2005

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Recommended Citation

J. Barrett et al 2005 *EPL* 72 685. doi:10.1209/epl/i2005-10297-5

Bell's Jump Process in Discrete Time

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September 27, 2005

Abstract

The jump process introduced by J. S. Bell in 1986, for defining a quantum field theory without observers, presupposes that space is discrete whereas time is continuous. In this letter, our interest is to find an analogous process in discrete time. We argue that a genuine analog does not exist, but provide examples of processes in discrete time that could be used as a replacement.

MSC (2000): 81T25, 60J10. PACS: 02.50.Ga; 03.65.Ta. Key words: Bell's jump process; Markov chain; quantum theory on a lattice.

One of the central challenges for "hidden variable" approaches to quantum mechanics, such as the de Broglie-Bohm pilot wave theory, is to provide an adequate account of relativistic quantum field theory. To address this, Bell introduced a jump process on a discrete lattice [4, 9, 10, 11], intended to reproduce the quantum mechanical predictions for fermion number density in space. The same method can be used to generate stochastic trajectories for any discrete observable, both in field theory and in nonrelativistic quantum mechanics. For a discretized position observable in nonrelativistic quantum mechanics, Bell's process reduces to the de Broglie-Bohm pilot wave theory in the continuum limit [16, 14], so it is a natural analog of this theory for discrete "beables" [2].

Although the "beables" in Bell's process are discrete, it still contains a continuous time parameter. However, there are several reasons for developing a discrete-time version of the process. Firstly, some approaches to quantum gravity are based on fundamentally discrete space-time structures, so a realist account of these theories along Bohmian lines would have to be fully discrete. Secondly, "hidden variable" theories, no matter whether they are realized in nature or not, can be useful for numerical simulations [12, 7],

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visualizations [8, 13], bookkeeping [8], and obtaining better intuitions about quantum phenomena. Numerical simulations are discrete by nature, and a fully discrete theory may also be useful when dealing with quantum phenomena usually described in a discrete setting, such as those considered in quantum information and computation. Thirdly, Valentini [15] has recently proposed that matter in quantum nonequilibrium, i.e. beables with distributions other than $|\Psi|^2$, if existant, may provide astonishing computational resources, enabling us to solve NP-complete problems in polynomial time. However, since classical analog computers can also outperform Turing machines if the continuous variables can be manipulated with perfect accuracy, this claim would be simpler to verify in a fully discrete model.

In this letter, we highlight the difficulties inherent in discretizing Bell's jump process, and propose two concrete discretized processes that circumvent them and converge to Bell's process as the time step τ goes to zero. Other possibilities exist, along the lines of recent proposals by Aaronson [1], and these will be developed in future work.

Bell's process is a Markovian pure jump process $(Q_t)_{t\in\mathbb{R}}$ on a lattice $\mathcal{Q}$ with rate for the jump $q' \to q$ given by

$$\sigma_t(q|q') = \frac{\left[\frac{2}{\hbar}\,\mathrm{Im}\,\langle\Psi_t|P(q)HP(q')|\Psi_t\rangle\right]^+}{\langle\Psi_t|P(q')|\Psi_t\rangle}\,, \tag{1}$$

where $x^+ = \max(x,0)$ denotes the positive part of $x\in\mathbb{R}$, Ψ_t is the state vector of a quantum (field) theory, evolving in some Hilbert space $\mathscr{H}$ according to

$$i\hbar\frac{d\Psi_t}{dt} = H\Psi_t\,, \tag{2}$$

H is the Hamiltonian, and $P(q)$ is the projection to the subspace $\mathscr{H}_q \subseteq \mathscr{H}$, where the $\mathscr{H}_q$ form an orthogonal decomposition, $\mathscr{H} = \bigoplus_{q\in\mathcal{Q}}\mathscr{H}_q$. Relevant properties of Bell's process are that at every time t, the distribution of Q_t is the quantum distribution

$$\langle\Psi_t|P(q)|\Psi_t\rangle\,, \tag{3}$$

and that its net probability current between q' and q, $\sigma_t(q|q')\mathbb{P}(Q_t = q') - \sigma_t(q'|q)\mathbb{P}(Q_t = q)$ where $\mathbb{P}$ denotes "probability," agrees with the quantum expression for the probability current,

