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LOGICAL PRE- AND POST-SELECTION PARADOXES, MEASUREMENT-DISTURBANCE AND CONTEXTUALITY

M. S. LEIFER AND R. W. SPEKKENS

ABSTRACT. Many seemingly paradoxical effects are known in the predictions for outcomes of measurements made on pre- and post-selected quantum systems. A class of such effects, which we call "logical pre- and post-selection paradoxes", bear a striking resemblance to proofs of the Bell-Kochen-Specker theorem, which suggests that they demonstrate the contextuality of quantum mechanics. Despite the apparent similarity, we show that such effects can occur in noncontextual hidden variable theories, provided measurements are allowed to disturb the values of the hidden variables.

1. INTRODUCTION

The study of quantum systems that are both pre- and post-selected was initiated by Aharonov, Bergmann and Lebowitz (ABL) (Aharonov et al., 1964), and has led to the discovery of many counter-intuitive results, which we call Pre- and Post-Selection (PPS) paradoxes[1]. These results have led to a long debate about the interpretation of the ABL probability rule[2].

An undercurrent in this debate has been the connection between PPS paradoxes and contextuality. Bub and Brown (1986) understood the first paper describing a PPS paradox, that of Albert et al. (1985), as a claim to a novel proof of the contextuality of Hidden Variable Theories (HVTs), that is, as a version of the Bell-Kochen Specker theorem (Bell, 1966; Kochen and Specker, 1967), and convincingly disputed this claim. Although the language of Albert et al. (1985) does suggest such a reading[3], in Albert et al. (1986) these authors clarified their position, stating that it was not their intention to conclude anything about HVTs. Nonetheless, discussions of PPS paradoxes continue to make use of a language that suggests implications for ontology (Vaidman, 1999) and there are explicit claims that PPS paradoxes are proofs of contextuality (Marchildon, 2003).

In this paper, we show that PPS paradoxes can be explained without invoking contextuality if one allows that measurement interactions can cause a disturbance to the values of the hidden variables of the system; a possibility that is often overlooked in analyses of PPS paradoxes. The paper is organized as follows. After introducing a general form of the ABL rule, we give a simple example of how the surprising

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[1]For a recent review of these results see Aharonov and Vaidman (2002).
[2]See Kastner (2003) and references therein.
[3]For instance, it is stated that "The assumption of Gleason and of Kochen and Specker [...] is not satisfied by quantum mechanical systems within the interval between two measurements!"

features of a particular PPS paradox, known as the three-box paradox, can be reproduced in a simple noncontextual model that involves measurement-disturbance. Thereafter, we consider the analogue of the ABL rule for HVTs, introduce the assumption of noncontextuality, and introduce the notion of a "logical" PPS paradox. We then show how PPS paradoxes of this type are consistent with noncontextuality if one allows for measurement-disturbance. We end the paper with a brief discussion of how a recent theorem that connects logical PPS paradoxes to proofs of contextuality (Leifer and Spekkens, 2004) appears in light of our results.

2. Pre- and Post-Selection in Quantum Theory

2.1. **Quantum Measurements.** We consider a finite dimensional Hilbert space and assume that no evolution occurs between measurements. The outcome of a quantum measurement, M, is a random variable, which we denote by X_{M}. We restrict attention to the case where the range of X_{M} is a discrete set labelled by j.

A measurement has both a *statistical aspect*, which specifies the probability of obtaining the different outcomes of the measurement for any given density operator, and a *transformation aspect*, which specifies how the quantum state is updated as a result of the measurement. We restrict our attention to *sharp* measurements, that is, those associated with projector valued measures (PVMs) (sets of projectors $\{P_j^{\mathrm{M}}\}$ that sum to the identity operator, $\sum_j P_j^{\mathrm{M}} = I$). The probability of obtaining outcome $X_{\mathrm{M}} = j$ when the initial density matrix is ρ is then given by $p_\rho(X_{\mathrm{M}} = j) = \mathrm{Tr}\left(P_j^{\mathrm{M}}\rho\right)$. The most general possible transformation aspect of M is given by a set of completely positive (CP) maps $\{\mathcal{E}_j^{\mathrm{M}}\}$ satisfying

$$\mathcal{E}_j^{\mathrm{M}\,\#}(I) = P_j^{\mathrm{M}} \tag{1}$$

where $\mathcal{E}^{\#}$ is the dual of $\mathcal{E}$ defined by $\mathrm{Tr}\left(\mathcal{E}^{\#}(A)B\right) = \mathrm{Tr}\left(A\mathcal{E}(B)\right)$. The updated state on obtaining outcome $X_{\mathrm{M}} = j$ is

$$\rho_{|X_{\mathrm{M}}=j} = \frac{\mathcal{E}_j^{\mathrm{M}}(\rho)}{\mathrm{Tr}\left(\mathcal{E}_j^{\mathrm{M}}(\rho)\right)}. \tag{2}$$

