5-29-2018

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Recommended Citation

Kh. Abu-Ghanem, D. Alpay, F. Colombo, I. Lewkowicz, and I. Sabadini. Herglotz functions of several quaternionic variables. J. Math. Anal. Appl., 466(1):169-182, 2018. doi: 10.1016/j.jmaa.2018.05.066

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HERGLOTZ FUNCTIONS OF SEVERAL QUATERNIONIC VARIABLES

KHALED ABU-GHANEM, DANIEL ALPAY, FABRIZIO COLOMBO, IZCHAK LEWKOWICZ, AND IRENE SABADINI

Abstract. We first review realizations of Herglotz functions in the unit ball of $\mathbb{C}^N$ and provide new insights. Then, we define the corresponding class and prove the extend the results in the case of several quaternionic variables.

AMS Classification: 47B32, 30G35.
Key words: Several complex variables, Several quaternionic variables, Gleason's problem, Herglotz functions.

1. Introduction

Functions $\Phi(z)$ which in the open unit disk $\mathbb{D}$ are analytic and with positive real part, are called Herglotz functions and play an important role in analysis, electrical engineering and other related topics. It is of therefore of interest to study their counterparts and refined Herglotz class realizations in other settings. The condition $\operatorname{Re}\Phi(z) \geq 0$ for $z \in \mathbb{D}$ is equivalent to the fact that the kernel

$$K_\Phi(z,w) = \frac{\Phi(z) + \overline{\Phi(w)}}{1 - z\overline{w}} \tag{1.1}$$

is positive definite in the sense of reproducing kernels (see e.g. [10] for this notion) in $\mathbb{D}$. More generally, a function defined on a uniqueness set $\Omega \subset \mathbb{D}$ and such that $K_\Phi(z,w)$ is positive definite in Ω is the restriction of a function, still denoted by Φ, for which the kernel K_Φ is positive definite in $\mathbb{D}$ and hence with a real positive part in $\mathbb{D}$.

When $N = 1$ the fact that a function analytic in $\mathbb{D}$ has a positive real part there, is equivalent to the fact that the kernel (1.1) is positive definite in $\mathbb{D}$. However, already for $N = 1$, a function Φ for which the kernel (1.1) is positive definite in $\mathbb{D}$ need not be bounded, as is illustrated by the example $\Phi(z) = \frac{1+z}{1-z}$. The corresponding multiplication operator M_Φ is not bounded from the Hardy space of the open unit disk into itself.

In the present paper we consider the counterparts of (1.1) in two different settings, namely for the unit ball of $\mathbb{C}^N$ and in the case of several quaternionic variables. In the latter case we pursue the approach introduced in [1]. For other approaches to the study of functions of several quaternionic variables case, see [14, 17]. We refer to the papers [11, 12, 13] for the case of the polydisk and the Herglotz-Agler classes, and to [19] for the case of poly-halfplanes. The case of the unit ball was considered, among other results, in the papers [18, 20]. We use a common method, namely in both cases we build a partial isometry adapted from the one introduced in [9] independently from the earlier isometry defined

Daniel Alpay thanks the Foster G. and Mary McGaw Professorship in Mathematical Sciences, which supported this research.

(and said to be isometric) in [20, (6.3), p. 69].

We here consider bounded linear operator-valued functions from a Krein space into itself. The definitions of Krein and Pontryagin spaces, which appear in the following discussion, are recalled in the sequel; see Definition 2.1 below. From the several complex variables side, this paper is a continuation of [5], where the case of Schur multipliers is considered. In the case of Schur multipliers, the relation used is between Pontryagin spaces and involves the coefficient space. More precisely, if S is a Schur multiplier, taking values from the Pontryagin space $\mathcal{P}_1$ into the Pontryagin space of same index $\mathcal{P}_2$, with associated kernel $\frac{I_{\mathcal{P}_2}-S(z)S(w)^*}{1-\langle z,w\rangle}$ positive definite in the open unit ball $\mathbb{B}$ of $\mathbb{C}^N$, the relation is between the spaces

$$\mathcal{H}(S)^N \oplus \mathcal{P}_1 \quad \text{and} \quad \mathcal{H}(S) \oplus \mathcal{P}_2. \tag{1.2}$$

The method uses a theorem of Shmulyan (see [6, 25]) on the extension of isometric relations in Pontryagin spaces; such a theorem does not hold in the setting of Krein spaces and one cannot take such spaces as coefficient spaces; the linear relation considered in [20] and in this work is between Hilbert spaces and does not involve the coefficient space. This allows us to take the more general case of Krein spaces as coefficient space. The paper contains some complements to the paper [20], such as uniqueness of a certain representation of an Herglotz multiplier and the solution of the Gleason problem in the associated reproducing kernel Hilbert space, and is closely related to the paper [23], which considers the non-commutative setting. We also refer to [22] for results in that setting. In the quaternionic setting this paper is a continuation of [1], where the case of Schur multipliers was considered.

2. The case of several complex variables

We start by recalling the definition of a Krein space.

