations of F_L and F_R are minimal and consider a change of coordinates as in (2.3), i.e. for some non-singular S_L, S_R ($n \times n$ and $m \times m$ respectively),

$$\begin{pmatrix} S_L & 0 \\ 0 & I_p \end{pmatrix}^{-1} \begin{pmatrix} A_L & B_L \\ C_L & D_L \end{pmatrix} \begin{pmatrix} S_L & 0 \\ 0 & I_p \end{pmatrix} \qquad \begin{pmatrix} S_R & 0 \\ 0 & I_n \end{pmatrix}^{-1} \begin{pmatrix} A_R & B_R \\ C_R & D_R \end{pmatrix} \begin{pmatrix} S_R & 0 \\ 0 & I_n \end{pmatrix}.$$

Substituting in Proposition 5.1 yields

(5.4)
$$\begin{aligned}
A_{\text{comp}} &= S_R^{-1} A_R S_R - S_R^{-1} B_R (D_R - S_L^{-1} A_L S_L)^{-1} C_R S_R = S_R^{-1} \big(A_R - B_R (D_R - S_L^{-1} A_L S_L)^{-1} C_R \big) S_R \\
B_{\text{comp}} &= \qquad S_R^{-1} B_R (D_R - S_L^{-1} A_L S_L)^{-1} S_L^{-1} B_L \qquad = \qquad S_R^{-1} \big(B_R (D_R - S_L^{-1} A_L S_L)^{-1} S_L^{-1} B_L \big) \\
C_{\text{comp}} &= \qquad -C_L S_L (D_R - S_L^{-1} A_L S_L)^{-1} C_R S_R \qquad = \qquad \big(-C_L S_L (D_R - S_L^{-1} A_L S_L)^{-1} C_R \big) S_R \\
D_{\text{comp}} &= \quad D_L + C_L S_L (D_R - S_L^{-1} A_L S_L)^{-1} S_L^{-1} B_L \qquad = \qquad D_L + C_L (S_L D_R S_L^{-1} - A_L)^{-1} B_L \ ,
\end{aligned}$$

which may be a different system.

In the special case where,

(5.5)
$$\begin{pmatrix} S_L & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A_L & B_L \\ C_L & D_L \end{pmatrix} = \begin{pmatrix} A_L & B_L \\ C_L & D_L \end{pmatrix} \begin{pmatrix} S_L & 0 \\ 0 & I_n \end{pmatrix},$$

(5.4) can be written as the following change of coordinates,

$$\begin{pmatrix} S_R & 0 \\ 0 & I_p \end{pmatrix}^{-1} \begin{pmatrix} A_{\text{comp}} & B_{\text{comp}} \\ C_{\text{comp}} & D_{\text{comp}} \end{pmatrix} \begin{pmatrix} S_R & 0 \\ 0 & I_p \end{pmatrix}.$$

We also remark that the set of invertible matrices S_L satisfying (5.5) forms a multiplicative group.

6. Stieltjes functions

Recall that in (4.1) we described positive functions $F(z)$ as those that

(6.1)
$$\forall z \in \mathbb{C}_r \qquad \begin{aligned} F(z) &\quad \text{analytic} \\ (F(z) + (F(z))^*) &\in \overline{\mathbf{P}}_p \ . \end{aligned}$$

The subset of positive functions in (6.1), where in addition

(6.2)
$$\forall z \in \mathbb{C}_r \qquad \left(\tfrac{1}{iz} F(z) + \left(\tfrac{1}{iz} F(z) \right)^* \right) \in \overline{\mathbf{P}}_p \ ,$$

are called Stieltjes functions[10] .

In the sequel we shall rely on the following result taken from[11] [24, Theorem 3.1] (where originally poles at infinity are allowed): Stieltjes functions are exactly those which be can be written in the form

(6.3)
$$F(z) = i\Delta + \int_0^\infty \frac{z}{t - iz} d\sigma(t), \quad \Delta \in \mathbf{P}_p \quad \forall z \in \mathbb{C}_r \ ,$$

where the $p \times p$-valued positive measure σ satisfies

$$\int_0^\infty \frac{d\sigma(t)}{1 + t} < \infty.$$

Here we focus on the rational case, namely where the measure σ has a finite number of jumps.

---

[10]Note that we are not consistent with [24, Definition 3.1] where instead of positive functions described in (6.1), they use Nevanlinna functions analytically mapping the *upper* half plane to itself.

