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Multi-type Display Calculus for Semi-De Morgan Logic

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Multi-type display caculus for Semi-De Morgan Logic

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Preliminaries $\mathbb{A} = (A, \cap, \cup, ^*, 1, 0)$ is a *De Morgan algebra* (*DM-algebra*) if: (D1) $(L, \cap, \cup, 1, 0)$ is a bounded distributive lattice; (D2) $0^* = 1, 1^* = 0$; (D3) $(a \cup b)^* = a^* \cap b^*$ for all $a, b \in A$; (D4) $(a \cap b)^* = a^* \cup b^*$ for all $a, b \in A$; (D5) $a = a^{**}$ for every $a \in A$. It was originally introduced by Bialynicki-Birula and Rasiowa [1].

$\mathbb{A} = (A, \wedge, \vee, ', \top, \bot)$ is a *Semi-De Morgan algebra* (*SM-algebra*) if: (S1) $(A, \wedge, \vee, 1, 0)$ is a bounded distributive lattice; (S2) $\bot' = \top, \top' = \bot$; (S3) $(a \vee b)' = a' \wedge b'$ for all $a, b \in A$; (S4) $(a \wedge b)'' = a'' \wedge b''$ for all $a, b \in A$; (S5) $a' = a'''$ for every $a \in A$. It was first introduced by H.P. Sankappanavar [15] as a common abstraction of De Morgan algebras and distributive pseudocomplemented lattices.

Problems D.Hobby [12] gave a topological duality based on Priestley spaces for semi-De Morgan algebras. Some subvarieties of semi-De Morgan algebras are also studied in the literature [14]. From a proof theoretical perspective, M. Ma and F. Liang [13] proposed a G3-style sequent calculus for semi-De Morgan algebras. In this talk, we will present a proper display calculi for semi-De Morgan logic. A formula is analytic iff it can be transformed into a structural rule which preserves cut-elimination in display calculi [3][9]. (S4) and (S5) are not analytic, that is to say, we cannot transform them into display structural rules which preserve cut-elimination.

Solutions We introduce a heterogeneous representation for semi-De Morgan algebra, in which all axioms are analytic. We proceed as follows:

For any SM-algebra $\mathbb{A} := (A, \wedge, \vee, ', \top, \bot)$, denoting by $\mathbb{L}$ the distributive lattice reduct of $\mathbb{A}$, there exists a De Morgan algebra $\mathbb{D} = (A'', \cap, \cup, ^*, 1, 0)$, where $A'' = \{a'' \mid a \in A\}$ such that there is a homomorphism h from $\mathbb{L}$ onto $\mathbb{D}$, and a one-to-one map f from $\mathbb{D}$ to $\mathbb{L}$ such that f is meet preserving and also preserves the bounded elements, moreover, $h \circ f = Id_D$.

An *heterogeneous SDM-algebra* [2] is a structure $(\mathbb{L}, \mathbb{D}, f, h)$ such that $\mathbb{L}$ is a distributive lattice, $\mathbb{D}$ is a De Morgan algebra, $f : \mathbb{D} \to \mathbb{L}$ and $h : \mathbb{L} \to \mathbb{D}$ such that ,

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h is a homomorphism from $\mathbb{L}$ onto $\mathbb{D}$, $h(f(\alpha)) = \alpha$ for every $\alpha \in \mathbb{D}$, and moreover, f is a one-to-one map, and it is meet preserving and also preserves the bounded elements. Then, we can define $\neg : \mathbb{L} \to \mathbb{L}$ by $\neg a := f((h(a))^*)$ for every $a \in \mathbb{L}$. Finally, we can show $(L, \neg)$ is a semi-De Morgan algebra. Then (S4) and (S5) will be:
(S4') $f((h(f((h(a \wedge b))^*)))^*) = f((h(f((h(a))^*)))^*) \wedge f((h(f((h(b))^*)))^*)$
(S5') $f((h(a))^*)) = f((h(f((h(f((h(a \wedge b))^*)))^*)))^*)$.
Now, both of them are analytic.

Since $h, f, ^*$ are normal operators, all of them can preserve canonicity see [7]. It suffices to show $(\mathbb{L}^\delta, \mathbb{A}^\delta, f^\pi, h^\pi)$ is a perfect heterogeneous SDM-algebra and $\neg^\delta := f^\pi \circ {}^{*\pi} \circ h^\pi$ is the canonical extension of $\neg$. By general results of the theory of multi-type calculi [10, 11, 5, 8, 4, 6], these ingredients suffice to generate a conservative proper display calculus for semi-De Morgan algebras.

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