Definition 2.1. *The complex vector space $\mathcal{V}$ endowed with an Hermitian form $[\cdot,\cdot]$ is said to be a Krein space if it can be written as*

$$\mathcal{V} = \mathcal{V}_+ [\dot{+}] \mathcal{V}_-$$

where $(\mathcal{V}_+, [\cdot,\cdot])$ and $(\mathcal{V}_-, -[\cdot,\cdot])$ are Hilbert spaces and where the sum is direct and orthogonal, that is:

$$\mathcal{V}_+ \cap \mathcal{V}_- = \{0\}$$

and

$$[v_+, v_-] = 0, \quad \forall h_+ \in \mathcal{V}_+ \quad \textit{and} \quad \forall h_- \in \mathcal{V}_-.$$

To prove the main result in this section, we will solve a Gleason problem in a suitable reproducing kernel Hilbert space, thus we recall the following, see [24]:

Gleason's problem. Let X be a space of holomorphic functions defined in a region $\Omega \subseteq \mathbb{C}^N$ and let $a \in \Omega$. Find, if possible, functions $g_1, \ldots, g_N \in X$ such that

$$f(z) - f(a) = \sum_{k=1}^{N} (z_k - a_k) g_k(z).$$

This section has a large intersection with the paper [20], and references to the results in that paper are mentioned in the text. We denote by $\mathbb{B}$ the open unit ball of $\mathbb{C}^N$. The kernel

$$\frac{1}{1-\langle z,w\rangle}=\frac{1}{1-z_1\bar{w}_1-\cdots-z_N\bar{w}_N}.$$

is positive definite in $\mathbb{B}$. We note that $z\in\mathbb{C}^N$ is a row vector $z=(z_1,\ldots,z_N)$ so that $z^*=(\bar{z}_1,\ldots\bar{z}_N)^T$ is a column vector and

$$\langle z,w\rangle=z_1\bar{w}_1+\cdots+z_N\bar{w}_N=zw^*.$$

The associated reproducing kernel Hilbert space is called the Drury-Arveson space, and denoted by $\mathcal{A}$. For a given Krein space $\mathcal{V}$ we denote by $\mathbf{L}(\mathcal{V},\mathcal{V})$ the space of linear bounded operators from $\mathcal{V}$ into itself. We consider $\mathbf{L}(\mathcal{V},\mathcal{V})$-valued functions $\Phi(z)$ defined in an open subset Ω of $\mathbb{B}$, and such that the kernel

$$K_\Phi(z,w)=\frac{\Phi(z)+\Phi(w)^*}{1-\langle z,w\rangle} \tag{2.1}$$

is positive definite for $z,w\in\Omega$.

Definition 2.2. *A $\mathbf{L}(\mathcal{V},\mathcal{V})$-valued functions $\Phi(z)$ defined in $\mathbb{B}$, and such that the kernel* (2.1) *is positive there is called a Herglotz function (these functions are called in* [20, §6, p. 68] *positive Schur classes). It is called a Herglotz multiplier if the multiplication operator by Φ is bounded from the corresponding vector-valued Arveson space into itself and has a positive real part.*

Herglotz functions are in particular Herglotz multipliers, and have positive real parts in $\mathbb{B}$; furthermore a Herglotz multiplier is uniformly pointwise bounded: $\sup_{w\in\mathbb{B}}\|\Phi(w)\|<\infty$. These various notions are in general different as we explain in the following remark.

Remark 2.3. When $N>1$, the positivity of the kernel $K_\Phi(z,w)$ in the unit ball $\mathbb{B}$ implies, but is not equivalent to, the positivity of the real part of Φ in $\mathbb{B}$. To see that, it is enough to consider the scalar case. Via Cayley transform, the claim is equivalent to the related statement for kernels of the form

$$K_s(z,w)=\frac{1-s(z)\overline{s(w)}}{1-\langle z,w\rangle}. \tag{2.2}$$

The fact is proved in [2] on general grounds. An explicit example can be found in Drury's paper [15] (see [20, Remark 3, p. 70]). Another explicit example was given in [7] and is recalled here for completeness. It uses the fact that the kernel (2.2) is positive definite in $\mathbb{B}$ if and only if the operator of multiplication by s is a contraction from the Drury-Arveson space into itself. Consider the polynomials of N complex variables (in fact they depend only on the variables z_1 and z_2)

$$p_m(z_1,z_2,\ldots,z_N)=z_1+c_1z_2^2+\cdots+c_mz_2^{2m},$$

where $c_1,c_2,c_3,\ldots$ are defined by

$$1-\sqrt{1-t}=\sum_{n=1}^\infty c_jt^j,\quad |t|<1.$$

Since the c_j are strictly positive and add up to 1, we have that $|p_m(z)|\le 1$ in $\mathbb{B}$ but $\|p_m1\|^2_{\mathcal{A}}>\|1\|_{\mathcal{A}}$, and so p_m is not a Schur multiplier. We refer to [7] for more details. We also refer to [24, p. 164] and [21, Example 4.4, p. 834] for related discussions.

Definition 2.4. *We let $\mathcal{L}(\Phi)$ denote the reproducing kernel Hilbert space with reproducing kernel $K_\Phi(z,w)$ given in* (2.1).