[11]A proof of this result is given in [31].

For example, a straightforward calculation reveals that all scalar rational Stieltjes functions of degree one, $f(z)$, may be parametrized as,

$$f(z) = i\left(\delta + \tfrac{\beta}{\alpha}\right) + \frac{\beta}{z + i\alpha} \qquad \begin{matrix} \alpha > 0 \\ \beta > 0 \\ \delta \geq 0. \end{matrix}$$

This observation is next generalized to all *rational* functions satisfying (6.3).

Proposition 6.1. *Let $F(z)$ be a $p \times p$-valued rational function, analytic at the origin and at infinity, of McMillan degree n.*
$F(z)$ is a Stieltjes function, satisfying (6.1) and (6.2), if and only if, it can be written as,

$$(6.4) \qquad F(z) = i\left(C\alpha^{-1}C^* + \delta\right) + C(zI_n + i\alpha)^{-1}C^* \qquad \text{with} \qquad \begin{matrix} C \in \mathbb{C}^{p \times n} \text{ full rank} \\ \alpha \in \mathbf{P}_n \\ \delta \in \overline{\mathbf{P}}_p . \end{matrix}$$

Proof : First recall, see e.g. [4, Lemma 1.1(II)], [6, Chapter 5], [23], that from the realization matrix formulation of the K-Y-P Lemma it follows that a rational function $F(z)$, analytic at infinity, is positive[12] if and only if, up to change of coordinates, its minimal realization satisfies

$$(6.5) \qquad \begin{pmatrix} -I_n & 0 \\ 0 & I_p \end{pmatrix} R_F + R_F^* \begin{pmatrix} -I_n & 0 \\ 0 & I_p \end{pmatrix} \in \overline{\mathbf{P}}_{n+p} .$$

Next, note that (6.3) gives an analytic extension of $F(z)$ to $\mathbb{C} \smallsetminus i\mathbb{R}_-$ such that

$$(6.6) \qquad (F(-z^*))^* = -F(z).$$

Recall now that positive functions which in addition satisfy (6.6) are called Positive Odd. In the real rational case they are known in electrical engineering as *Lossless* or *Foster*, see e.g. [6], [13], [40].

Furthermore, if in addition $F(z)$ is odd, i.e. (6.6) holds, then its realization array R_F may be chosen so that

$$(6.7) \qquad \begin{pmatrix} -I_n & 0 \\ 0 & I_p \end{pmatrix} R_F + R_F^* \begin{pmatrix} -I_n & 0 \\ 0 & I_p \end{pmatrix} = 0,$$

see e.g. [3, Theorem 4.1], [6, Section 5.2]. Note now that (6.7) means that the $(n+p) \times (n+p)$ matrix $\begin{pmatrix} -I_n & 0 \\ 0 & I_p \end{pmatrix} R_f$ is skew-Hermitian, namely,

$$\left(i \begin{pmatrix} -I_n & 0 \\ 0 & I_p \end{pmatrix} R_F\right) = \left(i \begin{pmatrix} -I_n & 0 \\ 0 & I_p \end{pmatrix} R_F\right)^* ,$$

which in turn can be written as,

$$R_F = \left(\begin{array}{c|c} -i\alpha & C^* \\ \hline C & i\Delta \end{array}\right) \qquad \text{with} \qquad \begin{matrix} \alpha = \alpha^* \\ C \in \mathbb{C}^{p \times n} \\ \Delta = \Delta^*. \end{matrix}$$

Thus far one can conclude that

$$(6.8) \qquad F(z) = i\Delta + C(zI_n + i\alpha)^{-1}C^* \qquad \text{with} \qquad \begin{matrix} \alpha = \alpha^* \\ C \in \mathbb{C}^{p \times n} \\ \Delta = \Delta^*. \end{matrix}$$

We next show that

$$(6.9) \qquad \alpha \in \mathbf{P}_n ,$$

---

[12]It may be complex or real.

and that

(6.10) $$\Delta = C\alpha^{-1}C^* + \delta \qquad \text{for some} \qquad \delta \in \overline{\mathbf{P}}_p .$$