$$\tfrac{2}{\hbar}\,\mathrm{Im}\,\langle\Psi|P(q)HP(q')|\Psi\rangle\,. \tag{4}$$

Since many constructions are easier in discrete time than in continuous time, one might have expected that there is an analogous Markov chain $(\tilde{Q}_t)_{t\in\tau\mathbb{Z}}$ on $\mathcal{Q}$ with discrete time step τ such that the probability $\mathbb{P}_t(q'\to q)$ for the transition $q'\to q$, i.e., the conditional probability $\mathbb{P}(\tilde{Q}_{t+\tau} = q|\tilde{Q}_t = q')$, is given by a formula similar to (1), with H replaced by a simple function of the unitary U defining the time evolution

$$\Psi_{t+\tau} = U\Psi_t\,, \tag{5}$$

and that one could arrive at this formula by a reasoning similar to the one leading to (1) from (3) and (4), as given in [9, Sec. 2.5].

However, this is not possible in any obvious way. The obstacle is that in the time-discrete case there is no obvious formula for the net probability current $J(q,q')$ between q' and q, replacing (4) of the continuous case. Given an expression for $J(q,q')$ in terms of Ψ, P, and U, we could set

$$\mathbb{P}_t(q' \to q) = \frac{J_t(q,q')^+}{\langle \Psi_t | P(q') | \Psi_t \rangle} \quad \text{for } q \neq q' \,, \tag{6}$$

which would define a Markov chain $(\tilde{Q}_t)_{t\in\tau\mathbb{Z}}$ whose probability current

$$\mathbb{P}_t(q' \to q)\, \mathbb{P}(\tilde{Q}_t = q') - \mathbb{P}_t(q \to q')\, \mathbb{P}(\tilde{Q}_t = q) \tag{7}$$

coincides with $J(q,q')$ and whose distribution at any time t coincides with the quantum distribution (3), provided $J(q,q')$ has the following properties:

$$J(q,q') \in \mathbb{R} \tag{8a}$$
$$J(q',q) = -J(q,q') \tag{8b}$$
$$\sum_{q\in\mathcal{Q}} J(q,q')^+ \leq \langle \Psi | P(q') | \Psi \rangle \tag{8c}$$
$$\sum_{q'\in\mathcal{Q}} J(q,q') = \langle \Psi | U^* P(q) U | \Psi \rangle - \langle \Psi | P(q) | \Psi \rangle \,. \tag{8d}$$

Currents of the form (7), with transition probabilities (6) and distribution (3), have these properties by construction. Property (8c) expresses that no greater amount of probability can get transported away from q' than present at q', and (8d) guarantees the quantum distribution (3) at the next time step. The obvious way of guessing a formula for $J(q,q')$ is to start from one of the expressions

$$\langle \Psi | U^* P(q) U P(q') | \Psi \rangle \tag{9a}$$
$$\langle \Psi | P(q) U P(q') | \Psi \rangle \,, \tag{9b}$$

to multiply it by any numerical constant, to take the real or imaginary parts to ensure (8a), and to anti-symmetrize in q and q' to ensure (8b). However, all expressions thus obtained generically violate (8d), except for the anti-symmetrization of 2Re(9a),

$$J(q,q') = \tfrac{1}{2} \langle \Psi | \big(U^* P(q) U P(q') + P(q') U^* P(q) U - U^* P(q') U P(q) - P(q) U^* P(q') U \big) | \Psi \rangle \,, \tag{10}$$

which can violate (8c) (numerically we found 46 examples of such violations among one thousand randomly chosen U and ψ in $\mathscr{H} = \mathbb{C}^3$ with fixed one-dimensional projections $P(q)$ and $P(q')$).

However, a different reasoning leads to a process in discrete time that has some features in common with Bell's process. Choose H such that

$$U = e^{-i\tau H}\,, \tag{11}$$

so that the evolution (2) generated by H is a continuation of the evolution (5) generated by U. (The degree of non-uniqueness of this choice is discussed later.) Then, consider Bell's process $(Q_t)_{t\in\mathbb{R}}$ in continuous time for this H. By restriction to just the integer times, we obtain a Markov process $\tilde{Q}_t := Q_t$ for $t \in \tau\mathbb{Z}$.

The process $(\tilde{Q}_t)_{t\in\tau\mathbb{Z}}$ has the quantum distribution (3) at every time. It is important for this that the two evolution laws (2) and (5) for Ψ lead to the same Ψ_t at every t that is an integer multiple of τ. It makes no sense to ask whether the probability current of this process, $\mathbb{P}(\tilde{Q}_{t+\tau} = q, \tilde{Q}_t = q') - \mathbb{P}(\tilde{Q}_{t+\tau} = q', \tilde{Q}_t = q)$, agrees with the one prescribed by quantum theory, since, as discussed above, quantum theory does not prescribe a unique current in the discrete-time case. Note that in the limit $\tau \to 0$ the process approaches Bell's process. This fact and the simple and straightforward construction of $(\tilde{Q}_t)_{t\in\tau\mathbb{Z}}$ suggest that this may be the closest one can get to an analog of Bell's process in the time-discrete case.

