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# REALIZATION OF TENSOR-PRODUCT AND OF TENSOR-FACTORIZATION OF RATIONAL FUNCTIONS

DANIEL ALPAY AND IZCHAK LEWKOWICZ

ABSTRACT. We here first study the state space realization of a tensor-product of a pair of rational functions. At the expense of “inflating” the dimensions, we recover the classical expressions for realization of a regular product of rational functions. Then, under an additional assumption that the limit at infinity of a given rational function exists and is equal to identity, an explicit formula for a *tensor-factorization* of this function, is introduced.

## 1. INTRODUCTION

The problem of minimal factorization of matrix-valued rational functions of one complex variable has long history; see for instance [1, 2, 6]. Less known seems to be the counterpart of this problem when matrix product is replaced by tensor product. More precisely, we study the following two problems: First, given two rational matrix-valued functions  $R_1$  and  $R_2$  analytic at infinity, write a realization of the tensor product  $R_1 \otimes R_2$  in terms of realizations of  $R_1$  and  $R_2$ . Next, given a matrix-valued rational function  $R$  analytic at infinity, find its representations as  $R_1 \otimes R_2$  where  $R_1$  and  $R_2$  are rational and analytic at infinity.

To provide some motivation we note the following. Tensor products play an important role in mathematics and quantum mechanics. In the latter case, a first example (see e.g. [4, p. 162]) is the product of two wave functions, each belonging to a given Hilbert space, which belongs to the tensor product of the given Hilbert spaces; see e.g. [8, Proposition 6.2, p. 111] for the latter. Another example is the case of quantum states (positive matrices with trace equal to 1; see e.g. [9]). Given two states  $M_1 \in \mathbb{C}^{N_1 \times N_1}$  and  $M_2 \in \mathbb{C}^{N_2 \times N_2}$ , of possibly different sizes, the tensor product  $M_1 \otimes M_2$  is still a state. Note that if  $M = M_1 \otimes M_2$ , one can recover  $M_1$  and  $M_2$  uniquely via the formula

$$(1.1) \quad d_1^* M_1 c_1 = \sum_{k=1}^{N_2} (d_1 \otimes f_k)^* M (c_1 \otimes f_k), \quad c_1, d_1 \in \mathbb{C}^{N_1},$$

where  $f_1, \dots, f_{N_2}$  denotes an orthonormal basis for  $\mathbb{C}^{N_2}$ , and similarly for  $M_2$ ,

$$(1.2) \quad d_2^* M_2 c_2 = \sum_{k=1}^{N_1} (e_k \otimes d_2)^* M (e_k \otimes c_2), \quad c_2, d_2 \in \mathbb{C}^{N_2},$$

where now  $e_1, \dots, e_{N_1}$  is an orthonormal basis for  $\mathbb{C}^{N_1}$ . See e.g. [9, eq. (9.2.1) p. 97].

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If one starts from an arbitrary state  $M \in \mathbb{C}^{N_1 N_2 \times N_1 N_2}$  the matrices defined by (1.1) and (1.2) will be states, called marginal states, but their tensor product need not be equal to  $M$ .

One can consider similar problems in the setting of functions. We focus the discussion on rational functions. If  $R(z)$  is a  $\mathbb{C}^{N \times N}$ -valued rational function and if  $N = N_1 N_2$ , formulas (1.1) and (1.2) now define two rational functions  $R_A$  and  $R_B$ , respectively  $\mathbb{C}^{N_1 \times N_1}$  and  $\mathbb{C}^{N_2 \times N_2}$ -valued, via

$$(1.3) \quad \begin{aligned} d_1^* R_A(z) c_1 &= \sum_{k=1}^{N_2} (d_1 \otimes f_k)^* R(z) (c_1 \otimes f_k), \\ d_2^* R_B(z) c_2 &= \sum_{k=1}^{N_1} (d_2 \otimes f_k)^* R(z) (c_2 \otimes f_k), \end{aligned}$$

If  $R = R_1 \otimes R_2$  where  $R_1$  is  $\mathbb{C}^{N_1 \times N_1}$ -valued and  $R_2$  is  $\mathbb{C}^{N_2 \times N_2}$ -valued, then these equations can be rewritten as

$$(1.4) \quad \begin{aligned} R_A(z) &= R_1(z) \cdot (\text{Tr } R_2(z)) \\ R_B(z) &= R_2(z) \cdot (\text{Tr } R_1(z)) \end{aligned}$$

and so these equations basically solve the tensor factorization problem.

The purpose of this work is in a somewhat different direction; we would like to express both tensor multiplication and tensor factorization of matrix-valued rational functions using state space representations.

In the rest of this section we cite some known results. Let  $z_l, z_r$  (the subscript stands for ‘‘left’’ and ‘‘right’’) be a pair of complex variables, and let  $F_l(z_l), F_r(z_r)$  be a pair of  $p_l \times m_l, p_r \times m_r$ -valued rational functions, respectively. Assume that neither has poles at infinity and denote by  $n_l, n_r$  the respective McMillan degrees. Thus, one can write the rational functions and the respective realization as

$$(1.5) \quad \begin{aligned} F_l(z_l) &= D_l + C_l(z_l I_{n_l} - A_l)^{-1} B_l & F_r(z_r) &= D_r + C_r(z_r I_{n_r} - A_r)^{-1} B_r \\ R_{F_l} &= \left( \begin{array}{c|c} A_l & B_l \\ \hline C_l & D_l \end{array} \right) & R_{F_r} &= \left( \begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array} \right). \end{aligned}$$