To this end, using the fact that by assumption, $F(z)$ is analytic at the origin, i.e. α is non-singular, we shall find it convenient to re-write the $F(z)$ in (6.8) as

$$F(z) = i\left(\Delta - C\alpha^{-1}C^*\right) + izC\alpha^{-1}(z\alpha^{-1} + iI_n)^{-1}\alpha^{-1}C^* \qquad \text{with} \qquad \begin{matrix} \alpha = \alpha^* \\ C \in \mathbb{C}^{p \times n} \\ \Delta = \Delta^*, \end{matrix}$$

and hence,

$$\tfrac{1}{iz}F(z) = \tfrac{1}{z}(\Delta - C\alpha^{-1}C^*) + C\alpha^{-1}(z\alpha^{-1} + iI_n)^{-1}\alpha^{-1}C^*.$$

We can now substitute the above $F(z)$ in (6.2) to obtain,

$$\left(\tfrac{1}{z}(\Delta - C\alpha^{-1}C^*) + C\alpha^{-1}(z\alpha^{-1}+iI_n)^{-1}\alpha^{-1}C^* + \left(\tfrac{1}{z}(\Delta - C\alpha^{-1}C^*) + C\alpha^{-1}(z\alpha^{-1}+iI_n)^{-1}\alpha^{-1}C^*\right)^*\right) \in \overline{\mathbf{P}}_p \qquad \forall z \in \mathbb{C}_r ,$$

i.e.

(6.11) $\; 2\mathrm{Re}(z)\left(\tfrac{1}{|z|^2}(\Delta - C\alpha^{-1}C^*) + \left(C\alpha^{-1}(z\alpha^{-1}+iI_n)^{-1}\right)\alpha^{-1}\left(C\alpha^{-1}(z\alpha^{-1}+iI_n)^{-1}\right)^*\right) \in \overline{\mathbf{P}}_p \qquad \forall z \in \mathbb{C}_r .$

Clearly, having (6.9) along with (6.10) implies that (6.11) holds. Thus, there is the converse direction to consider.

First, note that since (6.11) holds in particular for all points of $z \in \mathbb{C}_r$ (up to n points) so that the matrix $z\alpha^{-1} + iI_n$ is nearly singular, this in fact implies that $\alpha^{-1} \in \mathbf{P}_n$ i.e. (6.9) holds.

Similarly, as (6.11) is satisfied in particular for $z \in \mathbb{C}_r$ "sufficiently small", it implies that (6.10) holds as well, so the claim is established. $\qquad \square$

Remark 6.2. Eq. (6.4) may be viewed as a parametrization of all rational Stieltjes function analytic at the origin and at infinity.

We now next review this result. To this end we recall the following.

Remark 6.3. Consider the following statements for a full-rank matrix $z \in \mathbb{C}^{p \times n}$.

(i) $\quad \left(\begin{smallmatrix} X & Z^* \\ Z & Y \end{smallmatrix}\right) \in \overline{\mathbf{P}}_{n+p} .$

(ii) $\quad Y \in \mathbf{P}_p \quad$ and $\quad X - Z^*Y^{-1}Z \in \overline{\mathbf{P}}_n .$

(iii) $\quad X \in \mathbf{P}_n \quad$ and $\quad Y - ZX^{-1}Z^* \in \overline{\mathbf{P}}_p .$

Then, (ii) implies (i) and if $n \geq p$ then the converse is true as well.

Then, (iii) implies (i) and if $p \geq n$ then the converse is true as well.

Remark 6.3 leads to the conclusion that Proposition 6.1 and Remark 6.2 can be alternatively formulated as follows.

Remark 6.4. All $n \times n$-valued rational Stieltjes function $F(z)$, analytic at the origin and at infinity, of McMMillan degree m, with $n \geq m$, may be parametrized as

$$F(z) = i\Delta + C\left(zI_m + i(C^*\Delta^{-1}C + \eta)\right)^{-1}C^* \qquad \begin{matrix} \Delta \in \mathbf{P}_n \\ C \in \mathbb{C}^{n \times m} \text{ full rank} \\ \eta \in \overline{\mathbf{P}}_m , \end{matrix}$$

namely admitting a realization of the form,

$$R_F = \left(\begin{array}{c|c} -i(C^*\Delta^{-1}C + \eta) & C^* \\ \hline C & i\Delta \end{array} \right) \qquad \begin{array}{c} \Delta \in \mathbf{P}_n \\ C \in \mathbb{C}^{n \times m} \text{ full rank} \\ \eta \in \overline{\mathbf{P}}_m \ . \end{array}$$

In the sequel, we shall find it convenient to use the following.

