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Generalized Fock spaces and the Stirling numbers

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The Bargmann-Fock-Segal space plays an important role in mathematical physics and has been extended into a number of directions. In the present paper, we imbed this space into a Gelfand triple. The spaces forming the Fréchet part (i.e., the space of test functions) of the triple are characterized both in a geometric way and in terms of the adjoint of multiplication by the complex variable, using the Stirling numbers of the second kind. The dual of the space of test functions has a topological algebra structure, of the kind introduced and studied by the first named author and Salomon. *Published by AIP Publishing.* https://doi.org/10.1063/1.5035352

I. INTRODUCTION

The reproducing kernel Hilbert space $\mathcal{F}_1$ of entire functions with reproducing kernel $e^{z\overline{w}}$ is associated with the names of Bargmann, Segal, and Fock and will be called in this paper as the Fock space (more precisely, it is the symmetric Fock space associated with $\mathbb{C}$; see Ref. 10). It plays an important role in stochastic processes, mathematical physics, and quantum mechanics; for recent work on the topic see, e.g., Refs. 19 and 23. The space $\mathcal{F}_1$ is isometrically included in the Lebesgue space of the plane with weight $dA(z) := \frac{1}{\pi}e^{-|z|^2}dxdy$, and a key feature of $\mathcal{F}_1$ is that the adjoint of the operator of multiplication by the complex variable is the operator of differentiation. It is of interest to look at various generalizations of $\mathcal{F}_1$. One approach consists in slightly modifying the weight function, see, e.g., the studies of Refs. 17, 27, and 29, and another line is to change the kernel (that is, the norms of the monomials) in an appropriate way, for instance, replacing the exponential by the Mittag-Leffler function in the case of the gray noise theory; see, e.g., Ref. 28. Then too the weight is changed, but not always in an explicit way. Here we consider the family $(\mathcal{F}_m)_{m=1}^{\infty}$ of reproducing kernel Hilbert spaces with the reproducing kernel

$$k_m(z,\omega) = \sum_{n=0}^{\infty} \frac{z^n\overline{\omega}^n}{(n!)^m}, \quad m = 1, 2, \ldots \tag{1.1}$$

The space $\mathcal{F}_m$ can then be easily described as the space of all Taylor series of the form $f(z) = \sum_{n=0}^{\infty} f_n z^n$ for which

$$\sum_{n=0}^{\infty} |f_n|^2 (n!)^m < \infty.$$

For $m = 1$, the space is equal to the classical Fock space, and the case $m = 2$ was defined and studied in Ref. 8.

The main results are as follows: The first is a geometric characterization of the spaces $\mathcal{F}_m$ in terms of a weight; the second main result is the characterization of $\mathcal{F}_m$ in terms of the adjoint operator of multiplication by z, associated with the Stirling numbers of second kind; see (4.2). This generalizes the well known case of $m = 1$ and opens the ground for future applications such as interpolation and

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sampling theorems in the setting of $\mathcal{F}_m$; see, for instance, the papers[1,14] for the case of $\mathcal{F}_1$. The third main result is obtaining a structure of topological algebra for the inductive limit of the dual of the space $\cap_{m=1}^{\infty}\mathcal{F}_m$. This allows us to work locally in a Hilbert space rather than in the non-metrizable space $\cup_{m\in\mathbb{N}}\mathcal{F}_{2-m}$.

The outline of the paper is as follows. In Sec. II, we review some facts on the Mellin transform. In Sec. III, using the Mellin transform, we give a geometric characterization of the spaces $\mathcal{F}_m$ for $m\in\mathbb{N}$. A characterization of these spaces in terms of the adjoint of the operator of multiplication by z and using the Stirling numbers of the second kind is given in Sec. IV. A related Bargmann transform is defined in Sec. V. In Sec. VI, we define a Gelfand triple in which we imbed the Fock space. We observe that the intersection $\bigcap_{m=1}^{\infty}\mathcal{F}_m$ is a nuclear space, and its dual is an algebra of the type introduced in Ref. 9.

II. PRELIMINARIES

Let (a, b) an open interval of the real line, and let f and g be such that both $f(x)x^{c-1}$ and $g(x)x^{c-1}$ are summable on $[0, \infty)$ for $c \in (a, b)$. The Mellin transform of f, denoted by $\mathcal{M}(f)$, is given by

$$\mathcal{M}(f)(c) := \int_0^{\infty} x^{c-1}f(x)dx, \quad c\in(a,b).$$

In particular, the Mellin transform of the function $f_1(x) = e^{-x}$ is the Gamma function,

$$\mathcal{M}(f_1)(c) = \int_0^{\infty} x^{c-1}e^{-x}dx = \Gamma(c), \quad c > 0.$$

The Mellin convolution of f and g is defined by

$$(f * g)(x) := \int_0^{\infty} f(\frac{x}{t})g(t)\frac{dt}{t} = \int_0^{\infty} f(t)g(\frac{x}{t})\frac{dt}{t}, \quad x > 0.$$

An important relation between the Mellin transform and the Mellin convolution, see, e.g., Ref. 16, Theorem 3, is given by

$$\mathcal{M}(f * g)(c) = (\mathcal{M}(f)(c))(\mathcal{M}(g)(c)), \quad c\in(a,b).$$