When $\mathcal{E}_j^{\mathrm{M}}(\rho) = P_j^{\mathrm{M}}\rho P_j^{\mathrm{M}}$, the transformation is said to follow the Lüders rule (Lüders, 1951).

2.2. **Pre- and Post-Selected Systems.** Imagine an initial, an intermediate, and a final measurement occurring at times t_{pre}, t, and t_{post} respectively, where $t_{\mathrm{pre}} < t < t_{\mathrm{post}}$. We pre-select (post-select) by conditioning on a particular outcome of the initial (final) measurement. Denote the occurrence of this outcome by $\mathrm{A}_{\mathrm{pre}}$ ($\mathrm{A}_{\mathrm{post}}$), and suppose that it is associated with a projector Π_{pre} (Π_{post}). Let the intermediate measurement be denoted by M.

Assuming that the density operator prior to t_{pre} is I/d, and assuming Lüders rule for the initial measurement, the density operator after t_{pre} is $\rho_{\mathrm{pre}} = \Pi_{\mathrm{pre}}/\mathrm{Tr}(\Pi_{\mathrm{pre}})$. By Bayes' theorem, we can deduce that the probability of obtaining the outcome k for M is

$$p_{\mathrm{ABL}}(X_{\mathrm{M}} = k|\mathrm{A}_{\mathrm{pre}}, \mathrm{A}_{\mathrm{post}}, \mathrm{M}) = \frac{\mathrm{Tr}(\Pi_{\mathrm{post}}\mathcal{E}_k^{\mathrm{M}}(\Pi_{\mathrm{pre}}))}{\mathrm{Tr}(\Pi_{\mathrm{post}}\mathcal{E}_k^{\mathrm{M}}(\Pi_{\mathrm{pre}}) + \Pi_{\mathrm{post}}\mathcal{E}_{\neg k}^{\mathrm{M}}(\Pi_{\mathrm{pre}}))} \tag{3}$$

where $\mathcal{E}_{\neg k}^{\mathrm{M}}(\rho) = \sum_{j\neq k}\mathcal{E}_j^{\mathrm{M}}(\rho)$. We refer to this as the ABL probability rule (see Aharonov and Vaidman, 2002). From now on, unless stated otherwise, the Lüders update rule is assumed to apply for all intermediate measurements. In this case,

$\mathcal{E}_k^{\mathrm{M}}(\rho) = P_k^{\mathrm{M}} \rho P_k^{\mathrm{M}}$. Note that, unlike the standard Born rule probability, the ABL probability depends, through $\mathcal{E}_{\neg k}^{\mathrm{M}}$, on the entire PVM $\{P_j^{\mathrm{M}}\}$ and not just on the projector P_k^{M} that is associated with the outcome for which the probability is being computed. This will be critical further on.

2.3. **The Three-Box Paradox.** A simple example of the seemingly paradoxical predictions of the ABL rule is the three-box paradox (see Aharonov and Vaidman, 2002). Suppose we have a particle that can be in one of three boxes. We label the state where the particle is in box j by $|j\rangle$. The particle is pre-selected in the state $|\phi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle)$, i.e. $\Pi_{\mathrm{pre}} = |\phi\rangle\langle\phi|$, and post-selected in the state $|\psi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle - |3\rangle)$, i.e. $\Pi_{\mathrm{post}} = |\psi\rangle\langle\psi|$. At an intermediate time, we either determine whether the particle is in box 1 or not, or we determine whether it is in box 2 or not. The first measurement, M, corresponds to the PVM $\{P_1^{\mathrm{M}}, P_2^{\mathrm{M}}\}$, where

$$P_1^{\mathrm{M}} = |1\rangle\langle 1| \qquad P_2^{\mathrm{M}} = |2\rangle\langle 2| + |3\rangle\langle 3| . \tag{4}$$

For this measurement, $p_{\mathrm{ABL}}(X_{\mathrm{M}} = 1|\mathrm{A}_{\mathrm{pre}}, \mathrm{A}_{\mathrm{post}}, \mathrm{M}) = 1$.

