5-2018

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Peter Jipsen
Chapman University, jipsen@chapman.edu

Michael Kinyon
University of Denver

Recommended Citation

P. Jipsen and M. Kinyon, Nonassociative right hoops, *Algebras and Lattices in Hawai'i: honoring Ralph Freese, Bill Lampe, and JB Nation* (2018), 54-60

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Peter Jipsen and Michael Kinyon*

[1] School of Computational Sciences, Chapman University, Orange, CA 92866
jipsen@chapman.edu
[2] Department of Mathematics, University of Denver, Denver, CO 80208
mkinyon@du.edu

Abstract. The class of nonassociative right hoops, or narhoops for short, is defined as a subclass of right-residuated magmas, and is shown to be a variety. These algebras generalize both right quasigroups and right hoops, and we characterize the subvarieties in which the operation $x \wedge y = (x/y)y$ is associative and/or commutative. Narhoops with a left unit are proved to be integral if and only if $\wedge$ is commutative, and their congruences are determined by the equivalence class of the left unit. We also prove that the four identities defining narhoops are independent.

Extended Abstract

A *residuated magma* is a partially ordered algebra $(A, \leq, \cdot, /, \backslash)$ such that $(A, \leq)$ is a poset, $\cdot$ is a binary operation and $/, \backslash$ are the right and left residuals of $\cdot$, which means the residuation property

$$x \cdot y \leq z \qquad \Longleftrightarrow \qquad x \leq z/y \qquad \Longleftrightarrow \qquad y \leq x\backslash z$$

holds for all $x, y, z \in A$. As usual, we abbreviate $x \cdot y$ by xy and adopt the convention that $\cdot$ binds stronger than $/, \backslash$. If the operation $\backslash$ is omitted then the algebra $(A, \leq, \cdot, /)$ is called a *right-residuated magma*.

Define the term $x \wedge y = (x/y)y$ and consider the following two varieties:

– A *right quasigroup* is an algebra $(A, \cdot, /)$ satisfying the identities $x \wedge y = x = (xy)/y$. Right quasigroups are precisely those right-residuated magmas for which the partial order $\leq$ is the equality relation.

*Partially supported by Simons Foundation Collaboration Grant 359872.

– A *right hoop* is an algebra $(A, \cdot, /)$ satisfying the identities $x \wedge y = y \wedge x$, $(x/x)y = y$, and $x/(yz) = (x/z)/y$. Then it turns out that x/x is a constant (denoted by 1), the operation $\cdot$ is associative, the operation $\wedge$ is a semilattice operation, 1 is the top element with respect to the semilattice order $\leq$, and $/$ is the right residual of $\cdot$ with respect to $\leq$.

Right hoops were introduced by Bosbach [1,2] under the name "left complementary semigroups" and Büchi and Owens [3] studied the case where $\cdot$ is commutative, referring to these structures as "hoops". Note that the partial order is definable in both cases, which motivates the following definition. A *nonassociative right hoop* $(A, \leq, \cdot, /)$, or *narhoop* for short, is a right-residuated magma such that for all $x, y \in A$

$$\text{(N)} \qquad x \leq y \quad \Longleftrightarrow \quad x \wedge y = x = y \wedge x\,.$$

In any right-residuated magma $(x/y)y \leq x$ or equivalently $x \wedge y \leq x$ holds for all x, y, hence in a narhoop (N) implies that the identity $(x \wedge y) \wedge x = x \wedge y$ holds. This provides an alternative definition for narhoops: they are right-residuated magmas that satisfy the identity (N1) $(x \wedge y) \wedge x = x \wedge y$ and the bi-implication

$$\text{(N')} \qquad x \leq y \quad \Longleftrightarrow \quad x = y \wedge x$$

since in the presence of (N1), if $x = y \wedge x$ then multiplying by y on the right we have $x \wedge y = (y \wedge x) \wedge y = y \wedge x = x$.

A *nonassociative left hoop* or *nalhoop* $(A, \leq, \cdot, \backslash)$ is defined dually and a *nonassociative hoop* or *nahoop* is both a narhoop and a nalhoop. Here we consider only narhoops and save the two-sided case for future research.

The two motivating varieties fit into this framework as follows.