In the sequel, we denote by C the point evaluation at the origin. Note that

$$C^*c = K_\Phi(\cdot,0)c, \quad c\in\mathbb{C}^n. \tag{2.3}$$

We first need a preliminary result. From the equality

$$\|fw^*\|^2_{(\mathcal{L}(\Phi))^N} = \sum_{u=1}^n \|f\overline{w_u}\|^2_{\mathcal{L}(\Phi)} = \|f\|^2_{\mathcal{L}(\Phi)}\left(\sum_{u=1}^n |w_u|^2\right)$$

we get:

Lemma 2.5. *The operator M_{w^*}:*

$$f \;\mapsto\; fw^* = \begin{pmatrix} f\overline{w_1} \\ \vdots \\ f\overline{w_N}\end{pmatrix}$$

is a strict contraction from $\mathcal{L}(\Phi)$ into $(\mathcal{L}(\Phi))^N$, of norm less or equal to $\|w\|$, and its adjoint is given by M_w defined by:

$$\begin{pmatrix} f_1 \\ \vdots \\ f_N\end{pmatrix} \;\mapsto\; \sum_{u=1}^N w_u f_u. \tag{2.4}$$

Theorem 2.6. *Let $\Omega\subset\mathbb{B}$ be open and contain the origin, and let Φ defined in Ω be such that the associated kernel K_Φ is positive definite in Ω. Then, Φ is the restriction of a function analytic in $\mathbb{B}$, still denoted by Φ, of the form*

$$\Phi(w) = i\mathrm{Im}\,\Phi(0) + \frac{1}{2}C(I-M_wV)^{-1}(I+M_wV)C^*, \tag{2.5}$$

where V is a partial isometry. Furthermore, for $z,w\in\mathbb{B}$

$$\frac{\Phi(z)+\Phi(w)^*}{1-\langle z,w\rangle} = C(I-M_zV)^{-1}(I-M_{w^*}V^*)^{-1}C^*. \tag{2.6}$$

Conversely, any function of the form (2.5) *is a Herglotz function, and it has a (possibly different) realization of the form* (2.5) *for which* (2.6) *holds.*

Proof. Following [9, p. 708] we define a linear relation R on $(\mathcal{L}(\Phi)^N)\times\mathcal{L}(\Phi)$ to be the linear span of the pairs of the form

$$(K_\Phi(\cdot,w)w^*c, K_\Phi(\cdot,w)c-K_\Phi(\cdot,0)c)\,, \tag{2.7}$$

with $w\in\Omega$ and $c\in\mathcal{K}$.

STEP 1: *The relation is the graph of a partial isometry V.*

Let $w^{(1)},\ldots,w^{(m)}\in\Omega$ and $c_1,\ldots,c_m\in\mathcal{K}$, and set

$$f(\cdot) = \sum_{u=1}^m K_\Phi(\cdot,w^{(u)})(w^{(u)})^*c_u \quad\text{and}\quad g(\cdot) = \sum_{u=1}^m K_\Phi(\cdot,w^{(u)})c_u - K_\Phi(\cdot,0)(\sum_{u=0}^m c_u).$$

Then (and note that this equality will not hold in the quaternionic counterpart of the present argument)

$$\begin{aligned}\langle f, f\rangle_{(\mathcal{L}(\Phi))^N} &= \sum_{u,v=1}^{m} [K_\Phi(w^{(u)}, w^{(v)})(w^{(v)})^* c_v, (w^{(u)})^* c_u]_{\mathcal{K}^N} \\ &= \sum_{u,v=1}^{m} \langle w^{(u)}, w^{(v)}\rangle [K_\Phi(w^{(u)}, w^{(v)}) c_v, c_u]_{\mathcal{K}},\end{aligned} \tag{2.8}$$

while

$$\begin{aligned}\langle g, g\rangle_{\mathcal{L}(\Phi)} =& \sum_{u,v=1}^{m} [K_\Phi(w^{(u)}, w^{(v)}) c_v, c_u]_{\mathcal{K}} - \sum_{v=0}^{m} [K_\Phi(0, w^{(v)}) c_v, \left(\sum_{u=1}^{m} c_u\right)]_{\mathcal{K}} - \\ & - \sum_{u=1}^{m} [K_\Phi(w^{(u)}, 0) \left(\sum_{v=1}^{m} c_v\right), c_u]_{\mathcal{K}} + [K_\Phi(0,0)\left(\sum_{v=1}^{m} c_v\right), \left(\sum_{u=1}^{m} c_u\right)]_{\mathcal{K}} \\ =& \sum_{u,v=1}^{m} [K_\Phi(w^{(u)}, w^{(v)}) c_v, c_u]_{\mathcal{K}} - [\left(\sum_{v=1}^{m} \Phi(w^{(v)})^* c_v\right), \left(\sum_{u=1}^{m} c_u\right)]_{\mathcal{K}} - \\ & - [\left(\sum_{v=1}^{m} c_v\right), \left(\sum_{u=1}^{m} \Phi(w^{(u)})^* c_u\right)]_{\mathcal{K}}.\end{aligned}$$

Thus the isometry of the relation is equivalent to the fact that

$$\begin{aligned}&\sum_{u,v=1}^{m} [K_\Phi(w^{(u)}, w^{(v)}) c_v, c_u]_{\mathcal{K}} - \sum_{u,v=1}^{m} \langle w^{(u)}, w^{(v)}\rangle [K_\Phi(w^{(u)}, w^{(v)}) c_v, c_u]_{\mathcal{K}}, = \\ &= [\left(\sum_{v=1}^{m} \Phi(w^{(v)})^* c_v\right), \left(\sum_{u=1}^{m} c_u\right)]_{\mathcal{K}} + [\left(\sum_{v=1}^{m} c_v\right), \left(\sum_{u=1}^{m} \Phi(w^{(u)})^* c_u\right)]_{\mathcal{K}},\end{aligned} \tag{2.9}$$

which is easily checked. Note that the orthogonal complement of the domain of this relation is the space $\mathcal{N}$ of functions

$$\begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} \in (\mathcal{L}(\Phi))^N$$

such that

$$\sum_{u=1}^{N} w_u f_u(w) \equiv 0. \tag{2.10}$$

The relation R (defined by (2.7)) extends to the graph of an isometry, denoted by V^*, from the closure of its domain into $\mathcal{L}(\Phi)$. Extending this isometry to 0 on the space $\mathcal{N}$ defined by (2.10) , we obtain a partial isometry. Note that

$$V^*(K_\Phi(\cdot, w) w^* c) = K_\Phi(\cdot, w) c - K_\Phi(\cdot, 0) c. \tag{2.11}$$

STEP 2: *Gleason's problem is solvable in* $\mathcal{L}(\Phi)$.