The second measurement, N, corresponds to the PVM $= \{P_1^{\mathrm{N}}, P_2^{\mathrm{N}}\}$, where

$$P_1^{\mathrm{N}} = |2\rangle\langle 2| \qquad P_2^{\mathrm{N}} = |1\rangle\langle 1| + |3\rangle\langle 3| \tag{5}$$

In this case, $p_{\mathrm{ABL}}(X_{\mathrm{N}} = 1|\mathrm{A}_{\mathrm{pre}}, \mathrm{A}_{\mathrm{post}}, \mathrm{N}) = 1$.

Thus, if we ask whether or not the particle was in box 1, we find that it was in box 1 with certainty, and if we ask whether or not it was in box 2, we find that it was in box 2 with certainty!

2.4. **An Analogue of the three-box paradox.** Following the spirit of previous work by Kirkpatrick (2003), we present a simple toy model that parallels the features of the three-box paradox.

Consider an opaque box that can be divided into two opaque boxes by placing a double partition in the box and breaking it into two halves. It is also possible to put the two halves back together and to remove the partition. The partition can be placed in two possible positions, dividing the box either into front and back halves or into right and left halves. Suppose there is a ball inside the box. One can verify whether the ball is in the front half of the box by dividing the box into front and back halves and then shaking the front half of the box to hear whether the ball is inside. If the ball is found in the front then this action completely randomizes the left-right position of the ball. However, if it is not found in the front then its position remains undisturbed because the back half has not been shaken. A similar procedure can be used to verify whether the ball is in the back half of the box, only in this case the left-right position gets randomized if the ball is in the back and is left undisturbed if it is in the front. Two further procedures can be used to verify whether the ball is in the left or right half of the box, randomizing the front-back position of the ball whenever it is found in the half that one is checking.

Now imagine that one pre-selects on finding the ball in the front upon checking for it there, and one post-selects on finding the ball in the back upon checking for it there. Suppose that the two possible intermediate measurements are: 1) checking to see if the ball is on the left, and 2) checking to see if the ball is on the right. Clearly, if one checked to see if it was on the left, then one must have found it on the left, since otherwise there would have been no disturbance and consequently no way for the ball to have moved from the front to the back of the box. But, by the same argument, if one checked to see if it was on the right, then one must have found it on the right.

This model succeeds at reproducing the surprising feature of the three-box paradox, while being noncontextual according to the operational definition provided in Spekkens (2004a) (which we shall discuss further on). Moreover, it is clear the measurement-induced disturbance is critical to achieving the surprising results.

In the rest of the paper, we show that this result is generic; PPS paradoxes do not require contextuality for their explanation but do require measurement-disturbance. The demonstration requires us to examine explicitly how PPS scenarios are treated within an HVT.

3. Hidden Variable Theories

3.1. **Measurements in Hidden Variable Theories.** An HVT is an attempt to understand quantum measurements as revealing features of *ontic states*, by which we mean complete specifications of the state of reality. Let Ω be the set of all ontic states in an HVT. Although $\lambda \in \Omega$ are the states of reality, we may not have direct access to them (hence the term *hidden* variables), and so quantum mechanical preparation procedures generally correspond to probability distributions over the ontic states. We denote these by functions $\mu : \Omega \to \mathbb{R}_+$ satisfying $\int_\Omega \mu(\lambda) d\lambda = 1$. [4]

A common assumption that is made for HVTs is that the ontic state determines the outcomes of all possible sharp measurements uniquely. We refer to this assumption as *outcome determinism for sharp measurements*, and we presume it to hold in all that follows. Given this assumption, the statistical aspect of a measurement M is represented by a set of idempotent indicator functions $\chi_j^{\mathrm{M}} : \Omega \to \{0, 1\}$, that sum to the unit function $\sum_j \chi_j^{\mathrm{M}}(\lambda) = 1$. $\chi_j^{\mathrm{M}}(\omega)$ is the probability (0 or 1) of obtaining outcome j given that the ontic state is λ. More generally

$$p_\mu(X_{\mathrm{M}} = j) = \int_\Omega \chi_j^{\mathrm{M}}(\lambda)\mu(\lambda) d\lambda \tag{6}$$

is the probability of obtaining the outcome j given the distribution μ.