– A narhoop $(A, \cdot, /)$ is a right quasigroup if and only if $\leq$ is the equality relation.
– A narhoop $(A, \cdot, /)$ is a right hoop if and only if the quasiequation $x \wedge y = x \Rightarrow x \leq y$ holds.

The main result of this section is that narhoops form a finitely based variety of algebras. To reduce the need for parentheses, we assume that x/y binds stronger than $x \wedge y = (x/y)y$.

Theorem 1. *Let $(A, \leq, \cdot, /)$ be a narhoop. Then the following identities hold:*

(N1) $(x \wedge y) \wedge x = x \wedge y$
(N2) $xy/y \wedge x = x$
(N3) $xz \wedge (x \wedge y)z = (x \wedge y)z$

(N4) $(x/z) \wedge (x \wedge y)/z = (x \wedge y)/z.$

Conversely, let $(A, \cdot, /)$ *be an algebra with two binary operations satisfying* (N1)–(N4)*, and define* $x \le y \iff x = y \wedge x$. *Then the identies*

(N5) $x \wedge xy/y = x$
(N6) $(x \wedge y)/y = x/y$
(N7) $(x \wedge y) \wedge y = x \wedge y$

hold and $(A, \le, \cdot, /)$ *is a narhoop.*

Proof. Assume $(A, \le, \cdot, /)$ is a narhoop. As noted above, the identity (N1) holds in narhoops. Right-residuated magmas also satisfy $x \le xy/y$, hence (N2) follows from (N′).

Having a right residual implies that right-multiplication is order preserving, so $(x \wedge y)z \le xz$ holds in all narhoops, which produces (N3). Similarly the right residual is order preserving in the first argument, hence $(x \wedge y)/z \le x/z$ holds, and now (N4) follows from (N′).

For the converse, suppose $(A, \cdot, /)$ satisfies (N1)–(N4), and $\le$ is defined by (N′). From (N2), (N1) and (N2) again, we get (N5):

$$x \wedge (xy/y) = (xy/y \wedge x) \wedge (xy/y) = (xy/y) \wedge x = x.$$

For (N6), replace x in (N5) by x/y to get $x/y \wedge (x \wedge y)/y = x/y$ and then use (N4). To prove (N7) multiply (N6) on the right by y.

Now reflexivity of $\le$ follows from (N5) and (N1): $x \wedge x = (x \wedge xy/y) \wedge x = x \wedge (xy/y) = x$.

For antisymmetry, if $x \le y$ and $y \le x$, then $x \wedge y = x = y \wedge x$ and $y = x \wedge y$, hence $x = y$.

Transitivity is a bit more work. Suppose $x \le y$ and $y \le z$ so that $x \wedge y = x = y \wedge x$ and $y \wedge z = y = z \wedge y$. First, note that

$$z/x \wedge y/x = z/x \wedge (z \wedge y)/x = (z \wedge y)/x = y/x$$

using (N4) in the second equality. Now we compute

$$\begin{aligned}
z \wedge x &= (z \wedge x) \wedge x && \text{by (N6)}\\
&= (z \wedge x) \wedge (y \wedge x) = (z/x)x \wedge (y/x)x\\
&= (z/x)x \wedge (z/x \wedge y/x)x && \text{since } z/x \wedge y/x = y/x\\
&= (z/x \wedge y/x)x && \text{by (N3)}\\
&= (z/x \wedge (z \wedge y)/x)x = ((z \wedge y)/x)x && \text{by (N4)}\\
&= (z \wedge y) \wedge x = y \wedge x = x\,.
\end{aligned}$$

From $x = z \wedge x$ we deduce $x \wedge z = (z \wedge x) \wedge z = z \wedge x$ by (N1), hence $x \leq z$.

Finally, we prove $/$ is the right residual of $\cdot$ with respect to $\leq$. To do this, we verify $(x/y)y \leq x \leq xy/y$ and that $x \leq y$ implies $xz \leq yz$ and $x/z \leq y/z$ since the right residuation property is equivalent to these (quasi)identities. Note that (N2) and (N′) show $x \leq xy/y$. If $x \leq y$, then (N3) gives

$$yz \wedge xz = yz \wedge (y \wedge x)z = (y \wedge x)z = xz,$$

and so (N′) implies $xz \leq yz$. By the same argument, (N4) gives $x/z \leq y/z$.