III. GEOMETRIC DESCRIPTION OF $\mathcal{F}_m$

Recall that the Fock space $\mathcal{F}_1$ consists of those entire functions f for which

$$\iint_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dA(z) < \infty$$

and is the reproducing kernel Hilbert space with reproducing kernel $e^{z\overline{w}}$. In this section, we give for $m = 2, \ldots$ a geometric characterization for the space

$$\mathcal{F}_m = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ is entire with } \sum_{n=0}^{\infty} |a_n|^2 (n!)^m < \infty \right\}$$

which is the reproducing kernel Hilbert space with reproducing kernel (1.1), when equipped with the inner product

$$\langle f, g\rangle_{\mathcal{F}_m} := \sum_{n=0}^{\infty} f_n\overline{g_n}(n!)^m, \text{ where } f(z) = \sum_{n=0}^{\infty} f_n z^n, \; g(z) = \sum_{n=0}^{\infty} g_n z^n,$$

for every $f, g\in\mathcal{F}_m$. First, we use the properties of the Mellin transform to build the kernels $K_m(z)$, which are generalizations of the modified Bessel function of the second order, also called the Macdonald function. Let $K_1(x) = e^{-x}$ and for every integer $m > 1$ define the function

$$K_m(x) := (K_1 * \cdots * K_1)(x), \quad x\in\mathbb{R}_+, \tag{3.1}$$

that is, the function $K_1(x)$ Mellin-convoluted m many times with itself.

Lemma 3.1. Let m be an integer. The following properties hold:

(1) *For $m > 1$, the kernel K_m has the integral representations*

$$K_m(x) = \int_0^\infty \cdots \int_0^\infty \frac{e^{-\sum_{i=1}^{m-1} x_i - \frac{x}{\prod_{i=1}^{m-1} x_i}}}{\prod_{i=1}^{m-1} x_i} dx_1 \cdots dx_{m-1} \tag{3.2}$$

and

$$K_m(x) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-\sqrt[m]{x}(\sum_{i=1}^{m-1} e^{t_i} + e^{-\sum_{i=1}^{m-1} t_i})} dt_1 \cdots dt_{m-1}. \tag{3.3}$$

(2) *The function K_m is monotone decreasing in $(0, \infty)$.*

(3) *The Mellin transform of K_m is given by*

$$\mathcal{M}(K_m)(x) = \Gamma(x)^m, \quad x > 0,$$

and so

$$\int_0^\infty x^n K_m(x) dx = (n!)^m, \quad n \in \mathbb{N}. \tag{3.4}$$

Proof. Part 1 is proved by induction on m: if $m = 2$, we get

$$K_2(x) = \int_0^\infty e^{-x/t} e^{-t} \frac{dt}{t} = \int_0^\infty \frac{e^{-x_1 - \frac{x}{x_1}}}{x_1} dx_1.$$

Suppose formula (3.2) holds for m. Then

$$\begin{aligned} K_{m+1}(x) = (K_m * e^{-t})(x) &= \int_0^\infty K_m\left(\frac{x}{x_m}\right) e^{-x_m} \frac{dx_m}{x_m} \\ &= \int_0^\infty \cdots \int_0^\infty \frac{e^{-\sum_{i=1}^{m-1} x_i - \frac{\frac{x}{x_m}}{\prod_{i=1}^{m-1} x_i}}}{\prod_{i=1}^{m-1} x_i} \frac{e^{-x_m}}{x_m} dx_1 \cdots dx_m \\ &= \int_0^\infty \cdots \int_0^\infty \frac{e^{-\sum_{i=1}^{m} x_i - \frac{x}{\prod_{i=1}^{m} x_i}}}{\prod_{i=1}^{m} x_i} dx_1 \cdots dx_m, \end{aligned}$$

i.e., (3.2) holds for $m + 1$ and hence for every $m > 1$. Next, we use (3.2) and the change of variables $s_i = \ln(x_i)$, $1 \le i \le m - 1$, to obtain

$$K_m(x) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-\sum_{i=1}^{m-1} e^{s_i} - \frac{x}{e^{\sum_{i=1}^{m-1} s_i}}} ds_1 \cdots ds_{m-1},$$

and by another change of variables $t_i = s_i - \ln(\sqrt[m]{x})$, $1 \le i \le m - 1$, we get

$$K_m(x) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-\sqrt[m]{x}(\sum_{i=1}^{m-1} e^{t_i} + e^{-\sum_{i=1}^{m-1} t_i})} dt_1 \cdots dt_{m-1}.$$

From the representation (3.2), it is easily seen that $K_m(x)$ is a monotone decreasing function. Finally, the Mellin transform of K_m is given by

$$\mathcal{M}(K_m)(c) = \mathcal{M}(f_1)(c) \cdots \mathcal{M}(f_1)(c) = (\Gamma(c))^m, \quad c > 0;$$

therefore

$$\int_0^\infty x^{c-1} K_m(x) dx = (\Gamma(c))^m, \quad c > 0.$$

For $c = n + 1$, we have

$$\int_0^\infty x^n K_m(x) dx = (\Gamma(n+1))^m = (n!)^m.$$

∎

In the special case $m = 2$, we get that

$$K_2(x) = \int_{\mathbb{R}} e^{-\sqrt{x}2\cosh(t)}dt, \quad x \in \mathbb{R}_+$$

is the Bessel function of the second kind; see Ref. 8. For an arbitrary $m > 2$, the kernel $K_m(x)$ can be expressed in terms of the Meijer G-functions; see Ref. 20, Chap. 5 for the latter. We now show how the generalized Fock spaces $\mathcal{F}_m$ are obtained from the kernels $K_m(x)$ in a natural way.