We now turn to the transformation aspect of measurements. We must allow for the possibility that measurements cause a disturbance (possibly stochastic) to the ontic state of the system. Therefore, the disturbance induced by a measurement is described by a transition matrix $D_j^{\mathrm{M}} : \Omega \times \Omega \to \mathbb{R}_+$ satisfying $\int_\Omega D_j^{\mathrm{M}}(\lambda, \omega) d\lambda = 1$. $D_j^{\mathrm{M}}(\lambda, \omega)$ is the probability of a transition from ω to λ, given that $X_{\mathrm{M}} = j$.

Thus, the most general transformation aspect of a measurement M on obtaining outcome j is composed of both a Bayesian updating of the distribution and a disturbance, and is consequently represented by a norm-decreasing transition matrix

$$\Gamma_j^{\mathrm{M}}(\lambda, \omega) = D_j^{\mathrm{M}}(\lambda, \omega)\chi_j^{\mathrm{M}}(\omega). \tag{7}$$

This plays an analogous role in the HVT to the trace-decreasing CP map associated with a measurement outcome in quantum theory. In particular, it satisfies

$$\int_\Omega \Gamma_j^{\mathrm{M}}(\lambda, \omega) d\lambda = \chi_j^{\mathrm{M}}(\omega), \tag{8}$$

[4] We assume that Ω satisfies the measure-theoretic requirements necessary to make such integrals well-defined.

which is the analogue of Eq. (1). We have the following update rule for the probability density on obtaining $X_{\rm M} = j$

$$\mu(\lambda|X_{\rm M}=j) = \frac{\int_\omega \Gamma_j^{\rm M}(\lambda,\omega)\mu(\omega)d\omega}{\int_\Omega \Gamma_j^{\rm M}(\lambda,\omega)\mu(\omega)d\omega d\lambda}, \tag{9}$$

which is the analogue of Eq. (2).

3.2. **Pre- and Post-Selection in Hidden Variable Theories.** Pre- and post-selected systems can be described in an HVT as follows. The successful pre-selection event, $\rm A_{pre}$, is associated with a probability distribution $\mu_{\rm pre}(\lambda)$. The intermediate measurement, M, is described by a set of indicator functions $\{\chi_j^{\rm M}\}$ and corresponding transition matrices $\{\Gamma_j^{\rm M}\}$ and the successful post-selection event, $\rm A_{post}$ is associated with an indicator function $\chi_{\rm post}$. Applying Bayes' rule together with Eqs. (6) and (9) we obtain the PPS probability rule for HVTs

$$p_{\rm HVT}(X_{\rm M}=k|{\rm A_{pre}},{\rm A_{post}},{\rm M}) = \frac{\int_\Omega \chi_{\rm post}(\lambda)\Gamma_k^{\rm M}(\lambda,\omega)\mu_{\rm pre}(\omega)d\omega d\lambda}{\int_\Omega \chi_{\rm post}(\lambda)(\Gamma_k^{\rm M}(\lambda,\omega)+\Gamma_{\neg k}^{\rm M}(\lambda,\omega))\mu_{\rm pre}(\omega)d\omega d\lambda}. \tag{10}$$

where $\Gamma_{\neg k}^{\rm M}(\lambda,\omega) = \sum_{j\neq k}\Gamma_j^{\rm M}(\lambda,\omega)$. This is the analogue of Eq. (3).

3.3. **Measurement noncontextuality.** A particularly natural class of HVTs are the Measurement Noncontextual Hidden Variable Theories (MNHVTs) (see Spekkens, 2004a, for a discussion of different types of noncontextuality). The assumption of measurement noncontextuality is that if there is an outcome k of a measurement M and an outcome j of a measurement N that have the same probability for all possible preparations, then they must be represented by the same indicator function in the HVT. For quantum systems, this will be the case if and only if both the outcomes correspond to the same projector P, even though M and N may be associated with different PVMs. Thus, noncontextuality implies that the indicator functions in both cases depend only on the projector P, i.e.

$$\chi_k^{\rm M}(\lambda) = \chi_j^{\rm N}(\lambda) = \chi_P(\lambda). \tag{11}$$

Equivalently, in an MNHVT, every projector is associated with a unique indicator function (defining a unique subset of Ω) which specifies the property that is revealed by a measurement of that projector.

