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# Positive Subreducts in Finitely Generated Varieties of MV-algebras

## **Comments**

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# Positive Subreducts in Finitely Generated Varieties of MV-algebras

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## Abstract

Positive MV-algebras are negation-free and implication-free subreducts of MV-algebras. In this contribution we show that a finite axiomatic basis exists for the quasivariety of positive MV-algebras coming from any finitely generated variety of MV-algebras.

## 1 Positive subreducts of MV-algebras

Let  $\mathcal{MV}$  be the variety of MV-algebras [1] in the language containing all the usual definable operations and constants. Using this signature we denote an MV-algebra  $\mathbf{M} \in \mathcal{MV}$  as

$$\mathbf{M} = \langle M, \oplus, \odot, \vee, \wedge, \rightarrow, \neg, 0, 1 \rangle.$$

An algebra

$$\mathbf{A} = \langle A, \oplus, \odot, \vee, \wedge, 0, 1 \rangle$$

is a *positive subreduct* of  $\mathbf{M}$  if  $\mathbf{A}$  is a subreduct of  $\mathbf{M}$ .

**Definition 1.** Let  $F = \{\oplus, \odot, \vee, \wedge, 0, 1\}$  be a set of function symbols, where  $\oplus, \odot, \vee, \wedge$  are interpreted as binary operations and  $0, 1$  as constants. An algebra  $\mathbf{P}$  of type  $F$  is a *positive MV-algebra* if it is isomorphic to a positive subreduct of some MV-algebra.

Clearly, every MV-algebra gives rise to a positive MV-algebra and every bounded distributive lattice is a positive MV-algebra. In fact, positive MV-algebras are to MV-algebras as distributive lattices are to Boolean algebras.

**Example 1** (Lower Chang algebra). Let  $\mathbf{C}$  be Chang algebra and

$$\text{Rad } \mathbf{C} = \{0, \varepsilon, 2\varepsilon, \dots\}$$

be its radical, where the symbol  $\varepsilon$  denotes the least positive infinitesimal. Then the algebra  $\mathbf{C}_l$  having the universe  $\text{Rad } \mathbf{C} \cup \{1\}$  is a positive subreduct of  $\mathbf{C}$ .

**Example 2** (Non-decreasing McNaughton functions). For each natural number  $n$ , the free  $n$ -generated MV-algebra is isomorphic to the algebra  $\mathbf{F}_n$  of McNaughton functions  $[0, 1]^n \rightarrow [0, 1]$ . Then the algebra  $\mathbf{F}_n^{\leq}$  of nondecreasing McNaughton functions is a positive subreduct of  $\mathbf{F}_n$ .

The class of all positive MV-algebras is denoted by  $\mathcal{P}$ . Since  $\mathcal{P}$  is a class of algebras containing the trivial algebra and closed under isomorphisms, subalgebras, direct products and ultraproducts, it is a quasivariety. The following example shows that  $\mathcal{P}$  is not a variety.

**Example 3.** Let  $\theta$  be an equivalence relation on the algebra  $\mathbf{C}_l$  from Example 1 with classes  $\{0\}$ ,  $\{\varepsilon, 2\varepsilon, \dots\}$ , and  $\{1\}$ . Then  $\theta$  is a  $\mathcal{P}$ -congruence on  $\mathbf{C}_l$ . The quotient  $\mathbf{C}_l/\theta$  is isomorphic to the three-element algebra  $\{\bar{0}, \bar{\varepsilon}, \bar{1}\}$  that satisfies the identities  $\bar{\varepsilon} \oplus \bar{\varepsilon} = \bar{\varepsilon}$  and  $\bar{\varepsilon} \odot \bar{\varepsilon} = \bar{0}$ . However, the two equations cannot hold simultaneously in any MV-algebra. Hence,  $\mathbf{C}_l/\theta$  is not a positive MV-algebra.

It can be shown that the quasivariety  $\mathcal{P}$  is generated by the positive reduct of the standard MV-algebra  $[0, 1]$ . Moreover, the free  $n$ -generated positive MV-algebra is isomorphic to the positive subreduct  $\mathbf{F}_n^{\leq}$  from Example 2.

## 2 Axiomatization

We define a class  $\mathcal{Q}$  of algebras of type  $F = \{\oplus, \odot, \vee, \wedge, 0, 1\}$ . Specifically, an algebra  $\mathbf{A} = \langle A, \oplus, \odot, \vee, \wedge, 0, 1 \rangle$  belongs to  $\mathcal{Q}$  if  $\mathbf{A}$  satisfies the following identities and quasi-identities:

1.  $\langle A, \vee, \wedge, 0, 1 \rangle$  is a bounded distributive lattice
2.  $\langle A, \oplus, 0 \rangle$  and  $\langle A, \odot, 1 \rangle$  are commutative monoids
3.  $x \oplus 1 = 1$  and  $x \odot 0 = 0$
4.  $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$  and  $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$
5.  $x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z)$  and  $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$
6.  $x \oplus y = (x \oplus y) \oplus (x \odot y)$
7.  $x \oplus y = (x \vee y) \oplus (x \wedge y)$
8.  $x \odot y = (x \odot y) \odot (x \oplus y)$
9. If  $x \oplus y = x$ , then  $z \oplus y \leq z \vee x$
10. If  $x \oplus y = x \oplus z$  and  $x \odot y = x \odot z$ , then  $y = z$

Every positive MV-algebra is a member of  $\mathcal{Q}$  since 1.–10. are valid for any MV-algebra. The main open problem is to prove the opposite, that is, to show that any  $\mathbf{A} \in \mathcal{Q}$  is a positive MV-algebra. We solve this problem for those  $\mathbf{A} \in \mathcal{Q}$  satisfying additional identities of a special form. Namely let  $\mathcal{V}$  be any finitely generated variety of MV-algebras. Di Nola and Lettieri proved in [3] that there exists a finite set  $S$  of identities axiomatizing the variety  $\mathcal{V}$  within  $\mathcal{MV}$ , and every identity in  $S$  uses only terms of the language  $\{\oplus, \odot, \vee, \wedge, 0, 1\}$ .

**Theorem 1.** *The quasivariety of positive subreducts of  $\mathcal{V}$  is axiomatized by the quasi-identities 1.–10. and the identities from  $S$ .*

The essential ingredient of the proof of Theorem 1 is a certain non-trivial generalization of the technique of good sequences, which was introduced by Mundici [2]. It remains an open problem to extend this result beyond finitely generated varieties of MV-algebras, possibly using an axiomatization different from 1.–10.

## References

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