Theorem 3.2. For any integer $m \geq 1$, the space $\mathcal{F}_m$ is equal to the space of all entire functions $f : \mathbb{C} \to \mathbb{C}$ satisfying the condition

$$\iint_{\mathbb{C}} |f(z)|^2 K_m(|z|^2)dA(z) < \infty. \tag{3.5}$$

Moreover, the inner product of $\mathcal{F}_m$ is given by

$$\frac{1}{\pi}\iint_{\mathbb{C}} f(z)\overline{g(z)}K_m(|z|^2)dA(z) = \sum_{n=0}^{\infty} f_n\overline{g_n}(n!)^m, \quad f, g \in \mathcal{F}_m,$$

and $\mathcal{F}_m$ has the orthonormal basis $\left\{\frac{z^n}{(n!)^{m/2}}\right\}_{n=0}^{\infty}$.

Proof. A straightforward computation shows that

$$\begin{aligned}\iint_{\mathbb{C}} z^n\overline{z}^k K_m(|z|^2)dA(z) &= \int_0^{\infty}\int_0^{2\pi} r^n e^{in\theta} r^k e^{-ik\theta} K_m(r^2) r d\theta dr\\ &= \int_0^{2\pi} e^{i(n-k)\theta}d\theta \int_0^{\infty} r^{n+k+1}K_m(r^2)dr\\ &= 2\pi\delta_{n,k}\int_0^{\infty} r^{2n+1}K_m(r^2)dr\\ &= 2\pi\delta_{n,k}\int_0^{\infty} u^n K_m(u)\frac{du}{2}\\ &= \pi(n!)^m\delta_{n,k}.\end{aligned}$$

Let $f = \sum_{n=0}^{\infty} f_n z^n$ and $g = \sum_{n=0}^{\infty} g_n z^n$ be entire functions. Then

$$\begin{aligned}\pi\iint_{\mathbb{C}} f(z)\overline{g(z)}K_m(|z|^2)dA(z) &= \sum_{n,k=0}^{\infty} f_n\overline{g_k}\iint_{\mathbb{C}} z^n\overline{z}^k K_m(|z|^2)dA(z)\\ &= \sum_{n,k=0}^{\infty} f_n\overline{g_k}\delta_{n,k}(n!)^m = \sum_{n=0}^{\infty} f_n\overline{g_n}(n!)^m,\end{aligned}$$

which implies that $f \in \mathcal{F}_m$ if and only if condition (3.5) holds, i.e.,

$$\frac{1}{\pi}\sum_{n=0}^{\infty}|f_n|^2(n!)^m = \iint_{\mathbb{C}} |f(z)|^2 K_m(|z|^2)dA(z) < \infty$$

as wanted. Furthermore, the inner product in $\mathcal{F}_m$ is then given by

$$\langle f, g\rangle_{\mathcal{F}_m} = \sum_{n=0}^{\infty} f_n\overline{g_n}(n!)^m = \frac{1}{\pi}\iint_{\mathbb{C}} f(z)\overline{g(z)}K_m(|z|^2)dA(z).$$

■

In the case $m = 2$, similar yet different spaces related to other families of orthogonal polynomials, appear in Ref. 26, Lemma 4 and Ref. 25.

Remark 3.3. Let $0 < \epsilon < 1$. Then $\frac{\epsilon}{n!} < 1$ for every $n \geq 0$ and hence

$$\sum_{m=1}^{\infty}\epsilon^m k_m(z,\omega)=\sum_{m=1}^{\infty}\epsilon^m\left(\sum_{n=0}^{\infty}\frac{z^n\overline{\omega}^n}{(n!)^m}\right)=\sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty}\left(\frac{\epsilon}{n!}\right)^m\right)z^n\overline{\omega}^n$$
$$=\sum_{n=0}^{\infty}\frac{\epsilon}{n!}\left(\frac{1}{1-\frac{\epsilon}{n!}}\right)z^n\overline{\omega}^n=\epsilon\cdot\sum_{n=0}^{\infty}\frac{z^n\overline{\omega}^n}{n!-\epsilon}$$

and

$$\sum_{m=1}^{\infty}\frac{\epsilon^m}{m!}k_m(z,\omega)=\sum_{n=0}^{\infty}\left(e^{\frac{\epsilon}{n!}}-1\right)z^n\overline{\omega}^n.$$

IV. OPERATOR THEORETIC DESCRIPTION OF $\mathcal{F}_m$

Let $\mathfrak{a}$ and $\mathfrak{b}$ be respectively the operators of multiplication by z and of differentiation, i.e., $\mathfrak{a}=M_z$ and $\mathfrak{b}=\frac{\partial}{\partial z}$. Both $\mathfrak{a}$ and $\mathfrak{b}$ are defined on polynomials and more generally on entire functions. They satisfy the familiar commutation relation

$$[\mathfrak{b},\mathfrak{a}]=\mathfrak{b}\mathfrak{a}-\mathfrak{a}\mathfrak{b}=I.$$

In the Fock space $\mathcal{F}_1$, $\mathfrak{a}$ and $\mathfrak{b}$ are unbounded operators and satisfy

$$\mathfrak{a}^*=\mathfrak{b}\quad\text{and}\quad\mathfrak{b}^*=\mathfrak{a}.$$

This relation is very important, as the Fock space is the only space of entire functions for which $\mathfrak{a}$ and $\mathfrak{b}$ are adjoint to each other; see Ref. 10. We generalize this result by presenting a relation between the operators $\mathfrak{a}$ and $\mathfrak{b}$ in the space $\mathcal{F}_m$. That gives us another characterization of the space $\mathcal{F}_m$. We first introduce the Stirling numbers of the second kind $S(k,n)$, which appear naturally in the theory of ordering bosons.

Definition 4.1 (Stirling numbers of the second kind). *For $k\in\mathbb{N}_0$ and $n\in\mathbb{N}_0$, the numbers $S(k,n)$ are defined by the recurrence formula*

$$S(k,n)=nS(k-1,n)+S(k-1,n-1),\quad k,n\geq 1$$

with the initial values $S(k,0)=\delta_{k,0}$ and $S(k,n)=0$ if $k<n$.