Remark 6.5. Denoting

$$\gamma := -iC,$$

one can re-write the realization of $F(z)$ in Proposition 6.1 as

$$(6.12) \qquad\qquad R_F = \left(\begin{array}{c|c} -i\alpha & C^* \\ \hline C & i\Delta \end{array} \right) = i \left(\begin{smallmatrix} -I_n & 0 \\ 0 & I_p \end{smallmatrix} \right) \left(\begin{smallmatrix} \alpha & \gamma^* \\ \gamma & \Delta \end{smallmatrix} \right)$$

where

$$(6.13) \qquad\qquad \left(\begin{smallmatrix} \alpha & \gamma^* \\ \gamma & \Delta \end{smallmatrix} \right) \in \overline{\mathbf{P}}_{n+p} \qquad \text{and} \qquad \begin{array}{c} \alpha \in \mathbf{P}_n \\ \gamma \in \mathbb{C}^{p \times n} \text{ full rank} \\ n \geq p. \end{array}$$

To further emphasize the difference between Stieltjes functions and those discussed in Section 4, we have the following.

Remark 6.6. As already mentioned the family of Stieltjes functions is a proper subset of Positive Odd functions[13], which in turn is a proper subset of Positive functions. Consider the following properties.

•   Each of these three sets is closed under positive scaling and summation and thus is a convex cone.

•   Both Positive functions and its subset of Positive Odd functions are closed under inversion, namely if $F(z)$ is Positive (Odd) then $(F(z))^{-1}$ is well defined and is Positive (Odd). Thus, each of these two sets is a Convex Invertible Cone. In [21] this fact was explored in the framework of real functions.

•   If $F(z)$ is a Stieltjes function then $(F(z))^{-1}$ is well defined Positive Odd function, which can not be a Stieltjes function. Indeed, from Remark 6.5 it follows that

$$-i \left(\lim_{z \to \infty} F(z) \right) \in \mathbf{P}_p \qquad \text{but} \qquad +i \left(\lim_{z \to \infty} (F(z))^{-1} \right) \in \mathbf{P}_p \ .$$

Recall now that the K-Y-P Lemma, see e.g. [4] [6, Chapter 5], characterizes positive rational functions, along with some sub-families, through properties of their minimal realizations. We can now introduce an adaptation of the K-Y-P Lemma to Stieltjes functions and then use it to construct from a given realization *a whole family* of Stieltjes functions of various dimensions and McMillan degrees.

Corollary 6.7. *Let $F(z)$ be a rational function with p outputs, analytic at infinity, of McMillan degree n, as in (2.1) and (2.2) i.e.*

$$F(z) = D + C(zI_n - A)^{-1}B \qquad\qquad R_F = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right).$$

---

[13] For example $\frac{1}{s+1}$ is a positive function which is not odd and $\frac{1}{s+i}$ is a positive odd function which is not Stieltjes.

Then, $F(z)$ is a Stieltjes function, if and only if the realization R_F can be chosen so that each of the four blocks A, B, C, D is of a full rank and

$$(6.14) \qquad\qquad -i \begin{pmatrix} -I_n & 0 \\ 0 & I_p \end{pmatrix} R_F \in \overline{\mathbf{P}}_{n+p} \ .$$

Moreover, let $U \in \mathbb{C}^{\nu \times n}$ and $V \in \mathbb{C}^{\pi \times p}$ be full rank matrices, for some $\nu \in [1, n]$ and $\pi \in [1, p]$. Then,

$$R_{\hat{F}} = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_F \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}^* ,$$

is a realization of a $\pi \times \pi$-valued Stieltjes function $\hat{F}(z)$ (analytic at the origin) of McMillan degree ν.

Indeed the claim follows from Proposition 6.1, Remarks 6.3 and 6.5 along with the fact that T is a full rank matrix so that the $H = TMT^*$ is well defined, the product matrix H is positive (semi-)definite if and only if M is positive (semi-)definite.

Following Remark 6.5 and Corollary 6.7 we shall call a realization R_F of a Stieltjes function *canonical* if it satisfies (6.12) with (6.13), or equivalently (6.14).

Note that a realization remains *canonical* under *unitary* change of coordinates, i.e. in (2.3) $S^{-1} = S^*$.

We next show that the set of rational Stieltjes function is closed under the second version of composition of functions, see Section 5.