To check this claim, let $f \in \mathcal{L}(\Phi)$ at w; using (2.11) we have

$$\langle f, V^*(K_\Phi(\cdot, w)w^*c)\rangle = \langle f, K_\Phi(\cdot, w)c\rangle - \langle f, K_\Phi(\cdot, 0)c\rangle$$

which can be rewritten as

$$c^* f(w) - c^* f(0) = \sum_{u=1}^{N} c^* w_u g_u(w), \quad \text{where} \quad Vf = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix} \in (\mathcal{L}(\Phi))^N, \tag{2.12}$$

and so Gleason's problem is solvable in $\mathcal{L}(\Phi)$.

STEP 3: *The realization formula* (2.5) *holds for* Φ.

To show this we rewrite (2.7) using STEP 2 as

$$(I - V^* M_{w^*})^{-1}(K_\Phi(\cdot, 0)c) = K_\Phi(\cdot, w)c, \tag{2.13}$$

(see [20, p. 69]) which is the point evaluation in $\mathcal{L}(\Phi)$ at w in the direction of c. Applying C to the left on both sides of (2.13) we obtain

$$C(I - V^* M_{w^*})^{-1} C^* c = C(K_\Phi(\cdot, w)c) = K_\Phi(0, w)c,$$

and so

$$\Phi(w)^* + \Phi(0) = C(I - V^* M_{w^*})^{-1} C.$$

Taking into account that $CC^* = K_\Phi(0, 0) = \Phi(0) + \Phi(0)^*$ we can write:

$$\begin{aligned}\Phi(w)^* &= C(I - V^* M_{w^*})^{-1} C^* - \frac{\Phi(0) + \Phi(0)^*}{2} + \frac{\Phi(0)^* - \Phi(0)}{2} \\ &= C(I - V^* M_{w^*})^{-1} C^* - \frac{CC^*}{2} + \frac{\Phi(0)^* - \Phi(0)}{2} \\ &= \frac{\Phi(0)^* - \Phi(0)}{2} + \frac{1}{2} C(I - V^* M_{w^*})^{-1}(I + V^* M_{w^*})C^*,\end{aligned}$$

which ends the proof.

STEP 4: *Formula* (2.6) *holds.*

Let $z, w \in \Omega$ and $c, d \in \mathbb{C}^n$. We have from (2.13) and (2.3)

$$(I - V^* M_{w^*})^{-1} C^* c = K_\Phi(\cdot, w)c \quad \text{and} \quad (I - V^* M_{z^*})^{-1} C^* d = K_\Phi(\cdot, w)d$$

and taking inner products we obtain

$$\begin{aligned}d^* C(I - M_z V)^{-1}(I - V^* M_{w^*})^{-1} C^* c &= \langle (I - V^* M_{w^*})^{-1} C^* c, (I - V^* M_{z^*})^{-1} C^* d\rangle_{\mathcal{L}(\Phi)} \\ &= d^* K_\Phi(z, w)c.\end{aligned}$$

STEP 5: *Let* Φ *be of the form* (2.5), *where* V *is a partial isometry from a Hilbert space* $\mathcal{H}$ *into* $\mathcal{H}^N$, *and assume that* (2.6) *holds. Then* Φ *is a Herglotz function.*

We have

$$\begin{aligned}\Phi(z)+\Phi(w)^* = &\frac{1}{2}C(I-M_zV)^{-1}\times\\ &\times\{(I+M_zV)(I-V^*M_{w^*})+(I-M_zV)(I+V^*M_{w^*})\}\times\\ &\times(I-V^*M_{w^*})^{-1}C^*\\ =&\, C(I-M_zV)^{-1}\{2(I_n(1-\langle z,w\rangle+2M_z(I_n-VV^*)M_{w^*}\}(I-V^*M_{w^*})^{-1}C^*.\end{aligned}$$

Thus

$$\begin{aligned}K_\Phi(z,w)=&\,C(I-M_zV)^{-1}(I-V^*M_{w^*})^{-1}C^*+\\ &+\frac{C(I-M_zV)^{-1}(M_z(I_n-VV^*)M_{w^*})(I-V^*M_{w^*})^{-1}C^*}{1-\langle z,w\rangle}.\end{aligned}$$

Since K_Φ is a Herglotz function, applying the first part of the proof leads to another realization such that (2.6) is in force. □

Remark 2.7. We note that the relation R will not in general be an isometry, since it need not be densely defined, as illustrated by the case $\Phi=1$.

We now study the uniqueness of the realization (2.5). We first need some notation and a definition.