It is well known, see Refs. 12 and 13, that

$$(\mathfrak{a}\mathfrak{b})^k=\sum_{n=1}^{k}S(k,n)\mathfrak{a}^n\mathfrak{b}^n,\quad k\geq 1,$$

and this operator is called the Mellin derivative operator of order k (with $c=0$); see Ref. 16, Lemma 9.

Theorem 4.2. *Let $m\geq 1$ be an integer. The operators $\mathfrak{a}$ and $(\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}$ are closed densely defined operators on the space $\mathcal{F}_m$ and their domains coincide,*

$$Dom(\mathfrak{a})=Dom((\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b})=D,$$

where

$$D=\left\{f(z)=\sum_{n=0}^{\infty}f_nz^n:\sum_{n=0}^{\infty}|f_n|^2(n!)^mn^m<\infty\right\}\subseteq\mathcal{F}_m. \tag{4.1}$$

Moreover, the adjoint operator of $\mathfrak{a}$ in $\mathcal{F}_m$ is given by

$$\mathfrak{a}^*=(\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b},\quad\textit{with}\quad Dom(\mathfrak{a}^*)=Dom\left((\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}\right)=D.$$

Furthermore, let $\mathcal{H}$ be a Hilbert space of entire functions in which the polynomials are dense, and let $m\in\mathbb{N}$. If the adjoint operator of $\mathfrak{a}$ in $\mathcal{H}$ is equal to the operator $(\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}$, i.e., *if*

$$(M_z)^*=\frac{\partial}{\partial z}\left[\sum_{n=1}^{m-1}S(m-1,n)z^n\frac{\partial^n}{\partial z^n}\right], \tag{4.2}$$

then $\mathcal{H} = \mathcal{F}_m$ and there exists $c > 0$ for which

$$\langle f, g\rangle_{\mathcal{H}} = c \cdot \langle f, g\rangle_{\mathcal{F}_m}, \quad \forall f, g \in \mathcal{H}.$$

Proof. It is easy to see that $\mathfrak{a}$ and $(\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}$ are closed densely defined operators on $\mathcal{F}_m$. If $f(z) = \sum_{n=0}^{\infty} f_n z^n \in \mathcal{F}_m$, then

$$f \in Dom(\mathfrak{a}) \Longleftrightarrow \mathfrak{a}f = \sum_{n=0}^{\infty} f_n z^{n+1} \in \mathcal{F}_m \Longleftrightarrow \sum_{n=0}^{\infty} |f_n|^2((n+1)!)^m < \infty$$

and

$$f \in Dom((\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}) \Longleftrightarrow (\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}f = \sum_{n=1}^{\infty} f_n n^m z^{n-1} \in \mathcal{F}_m$$

$$\Longleftrightarrow \sum_{n=1}^{\infty} |f_n|^2 n^{2m}((n-1)!)^m = \sum_{n=1}^{\infty} |f_n|^2 (n!)^m n^m < \infty.$$

Therefore, $Dom(\mathfrak{a}) = Dom((\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}) = D$ as in (4.1). Next, if

$$g(z) = \sum_{n=0}^{\infty} g_n z^n \in Dom(\mathfrak{a}^*),$$

there exists

$$h(z) = \sum_{n=0}^{\infty} h_n z^n \in \mathcal{F}_m$$

such that $\langle \mathfrak{a}f, g\rangle_{\mathcal{F}_m} = \langle f, h\rangle_{\mathcal{F}_m}$ for every $f \in Dom(\mathfrak{a})$. In particular, for $f(z) = z^n$ $(n \geq 0)$, we get

$$\overline{g_{n+1}}((n+1)!)^m = \langle z^{n+1}, g\rangle_{\mathcal{F}_m} = \langle z^n, h\rangle_{\mathcal{F}_m} = \overline{h_n}(n!)^m,$$

and hence $h_n = g_{n+1}(n+1)^m$ for every $n \geq 0$. Thus,

$$h \in \mathcal{F}_m \Longrightarrow \sum_{n=0}^{\infty} |h_n|^2 (n!)^m = \sum_{n=0}^{\infty} |g_{n+1}|^2 (n+1)^{2m} (n!)^m < \infty$$

$$\Longrightarrow \sum_{n=1}^{\infty} |g_n|^2 (n!)^m n^m < \infty \Longrightarrow g \in D,$$

hence $Dom(\mathfrak{a}^*) \subseteq D$. Finally, if $g \in D = Dom((\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b})$, then

$$\langle f, (\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}g\rangle = \left\langle \sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} (n+1)^m g_{n+1} z^n \right\rangle = \sum_{n=0}^{\infty} f_n (n+1)^m \overline{g_{n+1}} (n!)^m$$

$$= \sum_{n=0}^{\infty} f_n \overline{g_{n+1}} ((n+1)!)^m = \left\langle \sum_{n=0}^{\infty} f_n z^{n+1}, \sum_{n=0}^{\infty} g_n z^n \right\rangle = \langle \mathfrak{a}f, g\rangle,$$

for every $f \in D = Dom(\mathfrak{a})$, which proves that $g \in Dom(\mathfrak{a}^*)$. Therefore, $D \subseteq Dom(\mathfrak{a}^*)$ and hence $Dom(\mathfrak{a}^*) = D$. By the previous calculation, we also know that $\mathfrak{a}^* = (\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}$. Now suppose that $\mathcal{H}$ is a Hilbert space which contains all polynomials such that