This implies stringent constraints on the probabilities that can be simultaneously assigned to commuting projectors. For instance, since any pair of orthogonal projectors Q_1, Q_2 can appear together in the same PVM, it follows that $\chi_{Q_1}+\chi_{Q_2}\leq 1$ and consequently that $\chi_{Q_1}\chi_{Q_2} = 0$ (where we leave the dependence of χ on λ implicit). Moreover, if $Q = Q_1 + Q_2$, then Q is a coarse-graining of Q_1 and Q_2, and therefore is represented in the HVT by $\chi_Q = \chi_{Q_1} + \chi_{Q_2}$. Given these identities, it follows that, for commuting projectors P and Q, $\chi_P\chi_Q = \chi_{PQ}$ and $\chi_P + \chi_Q - \chi_P\chi_Q = \chi_{P+Q-PQ}$. Finally, the projector onto the null space, $P_{\rm null}$, is represented by $\chi_{P_{\rm null}} = 0$ since the associated property never holds. Denoting the probability one assigns to P given a distribution $\mu(\lambda)$, by $p(P) = \int_\Omega \chi_P(\lambda)\mu(\lambda)d\lambda$, we obtain the following constraints

Algebraic conditions: For projectors P, Q such that $[P,Q]=0$,

$$0 \leq p(P) \leq 1 \tag{12}$$

$$p(I-P) = 1 - p(P), \tag{13}$$

$$p(I) = 1, \quad p(P_{\text{null}}) = 0, \tag{14}$$

$$p(PQ) \leq p(P), \quad p(PQ) \leq p(Q), \tag{15}$$

$$p(P+Q-PQ) = p(P) + p(Q) - p(PQ). \tag{16}$$

If one conditions on a particular ontic state ω, so that $\mu(\lambda) = \delta(\lambda - \omega)$ and $p(P) = \chi_P(\omega)$, then all of these probabilities must be either 0 or 1. However, the Bell-Kochen-Specker theorem shows that there are sets of projectors for which there are no assignments of probabilities 0 or 1 that satisfy the algebraic conditions. This is an example of a proof of contextuality.

We are now in a position to make precise what it is about PPS paradoxes that suggests that these might have something to do with measurement contextuality. The critical fact is that there exist sets of projectors that are each assigned probability 0 or 1 by the ABL rule such that the overall probability assignment violates the algebraic conditions. The three box paradox is an example of this. We call any such case a *logical* PPS paradox.

First of all, let us clarify why, in the absence of the measurement noncontextuality assumption , logical PPS paradoxes do not violate classical probability theory. The reason is that only assignments that are *similarly conditioned* need to respect the algebraic constraints, and when one conditions on the nature of the intermediate measurement, the ABL probabilities *do* satisfy the constraints. It is only if we consider ABL probabilities for different intermediate measurements together that we find that these probabilities do not satisfy the functional relations defined in Eqs. (12) to (16). (For instance, we considered two distinct intermediate measurements in the three box paradox.) Similarly, one can avoid a contradiction in a HVT by assuming that indicator functions depend on measurement context. In this case, a projector is not associated with a unique indicator function and consequently can be assigned different probabilities in different measurement contexts.

This way of describing things suggests that logical PPS paradoxes are themselves proofs of the impossibility of an MNHVT. This only follows however if the dependence of the ABL probabilities on context implies a similar dependence of the indicator functions on context, equivalently, if context dependence of probabilities that are conditioned on a pre- and post-selection of measurement outcomes implies context dependence for probabilities that are conditioned on a particular ontic state.

Since the HVT must reproduce the ABL probabilities, we can infer that $p_{\text{HVT}}(X_{\text{M}} = k|\text{A}_{\text{pre}}, \text{A}_{\text{post}}, \text{M})$ must be context-dependent. However, this does not immediately imply context-dependence for the indicator functions. One possible reason for this is that $p_{\text{HVT}}(X_{\text{M}} = k|\text{A}_{\text{pre}}, \text{A}_{\text{post}}, \text{M})$ depends on the transition matrices $\{\Gamma_j^{\text{M}}(\lambda, \omega)\}$, and thus, given Eq. (7), it depends not only on the indicator functions $\{\chi_j^{\text{M}}(\lambda)\}$, but also on the transition matrices for the disturbance, $\{D_j^{\text{M}}(\lambda, \omega)\}$. We must therefore begin by analysing the possibility of context-dependence of the $D_j^{\text{M}}(\lambda, \omega)$.