Definition 2.8. *We let $\widetilde{\ell}$ denote the space of finite sequences of elements of the form*

$$(\alpha_j,n_j)$$

where $\alpha_j\in\{1,\ldots,N\}$, $n_j\in\mathbb{N}$ and $\alpha_{j+1}\neq\alpha_j$. For $A=(A_1,\ldots,A_N)$ where the A_j are bounded operators from $\mathcal{L}(\Phi)$ into itself and $\alpha\in\widetilde{\ell}$ we set

$$A^\alpha=A_{\alpha_1}^{n_1}A_{\alpha_2}^{n_2}\cdots \tag{2.14}$$

Remark 2.9. We point out that in the previous definition one may also use a different notation: one may denote by $\mathcal{F}_d$ the words of length d in the letters $\{1,2,\ldots,N\}$ namely $\alpha=i_1\ldots i_d$, $i_k\in\{1,2,\ldots,N\}$. For $A=(A_1,\ldots,A_N)$ we can define $A^\alpha=A_{i_1}\ldots A_{i_d}$. With this notation we allow adjacent letters of words to repeat—hence we do not need a notation for the multiplicities n_k. We adopted the previous definition since it is consistent with the one we used in [1].

Definition 2.10. *Let $\mathcal{H}$ be some Hilbert space, $C\in\mathbf{L}(\mathcal{H},\mathcal{C}^N)$, $A_1,\ldots,A_N\in\mathbf{L}(\mathcal{H})$, and set $A=(A_1,\ldots,A_N)$. The pair (C,A) is called observable if*

$$\cap_{\alpha\in\widetilde{\ell}}\ker CA^\alpha=\{0\}\,. \tag{2.15}$$

Theorem 2.11. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, and let*

$$\Phi(z)=i\mathrm{Im}\,\Phi(0)+\frac{1}{2}C_j(I_{\mathcal{H}_j}-M_w(V^{(j)})^*)^{-1}(I_{\mathcal{H}_j}+M_w(V^{(j)})^*)C_j^*,\quad j=1,2 \tag{2.16}$$

be two realizations of the Herglotz function Φ, with observable pairs $(C_j,V^{(j)})$, the operators $V^{(j)}$ being partial isometries for $j=1,2$, and set $V^{(j)}=(V_1^{(j)},\ldots,V_N^{(j)})$, $j=1,2$, where the operators $V_u^{(j)}$ are linear and bounded from $\mathcal{H}_1$ into $\mathcal{H}_2$. Then there is a unique unitary operator S such that

$$C_1=C_2S\quad\textit{and}\quad SV_k^{(1)}S^{-1}=V_k^{(2)},\quad k=1,\ldots,N. \tag{2.17}$$

Proof. The relation defined by the linear span of the pairs

$$((I-(V^{(1)})^*M_{w^*})^{-1}C_1^*c\,,\,(I-(V^{(2)})^*M_{w^*})^{-1}C_2^*c),\quad w\in\Omega,\ c\in\mathbb{C}^N$$

has dense domain and dense range. It is isometric since (2.6) holds for both realizations. It extends therefore to the graph of a unitary map from $\mathcal{H}_1$ onto $\mathcal{H}_2$:

$$S\left((I-(V^{(1)})^*M_{w^*})^{-1}C_1^*c\right)=(I-(V^{(2)})^*M_{w^*})^{-1}C_2^*c.$$

By taking the power series expansion in w in both sides and comparing the coefficients, it follows that

$$S((V^{(1)})^*)^{\alpha}C_1^*=((V^{(2)})^*)^{\alpha}C_2^*,\quad \alpha\in\widetilde{\ell}.$$

Taking $\alpha=(k,\beta)$, where β itself runs through $\widetilde{\ell}$ and $k\in\{1,\ldots,N\}$ is fixed, we obtain

$$S(V_k^{(1)})^*S^{-1}S((V^{(1)})^*)^{\beta}C_1^*=(V_k^{(2)})^*((V^{(2)})^*)^{\beta}C_2^*,\quad \beta\in\widetilde{\ell},$$

and so

$$S(V_k^{(1)})^*S^{-1}((V^{(2)})^*)^{\beta}C_2^*=(V_k^{(2)})^*((V^{(2)})^*)^{\beta}C_2^*,\quad \beta\in\widetilde{\ell}.$$

In view of the observability of the pair $(C_2,V^{(2)})$ we get $S(V_k^{(1)})^*S^{-1}=(V_k^{(2)})^*$, and hence (2.17) holds. □

In the previous theorems we assumed that $0\in\Omega$. One can reduce the general case to this one using the automorphism of the ball

$$\varphi_a(z)=\frac{(1-\langle a,a\rangle)^{1/2}}{1-\langle z,a\rangle}(z-a)(I_N-a^*a)^{-1/2},$$

sending some point $a\in\Omega$ to the origin. The positivity of the kernel associated to $\Phi\circ\varphi_a$ is not changed since

$$\frac{1-\varphi_a(z)\varphi_a(w)^*}{1-\langle z,w\rangle}=\frac{1-\langle a,a\rangle}{(1-\langle z,a\rangle)(1-\langle a,w\rangle)}. \tag{2.18}$$

See [24, Theorem 2.2.2, p. 26] for this equality, and [7, p. 11] for another proof of it. The realization of Φ now takes the form

$$\Phi(z)=i\mathrm{Im}\,\Phi(0)+\frac{1}{2}C(I-M_{\varphi_a(z)}V)^{-1}(I+M_{\varphi_a(z)}V)C^*,$$

Example 2.12. Let $d\mu$ be a positive finite measure on the closed unit ball $\overline{\mathbb{B}}$ of $\mathbb{C}^N$ (the case of the unit sphere is of special interest; see [20]). The functions

$$\Phi_\mu(z)=\int_{\overline{\mathbb{B}_1}}\frac{d\mu(s)}{1-\langle z,s\rangle} \tag{2.19}$$

and

$$\Psi_\mu(z)=\int_{\overline{\mathbb{B}_1}}d\mu(s)\frac{1+\langle z,s\rangle}{1-\langle z,s\rangle} \tag{2.20}$$

are Herglotz functions.