Recall that whenever  $m_l = p_r$  the product  $F_l(z_l)F_r(z_r)$  is well-defined and its realization is given by<sup>1</sup> (see e.g. [3, Section 2.5])

$$(1.6) \quad R_{F_l F_r} = \left( \begin{array}{cc|c} A_l & B_l C_r & B_l D_r \\ \hline 0 & A_r & B_r \\ \hline C_l & D_l C_r & D_l D_r \end{array} \right) = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \begin{pmatrix} A_l & 0 & B_l \\ 0 & I_{n_r} & 0 \\ C_l & 0 & D_l \end{pmatrix} \begin{pmatrix} I_{n_l} & 0 & 0 \\ 0 & A_r & B_r \\ 0 & C_r & D_r \end{pmatrix},$$

in the sense that

$$(1.7) \quad F_1(z_1)F_2(z_2) = D_1 D_2 + (C_1 \quad D_1 C_2) \left( \begin{pmatrix} z_1 I_{n_1} & 0 \\ 0 & z_2 I_{n_2} \end{pmatrix} - \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \right)^{-1} \begin{pmatrix} B_1 D_2 \\ B_2 \end{pmatrix}.$$

If  $z_l = z_r$  the sought realization in (1.6) is of McMillan degree

$$n_l + n_r.$$

<sup>1</sup> Strictly speaking, in the references it was formulated for  $z_l = z_r = z$  i.e. for  $F_l(z)F_r(z)$

when minimal (roughly speaking when there is no pole-zero cancelation). We next address ourselves to the *tensor product*<sup>2</sup> of  $F_l(z_l)$  and  $F_r(z_r)$ , resulting in  $F_l \otimes F_r$ , a  $p_l p_r \times m_l m_r$ -valued rational function. Tensor product of rational functions is discussed in [5, Section 5.2].

So far for known results. In the next section we focus on  $R_{F_l \otimes F_r}$ , the state space realization of  $F_l \otimes F_r$ . In Section 3 we set the framework for the main result, which is the factorization result presented in Section 4.

## 2. REALIZATION OF A TENSOR-PRODUCT OF RATIONAL FUNCTIONS

We start with technicalities: We denote by boldface characters, “inflated version” of the original ones, i.e.

$$(2.8) \quad \begin{aligned} \mathbf{A}_l &:= A_l \otimes I_{p_r} & \mathbf{A}_r &:= I_{m_l} \otimes A_r \\ \mathbf{B}_l &:= B_l \otimes I_{p_r} & \mathbf{B}_r &:= I_{m_l} \otimes B_r \\ \mathbf{C}_l &:= C_l \otimes I_{p_r} & \mathbf{C}_r &:= I_{m_l} \otimes C_r \\ \mathbf{D}_l &:= D_l \otimes I_{p_r} & \mathbf{D}_r &:= I_{m_l} \otimes D_r \end{aligned}$$

$$\mathbf{F}_l(z_l) := \mathbf{C}_l (z_l I_{n_l p_r} - \mathbf{A}_l)^{-1} \mathbf{B}_l + \mathbf{D}_l \quad \mathbf{F}_r(z_r) := \mathbf{C}_r (z_r I_{m_l n_r} - \mathbf{A}_r)^{-1} \mathbf{B}_r + \mathbf{D}_r .$$

We then show that at the expense of “inflating” the dimensions one can replace a *tensor product* by a *usual product*.

**Proposition 2.1.** *Let  $F_l(z_r)$ ,  $F_r(z_r)$  be a pair of  $p_l \times m_l$ ,  $p_r \times m_r$ -valued rational functions, of McMillan degree  $n_l$ ,  $n_r$ , respectively, whose realization is given in Eq. (1.5). Following Eqs. (1.6) and (2.8), one has that,*

$$(2.9) \quad R_{F_l \otimes F_r} = R_{\mathbf{F}_l \mathbf{F}_r} .$$

In order to go into details we shall repeatedly use the fact, see e.g. [7, Lemma 4.2.10], that for matrices  $T \in \mathbb{C}^{n \times m}$ ,  $X \in \mathbb{C}^{m \times l}$ ,  $Y \in \mathbb{C}^{l \times p}$ ,  $Z \in \mathbb{C}^{p \times q}$  one has that

$$(2.10) \quad TX \otimes YZ = (T \otimes Y)(X \otimes Z).$$

We now explicitly compute the tensor product of  $F_l(z_l)$  and  $F_r(z_r)$ ,

$$\begin{aligned} F_l \otimes F_r &= (D_l + C_l(z_l I_{n_l} - A_l)^{-1} B_l) \otimes (D_r + C_r(z_r I_{n_r} - A_r)^{-1} B_r) \\ &= D_l \otimes D_r + D_l \otimes (C_r(z_r I_{n_r} - A_r)^{-1} B_r) + (C_l(z_l I_{n_l} - A_l)^{-1} B_l) \otimes D_r + (C_l(z_l I_{n_l} - A_l)^{-1} B_l) \otimes (C_r(z_r I_{n_r} - A_r)^{-1} B_r) \end{aligned}$$

We next separately examine each block

$$\begin{aligned} D_l \otimes (C_r(z_r I_{n_r} - A_r)^{-1} B_r) &= D_l I_{m_l} \otimes (C_r(z_r I_{n_r} - A_r)^{-1} B_r) \\ &= (D_l \otimes (C_r(z_r I_{n_r} - A_r)^{-1})) (I_{m_l} \otimes B_r) \\ &= (D_l I_{m_l} \otimes (C_r(z_r I_{n_r} - A_r)^{-1})) (I_{m_l} \otimes B_r) \\ &= (D_l \otimes C_r) (I_{m_l} \otimes ((z_r I_{n_r} - A_r)^{-1})) (I_{m_l} \otimes B_r) \\ &= (D_l \otimes I_{p_r}) (I_{m_l} \otimes C_r) (I_{m_l} \otimes ((z_r I_{n_r} - A_r)^{-1})) (I_{m_l} \otimes B_r) \\ &= \underbrace{(D_l \otimes I_{p_r})}_{\mathbf{D}_l} \underbrace{(I_{m_l} \otimes C_r)}_{\mathbf{C}_r} \left( z_r I_{m_l n_r} - \underbrace{I_{m_l} \otimes A_r}_{\mathbf{A}_r} \right)^{-1} \underbrace{(I_{m_l} \otimes B_r)}_{\mathbf{B}_r} \\ &= \mathbf{D}_l \mathbf{C}_r (z_r I_{m_l n_r} - \mathbf{A}_r)^{-1} \mathbf{B}_r \end{aligned}$$