3.4. **Transformation noncontextuality.** In Spekkens (2004a), the notion of noncontextuality is generalized to transformations. Transformation noncontextuality is the assumption that equivalent transformations (i.e. those represented by the same CP map in the quantum formalism) are associated with the same transition matrix in an HVT. Thus, if outcome k of a measurement M and outcome j of a measurement N are both associated with the CP-map $\mathcal{E}$, then

$$\Gamma_k^{\mathrm{M}}(\lambda,\omega)=\Gamma_j^{N}(\lambda,\omega)=\Gamma_{\mathcal{E}}(\lambda,\omega). \tag{17}$$

If the measurement is sharp, that is, associated with a projector P, and the CP map $\mathcal{E}$ corresponds to the Lüders rule, that is, $\mathcal{E}(\rho)=P\rho P$, then a dependence on $\mathcal{E}$ is simply a dependence on P. In this case, the assumption of transformation noncontextuality is that $\Gamma_k^{\mathrm{M}}(\lambda,\omega)=\Gamma_j^{\mathrm{N}}(\lambda,\omega)=\Gamma_P(\lambda,\omega)$. By Eq. (8), it follows that $\chi_k^{\mathrm{M}}(\lambda)=\chi_j^{\mathrm{N}}=\chi_P(\lambda)$. Thus, for Lüders rule measurements, transformation noncontextuality implies measurement noncontextuality. So if we can show that transformation noncontextuality is consistent with the existence of logical PPS paradoxes, then we have also shown that measurement noncontextuality is consistent with their existence.

4. Main results

As noted below eq. (3), $p_{\mathrm{ABL}}(X_{\mathrm{M}}=k|\mathrm{A}_{\mathrm{pre}},\mathrm{A}_{\mathrm{post}},\mathrm{M})$ depends on the identity of the entire PVM $\{P_j^{\mathrm{M}}\}$ associated with M and not just on the projector P_k^{M} associated with the outcome k. Consequently, if the HVT is to reproduce the ABL predictions, $p_{\mathrm{HVT}}(X_{\mathrm{M}}=k|\mathrm{A}_{\mathrm{pre}},\mathrm{A}_{\mathrm{post}},\mathrm{M})$ must also depend on $\{P_j^{\mathrm{M}}\}$ and not just on P_k^{M}. We will show that such a dependence can be achieved even under the assumption of transformation noncontextuality.

First note that the presence of $\Gamma_{\neg k}^{\mathrm{M}}(\lambda,\omega)$ in Eq. (10) shows that the PPS probability for HVTs *can* have a dependence on the PVM even under the assumption of transformation noncontextuality, since $\Gamma_{\neg k}^{\mathrm{M}}(\lambda,\omega)=\sum_{j\neq k}\Gamma_j^{\mathrm{M}}(\lambda,\omega)=\sum_{j\neq k}\Gamma_{P_j^{\mathrm{M}}}(\lambda,\omega)$ depends on the entire PVM. We now show that it *must* have such a dependence. Suppose M and N are associated with distinct PVMs, $\{P_j^{\mathrm{M}}\}$ and $\{P_j^{\mathrm{N}}\}$ where $P_k^{\mathrm{M}}=P_k^{\mathrm{N}}=P$ for some k. Suppose further that the CP maps for each outcome are described by the Lüders rule, so that the effective CP maps associated with the "not k" outcome are $\mathcal{E}_{\neg k}^{\mathrm{M}}(\cdot)=\sum_{j\neq k}P_j^{\mathrm{M}}(\cdot)P_j^{\mathrm{M}}$ and $\mathcal{E}_{\neg k}^{\mathrm{N}}(\cdot)=\sum_{j\neq k}P_j^{\mathrm{N}}(\cdot)P_j^{\mathrm{N}}$. The distinctness of $\{P_j^{\mathrm{M}}\}$ and $\{P_j^{\mathrm{N}}\}$ then implies that $\mathcal{E}_{\neg k}^{\mathrm{M}}$ and $\mathcal{E}_{\neg k}^{\mathrm{N}}$ are distinct, which in turn implies that there is some density operator ρ that is mapped to distinct density operators by $\mathcal{E}_{\neg k}^{\mathrm{M}}$ and $\mathcal{E}_{\neg k}^{\mathrm{N}}$. Consequently, there must be some distribution $\mu(\lambda)$ that is mapped to distinct distributions by $\Gamma_{\neg k}^{\mathrm{M}}(\lambda,\omega)$ and $\Gamma_{\neg k}^{\mathrm{N}}(\lambda,\omega)$. However, this is only possible if $\Gamma_{\neg k}^{\mathrm{M}}(\lambda,\omega)$ and $\Gamma_{\neg k}^{\mathrm{N}}(\lambda,\omega)$ are themselves distinct transition matrices. Being distinct, they cannot depend only on P, but must depend on the full PVM. This concludes the proof.