Proposition 6.8. *Consider a pair of rational Stieltjes functions, $F_L(z)$, $F_R(z)$, analytic at the origin and at infinity: $F_L(z)$ is $p \times p$-valued of McMillan degree n and $F_R(z)$ is $n \times n$-valued of McMillan degree m with $n \geq m$.*

If the realization of both $F_L(z)$ and $F_R(z)$ is canonical[14], then the composition $F_L(F_R)$ in (5.1) is a $p \times p$-valued rational Stieltjes function of McMillan degree m, given in a canonical realization.

Proof : Using Proposition 6.1 along with Remark 6.5 below left, we have R_{F_L} a realization array of $F_L(z)$ and using Remark 6.4, below right, we have R_{F_R} a realization array of $F_R(z)$,

$$R_{F_L} = \left(\begin{array}{c|c} -i\alpha_L & (i\gamma_L)^* \\ \hline i\gamma_L & i(\gamma_L \alpha_L{}^{-1}\gamma_L{}^* + \delta) \end{array} \right) \qquad R_{F_R} = \left(\begin{array}{c|c} -i(\gamma_R{}^* \Delta_R{}^{-1}\gamma_R + \eta) & (i\gamma_R)^* \\ \hline i\gamma_R & i\Delta_R \end{array} \right)$$

$$\gamma_L \in \mathbb{C}^{p \times n} \quad \text{full rank} \qquad\qquad \gamma_R \in \mathbb{C}^{n \times m} \quad \text{full rank}$$
$$\alpha_L \in \mathbf{P}_n \qquad\qquad\qquad \Delta_R \in \mathbf{P}_n$$
$$\delta \in \overline{\mathbf{P}}_p \qquad\qquad\qquad \eta \in \overline{\mathbf{P}}_m \ .$$

Substituting in Proposition 5.1 yields that a realization of $F_{\text{comp}} = F_L(F_R)$ in (5.1) is given by

$$R_{F_{\text{comp}}} = i \left(\begin{array}{c|c} -(\gamma_R^* \Delta_R{}^{-1}\gamma_R + \eta) + \gamma_R^* \left(\Delta_R + \alpha_L \right)^{-1} \gamma_R & \gamma_R^* \left(\Delta_R + \alpha_L \right)^{-1} \gamma_L{}^* \\ \hline -\gamma_L \left(\Delta_R + \alpha_L \right)^{-1} \gamma_R & \gamma_L \alpha_L{}^{-1}\gamma_L{}^* + \delta - \gamma_L \left(\Delta_R + \alpha_L \right)^{-1} \gamma_L{}^* \end{array} \right)$$

---

[14]Thus the condition in (5.2) is trivially satisfied.

A straightforward exercise enables one to re-write this realization as,

$$(6.15) \quad R_{F_{\text{comp}}} = i \begin{pmatrix} -I_m & 0 \\ 0 & I_p \end{pmatrix} \underbrace{\left(\begin{pmatrix} \eta & 0 \\ 0 & \delta \end{pmatrix} + \begin{pmatrix} \gamma_R^* \\ -\gamma_L \alpha_L^{-1} \Delta_R \end{pmatrix} \underbrace{\left(\Delta_R^{-1} - \left(\Delta_R + \alpha_L \right)^{-1} \right)}_{M} \begin{pmatrix} \gamma_R^* \\ -\gamma_L \alpha_L^{-1} \Delta_R \end{pmatrix}^* \right)}_{W} .$$

Now as by assumption, both α_L and Δ_R are in $\mathbf{P}_n$, it implies that

$$M := \left(\Delta_R^{-1} - \left(\Delta_R + \alpha_L \right)^{-1} \right) \in \mathbf{P}_n \ ,$$

as well. From the structure it follows that[15]

$$W := \left(\begin{pmatrix} \eta & 0 \\ 0 & \delta \end{pmatrix} + \begin{pmatrix} \gamma_R^* \\ -\gamma_L \alpha_L^{-1} \Delta_R \end{pmatrix} M \begin{pmatrix} \gamma_R^* \\ -\gamma_L \alpha_L^{-1} \Delta_R \end{pmatrix}^* \right) \in \overline{\mathbf{P}}_{m+p} \ .$$

From Remarks 6.3 and 6.5 it thus follows that the resulting $F_{\text{comp}} = F_L(F_R)$ in (5.1), is a Stieltjes function.   □

7. FUTURE WORK

In this work we focused on composition of rational functions their state-space realization and applications. Yet, this research area is mostly open. We here point out at three sample problems of various level of importance.