<sup>2</sup>In matrix theory circles known as the “Kronecker product”, see e.g. [7, Section 4.2].

$$\begin{aligned}
(C_l(z_l I_{n_l} - A_l)^{-1} B_l) \otimes D_r &= (C_l(z_l I_{n_l} - A_l)^{-1} B_l) \otimes I_{p_r} D_r \\
&= (C_l \otimes I_{p_r}) \left( (z_l I_{n_l} - A_l)^{-1} B_l \right) \otimes D_r \\
&= (C_l \otimes I_{p_r}) \left( (z_l I_{n_l} - A_l)^{-1} B_l \right) \otimes I_{p_r} D_r \\
&= (C_l \otimes I_{p_r}) \left( (z_l I_{n_l} - A_l)^{-1} \otimes I_{p_r} \right) (B_l \otimes D_r) \\
&= (C_l \otimes I_{p_r}) \left( (z_l I_{n_l} - A_l)^{-1} \otimes I_{p_r} \right) (B_l \otimes I_{p_r}) (I_{m_l} \otimes D_r) \\
&= \underbrace{(C_l \otimes I_{p_r})}_{\mathbf{C}_l} \left( (z_l I_{n_l p_r} - \underbrace{A_l \otimes I_{p_r}}_{\mathbf{A}_l}) \right)^{-1} \underbrace{(B_l \otimes I_{p_r})}_{\mathbf{B}_l} \underbrace{(I_{m_l} \otimes D_r)}_{\mathbf{D}_r} \\
&= \mathbf{C}_l \left( (z_l I_{n_l p_r} - \mathbf{A}_l)^{-1} \mathbf{B}_l \mathbf{D}_r \right) \\
(C_l(z_l I_{n_l} - A_l)^{-1} B_l) \otimes (C_r(z_r I_{n_r} - A_r)^{-1} B_r) &= (C_l(z_l I_{n_l} - A_l)^{-1} B_l I_{m_l}) \otimes (I_{p_r} C_r(z_r I_{n_r} - A_r)^{-1} B_r) \\
&= (C_l \otimes I_{p_r}) \left( (z_l I_{n_l} - A_l)^{-1} B_l \right) \otimes (C_r(z_r I_{n_r} - A_r)^{-1}) (I_{m_l} \otimes B_r) \\
&= (C_l \otimes I_{p_r}) \left( (z_l I_{n_l} - A_l)^{-1} B_l I_{m_l} \right) \otimes (I_{p_r} C_r(z_r I_{n_r} - A_r)^{-1}) (I_{m_l} \otimes B_r) \\
&= (C_l \otimes I_{p_r}) \left( (z_l I_{n_l} - A_l)^{-1} \otimes I_{p_r} \right) (B_l \otimes C_r) (I_{m_l} \otimes (z_r I_{n_r} - A_r)^{-1}) (I_{m_l} \otimes B_r) \\
&= (C_l \otimes I_{p_r}) \left( (z_l I_{n_l p_r} - A_l \otimes I_{p_r})^{-1} (B_l \otimes I_{p_r}) (I_{m_l} \otimes C_r) \right) \left( (z_r I_{m_l n_r} - I_{m_l} \otimes A_r)^{-1} (I_{m_l} \otimes B_r) \right) \\
&= \mathbf{C}_l \left( (z_l I_{n_l p_r} - \mathbf{A}_l)^{-1} \mathbf{B}_l \mathbf{C}_r \left( (z_r I_{m_l n_r} - \mathbf{A}_r)^{-1} \mathbf{B}_r \right) \right)
\end{aligned}$$

Thus, one can write

$$\begin{aligned}
F_l \otimes F_r &= \underbrace{D_l \otimes D_r}_{\mathbf{D}} + (\mathbf{C}_l \ \mathbf{D}_l \mathbf{C}_r) \begin{pmatrix} ((z_l I_{n_l p_r} - \mathbf{A}_l)^{-1} (z_l I_{n_l p_r} - \mathbf{A}_l)^{-1} \mathbf{B}_l \mathbf{C}_r (z_r I_{m_l n_r} - \mathbf{A}_r)^{-1}) \\ 0 \qquad \qquad \qquad (z_r I_{m_l n_r} - \mathbf{A}_r)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{B}_l \mathbf{D}_r \\ \mathbf{B}_r \end{pmatrix} \\
&= \mathbf{D} + (\mathbf{C}_l \ \mathbf{D}_l \mathbf{C}_r) \left( \begin{pmatrix} z_l I_{n_l p_r} & 0 \\ 0 & z_r I_{m_l n_r} \end{pmatrix} - \begin{pmatrix} \mathbf{A}_l & \mathbf{B}_l \mathbf{C}_r \\ 0 & \mathbf{A}_r \end{pmatrix} \right)^{-1} \begin{pmatrix} \mathbf{B}_l \mathbf{D}_r \\ \mathbf{B}_r \end{pmatrix}.
\end{aligned}$$

Note that in particular

$$D_l \otimes D_r = (D_l I_{m_l}) \otimes (I_{p_r} D_r) = \underbrace{(D_l \otimes I_{p_r})}_{\mathbf{D}_l} \underbrace{(I_{m_l} \otimes D_r)}_{\mathbf{D}_r} = \mathbf{D}_l \mathbf{D}_r = \mathbf{D}.$$