The transition probability $\mathbb{P}_t(q' \to q) = \mathbb{P}(\tilde{Q}_{t+\tau} = q|\tilde{Q}_t = q')$ does not, however, possess a simple formula in terms of Ψ_t, U, and $P(\cdot)$ analogous to (1), only the following one:

$$\begin{aligned}\mathbb{P}_{t_0}(q' \to q) = \sum_{n=0}^{\infty} \sum_{q_0,\ldots,q_n\in\mathcal{Q}} \delta_{q',q_0}\,\delta_{q,q_n} \int\limits_{t_0}^{t_0+\tau} dt_1 \int\limits_{t_1}^{t_0+\tau} dt_2 \cdots \int\limits_{t_{n-1}}^{t_0+\tau} dt_n \,\times \\ \times\, \exp\Bigg(-\int\limits_{t_0}^{t_0+\tau} \sigma_s(\mathcal{Q}|q_{\max\{k:t_k<s\}})\,ds\Bigg) \prod_{k=1}^{n} \sigma_{t_k}(q_k|q_{k-1})\,,\end{aligned} \tag{12}$$

with $\sigma_s(q|r)$ given by (1) and $\sigma_s(\mathcal{Q}|r) := \sum_{q\in\mathcal{Q}} \sigma_s(q|r)$. Eq. (12) is a fact about any jump process in continuous time with jump rates σ (applied here to Bell's process Q_t).[1]

The process $\tilde{Q}$ is not completely determined by Ψ_0, U, and $P(\cdot)$ since H is not completely determined by (11), even though in many cases there may be a natural choice

[1] To get a grasp of (12), begin with noting that $\sigma_s(\mathcal{Q}|r)$ is the total jump rate at time s in the configuration r. The probability that no jump takes place before time t, if the process starts at t_0 in q_0, is $\exp\big(-\int_{t_0}^{t} \sigma_s(\mathcal{Q}|q_0)\,ds\big)$. Thus, the probability that the first jump takes place between time t and $t+dt$ is $\exp\big(-\int_{t_0}^{t} \sigma_s(\mathcal{Q}|q_0)\,ds\big)\,\sigma_t(\mathcal{Q}|q_0)\,dt$. The probability that the destination of the first jump is q_1, given that the jump takes place at time t, is $\sigma_t(q_1|q_0)/\sigma_t(\mathcal{Q}|q_0)$. Conditional on that the first jump occurs at t and leads to q_1, the distribution of the times and destinations of the further jumps is the same as for a process starting at time t in q_1. Thus, the probability of a path $q_0, \ldots, q_n$ with the k-th jump between t_k and $t_k + dt_k$ and no further jump before $t_0 + \tau$ is the integrand of (12) times $dt_1 \cdots dt_n$. Now add (respectively integrate) the probabilities of all ways the process can move from q' to q in the time interval $[t_0, t_0 + \tau]$, namely by means of n jumps at times $t_1, \ldots, t_n$ with destinations $q_1, \ldots, q_n$. For a more detailed discussion of such probability formulas, see [10] and [6].

of H. For example, if U has an eigenvalue $e^{-i\theta}$, then H may have as the corresponding eigenvalue any of the numbers $\frac{\theta}{\tau} + \frac{2\pi}{\tau}k$ with $k \in \mathbb{Z}$. More generally, for any self-adjoint operator S with spectrum contained in $\frac{2\pi}{\tau}\mathbb{Z}$ and commuting with H (in the sense of commuting spectral projections), $H + S$ is another solution of (11) for given U. A unique H could be selected by the additional condition that the spectrum of H be contained in $(-\frac{\pi}{\tau}, \frac{\pi}{\tau}]$.