To prove $(x/y)y \leq x$, or equivalently $x \wedge y \leq x$, substitute x/x for x, x for y, and $(x \wedge y)/x$ for z in (N3) to get

$$(x/x)x \wedge (x/x \wedge (x \wedge y)/x)x = (x/x \wedge (x \wedge y)/x)x.$$

Using (N4) this simplifies to $(x \wedge x) \wedge ((x \wedge y)/x)x = ((x \wedge y)/x)x$, so by (N1), (N′) and reflexivity we have $x \wedge y \leq x \wedge x = x$. □

The equational basis (N1)–(N4) for narhoops is independent as can be seen from algebras $A_i = \{0, 1\}$ ($i = 1, 2, 3, 4$) that each satisfy the axioms except for (N*i*).

– In A_1, $\cdot$ is ordinary multiplication and $x/y = y$.
– In A_2, $x \cdot y = x$ and $x/y = 1$.
– In A_3, $x \cdot y$ is addition modulo 2 and $x/y = 0$ except that $1/0 = 1$.
– In A_4, $x \cdot y$ is the max operation and x/y is addition modulo 2.

In general, neither $\cdot$ nor the term operation $\wedge$ of a narhoop is associative. However $\wedge$ is associative both in right quasigroups and in right hoops. In right quasigroups, this follows from the identity $x \wedge y = x$. In right hoops, $\wedge$ turns out be a semilattice operation ([4], Lem. 4). In both cases the reduct $(A, \wedge)$ is a *left normal band*, that is, an idempotent semigroup satisfying the identity $x \wedge y \wedge z = x \wedge z \wedge y$.

If $(A, \cdot, /)$ is a narhoop and $B \subseteq A$ is closed under $\wedge$, then B inherits the order $\leq$ from A. We state the next two results in the slightly more general context of such subsets because we will need them in Theorem 4.

Theorem 2. *Let $(A, \cdot, /)$ be a narhoop and let $B \subseteq A$ be closed under $\wedge$. The following are equivalent.*

1. *$(B, \wedge)$ is a right normal band;*
2. *$(B, \wedge)$ is a semigroup;*

3. *For all* $x, y \in B$, $x \wedge (y \wedge x) = x \wedge y$

As noted, $\wedge$-reducts of right hoops are semilattices. A natural nonassociative generalization of right hoops is the variety of narhoops described in the next result. The description of the $\wedge$-reduct generalizes ([4], Lem. 4).

Theorem 3. *Let* $(A, \cdot, /)$ *be a narhoop and let* $B \subseteq A$ *be closed under* $\wedge$. *The following are equivalent.*

1. $(B, \wedge)$ *is commutative;*
2. *For all* $x, y \in B$, $x \wedge (y \wedge x) = y \wedge x$.

When these equivalent conditions hold, $(B, \wedge)$ *is a semilattice.*

In a left normal band, the identity $x \wedge y \wedge z = x \wedge z \wedge y$ essentially expresses the fact that every downset $(a] = \{x \in A \mid x \leq a\} = \{a \wedge x \mid x \in A\}$ is a subsemilattice. The same role is played by $(x \wedge y) \wedge z = (x \wedge z) \wedge y$ in narhoops.

Theorem 4. *Let* $(A, \cdot, /)$ *be a narhoop and fix* $a \in A$. *Then the downset* $(a]$ *is closed under* $\wedge$ *and is a semilattice.*

We now consider narhoops which have a left identity element.

Lemma 1. *Let* $(A, \leq, \cdot, /)$ *be a right-residuated magma such that* $x \leq y \iff x = y \wedge x$ *holds for all* $x, y \in A$. *Then*

1. x/x *is a maximal element for all* $x \in A$,
2. *the identity* $(x/x)y/y = x/x$ *holds in* A, *and*
3. *if* A *has a top element then the term* x/x *is this top element.*

Lemma 2. *Let* $(A, \cdot, /)$ *be a narhoop. The following are equivalent.*

1. $x/x = y/y$ *for all* $x, y \in A$;
2. $(x/x)y = y$ *for all* $x, y \in A$;
3. *There exists* $e \in A$ *such that* $ey = y$ *for all* $y \in A$.