The realization of  $F_l(z_l) \otimes F_r(z_r)$  can be compactly written as

$$(2.11) \quad R_{F_l \otimes F_r} = \left( \begin{array}{cc|c} \mathbf{A}_l & \mathbf{B}_l \mathbf{C}_r & \mathbf{B}_l \mathbf{D}_r \\ 0 & \mathbf{A}_r & \mathbf{B}_r \\ \hline \mathbf{C}_l & \mathbf{D}_l \mathbf{C}_r & \mathbf{D}_l \mathbf{D}_r \end{array} \right) = \left( \begin{array}{c|c} \mathbf{A}_o & \mathbf{B}_o \\ \hline \mathbf{C}_o & \mathbf{D} \end{array} \right) = \mathbf{R},$$

which is indeed in form of (1.6), (2.8). If  $z_l = z_r$  and there is no pole-zero cancelation, the sought realization in (2.11) is of McMillan degree

$$n_l p_r + m_l n_r.$$

Note now that in a way similar to (1.6), one can factorize the realization in (2.11) as follows,

$$(2.12) \quad \mathbf{R} = \left( \begin{array}{cc|c} \mathbf{A}_l & \mathbf{B}_l \mathbf{C}_r & \mathbf{B}_l \mathbf{D}_r \\ 0 & \mathbf{A}_r & \mathbf{B}_r \\ \hline \mathbf{C}_l & \mathbf{D}_l \mathbf{C}_r & \mathbf{D} \end{array} \right) = \begin{pmatrix} \mathbf{A}_l & 0 & \mathbf{B}_l \\ 0 & I_{m_l n_r} & 0 \\ \hline \mathbf{C}_l & 0 & \mathbf{D}_l \end{pmatrix} \begin{pmatrix} I_{n_l p_r} & 0 & 0 \\ 0 & \mathbf{A}_r & \mathbf{B}_r \\ 0 & \mathbf{C}_r & \mathbf{D}_r \end{pmatrix}.$$

We conclude this section by pointing out that Proposition 2.1 can be easily extended to more elaborate cases like

$$F_a(z_a) \otimes F_b(z_b) \otimes F_c(z_c) \cdots$$

## 3. REALIZATION OF THE INVERSE OF A TENSOR PRODUCT OF RATIONAL FUNCTIONS

For future reference, in this section we examine the realization of the inverse of rational functions of the form  $F_l(z_l) \otimes F_r(z_r)$  studied in the previous section.

We first recall, see e.g. [3, Theorem 2.4], in the realization of the inverse a rational function: Namely if

$$R_F = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right),$$

is a realization of a square matrix-valued rational function  $F(z)$ , then whenever  $D$  is non-singular,  $(F(z))^{-1}$  is well-defined almost everywhere, and a corresponding realization is given by,

$$(3.13) \quad R_{F^{-1}} = \left( \begin{array}{c|c} A^\times & B^\times \\ \hline C^\times & D^\times \end{array} \right) = \left( \begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array} \right).$$

Next, whenever the above  $F_l(z)$  and  $F_r(z)$  are so that

$$m_l = p_r \quad \text{and} \quad p_l = m_r$$

the product  $F_l(z)F_r(z)$  is square, and whenever  $D_l D_r$  is non-singular<sup>3</sup>,  $(F_l(z)F_r(z))^{-1}$  is well-defined almost everywhere, and by combining (1.6) together with (3.13) a corresponding realization is given by

$$(3.14) \quad R_{(F_l F_r)^{-1}} = \left( \begin{array}{cc|c} A_l^\times & 0 & B_l^\times \\ \hline B_r^\times C_l^\times & A_r^\times & B_r^\times D_l^{-1} \\ \hline D_r^{-1} C_l^\times & C_r^\times & D_r^{-1} D_l^{-1} \end{array} \right) = \begin{pmatrix} I_{m_l} & 0 & 0 \\ 0 & A_r^\times & B_r^\times \\ 0 & C_r^\times & D_r^{-1} \end{pmatrix} \begin{pmatrix} A_l^\times & 0 & B_l^\times \\ 0 & I_{m_r} & 0 \\ C_l^\times & 0 & D_l^{-1} \end{pmatrix}.$$

Similarly, whenever

$$m_l m_r = p_l p_r,$$

the rational function  $F_l(z) \otimes F_r(z)$  is square and if  $D_l \otimes D_r = \mathbf{D}_l \mathbf{D}_r = \mathbf{D}$  is non-singular, then  $D_l, D_r$  are square, i.e.

$$m_l = p_l \quad m_r = p_r$$

and non-singular, see e.g. [7, Theorem 4.2.15]. Thus, we shall denote hereafter by  $m_l \times m_l, m_r \times m_r$  the dimensions of  $F_l, F_r$ , respectively.

Under these conditions, the  $m_l m_r \times m_l m_r$ -valued rational function,  $(F_l(z) \otimes F_r(z))^{-1}$  is almost everywhere defined. (2.11), we next compute the realization of  $(F_l \otimes F_r)^{-1}$ ,

$$R_{(F_l \otimes F_r)^{-1}} = \begin{pmatrix} (A_l \otimes I_{p_r}) - (B_l \otimes D_r)(D_l \otimes D_r)^{-1}(C_l \otimes I_{p_r}) & B_l \otimes C_r - (B_l \otimes D_r)(D_l \otimes D_r)^{-1}(D_l \otimes C_r) & -(B_l \otimes D_r)(D_l \otimes D_r)^{-1} \\ -(I_{m_l} \otimes B_r)(D_l \otimes D_r)^{-1}(C_l \otimes I_{p_r}) & (I_{m_l} \otimes A_r) - (I_{m_l} \otimes B_r)(D_l \otimes D_r)^{-1}(D_l \otimes C_r) & -(I_{m_l} \otimes B_r)(D_l \otimes D_r)^{-1} \\ (D_l \otimes D_r)^{-1}(C_l \otimes I_{p_r}) & (D_l \otimes D_r)^{-1}(D_l \otimes C_r) & (D_l \otimes D_r)^{-1} \end{pmatrix}.$$

<sup>3</sup>this implies that  $m_l = p_r \geq \text{rank}(D_l D_r) = p_l = m_r$ .