Indeed,

$$\begin{aligned}\frac{1}{1-za^*}+\frac{1}{1-aw^*}&=\frac{2-za^*-aw^*}{(1-za^*)(1-aw^*)}\\&=\frac{(1-za^*)(1-aw^*)+1-za^*aw^*}{(1-za^*)(1-aw^*)}\\&=1+\frac{1-zw^*+z(I_N-a^*a)w^*}{(1-za^*)(1-aw^*)}\end{aligned}$$

and so

$$\begin{aligned}\frac{1}{1-zw^*}\left(\frac{1}{1-za^*}+\frac{1}{1-aw^*}\right)&=\frac{1}{1-zw^*}+\frac{1}{(1-za^*)(1-aw^*)}\\&\quad+\frac{z(I_N-a^*a)w^*}{(1-zw^*)(1-za^*)(1-aw^*)}\end{aligned}$$

which is positive definite in $\mathbb{B}_1$, and so K_{Φ_μ} is positive definite, as a positive measure of such functions.

Similarly,

$$\begin{aligned}\frac{1+za^*}{1-za^*}+\frac{1+aw^*}{1-aw^*}&=\frac{2-2za^*aw^*}{(1-za^*)(1-aw^*)}\\&=\frac{2(1-zw^*)+2z(I_N-a^*a)w^*}{(1-za^*)(1-aw^*)}\end{aligned}$$

and so

$$\frac{1}{1-zw^*}\left(\frac{1+za^*}{1-za^*}+\frac{1+aw^*}{1-aw^*}\right)=2+2\frac{z(I_N-a^*a)w^*}{(1-zw^*)(1-za^*)(1-aw^*)}$$

which is also positive definite in $\mathbb{B}_1$. Hence, K_{Ψ_μ} is positive definite.

We note that these equalities do not imply that both Φ_μ and Ψ_μ are Herglotz multipliers. In fact the associated multiplication operator is not necessarily bounded. We also note that, when $d\mu$ has support on the unit sphere, Ψ_μ is of the form (2.5) with $\mathcal{H}=\mathbf{L}_2(d\mu)$, and V and C defined by

$$(Vf)(a)=a^*f(a)\ :\ \mathbf{L}_2(d\mu)\ \longrightarrow\ (\mathbf{L}_2(d\mu))^N$$

and

$$C1=1\ :\ \mathbb{C}\ \longrightarrow\ \mathbf{L}_2(d\mu).$$

Then,

$$C^*f=\int_{\partial\mathbb{B}_1}f(a)d\mu(a).$$

3. The setting of several quaternionic variables: Gleason's problem

We now consider the quaternionic setting, which is the new part of the paper. We denote by $\mathbb{H}$ the skew field of quaternions. It consists of elements $q = x_0 + x_1 i + x_2 j + x_3 k$ where i, j, k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$ and $x_\ell \in \mathbb{R}$, $\ell = 0, \ldots, 3$. Given $q = x_0 + x_1 i + x_2 j + x_3 k$, its conjugate is defined by $q = x_0 - x_1 i - x_2 j - x_3 k$. Due to the non-commutativity, the definition of linear vector spaces over the quaternions has to be made precise; one can consider left-sided, right-sided or two-sided spaces with respect to the multiplication by a scalar in $\mathbb{H}$. The spaces of functions we build will be right-sided, but the coefficient space $\mathcal{K}$ will be a two-sided quaternionic Krein space (the coefficient space).
In a quaternionic right-linear space $\mathcal{V}$ we can introduce a notion of inner product denoted by $[\cdot,\cdot]$; an inner product is Hermitian, additive, positive and satisfies the condition

$$[fa, gb] = \overline{b}[f, g]a, \quad \forall a, b \in \mathbb{H}, \quad \forall f, g \in \mathcal{V}.$$

for any $f, g \in \ V$, $a, b \in \mathbb{H}$.
When considering a two-sided linear space $\mathcal{V}$, one has to fix on which side a linear operator is acting. For us an operator T from $\mathcal{V}$ to itself will act on the right, namely $T(va) = T(v)a$, for all $v \in \mathcal{V}$, $a \in \mathbb{H}$. However, the left structure on $\mathcal{V}$ is needed in order to define $(aT)(v) = aT(v)$, for all $v \in \mathcal{V}$, $a \in \mathbb{H}$ and to have to have a linear structure on the set of linear operators.
The inner product $[\cdot,\cdot]$ in the two-sided linear space and, in particular, for the Krein space $\mathcal{K}$ that we will consider below, satisfies by the definition the following additional condition with respect to the left multiplication:

$$[f, ag] = [\overline{a}f, g], \quad \forall a \in \mathbb{H}, \quad \forall f, g \in \mathcal{K}. \tag{3.1}$$

For the study of quaternionic linear spaces and their (right or left) linear operators in the quaternionic setting we refer to [3, 4, 8].

Given a matrix $A = (a_{jk})_{j,k=1}^n \in \mathbb{H}^{n\times n}$ its adjoint A^* is defined by $A^* = (\overline{a_{kj}})_{j,k=1}^n \in \mathbb{H}^{n\times n}$. The Hermitian matrix is called positive if $c^*Ac \geq 0$ for every $c \in \mathbb{H}^n$. This definition allows to extend the notion of positive definite functions in the sense of reproducing kernels to the quaternionic setting, see [4]. We also recall that the spectral theorem holds for Hermitian functions (i.e. A can be written as $A = UDU^*$ where U is unitary and D is diagonal with real entries; see [26]). One can thus also define the notion of negative squares and associated reproducing kernel Pontryagin spaces in this setting. The one-to-one correspondence between reproducing kernel Hilbert spaces (resp. Pontryagin spaces) and positive definite functions (resp. functions having a finite number of negative squares) extend to the quaternionic setting; see [8].