When these conditions hold, the element $1 = x/x$ *is the maximum left identity element in* $(A, \leq)$.

A narhoop $(A, \cdot, /)$ is *unital* if the equivalent conditions of Lemma 2 hold. In this case as in the lemma, we denote by $1 = x/x$ the distinguished left identity element. Note that the lemma does not claim that 1 is the unique left identity element, even when $\wedge$ is commutative.

Theorem 5. *If A is finite and unital, then 1 is the unique left identity element.*

Note that in a unital narhoop A the partial order $\leq$ can be characterized in terms of 1 and $/$:

$$x \leq y \quad \Longleftrightarrow \quad x/y = 1$$

for all $x, y \in A$. Furthermore, the left identity 1 can also be used to characterize the commutativity of $\wedge$.

Theorem 6. *Let $(A, \cdot, /, 1)$ be a unital narhoop. Then:*

1. *The left unit 1 is the top element of $(A, \leq)$ if and only if $\wedge$ is commutative.*
2. *The downset $(1]$ is a subnarhoop and $((1], \wedge)$ is a semilattice.*

A congruence θ on a narhoop $(A, \cdot, /)$ is said to be *unital* if the factor narhoop A/θ is unital. In other words, θ is unital if and only if $x/x \;\theta\; y/y$ for all $x, y \in A$. If A itself is unital then every congruence on A is unital. For a unital congruence on an arbitrary narhoop, set

$$\begin{aligned} N_\theta &= \{x \in A \mid x \;\theta\; y/y \text{ for some } y \in A\} \\ &= \{x \in A \mid x \;\theta\; y/y \text{ for all } y \in A\}\,, \end{aligned}$$

where the second equality follows since θ is unital. Analogous to the relationship between congruences and normal subgroups in group theory, we now show that θ is determined by the congruence class N_θ. In order to state our characterizations concisely, we introduce six families of mappings on a narhoop $(A, \cdot, /)$. For each $x, y \in A$, $i = 1, \ldots, 6$, define $\phi_{i,x,y} : A \to A$ by

$$\begin{aligned} \phi_{1,x,y}(z) &= (zx \cdot y)/xy\,, & \phi_{2,x,y}(z) &= (zx/y)/(x/y)\,, \\ \phi_{3,x,y}(z) &= (x \cdot zy)/xy\,, & \phi_{4,x,y}(z) &= (x/zy)/(x/y)\,, \\ \phi_{5,x,y}(z) &= xy/(x \cdot zy)\,, & \phi_{6,x,y}(z) &= (x/y)/(x/zy)\,. \end{aligned}$$

Again keeping analogies with group theory in mind, let $\mathrm{Inn}(A)$ denote the transformation semigroup on A generated by these six families of mappings.

Theorem 7. *Let θ be a unital congruence on a narhoop $(A, \cdot, /)$. Then:*

1. *N_θ is a subnarhoop of A;*
2. *For all $x, y \in A$, if $x \leq y$ and $x \in N_\theta$, then $y \in N_\theta$;*
3. *N_θ is invariant under $\mathrm{Inn}(A)$.*

Let $(A, \cdot, /)$ be a narhoop. A nonempty subset N of A is said to be a *normal subnarhoop* of A, denoted $N \trianglelefteq A$, if the following hold:

1. N is a subnarhoop of A;
2. For all $x, y \in A$, if $x \leq y$ and $x \in N$, then $y \in N$;
3. N is invariant under $\mathrm{Inn}(A)$.

Theorem 8. *Let $(A, \cdot, /)$ be a narhoop and assume $N \trianglelefteq A$ is nonempty. Define θ_N on A by $x\ \theta_N\ y$ if and only if $x/y, y/x \in N$. Then θ_N is a unital congruence and $N_{\theta_N} = N$.*

Acknowledgments

This research was supported by the automated theorem prover PROVER9 and the finite model builder MACE4, both created by McCune [5]. We would like to thank Bob Veroff for hosting the 2016 Workshop on Automated Deduction and its Applications to Mathematics (ADAM) which is where our collaboration began.

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