Taking into account the fact that  $D_l$  and  $D_r$  are square and non-singular, the realization  $R_{(F_l \otimes F_r)^{-1}}$  takes the form

$$\begin{aligned}
R_{(F_l \otimes F_r)^{-1}} &= \left( \begin{array}{c|c} \begin{array}{c} (A_l - B_l D_l^{-1} C_l) \otimes I_{p_r} \\ (D_l^{-1} C_l) \otimes (-B_r D_r^{-1}) \end{array} & 0 \\ \hline \begin{array}{c} D_l^{-1} C_l \otimes D_r^{-1} \\ I_{m_l} \otimes D_r^{-1} C_r \end{array} & \begin{array}{c} (-B_l D_l^{-1}) \otimes I_{p_r} \\ D_l^{-1} \otimes (-B_r D_r^{-1}) \end{array} \end{array} \right) \\
&= \left( \begin{array}{c|c} \begin{array}{c} A_l^\times \otimes I_{p_r} \\ C_l^\times \otimes B_r^\times \end{array} & 0 \\ \hline \begin{array}{c} C_l^\times \otimes D_r^{-1} \\ I_{m_l} \otimes C_r^\times \end{array} & \begin{array}{c} B_l^\times \otimes I_{p_r} \\ D_l^{-1} \otimes B_r^\times \end{array} \end{array} \right) \\
&= \left( \begin{array}{c|c} \begin{array}{c} A_l^\times \otimes I_{p_r} \\ (I_{n_l} \otimes B_r^\times) (C_l^\times \otimes I_{p_r}) \end{array} & 0 \\ \hline \begin{array}{c} (I_{n_l} \otimes D_r^{-1}) (C_l^\times \otimes I_{p_r}) \\ I_{m_l p_r} \otimes (I_{m_l} \otimes C_r^\times) \end{array} & \begin{array}{c} (B_l^\times \otimes I_{p_r}) \\ (I_{m_l} \otimes B_r^\times) (D_l^{-1} \otimes I_{n_r}) \end{array} \end{array} \right) \\
&= \left( \begin{array}{c|c} \begin{array}{c} \mathbf{A}_l^\times \\ \mathbf{B}_r^\times \mathbf{C}_l^\times \end{array} & 0 \\ \hline \begin{array}{c} \mathbf{B}_r^\times \mathbf{C}_l^\times \\ \mathbf{D}_r^{-1} \mathbf{C}_l^\times \end{array} & \begin{array}{c} \mathbf{B}_l^\times \\ \mathbf{B}_r^\times \mathbf{D}_l^{-1} \end{array} \end{array} \right) = \left( \begin{array}{c|c} \mathbf{A}_o^\times & \mathbf{B}_o^\times \\ \hline \mathbf{C}_o^\times & \mathbf{D}^\times \end{array} \right) = \mathbf{R}^\times,
\end{aligned}$$

where the boldface entries are given by

$$\begin{aligned}
(3.15) \quad \mathbf{A}_l^\times &:= A_l^\times \otimes I_{p_r} & \mathbf{A}_r^\times &:= I_{m_l} \otimes A_r^\times \\
\mathbf{B}_l^\times &:= B_l^\times \otimes I_{p_r} & \mathbf{B}_r^\times &:= I_{m_l} \otimes B_r^\times \\
\mathbf{C}_l^\times &:= C_l^\times \otimes I_{p_r} & \mathbf{C}_r^\times &:= I_{m_l} \otimes C_r^\times \\
\mathbf{D}_l^{-1} &:= D_l^{-1} \otimes I_{p_r} & \mathbf{D}_r^{-1} &:= I_{m_l} \otimes D_r^{-1}.
\end{aligned}$$

One can conclude that

$$R_{(F_l \otimes F_r)^{-1}} = R_{(\mathbf{F}_l \mathbf{F}_r)^{-1}},$$

and in a way similar to (2.12), one can factorize the above realization as follows,

$$(3.16) \quad \mathbf{R}^\times = \left( \begin{array}{c|c} \begin{array}{c} \mathbf{A}_l^\times \\ \mathbf{B}_r^\times \mathbf{C}_l^\times \end{array} & 0 \\ \hline \begin{array}{c} \mathbf{B}_r^\times \mathbf{C}_l^\times \\ \mathbf{D}_r^{-1} \mathbf{C}_l^\times \end{array} & \begin{array}{c} \mathbf{B}_l^\times \\ \mathbf{B}_r^\times \mathbf{D}_l^{-1} \end{array} \end{array} \right) = \begin{pmatrix} I_{m_l p_r} & 0 & 0 \\ 0 & \mathbf{A}_r^\times & \mathbf{B}_r^\times \\ 0 & \mathbf{C}_r^\times & \mathbf{D}_r^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{A}_l^\times & 0 & \mathbf{B}_l^\times \\ 0 & I_{m_l n_r} & 0 \\ \mathbf{C}_l^\times & 0 & \mathbf{D}_l^{-1} \end{pmatrix}.$$

#### 4. TENSOR-FACTORIZATION OF RATIONAL FUNCTIONS

We now address a more challenging question: Given  $\mathbf{F}(z)$  and  $(\mathbf{F}(z))^{-1}$  (assuming that  $\det \mathbf{F}(z) \neq 0$ ), under what conditions and how, can it be ‘‘tensor-factorized’’ to *some*  $F_l(z)$  and  $F_r(z)$ , namely the following relation holds,

$$(4.1) \quad \mathbf{F}(z) = F_l(z) \otimes F_r(z).$$

Note that here, we confine the discussion to a single complex variable, i.e.  $z_l = z_r = z$ .

Note also that if (4.1) holds, this is true up to complex scaling i.e.,

$$F_l(z) \otimes F_r(z) = c(z) F_l(z) \otimes \frac{1}{c(z)} F_r(z) \quad 0 \neq c(z) \in \mathbb{C}.$$

We shall use this degree of freedom in the sequel.

We next recall in the following fact from matrix theory.

Let  $\Pi_\alpha, \Pi_\beta$  be a pair of supporting projections of the space  $\mathbb{C}^{(\alpha+\beta) \times (\alpha+\beta)}$ , i.e.

$$\begin{aligned}
(4.2) \quad \Pi_\alpha^2 &= \Pi_\alpha & \Pi_\alpha \Pi_\beta &= 0_{\alpha+\beta} = \Pi_\beta \Pi_\alpha \\
\Pi_\beta^2 &= \Pi_\beta & \Pi_\alpha + \Pi_\beta &= I_{\alpha+\beta}.
\end{aligned}$$



Such a pair of projections can be obtained by partitioning an arbitrary non-singular  $T \in \mathbb{C}^{(\alpha+\beta) \times (\alpha+\beta)}$  as follows.

$$(4.3) \quad \begin{aligned} T^{-1} \begin{pmatrix} I_\alpha & 0 \\ 0 & 0_\beta \end{pmatrix} T &:= \Pi_\alpha \\ T^{-1} \begin{pmatrix} 0_\alpha & 0 \\ 0 & I_\beta \end{pmatrix} T &= \Pi_\beta . \end{aligned}$$

By using an isometry-like relation, we next offer a simple way to “deflate” matrix dimensions.