Following Definition 2.8 we set for $p = (p_1, \ldots, p_N) \in \mathbb{H}^N$ and $\alpha \in \widetilde{\ell}$

$$p^\alpha = p_{\alpha_1}^{n_1} p_{\alpha_2}^{n_2} \cdots . \tag{3.2}$$

Let $\Phi(p)$ be a $\mathbf{L}(\mathcal{K},\mathcal{K})$-valued function defined for $p = (p_1, \ldots, p_N) \in \mathbb{B}_1$, where $\mathbb{B}_1$ denotes the open unit ball of $\mathbb{H}$. Then the series

$$K_\Phi(p, q) = \sum_{\alpha\in\widetilde{\ell}} p^\alpha(\Phi(p) + \Phi(q)^*)\overline{q^\alpha} \tag{3.3}$$

converges pointwise since p and q are in $\mathbb{B}_1^N$ and $\Phi(p)$ is a bounded operator for $p \in \mathbb{B}_1$. One sees that, equivalently, $K_\Phi(p,q)$ is the unique solution of

$$\Phi(p) + \Phi(q)^* = K_\Phi(p,q) - \sum_{n=1}^N p_n K_\Phi(p,q)\overline{q_n} \tag{3.4}$$

by iterating (3.4).

Definition 3.1. *A function Φ such that K_Φ is positive definite in $\mathbb{B}_1^N$ will be called an Herglotz function of N quaternionic variables $p_1, \ldots, p_N$. We denote by $\mathcal{L}(\Phi)$ the associated reproducing kernel Hilbert space of $\mathcal{K}$-valued functions.*

In this section we study the realization of Herglotz functions of N quaternionic variables $p_1, \ldots, p_N$. Given a quaternionic right linear space $\mathcal{V}$ of non-commutative quaternionic power series of the form

$$f(p) = \sum_{\alpha \in \widetilde{\ell}} p^\alpha f_\alpha$$

there are uniquely determined non-commutative power quaternionic power series $f_1, \ldots, f_N$ such that

$$f(p) - f(0) = \sum_{n=1}^N p_n f_n.$$

Gleason's problem is said to be solvable in $\mathcal{V}$ if $f_1, \ldots, f_N \in \mathcal{V}$. In the present setting we have:

Theorem 3.2. *Gleason's problem is solvable in $\mathcal{L}(\Phi)$.*

Proof. In a way similar to the previous section we define a linear relation R on $(\mathcal{L}(\Phi)^N) \times \mathcal{L}(\Phi)$ to be the right linear span of the pairs of the form

$$(K_\Phi(\cdot, p)p^* c, K_\Phi(\cdot, p)c - K_\Phi(\cdot, 0)c)\,.$$

STEP 1: *The relation extends to the graph of an isometry.*

The proof follows the proof of STEP 1 of Theorem 2.6. We now take $p^{(1)}, \ldots, p^{(m)} \in \mathbb{B}$ (and set $p^{(u)} = (p_1^{(u)}, \ldots, p_N^{(u)})$) and $c_1, \ldots, c_m \in \mathcal{K}$, so that now

$$f(\cdot) = \sum_{u=1}^m K_\Phi(\cdot, p^{(u)})(p^{(u)})^* c_u \quad \text{and} \quad g(\cdot) = \sum_{u=1}^m K_\Phi(\cdot, p^{(u)})(p^{(v)})^* c_u - K_\Phi(\cdot, 0)(\sum_{u=0}^m c_u).$$

In view of the non commutativity, equation (2.8) becomes

$$\begin{aligned} \langle f, f\rangle_{(\mathcal{L}(\Phi))^N} &= \sum_{u,v=1}^m [K_\Phi(p^{(u)}, p^{(v)})(w^{(v)})^* c_v, (w^{(u)})^* c_u]_{\mathcal{K}^N} \\ &= \sum_{k=1}^N \sum_{u,v=1}^m w_k^{(u)}([K_\Phi(w^{(u)}, w^{(v)})c_v, c_u]_{\mathcal{K}})\overline{w_k^{(v)}}, \end{aligned} \tag{3.5}$$

and (2.9) becomes

(3.6)

$$\sum_{u,v=1}^{m}[K_\Phi(p^{(u)},p^{(v)})c_v,c_u]_{\mathcal{K}}-\sum_{k=1}^{N}\sum_{u,v=1}^{m}w_k^{(u)}([K_\Phi(w^{(u)},w^{(v)})c_v,c_u]_{\mathcal{K}})\overline{w_k^{(v)}}=$$
$$=[\left(\sum_{v=1}^{m}\Phi(w^{(v)})^*c_v\right),\left(\sum_{u=1}^{m}c_u\right)]_{\mathcal{K}}+[\left(\sum_{v=1}^{m}c_v\right),\left(\sum_{u=1}^{m}\Phi(w^{(u)})^*c_u\right)]_{\mathcal{K}},$$

which follows from (3.4).