$$\mathfrak{a}^* = (\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}$$

in $\mathcal{H}$. Then for every $f \in Dom(\mathfrak{a}) \cap \mathcal{H}$ and $g \in Dom((\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}) \cap \mathcal{H}$,

$$\langle \mathfrak{a}f, g\rangle_{\mathcal{H}} = \langle f, (\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}g\rangle_{\mathcal{H}}, \tag{4.3}$$

and as both $Dom(\mathfrak{a})$ and $Dom((\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b})$ contain all polynomials, we apply (4.3) for the choice $f(z) = z^l$, $g(z) = z^k$ $(k, l \geq 0)$; thus

$$\begin{aligned}\langle z^{l+1}, z^k\rangle_{\mathcal{H}} &= \langle \mathfrak{a} f, g\rangle_{\mathcal{H}} = \langle f, (\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b} g\rangle_{\mathcal{H}}\\ &= \langle z^l, k^m z^{k-1}\rangle_{\mathcal{H}} = k^m \langle z^l, z^{k-1}\rangle_{\mathcal{H}}, \quad k, l \geq 0.\end{aligned}$$

We now prove by induction that for every $k \geq 0$ and $l \geq k$,

$$\langle z^{l+1}, z^k\rangle_{\mathcal{H}} = 0:$$

- If $k = 0$, we know that $\langle z^{l+1}, 1\rangle_{\mathcal{H}} = 0$ for every $l \geq 0$.
- Assume that for some $k \geq 0$, we have $\langle z^{l+1}, z^k\rangle_{\mathcal{H}} = 0$ for every $l \geq k$. Therefore, $\langle z^{l+2}, z^{k+1}\rangle_{\mathcal{H}} = (k+1)^m \langle z^{l+1}, z^k\rangle_{\mathcal{H}} = 0$ for every $l \geq k$, which means that

$$\langle z^{l+1}, z^{k+1}\rangle_{\mathcal{H}} = 0$$

  for every $l \geq k+1$, as wanted.

Thus the family $\{z^k\}_{k=0}^{\infty}$ is orthogonal in $\mathcal{H}$ and one can easily see that

$$\langle z^k, z^k\rangle_{\mathcal{H}} = k^m \langle z^{k-1}, z^{k-1}\rangle_{\mathcal{H}}, \quad \forall k \geq 1,$$

which implies that

$$\langle z^k, z^k\rangle_{\mathcal{H}} = (k!)^m \langle 1, 1\rangle_{\mathcal{H}}.$$

To conclude, if $f(z) = \sum_{k=0}^{\infty} f_k z^k$ and $g(z) = \sum_{k=0}^{\infty} g_k z^k \in \mathcal{H}$, then

$$\langle f, g\rangle_{\mathcal{H}} = \sum_{k,l=0}^{\infty} f_k \overline{g_l} \langle z^k, z^l\rangle_{\mathcal{H}} = \sum_{k=0}^{\infty} f_k \overline{g_k} (k!)^m \langle 1, 1\rangle_{\mathcal{H}},$$

i.e., the inner product in $\mathcal{H}$ is equal to the one in $\mathcal{F}_m$, up to a positive multiplicative constant $c = \langle 1, 1\rangle_{\mathcal{H}}$. As $\mathcal{H}$ is a Hilbert space which contains all the polynomials, it follows that

$$\mathcal{H} = \left\{ f = \sum_{n=0}^{\infty} f_n z^n : \langle f, f\rangle_{\mathcal{H}} = c \sum_{n=0}^{\infty} |f_n|^2 (n!)^m < \infty \right\} = \mathcal{F}_m. \qquad \blacksquare$$

In the previous theorem, we proved that $\mathcal{F}_m$ is the only Hilbert space which contains all polynomials and in which the adjoint operator of $\mathfrak{a} = M_z$ is equal to the operator

$$(\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b} = \frac{\partial}{\partial z}\left[\sum_{n=1}^{m-1} S(m-1, n) z^n \frac{\partial^n}{\partial z^n}\right].$$

One can see that we have the relations

$$\mathfrak{b}^n \mathfrak{a} = \mathfrak{a}\mathfrak{b}^n + n\mathfrak{b}^{n-1} \quad \text{and} \quad \mathfrak{b}\mathfrak{a}^n = \mathfrak{a}^n \mathfrak{b} + n\mathfrak{a}^{n-1}$$

for every $n \in \mathbb{N}$, and, in particular, the operators $\mathfrak{a}$ and $\mathfrak{a}^*$ do not satisfy the commutation relation. However we have the following result.

Proposition 4.3. The commutator of $\mathfrak{a}$ *and* $\mathfrak{a}^* = (\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b}$ *is equal to*

$$[\mathfrak{a}^*, \mathfrak{a}] = I + \sum_{n=1}^{m-1} (n+1) S(m, n+1) \mathfrak{a}^n \mathfrak{b}^n. \tag{4.4}$$