**Observation 4.1.** *Given  $M \in \mathbb{C}^{s \times q}$ , denote*

$$\mathbf{M}_l := M \otimes I_p \quad \mathbf{M}_r := I_m \otimes M.$$

*For arbitrary  $u \in \mathbb{C}^p$ ,  $v \in \mathbb{C}^m$  normalized so that  $u^*u = 1$ ,  $v^*v = 1$ , one has that*

$$(I_s \otimes u^*) \mathbf{M}_l (I_q \otimes u) = M \quad \text{and} \quad (v^* \otimes I_s) \mathbf{M}_r (v \otimes I_q) = M.$$

Indeed, by twice applying (2.10) one obtains,

$$\begin{aligned} (I_s \otimes u^*) \underbrace{(M \otimes I_p)}_{\mathbf{M}_l} (I_q \otimes u) &= (I_s M I_q) \otimes \underbrace{(u^* I_p u)}_{=1} = M \\ (v^* \otimes I_s) \underbrace{(I_m \otimes M)}_{\mathbf{M}_r} (v \otimes I_q) &= \underbrace{(v^* I_m v)}_{=1} \otimes (I_s M I_q) = M. \end{aligned}$$

We next apply the last observation to the variables here.

**Corollary 4.2.** *For  $u \in \mathbb{C}^{p_r}$ ,  $v \in \mathbb{C}^{m_l}$ , normalized so that  $u^*u = 1$  and  $v^*v = 1$ , the boldface characters in (2.8) satisfy*

$$\begin{aligned} A_l &= (I_{n_l} \times u^*) \mathbf{A}_l (I_{n_l} \otimes u) & A_r &= (v^* \otimes I_{n_l}) \mathbf{A}_r (v \otimes I_{n_r}) \\ B_l &= (I_{n_l} \times u^*) \mathbf{B}_l (I_{m_l} \otimes u) & B_r &= (v^* \otimes I_{n_l}) \mathbf{B}_r (v \otimes I_{m_r}) \\ C_l &= (I_{p_l} \times u^*) \mathbf{C}_l (I_{n_l} \otimes u) & C_r &= (v^* \otimes I_{p_l}) \mathbf{C}_r (v \otimes I_{n_r}) \\ D_l &= (I_{p_l} \times u^*) \mathbf{D}_l (I_{m_l} \otimes u) & D_r &= (v^* \otimes I_{p_l}) \mathbf{D}_r (v \otimes I_{m_r}). \end{aligned}$$

We now return to the problem of “tensor-factorization” in (4.1). We note that in place of  $\mathbf{R}$  in (2.11) and  $\mathbf{R}^\times$  in (3.16), the realization arrays associated with  $\mathbf{F}$  and  $\mathbf{F}^{-1}$ , are known only *up to a coordinate transformation*, i.e. there exists, a non-singular matrix  $T \in \mathbb{C}^{(n_l p_r + m_l n_r) \times (n_l p_r + m_l n_r)}$  namely in (4.2) and (4.3)

$$\alpha = n_l p_r \quad \text{and} \quad \beta = m_l n_r ,$$

so that the actual realization array is given by

$$(4.4) \quad \begin{pmatrix} T & 0 \\ 0 & I_{p_l m_r} \end{pmatrix}^{-1} \mathbf{R} \begin{pmatrix} T & 0 \\ 0 & I_{p_l m_r} \end{pmatrix} = \left( \begin{array}{c|c} T^{-1} \mathbf{A}_o T & T^{-1} \mathbf{B}_o \\ \hline \mathbf{C}_o T & \mathbf{D} \end{array} \right) = \left( \begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right),$$

and

$$(4.5) \quad \begin{pmatrix} T & 0 \\ 0 & I_{p_l m_r} \end{pmatrix}^{-1} \mathbf{R}^\times \begin{pmatrix} T & 0 \\ 0 & I_{p_l m_r} \end{pmatrix} = \left( \begin{array}{c|c} T^{-1} \mathbf{A}_o^\times T & T^{-1} \mathbf{B}_o^\times \\ \hline \mathbf{C}_o^\times T & \mathbf{D}^{-1} \end{array} \right) = \left( \begin{array}{c|c} \mathbf{A}^\times & \mathbf{B}^\times \\ \hline \mathbf{C}^\times & \mathbf{D}^{-1} \end{array} \right).$$

As in reality, the specific coordinate transformation,  $T$  in (4.4) and (4.5) is *unknown* one can conclude that to extract  $F_l(z)$  and  $F_r(z)$  from (4.1) along with the realization arrays in (4.4), (4.5), additional conditions are needed.

**Theorem 4.3.** *Let  $\mathbf{F}(z)$  be a given square matrix-valued rational function. Assume that*

$$\lim_{z \rightarrow \infty} \mathbf{F}(z) = I.$$

Let  $\left( \begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & I \end{array} \right)$ , see (4.4), and  $\left( \begin{array}{c|c} \mathbf{A}^\times & \mathbf{B}^\times \\ \hline \mathbf{C}^\times & I \end{array} \right)$ , see (4.5), be realizations of  $\mathbf{F}(z)$  and of  $(\mathbf{F}(z))^{-1}$ , respectively.