•   Assume having a small set of "simple" (e.g. low degree) rational functions as "building blocks".
Synthesis: What functions can be generated from these building blocks.
Analysis: Given a complicated rational function, can it be, and if yes how, "factorized" or "decomposed" into a composition of simpler building blocks.

Synthesis is further discussed below. To emphasize the the importance of analysis, recall that in Remark 3.8 it was pointed out that in the first version composition, the McMillan degree of $F_L(F_R)$ is equal to the *product* of the McMillan degrees of the original functions $F_L(z)$ and $F_R(z)$. Thus "decomposition" may significantly simplify the functions at hand.

As an illustration consider the following rational function of two variables

$$(7.1) \qquad\qquad \phi(F, G) := \left(F + G^{-1} \right)^{-1} .$$

Note that this is function on *non-commuting* variables, in fact,

$$\phi(G^{-1}, F^{-1}) = \phi(F, G).$$

Note now that the driving point impedance in Figure 3 can be written as,

$$Z_{\text{in}} = \phi(Y_F, \ Z_G),$$

and a basic feedback loop in Figure 4, may be viewed as

$$\text{Out} = \phi(F, G) \cdot \text{In}.$$

Now composition of such ϕ functions yields

$$\phi(F_c, G_c) = \left(\underbrace{F_a + G_a^{-1}}_{F_c} + \underbrace{\left(F_b + G_b^{-1} \right)^{-1}}_{G_c^{-1}} \right)^{-1} \qquad \text{where} \qquad \begin{matrix} F_c = \phi(F_a, G_a) \\ G_c = \phi(F_b, G_b). \end{matrix}$$

---

[15]To be precise, $\text{rank}(W) = \min(n, \ m+p)$.

An illustration of the converse problem let the starting point be the above function

$$\left(F_a + G_a^{-1} + \left(F_b + G_b^{-1}\right)^{-1}\right)^{-1},$$

and using ϕ from (7.1), one needs to rewrite it in the form of

$$\phi\left(\ \phi(F_a, G_a),\ (\phi(F_a, G_a))^{-1}\right).$$

● In Section 4 we presented inter-relations between (i) positive real rational functions (ii) driving point inpedance of $R - L - C$ networks and (iii) feedback loops.

As already mentioned, identifying items (i) with (ii) is classical. Moreover, there is a whole list of synthesis schemes how to construct an $R - L - C$ circuit whose driving point impedance realizes a prescribed positive real rational function: Bott-Duffin, Brune, Darlington, Foster to name but few, see e.g. [6], [20].

The inter-relation between (i) and (iii) suggests that one can exploit these electrical circuit synthesis schemes to construct, out of simple building blocks, an elaborate network of feedback loops. As a potential application, see Figures 6 above or 8 below.
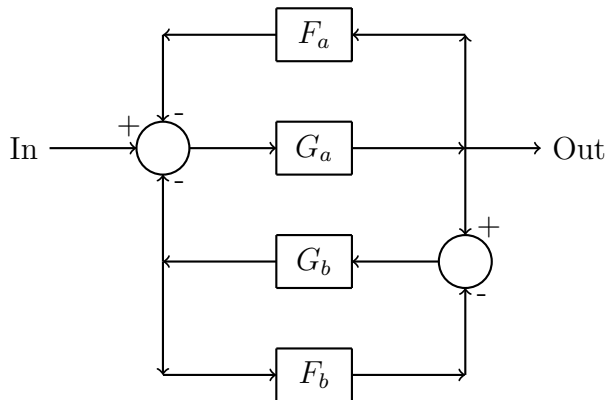

FIGURE 8. $\text{Out} = \left(F_a + G_a^{-1} + \left(F_b + G_b^{-1}\right)^{-1}\right)^{-1} \cdot \text{In}$

For example the above ϕ in (7.1) is positive real in the sense that if $F(z)$ and $G(z)$ are positive real, ϕ satisfies (4.1) or (6.1). However, this study requires caution in at least two ways:
(i) The application to constructing feedback loop networks, *transcends* the framework where the building blocks, like $F(z)$ or $G(z)$, are rational positive. For instance, in Figure 4 the functions $F(z)$ and $G(z)$ are only required to satisfy $\det(G) \not\equiv 0$ and $\det(F + G^{-1}) \not\equiv 0$. They need not be positive and in principle even not necessarily rational.
(ii) Study of rational functions of *non-commuting* variables in general and positive real is praticular, has been flourishing recently, as sample references see e.g. [1], [7], [8], [9], [33], [34], [35], [30]. Nevertheless many properties of these functions are yet to be explored. For example, there is a long way to go to extend (as proposed above) some of the known electrical circuits synthesis schemes to the framework of non-commuting variables in order to render it an engineering tool for designing multi-inputs multi-outouts feedback networks.