Proof. As

$$\mathfrak{a}^* = (\mathfrak{b}\mathfrak{a})^{m-1}\mathfrak{b} = \mathfrak{b} \sum_{n=1}^{m-1} S(m-1, n) \mathfrak{a}^n \mathfrak{b}^n,$$

we have

$$
\begin{aligned}
[\mathfrak{a}^*, \mathfrak{a}] &= \mathfrak{b}\sum_{n=1}^{m-1} S(m-1,n)\mathfrak{a}^n\mathfrak{b}^n\mathfrak{a} - \mathfrak{a}\mathfrak{b}\sum_{n=1}^{m-1} S(m-1,n)\mathfrak{a}^n\mathfrak{b}^n \\
&= \mathfrak{b}\sum_{n=1}^{m-1} S(m-1,n)\mathfrak{a}^n(\mathfrak{a}\mathfrak{b}^n + n\mathfrak{b}^{n-1}) - \mathfrak{a}\mathfrak{b}\sum_{n=1}^{m-1} S(m-1,n)\mathfrak{a}^n\mathfrak{b}^n \\
&= (\mathfrak{b}\mathfrak{a} - \mathfrak{a}\mathfrak{b})\sum_{n=1}^{m-1} S(m-1,n)\mathfrak{a}^n\mathfrak{b}^n + \mathfrak{b}\sum_{n=1}^{m-1} nS(m-1,n)\mathfrak{a}^n\mathfrak{b}^{n-1} \\
&= \sum_{n=1}^{m-1} S(m-1,n)\mathfrak{a}^n\mathfrak{b}^n + \sum_{n=1}^{m-1} nS(m-1,n)(\mathfrak{a}^n\mathfrak{b} + n\mathfrak{a}^{n-1})\mathfrak{b}^{n-1} \\
&= \sum_{n=1}^{m-1}(n+1)S(m-1,n)\mathfrak{a}^n\mathfrak{b}^n + \sum_{n=1}^{m-1} n^2 S(m-1,n)\mathfrak{a}^{n-1}\mathfrak{b}^{n-1},
\end{aligned}
$$

and as $S(m-1, 1) = S(m-1, m-1) = S(m, m) = 1$, we have

$$
\begin{aligned}
[\mathfrak{a}^*, \mathfrak{a}] &= I + m\mathfrak{a}^{m-1}\mathfrak{b}^{m-1} + \sum_{n=1}^{m-2}(n+1)[S(m-1,n) + (n+1)S(m-1,n+1)]\mathfrak{a}^n\mathfrak{b}^n \\
&= I + m\mathfrak{a}^{m-1}\mathfrak{b}^{m-1} + \sum_{n=1}^{m-2}(n+1)S(m,n+1)\mathfrak{a}^n\mathfrak{b}^n \\
&= I + \sum_{n=1}^{m-1}(n+1)S(m,n+1)\mathfrak{a}^n\mathfrak{b}^n.
\end{aligned}
$$

■

Sequentially, a straightforward calculation shows that

$$
\|\mathfrak{a}f\|^2_{\mathcal{F}_m} = \|\mathfrak{a}^*f\|^2_{\mathcal{F}_m} + \|f\|^2_{\mathcal{F}_m} + \sum_{k=1}^{m-1}\binom{m}{k}\left[\sum_{n=0}^{\infty}|f_n|^2(n!)^m n^k\right]
$$

for every $f \in D$, which guarantees that all the terms in the identity are finite. It is tempting to write the last identity (with some abuse of notation) as

$$
\|\mathfrak{a}f\|^2_{\mathcal{F}_m} = \|\mathfrak{a}^*f\|^2_{\mathcal{F}_m} + \|f\|^2_{\mathcal{F}_m} + \sum_{k=1}^{m-1}\binom{m}{k}\langle f, (\mathfrak{a}\mathfrak{b})^k f\rangle_{\mathcal{F}_m};
$$

however $f \in D$ does not necessarily imply that $f \in Dom((\mathfrak{a}\mathfrak{b})^k)$.

Finally, we have the following relation between the operators $\mathfrak{a}, \mathfrak{b}$ and the family of spaces $(\mathcal{F}_m)_{m\in\mathbb{Z}}$. For every $n \geq 1$,

- the Fock space $\mathcal{F}_1$ satisfies
$$\mathfrak{a}^n(\mathcal{F}_1) \subseteq \mathcal{F}_0 \quad \text{and} \quad \mathfrak{b}^n(\mathcal{F}_1) \subseteq \mathcal{F}_0;$$
- if $m > 1$, then
$$\mathfrak{a}^n(\mathcal{F}_m) \subseteq \mathcal{F}_{m-1} \quad \text{and} \quad \mathfrak{b}^n(\mathcal{F}_m) \subseteq \mathcal{F}_m,$$
- if $m < 1$, then
$$\mathfrak{a}^n(\mathcal{F}_m) \subseteq \mathcal{F}_m \quad \text{and} \quad \mathfrak{b}^n(\mathcal{F}_m) \subseteq \mathcal{F}_{m-1}.$$

Remark 4.4. Unlike the situation in the Fock space, where the adjoint of $\mathfrak{b}$ *is equal to* $\mathfrak{a}$, *in the space* $\mathcal{F}_m$, *the adjoint operator of* $\mathfrak{b}$ *is equal to*

$$
\mathfrak{b}^*\left(\sum_{k=0}^{\infty} f_k z^k\right) = \sum_{k=0}^{\infty}\frac{f_k}{(k+1)^{m-1}}z^{k+1};
$$

thus $\mathfrak{b}^* \neq \mathfrak{a}$ *if* $m > 1$.

V. GENERALIZED BARGMANN TRANSFORM

Recall that the normalized Hermite functions are defined by

$$\eta_n(t) = \frac{1}{\pi^{1/4} 2^{n/2} \sqrt{n!}} e^{\frac{t^2}{2}} \left(e^{-t^2}\right)^{(n)}, \quad n \in \mathbb{N}_0.$$

The family $\{\eta_n\}_{n=0}^{\infty}$ is an orthonormal basis of the Lebesgue space $\mathbf{L}_2(\mathbb{R}, dt)$. Furthermore, see Ref. 24, p. 436, the η_n are uniformly bounded by some constant, i.e.,

$$\exists C > 0 \text{ such that } |\eta_n(t)| \leq C, \text{ for every } n \in \mathbb{N} \text{ and } t \in \mathbb{R}.$$

Similarly to the symmetric Fock space associated with $\mathbb{C}$, see, e.g., Ref. 10, that is, $\mathcal{F}_1$, there is a fourth characterization of the space $\mathcal{F}_m$, given by a mapping from $\mathbf{L}_2(\mathbb{R}, dt)$ into $\mathcal{F}_m$, presented in the following proposition.