Substituting in (4.2),  $\alpha = n_l m_r$  and  $\beta = m_l n_r$ , assume also that there exists a pair of supporting projection to  $\mathbb{C}^{n_l m_r + m_l n_r}$  denoted by  $\Pi_{n_l m_r}$  and  $\Pi_{m_l n_r}$  so that

$$(4.6) \quad \mathbf{A}\Pi_{n_l m_r} = \Pi_{n_l m_r}\mathbf{A}\Pi_{n_l m_r} \quad \mathbf{A}^\times\Pi_{m_l n_r} = \Pi_{m_l n_r}\mathbf{A}^\times\Pi_{m_l n_r} .$$

Following the definition of the projections  $\Pi_{n_l m_r}$  and  $\Pi_{m_l n_r}$ , see (4.3) and (4.6), along with Corollary 4.2, for arbitrary  $u \in \mathbb{C}^{m_r}$ ,  $v \in \mathbb{C}^{m_l}$ , normalized so that  $u^*u = 1$  and  $v^*v = 1$ , we find it convenient to introduce the following related projections<sup>4</sup>,

$$(4.7) \quad \hat{\Pi}_{n_l m_r} = T^{-1} \begin{pmatrix} I_{n_l} \otimes uu^* & 0 \\ 0 & 0_{m_l n_r} \end{pmatrix} T \quad \hat{\Pi}_{m_l n_r} = T^{-1} \begin{pmatrix} 0_{n_l m_r} & 0 \\ 0 & vv^* \otimes I_{m_l} \end{pmatrix} T$$

Then, using (2.11) and (4.4), one can take in (4.1)  $\mathbf{F} = F_l \otimes F_r$  where,

$$F_l(z) = (I_{m_l} \otimes u^*)\mathbf{C}\hat{\Pi}_{n_l m_r} (zI_{n_l m_r + m_l n_r} - \mathbf{A})^{-1} \hat{\Pi}_{n_l m_r} \mathbf{B}(I_{m_l} \otimes u) + I_{m_l}$$

$$F_r(z) = (v^* \otimes I_{m_r})\mathbf{C}\hat{\Pi}_{m_l n_r} (zI_{n_l m_r + m_l n_r} - \mathbf{A})^{-1} \hat{\Pi}_{m_l n_r} \mathbf{B}(v \otimes I_{m_r}) + I_{m_r}$$

**Proof :** First, recall (see Section 3) that the assumption that  $D_l \otimes D_r = \mathbf{D}_l \mathbf{D}_r = \mathbf{D}$  is square non-singular, it implies that both  $D_l$  and  $D_r$  are square non-singular. We shall thus denote the dimensions of  $F_l$  and  $F_r$ , by  $m_l \times m_l$  and  $m_r \times m_r$ , respectively.

The assumption here that  $\mathbf{D} = I_{m_l m_r}$  implies (see e.e. [7, Theorem 4.2.12]) that

$$D_l = cI_{m_l} \quad D_r = \frac{1}{c}I_{m_r} \quad \text{for some non-zero } c \in \mathbb{C}.$$

As already mentioned after (4.1), to simplify the exposition we shall take  $c = 1$ .

Next, let  $T$  in (4.3), (4.4), (4.5) be the same so that the supporting projections are  $\Pi_{n_l m_r}$  and  $\Pi_{m_l n_r}$ . Next note that substituting (2.11), (3.16), (4.4) and (4.5) in condition (4.6) yields,

$$\mathbf{A}\Pi_{n_l m_r} = T^{-1} \begin{pmatrix} \mathbf{A}_l & 0 \\ 0 & 0_{m_l n_r} \end{pmatrix} T \quad \Pi_{m_l n_r} \mathbf{A} = T^{-1} \begin{pmatrix} 0_{n_l m_r} & 0 \\ 0 & \mathbf{A}_r \end{pmatrix} T$$

$$\mathbf{A}^\times\Pi_{m_l n_r} = T^{-1} \begin{pmatrix} 0_{n_l m_r} & 0 \\ 0 & \mathbf{A}_r^\times \end{pmatrix} T \quad \Pi_{n_l m_r} \mathbf{A}^\times = T^{-1} \begin{pmatrix} \mathbf{A}_l^\times & 0 \\ 0 & 0_{m_l n_r} \end{pmatrix} T$$

and thus in the sequel we shall use the two upper relations, i.e.

$$\Pi_{n_l m_r} \mathbf{A}\Pi_{n_l m_r} = T^{-1} \begin{pmatrix} \mathbf{A}_l & 0 \\ 0 & 0_{m_l n_r} \end{pmatrix} T \quad \Pi_{m_l n_r} \mathbf{A}\Pi_{m_l n_r} = T^{-1} \begin{pmatrix} 0_{n_l m_r} & 0 \\ 0 & \mathbf{A}_r \end{pmatrix} T.$$

<sup>4</sup>note that  $\hat{\Pi}_{n_l m_r} \Pi_{n_l m_r} = \hat{\Pi}_{n_l m_r} \Pi_{n_l m_r} = \hat{\Pi}_{n_l m_r}$  and  $\hat{\Pi}_{m_l n_r} \Pi_{m_l n_r} = \Pi_{m_l n_r} \hat{\Pi}_{m_l n_r} = \hat{\Pi}_{m_l n_r}$ .