We extend this isometry to $(\mathcal{L}(\Phi))^N$, and denote it by V^*. Its adjoint V maps $\mathcal{L}(\Phi)$ into $(\mathcal{L}(\Phi))^N$. We write $V=\begin{pmatrix}V_1 & V_2 & \cdots & V_N\end{pmatrix}$, where V_a maps $\mathcal{L}(\Phi)$ into itself for $a=1,2\ldots,N$.

STEP 2: *The maps $V_1,\ldots V_N$ solve Gleason's problem.*

Indeed, by definition of V^* we have

$$V^*(K_\Phi(\cdot,p)p^*c)=K_\Phi(\cdot,p)c-K_\Phi(\cdot,0)c. \tag{3.7}$$

Thus, for $f\in\mathcal{L}(\Phi)$ we have

$$\langle f,V^*(K_\Phi(\cdot,p)p^*c\rangle_{\mathcal{L}(\Phi)}=\langle Vf,(K_\Phi(\cdot,p)p^*c\rangle_{\mathcal{L}(\Phi)^N}=\langle f,K_\Phi(\cdot,p)c-K_\Phi(\cdot,0)c\rangle_{\mathcal{L}(\Phi)}$$

and so

$$\sum_{n=1}^{N}p_nf_n(p)=f(p)-f(0), \tag{3.8}$$

with $f_n=V_nf$, $n=1,\ldots,N$. □

The proof of the preceding theorem implies:

Corollary 3.3. *Elements in $\mathcal{L}(\Phi)$ are non-commutative power series.*

4. The setting of several quaternionic variables: Realization

We now extend Theorem 2.6. to the framework of (non-commuting) quaternionic variables.

Theorem 4.1. *The function Φ admits a converging power series expansion of the form $\Phi(p)=\sum_{\alpha\in\widetilde{\ell}}p^\alpha\Phi_\alpha$ where the coefficients Φ_α can be represented as follows: Let C denote the evaluation at 0 in $\mathcal{L}(\Phi)$. Then there is a co-isometry $V=\begin{pmatrix}V_1 & V_2 & \cdots & V_N\end{pmatrix}$ from $\mathcal{L}(\Phi)$ into $(\mathcal{L}(\Phi))^N$ such that*

$$\Phi_\alpha=CV^\alpha C^*. \tag{4.1}$$

Proof. We divide the proof into a number of steps.

STEP 1: *Let C denote the operator $f\mapsto f(0)$ from $\mathcal{L}(\Phi)$ into $\mathcal{C}$. Then*

$$C^*c=K_\Phi(\cdot,0)c,\quad c\in\mathcal{C}. \tag{4.2}$$

STEP 2: *Formula* (4.1) *holds.*

Let $c \in \mathcal{C}$. We apply Gleason's problem to $f(\cdot) = K_\Phi(\cdot, 0)c = (\Phi(\cdot) + \Phi(0)^*)c = C^*c$ and obtain

$$\begin{aligned}\Phi(p)c + \Phi(0)^*c - \Phi(0)c - \Phi(0)^*c &= \sum_{n=1}^{N} p_n (V_n f)(0) \\ &= \sum_{n=1}^{N} p_n C V_n C^* c,\end{aligned}$$

i.e.

$$\Phi(p) = \Phi(0) + \sum_{n=1}^{N} p_n C V_n C^*.$$

Reiterating this formula we obtain the required expression for the coefficient Φ_α. □

Remark 4.2. We here offer an argument which sheds light on the non-commutative setting. Let us rewrite (4.1) in a formal way as

$$\Phi(p) = \sum_{\alpha \in \widetilde{\ell}} \frac{\Phi(0) - \Phi(0)^*}{2} + \frac{1}{2} C(I + M_p V)(I - M_p V)^{-1} C^*. \tag{4.3}$$

The formal, and incorrect proof is then as follows: start from (3.7) and write

$$(I - V^* p^*)^{-1} K_\Phi(\cdot, 0)c = K_\Phi(\cdot, p)c$$

or, in view of STEP 2,

$$(I - V^* p^*)^{-1} C^* c = K_\Phi(\cdot, p)c.$$

Thus

$$C(I - V^* p^*)^{-1} C^* = K_\Phi(0, p) = \Phi(0) + \Phi(p)^*,$$

and so, since $CC^* = K_\Phi(0, 0) = \Phi(0) + \Phi(0)^*$,

$$\begin{aligned}\Phi(p)^* &= C(I - V^* p^*)^{-1} C^* - \frac{CC^*}{2} + \frac{CC^*}{2} - \Phi(0) \\ &= \frac{1}{2} C(I + V^* p^*)(I - V^* p^*)^{-1} C^* + \frac{\Phi(0) - \Phi(0)^*}{2}.\end{aligned}$$

Remark 4.3. In the setting of N quaternionic variables, the Fock space $\mathcal{F}$ of $\mathcal{K}$-valued functions is the right quaternionic Hilbert space with reproducing kernel

$$K_{\mathcal{F}}(p, q) = \left(\sum_{\alpha \in \widetilde{\ell}} p^\alpha \overline{q^\alpha} \right) I_{\mathcal{K}},$$

where the dependence on $\mathcal{K}$ is omitted to lighten the notation. An important particular case where the kernel (3.3) is positive definite is when the operator of Cauchy multiplication (that is, the convolution on the coefficients; see [16, p. 199]) by Φ is a bounded linear operator from $\mathcal{F}$ into itself, meaning also that Φ defines an Herglotz multiplier, and some of the arguments can then be simplified.

Acknowledgments: It is a pleasure to thank the referee for his/her constructive remarks on the previous version of the paper.

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