- Assuming the dimensions of all matrices involved are suitable and that M is non-singular, the results in Remarks 2.2, 3.2 and 5.2 are all in the framework of,

$$\begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix} + \begin{pmatrix} \pm B_R \\ C_L \end{pmatrix} M^{-1} \begin{pmatrix} \pm C_R & B_L \end{pmatrix}.$$

This observation calls for further investigation.

References

[1] K. Abu-Ghanem, D. Alpay, F. Colombo, I. Lewkowicz & I. Sabadini, "Herglotz Functions of Several Quaternionic Variables", to appear in *J. Math. Anal. & Appl.*

[2] D. Alpay, J.A. Ball, I.Gohberg & L. Rodman, "The two-sided residie interpolation in the Stieltjes class for matrix functionss", *Lin. Alg. & Appl.*, vol. 208/209, pp. 485-521, 1994.

[3] D. Alpay & I. Gohberg, "Unitary rational Matrix Functions", *Oper. Th. Advan. & Appl.*, Vol. 33, pp. 175-222, Birkhäuser, 1988.

[4] D. Alpay & I. Lewkowicz, "The Positive Real Lemma and Construction of all Realizations of Generalized Positive Rational Functions", *Sys. Cont Lett*, Vol. 60, pp. 985-993, 2011.

[5] D. Alpay, P.E.T. Jorgensen, I. Lewkowicz & I. Marziano, "Infinite Product Representation for Kernels and Iterations of Schur Functions", *Recent Advances in Inverse Scattering Schur Analysis and Strocastic Processes*, series *Oper. The. Adv. & Appl.*, Vol. 244, pp. 67-87, Birkhäuser, 2015.

[6] B.D.O. Anderson & S. Vongpanitlerd, *Networks Analysis and Synthesis, A Modern Systems Theory Approach*, Prentice-Hall, New Jersey, 1973.

[7] J.A. Ball, G.J. Groenewald & S. Ter Horst, "Standard Versus Strict Bounded Real Lemma with Infinite-Dimensional State Space I: The State Space Similarity Approach", to appear in *J. Operator Theory*.

[8] J.A. Ball, G.J. Groenewald & S. Ter Horst, "Standard Versus Strict Bounded Real Lemma with Infinite-Dimensional State Space II: The Storage Function Approach", to appear in *Oper. The. Adva. & Appl.*

[9] J.A. Ball & D.S. Kaluzhnyi-Verbovetskyi, "Schur-Agler and Herglotz-Agler Classes of Functiobns: Positive Kernel Decompositions and Transfer Function Realizations", *Adv. Math.*, Vol. 280, pp. 121-187, 2015.

[10] H. Bart, I. Gohberg & M.A. Kaashoek, *Minimal Factorization of Matrix and Operator Functions*, *Oper. Th.dvan. & Appl.*, Vol. 1, Birkhäuser, 1979.

[11] H. Bart, I. Gohberg, M.A. Kaashoek & A.C. Ran, *A State Space Approach to Canonical Factorization with Applications*, Operator Theory Advances & Applications, Vol. 200, Birkhäuser, 2010.

[12] A.F. Beardon, *Iterations of Rational Functions*, series Graduate Texts in Mathematics, Vol. 132. Springer 1991.

[13] V. Belevich, *Classical Network Theory*, Holden-Day, San-Francisco, 1968.

[14] V. Bolotnikov, "Bitangential Nevanlinna-Pick problem in the Stieltjes class" (in Russian), Kharkov Univ., 1984

[15] V. Bolotnikov, "The two-sided Nevanlinna-Pick problem in the Stieltjes class", in *Contributions to operator theory and its Applications* (T. Furuta, I. Gohberg, and T. Nakazi, Eds.), Oper. Theory Adv. Appl. 61, pp. 15-37, 1993.