We are now ready to recover  $F_l(z)$ ,

$$\begin{aligned}
 F_l(z) &= C_l (zI_{n_l} - A_l)^{-1} B_l + I_{m_l} \\
 &= \underbrace{(I_{m_l} \otimes u^*) \mathbf{C}_l (I_{n_l} \otimes u)}_{C_l} \underbrace{(I_{n_l} \otimes u^*) (zI_{n_l m_r} - \mathbf{A}_l)^{-1} (I_{n_l} \otimes u)}_{(zI_{n_l} - A_l)^{-1}} \underbrace{(I_{n_l} \otimes u^*) \mathbf{B}_l (I_{m_l} \otimes u)}_{B_l} + I_{m_l} \\
 &= (I_{m_l} \otimes u^*) \mathbf{C}_l (I_{n_l} \otimes uu^*) (zI_{n_l m_r} - \mathbf{A}_l)^{-1} (I_{n_l} \otimes uu^*) \mathbf{B}_l (I_{m_l} \otimes u) + I_{m_l} \\
 &= (I_{m_l} \otimes u^*) \mathbf{C}_o \begin{pmatrix} I_{n_l m_r} & \\ 0_{m_l n_r \times n_l m_r} & \end{pmatrix} (I_{n_l} \otimes uu^*) \begin{pmatrix} I_{n_l m_r} & 0_{n_l m_r \times m_l n_r} \\ & \end{pmatrix} (zI_{n_l m_r + m_l n_r} - \mathbf{A}_o)^{-1} \\
 &\quad \times \begin{pmatrix} I_{n_l m_r} & \\ & 0 \end{pmatrix} (I_{n_l} \otimes uu^*) \mathbf{B}_l (I_{m_l} \otimes u) + I_{m_l} \\
 &= (I_{m_l} \otimes u^*) \mathbf{C}_o \begin{pmatrix} I_{n_l} \otimes uu^* & 0 \\ 0 & 0_{m_l n_r} \end{pmatrix} (zI_{n_l m_r + m_l n_r} - \mathbf{A}_o)^{-1} \begin{pmatrix} I_{n_l} \otimes uu^* & 0 \\ 0 & 0_{m_l n_r} \end{pmatrix} \mathbf{B}_o (I_{m_l} \otimes u) + I_{m_l} \\
 &= (I_{m_l} \otimes u^*) \underbrace{\mathbf{C}_o T^{-1}}_{\mathbf{C}} \underbrace{\begin{pmatrix} I_{n_l} \otimes uu^* & 0 \\ 0 & 0_{m_l n_r} \end{pmatrix} T}_{\hat{\Pi}_{n_l m_r}} \underbrace{T T^{-1} (zI_{n_l m_r + m_l n_r} - \mathbf{A}_o)^{-1} T}_{(zI_{n_l m_r + m_l n_r} - \mathbf{A})^{-1}} \underbrace{T^{-1} \begin{pmatrix} I_{n_l} \otimes uu^* & 0 \\ 0 & 0_{m_l n_r} \end{pmatrix} T}_{\hat{\Pi}_{n_l m_r}} \\
 &\quad \times \underbrace{T^{-1} \mathbf{B}_o}_{\mathbf{B}} (I_{m_l} \otimes u) + I_{m_l} \\
 &= (I_{m_l} \otimes u^*) \mathbf{C} \hat{\Pi}_{n_l m_r} (zI_{n_l m_r + m_l n_r} - \mathbf{A})^{-1} \hat{\Pi}_{n_l m_r} \mathbf{B} (I_{m_l} \otimes u) + I_{m_l} .
 \end{aligned}$$

Similarly, for  $F_r(z)$

$$\begin{aligned}
 F_r(z) &= C_r (zI_{n_r} - A_r)^{-1} B_r + I_{m_r} \\
 &= \underbrace{(v^* \otimes I_{m_r}) \mathbf{C}_r (v \otimes I_{n_r})}_{C_r} \underbrace{(v^* \otimes I_{n_l}) (zI_{n_r} - \mathbf{A}_r)^{-1} (v \otimes I_{n_r})}_{(zI_{n_r} - A_r)^{-1}} \underbrace{(v^* \otimes I_{n_l}) \mathbf{B}_r (v \otimes I_{m_r})}_{B_r} + I_{m_r} \\
 &= (v^* \otimes I_{m_r}) \mathbf{C}_r (vv^* \otimes I_{m_l}) (zI_{n_r} - \mathbf{A}_r)^{-1} (vv^* \otimes I_{m_l}) \mathbf{B}_r (v \otimes I_{m_r}) + I_{m_r} \\
 &= (v^* \otimes I_{m_r}) \mathbf{C}_o \begin{pmatrix} 0_{n_l m_r \times m_l n_r} & \\ I_{m_l n_r} & \end{pmatrix} (vv^* \otimes I_{m_l}) \begin{pmatrix} 0_{m_l n_r \times n_l m_r} & I_{m_l n_r} \\ & \end{pmatrix} (zI_{n_l m_r + m_l n_r} - \mathbf{A}_o)^{-1} \\
 &\quad \times \begin{pmatrix} 0 & \\ I_{m_l n_r} & \end{pmatrix} (vv^* \otimes I_{m_l}) \begin{pmatrix} 0 & I_{m_l n_r} \\ & \end{pmatrix} \mathbf{B}_o (v \otimes I_{m_r}) + I_{m_r} \\
 &= (v^* \otimes I_{m_r}) \underbrace{\mathbf{C}_o T^{-1}}_{\mathbf{C}} \underbrace{\begin{pmatrix} 0_{n_l m_r} & 0 \\ 0 & vv^* \otimes I_{m_l} \end{pmatrix} T}_{\hat{\Pi}_{m_l n_r}} \underbrace{T T^{-1} (zI_{n_l m_r + m_l n_r} - \mathbf{A}_o)^{-1} T}_{(zI_{n_l m_r + m_l n_r} - \mathbf{A})^{-1}} \underbrace{T^{-1} \begin{pmatrix} 0_{n_l m_r} & 0 \\ 0 & vv^* \otimes I_{m_l} \end{pmatrix} T}_{\hat{\Pi}_{m_l n_r}} \\
 &\quad \times \underbrace{T^{-1} \mathbf{B}_o}_{\mathbf{B}} (v \otimes I_{m_r}) + I_{m_r} \\
 &= (v^* \otimes I_{m_r}) \mathbf{C} \hat{\Pi}_{m_l n_r} (zI_{n_l m_r + m_l n_r} - \mathbf{A})^{-1} \hat{\Pi}_{m_l n_r} \mathbf{B} (v \otimes I_{m_r}) + I_{m_r} .
 \end{aligned}$$

□

**Remark 4.4.** At first sight, the assumptions in Theorem 4.3 seem very restrictive. For perspective recall that to factorize a given rational function  $F(z)$  to  $F(z) = F_l(z)F_r(z)$ , the assumptions are virtually the same<sup>5</sup>, see [3, Section 2.5]).

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<sup>5</sup>There they only assume  $D$  is square non-singular, but then only  $F_l(z)D_r$  and  $D_lF_r(z)$  are obtained.