Finally, we prove that any transformation noncontextual HVT that can reproduce the ABL predictions (of which logical PPS paradoxes are an example) must involve measurement-disturbance.

We assume the contrary and derive a contradiction. If a measurement involves no disturbance in a HVT, then $\Gamma_k^{\mathrm{M}}(\lambda,\omega)=\delta(\lambda-\omega)\chi_k^{\mathrm{M}}(\omega)$ (which is simply Bayesian updating). As noted above, $p_{\mathrm{HVT}}(X_{\mathrm{M}}=k|\mathrm{A}_{\mathrm{pre}},\mathrm{A}_{\mathrm{post}},\mathrm{M})$ must depend on the full PVM rather than just P_k^{M} to reproduce $p_{\mathrm{ABL}}(X_{\mathrm{M}}=k|\mathrm{A}_{\mathrm{pre}},\mathrm{A}_{\mathrm{post}},\mathrm{M})$. Now, consider the pair of measurements M and N introduced above. Since Γ_k^{M} and Γ_k^{N} depend only on P (by virtue of transformation noncontextuality), it follows that $\Gamma_{\neg k}^{\mathrm{M}}$ and $\Gamma_{\neg k}^{\mathrm{N}}$ must depend on the PVM. The absence of measurement-disturbance would imply $\Gamma_{\neg k}^{\mathrm{M}}(\lambda,\omega)=\delta(\lambda-\omega)\sum_{j\neq k}\chi_j^{\mathrm{M}}(\lambda)$ and $\Gamma_{\neg k}^{\mathrm{N}}=\delta(\lambda-\omega)\sum_{j\neq k}\chi_j^{\mathrm{N}}(\lambda)$. But by the assumption of measurement noncontextuality, $\sum_{j\neq k}\chi_j^{\mathrm{M}}(\lambda) = \sum_{j\neq k}\chi_j^{\mathrm{N}}(\lambda) = \chi_{I-P}(\lambda)$, which implies that $\Gamma^{\mathrm{M}}_{\neg k}(\lambda,\omega) = \Gamma^{\mathrm{N}}_{\neg k}(\lambda,\omega)$. However, if the transition matrices are the same, then they do not depend on the PVM context, and this contradicts the assumption that the HVT reproduces the predictions of the ABL rule. This concludes the proof.

5. Outlook

We have shown that the existence of logical PPS paradoxes in a theory does not imply contextuality, which would seem to suggest that the two phenomena are unrelated. However, we recently proved a theorem (Leifer and Spekkens, 2004) showing that the mathematical structure of every logical PPS paradox in quantum mechanics is sufficient to construct a proof of contextuality. This does not contradict the results presented here because we did not show that a logical PPS paradox is *itself* a proof of contextuality; measurements that are temporal successors in the PPS paradox must be treated as counterfactual alternatives in the proof of contextuality. This distinction is critical, since an earlier measurement can cause a disturbance to the ontic state that is monitored by a later measurement, whereas the possibility of having implemented a different measurement cannot disturb the ontic state of the system in the actual measurement.

Nonetheless, there is some evidence that within the framework of HVTs satisfying additional constraints, one might find a closer connection between PPS paradoxes and contextuality. For instance, in the model of §2.4 there is no analogue of the uncertainty principle, since there is a nonzero probability of ascertaining both the left-right and front-back position of the ball without disturbing it in any way. Meanwhile, a toy theory that is noncontextual (by the operational definition of Spekkens, 2004a) and that *does* include a strong analogue of the uncertainty principle, that of Spekkens (2004b), seems to be devoid of logical PPS paradoxes. This suggests that there may be a natural set of conditions that both quantum theory and the toy theory of Spekkens (2004b) satisfy, but that the model of §2.4 and other HVTs do *not* satisfy, under which contextuality and logical PPS paradoxes are either both present or both absent. We expect that some of the quantum structures discussed at this conference might provide the appropriate setting to address this question.

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