8-2018

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Peter Jipsen
Chapman University, jipsen@chapman.edu

Fei Liang
University of Delft

M. Andrew Moshier
Chapman University, moshier@chapman.edu

Apostolos Tzimoulis
University of Delft

Recommended Citation

Jipsen, P., Liang, F., Moshier, M. A., Tzimoulis, A. (2017) Partially-ordered multi-type algebras, display calculi and the category of weakening relations. Proceedings of LATD 2018.

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Comments
This paper was presented at the Logic, Algebra and Truth Degrees conference in August 2018.

Partially-ordered multi-type algebras, display calculi and the category of weakening relations

Peter Jipsen*[1], Fei Liang[2], M. Andrew Moshier[1] and Apostolos Tzimoulis[2]

[1] Chapman University, California, USA
jipsen@chapman.edu
moshier@chapman.edu
[2] University of Delft, The Netherlands
f.liang@tudelft.nl
apostolos@tzimoulis.eu

We define partially-ordered multi-type algebras and use them as algebraic semantics for multi-type display calculi that have recently been developed for several logics, including dynamic epistemic logic [7], linear logic[10], lattice logic [11], bilattice logic [9] and semi-De Morgan logic [8].

Multi-type algebras, also called many-sorted or heterogeneous algebras [1] have been studied in the setting of universal algebra and they have applications, e.g., as abstract data types in computer science [2] and in algebraic logic [4].

A multi-type algebra is of the form $\mathbb{A} = ((A_\tau)_{\tau\in\mathcal{T}}, \mathcal{F})$ where each $f \in \mathcal{F}$ is a function $f : A_{\tau_1} \times \cdots \times A_{\tau_n} \to A_\tau$ for some $\tau_1, \ldots, \tau_n, \tau \in \mathcal{T}$. The set of types $\mathcal{T}$ and the sequences $\tau_1, \ldots, \tau_n, \tau$ for each operation $f \in \mathcal{F}$ determine the signature Σ of the algebra. A partially-ordered multi-type algebra (pom-algebra for short) replaces the carrier sets A_τ by partially-ordered sets $(A_\tau, \leq_\tau)$ and insists that the operations are order-preserving or order-reversing in each argument. This is recorded in the signature Σ by a sequence $\varepsilon \in \{1, \partial\}^n$ for each n-ary operation f such that $A_{\tau_i}^{\varepsilon_i} = A_{\tau_i}$ for $\varepsilon_i = 1$ if f is order-preserving in the ith coordinate, and $A_{\tau_i}^{\varepsilon_i} = A_{\tau_i}^{\partial}$, the dual poset, otherwise. Pom-algebras are a generalization of partially ordered (unitype) algebras. Varieties and quasivarieties of partially ordered algebras have been studied by Pigozzi in [12], and these universal algebraic concepts extend smoothly to pom-algebras of a given signature.

In the setting of this talk we mostly consider pom-algebras in which each carrier set has lattice operations $\vee_\tau, \wedge_\tau$ defined on it. In this case $\leq_\tau$ is assumed to be the lattice order and such algebras are called lattice-ordered multi-type algebras, or ℓm-algebras. An important insight of multi-type display calculi is that certain unitype lattice-ordered algebras can be recast as ℓm-algebras where each of the carriers support simpler algebraic structure. The decomposition of lattice-ordered unitype algebras into simpler loosely connected components can lead to the definition of uniform decision procedures, in the form of display calculi, for the equational theory or even the universal theory of the original unityped algebras. In some cases the unitype algebras satisfy identities that cannot be captured by display calculus rules, but for their pom-algebra counterparts this difficulty is resolved since the carriers of each type satisfy simpler identities.

Examples and results

Every lattice decomposes as a pom-algebra of a join-semilattice and a (disjoint) meet-semilattice, with the connection given by an inverse pair of order-isomorphisms. The decision procedure given by the display calculus of this variety of pom-algebras is Whitmann's solution of the word problem for free lattices, originally due to Skolem (see [3]).

A semi-De Morgan lattice (not necessarily distributive) decomposes as a lattice and a De Morgan lattice, connected by two unary order-preserving maps as in [8]. The display calculus for semi-De Morgan lattices interleaves Whitmann's solution for lattices with a similar algorithm for De Morgan lattices.

A linear logic algebra with exponentials decomposes into a residuated lattice and a Heyting algebra connected with two adjunctions. Again this leads to a display calculus for linear logic.

The concept of *residuated frame* from [6] is extended to ℓm-algebras and provides multi-type frame semantics for display calculi. This allows many of the techniques for residuated frames to be applied in the more general setting of ℓm-algebras.

Display calculi use sequents of the form $s \leq_\tau t$ as ingredients for the rules of the calculus, with both terms s, t having the same result type τ. However if $t = f(t_1, \ldots, t_n)$ one can also consider a new sequent symbol $s \leq_\tau^f (t_1, \ldots, t_n)$. In this case $\leq_\tau^f$ is a relation from A_τ to $A_{\tau_1}^{\varepsilon_1} \times \cdots \times A_{\tau_n}^{\varepsilon_n}$. Similarly the top-level operation symbol of the left-hand term of a sequent can be used to define a new sequent symbol. From this point of view sequent separator symbols are morphisms in the category of posets, similar to the adjunctions that map between carrier posets in pom-algebras.

For posets $\mathbb{P} = (P, \leq^\mathbb{P})$ and $\mathbb{Q} = (Q, \leq^\mathbb{Q})$ a binary relation $R \subseteq P \times Q$ is a *weakening relation* if $\leq^\mathbb{P} \circ R \circ \leq^\mathbb{Q} \subseteq R$. The class of posets with with weakening relations as morphisms forms a category $\overline{\mathsf{Pos}}$ that contains the category Pos of posets with order-preserving maps. We characterize the display calculus morphisms as weakening relations in the category $\overline{\mathsf{Pos}}$.

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