Proposition 5.1. Let $m \geq 2$. For every $t \in \mathbb{R}$ and $z \in \mathbb{C}$ define the function

$$h_m(z,t) := \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{m/2}} \eta_n(t). \tag{5.1}$$

Then,

1. *for every $t \in \mathbb{R}$, the function $h_m(\cdot, t)$ is entire.*
2. *$f \in \mathcal{F}_m$ if and only if there exists $g \in \mathbf{L}_2(\mathbb{R}, dt)$ such that*

$$f(z) = \int_{\mathbb{R}} h_m(z,t) g(t) dt = \langle g, \overline{h_m(z,\cdot)} \rangle_{\mathbf{L}_2(\mathbb{R},dt)}. \tag{5.2}$$

Proof. Since the functions $\eta_n(t)$ are all bounded by C, the sum in (5.1) converges, and so $h_m(\cdot, t)$ is entire. Next, let $f(z) = \langle g, \overline{h_m(z,\cdot)} \rangle_{\mathbf{L}_2(\mathbb{R},dt)}$ for some $g \in \mathbf{L}_2(\mathbb{R}, dt)$. Then,

$$f(z) = \int_{\mathbb{R}} \left(\sum_{n=0}^{\infty} \frac{z^n}{(n!)^{m/2}} \eta_n(t) g(t) \right) dt = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{m/2}} \int_{\mathbb{R}} \eta_n(t) g(t) dt.$$

As the system $\{\eta_n\}_{n=0}^{\infty}$ forms an orthonormal basis of $\mathbf{L}_2(\mathbb{R}, dt)$, we have Parseval's equality

$$\sum_{n=0}^{\infty} \left| \int_{\mathbb{R}} \eta_n(t) g(t) dt \right|^2 = \int_{\mathbb{R}} |g(t)|^2 dt,$$

and hence $f \in \mathcal{F}_m$ since

$$\sum_{n=0}^{\infty} \left| \frac{1}{(n!)^{m/2}} \int_{\mathbb{R}} \eta_n(t) g(t) dt \right|^2 (n!)^m = \|g\|^2_{\mathbf{L}_2(\mathbb{R},dt)} < \infty.$$

Finally, let $f \in \mathcal{F}_m$. It can be written as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\sum_{n=0}^{\infty} |a_n|^2 (n!)^m < \infty$. Setting

$$g(t) = \sum_{n=0}^{\infty} (n!)^{m/2} a_n \eta_n(t),$$

we observe that

$$\|g\|^2_{\mathbf{L}_2(\mathbb{R},dt)} = \sum_{n=0}^{\infty} |a_n|^2 (n!)^m < \infty$$

and finally that

$$\langle h_m(z,\cdot), g \rangle_{\mathbf{L}_2(\mathbb{R},dt)} = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{m/2}} (n!)^{m/2} a_n = f(z).$$

■

This characterization of $\mathcal{F}_m$ motivates us to consider an associated Bargmann transform. For any $g \in \mathbf{L}_2(\mathbb{R}, dt)$, we define the Bargmann transform of g to be

$$\mathfrak{B}_{\mathrm{m}}(g) := \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{m/2}} \int_{\mathbb{R}} \eta_n(t) g(t) dt = \langle g, \overline{h_m(z, \cdot)} \rangle_{\mathbf{L}_2(\mathbb{R}, dt)}.$$

The mapping $\mathfrak{B}_{\mathrm{m}} : \mathbf{L}_2(\mathbb{R}, dt) \to \mathcal{F}_m$ is unitary; it satisfies

$$\mathfrak{B}_{\mathrm{m}}(\eta_n)(z) = \frac{z^n}{(n!)^{m/2}} \quad \text{and} \quad \|g\|_{\mathbf{L}_2(\mathbb{R}, dt)} = \|\mathfrak{B}_{\mathrm{m}}(g)\|_{\mathcal{F}_m}$$

for every $g \in \mathbf{L}_2(\mathbb{R}, dt)$.

Remark 5.2. In case where $m = 1$, $\mathfrak{B}_1$ *is the well-known Bargmann transform and the function* $h_1(z, t)$ *can be written in closed form as*

$$h_1(z, t) = e^{2tz - t^2 - z^2/2}.$$

When $m > 1$, *finding an explicit closed formula for the function* $h_m(z, t)$ *might involve new generalizations of the exponential function.*

VI. A GELFAND TRIPLE ASSOCIATED WITH THE FAMILY $(\mathcal{F}_m)_{m \in \mathbb{Z}}$

The reproducing kernel Hilbert spaces $\{\mathcal{F}_m\}_{m=1}^{\infty}$, starting from the Fock space $\mathcal{F}_1$, form a decreasing sequence, i.e.,

$$\mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots \supset \mathcal{F}_m \supset \mathcal{F}_{m+1} \supset \ldots.$$

So it makes sense, in the spirit of the theory of Gelfand triples (as developed, for instance, in the books[21,22]) to consider the intersection space

$$\begin{aligned}\mathcal{F} &= \bigcap_{m=1}^{\infty} \mathcal{F}_m \\ &= \left\{ f = \sum_{n=0}^{\infty} a_n z^n \text{ such that } \|f\|_m = \sum_{n=0}^{\infty} |a_n|^2 (n!)^m < \infty, \forall m \in \mathbb{N} \right\},\end{aligned}$$

which consists of entire functions and its dual. We consider the dual space of each $\mathcal{F}_m$, with respect to the Fock space $\mathcal{F}_1$.