[16] O. Brune, "Synthesis of a finite two terminal network whose driving point impedance is a prescribed function of frequency", *J. Math. Phys.*, Vol. 10, pp. 191-236, 1931.

[17] O. Brune, *Synthesis of a finite two-terminal network whose driving point impedance is a prescribed function of frequency*, Thesis (MIT), 1931.

[18] W. Cauer, "The realization of impedances of Prescribed frequency dependence" (in German), *Archiv für Elektrotechnik*, Vol. 17, pp. 355-388, 1926.

[19] W. Cauer, "Über Funktionen mit positivem Realteil" (in German), *Mathematische Annalen*, Vol. 106, pp. 369-394, 1932

[20] W.H. Chen, *Linear Networks Design and Synthesis*, McGraw-Hill, Electrical and Electronic Engineering Series, 1964.

[21] N. Cohen and I. Lewkowicz, "Convex Invertible Cones and Positive Real Analytic Functions", *Lin. Alg. & Appl.*, Vol. 425, pp. 797-813, 2007.

[22] C.C. Cowen, Jr. & B.I. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, 1995.

[23] Ph. Delsartre, Y. Genin &d Y. Kamp, "On the Role of the Nevanlinna-Pick Problem in Circuit and System Theory", *Circuit Theory & Appl.*, Vol. 9, pp. 177-187, 1981.

[24] Dyukarev Yu.M. & Katsnelson V.E. "Multiplicative and Additive Cases of Stieltjes Analytic Matrix-Valued Functions and Interpolation Problems Associated with Them. I" *Anal. Math. Soc. Transl.*, Vol. 131, pp. 55-70, 1988.

[25] K. Hoffman & R. Kunze, *Linear Algebra*, Prentice-Hall, New Delhi, 1967.

[26] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[27] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[28] T. Kailath, *Linear Systems*, Prentice-Hall, 1980.

[29] M. Kaltenbäck, "Model Spaces for Composition of Q-functions", *Math. Nachr.*, Vol. 195, pp. 159-170, 1998.

[30] D.S. Kaluzhny-Verbovetskyi & V. Vinnikov, *Foundations of Free Noncommutative Function Theory*, series Math. Surveys and Monographs, Vol. 199, Aer. Math. Soc., 2014.

[31] M.G. Krein & A.A. Nudel'man, "The Markov Moment Problem and Extremal Problems", *Nauka Moskow* 1973. English Transl. Amer. Math. Soc., 1977.

[32] B.H. Lavenda. *Statistical physics: a probabilistic approach. Reprint of the 1991 original.* Mineola, NY: Dover Publications, reprint of the 1991 original edition, 2016.

[33] A-M. Luger & M. Nedic, "A Characterization of Herglotz-Nevanlinna Functions in Two Variables via Integral Representation", *Ark. Mat.*, Vol. 55, pp. 199-216, 2017.

[34] J.E. Pascoe, B. Passer & R. Tully-Doyle, "Representation of Free Herglotz Functions", arXiv:1607.00407v3.

[35] G. Popescu, "Noncommutative Multivariables Operator Theory", *Intg. Equ. & Oper. The.*, Vol. 75, pp. 87-133, 2013.

[36] F.M. Reza, "The Concept of Power Dominance Systems", *Lecture Notes in Control & Information Sciences* Vol. 58, pp. 787-795, Springer 1984.

[37] H.H. Rosenbrock, *Computer-Aided Control System Design*, Academic Press, 1974.

[38] E.D. Sontag, *Mathematical Control Theory*, Deterministic Finite-Dimensional Systems, Second edition, Springer 1998.

[39] G.H. Wannier. *Statistical physics. Reprint of the 1966 ed.* New York, NY: Dover Publications, reprint of the 1966 ed. edition, 1987.

[40] M. R. Wohlers, *Lumped and Distributed Passive Networks*, Acad. Press 1969.

(DA) FACULTY OF MATHEMATICS, PHYSICS, AND COMPUTATION, SCHMID COLLEGE OF SCIENCE AND TECHNOLOGY, CHAPMAN UNIVERSITY, ONE UNIVERSITY DRIVE ORANGE, CALIFORNIA 92866, USA
E-mail address: alpay@chapman.edu

(IL) DEPARTMENT OF ELECTRICAL ENGINEERING BEN-GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BEER-SHEVA, 84105, ISRAEL
E-mail address: izchak@bgu.ac.il