Lemma 6.1. For every $m \geq 1$, *the dual space of* $\mathcal{F}_m$, *with respect to* $\mathcal{F}_1$ *is*

$$\mathcal{F}_{2-m} := (\mathcal{F}_m)' = \left\{ b = (b_n)_{n \in \mathbb{N}_0} : \|b\|_{2-m}^2 := \sum_{n=0}^{\infty} |b_n|^2 (n!)^{2-m} < \infty \right\}.$$

Therefore, we have the Gelfand triple

$$\bigcap_{m=1}^{\infty} \mathcal{F}_m \subset \mathcal{F}_1 \subset \bigcup_{m=1}^{\infty} \mathcal{F}_{2-m}. \tag{6.1}$$

The inclusion map from $\mathcal{F}_m$ into $\mathcal{F}_{m+1}$ is nuclear, and it follows that $\cap_{m=1}^{\infty} \mathcal{F}_{2-m}$ is a Fréchet nuclear space, and, in particular, a perfect space in the terminology of Gelfand and Shilov; see Ref. 22. The dual space $\cup_{m=1}^{\infty} \mathcal{F}_{2-m}$ has two different sets of properties, topological and algebraic; the first follows from the theory of perfect spaces, and the structure algebra comes from the form of the weights. The fact that the product is jointly continuous comes from the theory of reflexive Fréchet spaces; see Ref. 15, Sec. IV.26, Theorem 2.

We begin with the topological properties. Although not metrizable, the space $\cup_{m=1}^{\infty} \mathcal{F}_{2-m}$ behaves well with respect to sequences and compactness:

(1) A sequence converges in the strong (or weak) topology of the dual if and only if its elements are in one of the spaces $\mathcal{F}_{2-m}$ and converges in the topology of the latter; see Ref. 22, p. 56.
(2) A subset of $\cup_{m=1}^{\infty}\mathcal{F}_{2-m}$ is compact in the strong topology of the dual if and only if it is included in one the spaces $\mathcal{F}_{2-m}$ and compact in the topology of the latter; see Ref. 22, p. 58.

These properties allow us to reduce to the Hilbert space setting and sequences the study of continuous functions from a compact metric space into $\cup_{m=1}^{\infty}\mathcal{F}_{2-m}$.

The algebra structure is given by the convolution product (or Cauchy product) defined as follows:

$$a * b := \left(\sum_{k=0}^{n} a_k b_{n-k}\right)_{n\in\mathbb{N}_0}, \tag{6.2}$$

where $a = (a_n)_{n\in_0\mathbb{N}}$ and $b = (b_n)_{n\in_0\mathbb{N}}$ belong to the dual.

Proposition 6.2. The space

$$\mathcal{F}' := \bigcup_{m=1}^{\infty}\mathcal{F}_{2-m} = \left\{b = (b_n)_{n\in\mathbb{N}_0} : \exists m \geq 1, \|b\|_{2-m} := \sum_{n=0}^{\infty}\frac{|b_n|^2}{(n!)^{m-2}} < \infty\right\}$$

is a topological algebra; the convolution product is jointly continuous with respect to the two variables and satisfies

$$\|a * b\|_{2-p} \leq A(q-p)\|a\|_{2-q}\|b\|_{2-p}, \tag{6.3}$$

for every $a \in \mathcal{F}_{2-q}$ and $b \in \mathcal{F}_{2-p}$, where $p, q \in \mathbb{N}$ such that $q \geq p + 1$.

The weights $\alpha_n = n!$ satisfy

$$\alpha_{m+n} = \sqrt{(m+n)!} \geq \sqrt{m!n!} = \alpha_m\alpha_n$$

for every $m, n \in \mathbb{N}_0$ and $\sum_{n=0}^{\infty}(\alpha_n)^{-2} = \sum_{n=0}^{\infty}\frac{1}{n!} = e < \infty$. Using these properties of the weight, the statements in the proposition follow then from Ref. 9 or, in a maybe more explicit way, from Ref. 2, Exercise 5.4.8, p. 260–261, with

$$A(q-p) = \left(\sum_{n=0}^{\infty}\alpha_n^{2(p-q)}\right)^{1/2} = \left(\sum_{n=0}^{\infty}\left(\frac{1}{n!}\right)^{q-p}\right)^{1/2} < \infty$$

for $q - p \geq 1$.

We note that (6.3) is called the Väge inequality and originates with the work of Väge; see Refs. 11 and 30.

Consider now a $\mathcal{F}_1$-valued function, say, f, defined a compact set (for instance [0, 1]). When viewing f as $\cup_{m=1}^{\infty}\mathcal{F}_{2-m}$-valued, one can define differentiability and compute explicitly the derivative, which will take values in one of the spaces $\mathcal{F}_{2-m}$ rather than in the Fock space itself. Using the Väge inequality one can also consider stochastic type integrals of the form

$$\int_0^1 f(t) * g(t)dt,$$

where f and g are continuous from [0, 1] into $\mathcal{F}'$ as Riemann integrals. The image of [0, 1] under the function $f * g$ is then compact, and the integral is computed in one of the spaces $\mathcal{F}_{2-m}$. See Refs. 3, 4, 7, and 5 for similar arguments and applications, Ref. 5 being in the setting of quaternionic stochastic processes. Finally, we refer to Ref. 6 for the study of the quaternionic Fock space and to Ref. 18 for some of its generalizations in the quaternionic setting.

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