In the particularly simple situation $|\mathcal{Q}| = 2$, there does exist a time-discrete analog $(\hat{Q}_t)_{t\in\tau\mathbb{Z}}$ to Bell's process. In this case, the expression (10) satisfies (8) and thus defines a process; in fact, the net probability current between the two configurations q' and q is already determined by the distribution (3) and must be

$$\langle \Psi_t | U^* P(q) U | \Psi_t \rangle - \langle \Psi_t | P(q) | \Psi_t \rangle \,, \tag{13}$$

since any increase or decrease can occur only by transitions from or to the other configuration. Just as Bell's process has the smallest jump rates compatible with the current (4) [9, 11], we may choose now the smallest transition probabilities compatible with the current (13), which are

$$\mathbb{P}_t\big(\hat{Q}_{t+\tau} \neq q \big| \hat{Q}_t = q\big) = \frac{\langle \Psi_t | (P(q) - U^* P(q) U) | \Psi_t \rangle^+}{\langle \Psi_t | P(q) | \Psi_t \rangle} \,. \tag{14}$$

This need not coincide with the transition probability (12) of $(\tilde{Q}_t)$, even though in the limit $\tau \to 0$ also $(\hat{Q}_t)$ converges to Bell's process. The same construction can be applied to the case $|\mathcal{Q}| > 2$ if U involves only pairs of configurations, i.e., if there is a partition of $\mathcal{Q}$ into subsets, all of which are either pairs or singlets, such that $P(q)UP(q') = 0$ whenever q and q' do not belong to the same subset. Then (10) still satisfies (8) and thus defines a process. An example of this is a quantum computing circuit, realized through a time sequence of single qubit unitaries and CNOT gates. (Here, a configuration q corresponds to a definite value for the computational basis observable for each qubit.)

To contrast the previous processes with an example of a process that does not converge to Bell's process in the limit $\tau \to 0$ but has the quantum distribution (3) at every time, we define the process $(Q^*_t)_{t\in\tau\mathbb{Z}}$ by the transition probability

$$\mathbb{P}(Q^*_{t+\tau} = q | Q^*_t = q') = \langle \Psi_{t+\tau} | P(q) | \Psi_{t+\tau} \rangle \,. \tag{15}$$

This means that for every t, Q^*_t is independent of the past and has the quantum distribution. Its limit as $\tau \to 0$, in a suitable sense, is simply the process $(\tilde{Q}^*_t)_{t\in\mathbb{R}}$ for which every $\tilde{Q}^*_t$ is independent of the past and has the quantum distribution, a process reminiscent of Bell's [3] description of a precise version of the "many worlds" interpretation of quantum mechanics: "[I]nstantaneous classical configurations [Q] are supposed to exist, and to be distributed [...] with probability $|\psi|^2$. But no pairing of configurations at different times, as would be effected by the existence of trajectories, is supposed."

Acknowledgments. We thank Shelly Goldstein of Rutgers University (USA) for helpful discussions and two anonymous referees for their comments. R.T. thanks the Institut des Hautes Études Scientifiques at Bures-sur-Yvette, France, for hospitality. This work has been partially supported by the European Commission through its 6th Framework Programme "Structuring the European Research Area" and the contract Nr. RITA-CT-2004-505493 for the provision of Transnational Access implemented as Specific Support Action.

References

[1] Aaronson, S.: Quantum Computing and Hidden Variables I: Mapping Unitary to Stochastic Matrices. quant-ph/0408035

[2] Bell, J. S.: The Theory of Local Beables. *Epistemological Letters*, March 1976. Reprinted in [5], Chapter 7: pp. 52-62

[3] Bell, J. S.: Quantum mechanics for cosmologists. In C. Isham, R. Penrose, D. Sciama (editors), *Quantum Gravity 2*, Oxford: Clarendon Press (1981), pp. 611-637. Reprinted in [5], Chapter 15: pp. 117-138

[4] Bell, J. S.: Beables for quantum field theory. *Phys. Rep.* **137**: 49-54 (1986). Reprinted in [5], Chapter 19: pp. 173-180

[5] Bell, J. S.: *Speakable and Unspeakable in Quantum Mechanics*, 2nd edition, Cambridge University Press (2004)

[6] Breiman, L.: *Probability.* Reading MA: Addison-Wesley (1968)

[7] Dennis, E.: Purifying Quantum States: Quantum and Classical Algorithms. Ph.D. thesis, University of California, Santa Barbara (2003)

[8] Dennis, E., Rabitz, H.: Bell trajectories for revealing quantum control mechanisms. *Phys. Rev. A.* **67**: 033401 (2003). quant-ph/0208109

[9] Dürr, D., Goldstein, S., Tumulka, R., Zanghì, N.: Quantum Hamiltonians and Stochastic Jumps. *Commun. Math. Phys.* **254**: 129-166 (2005). quant-ph/0303056

[10] Georgii, H.-O., Tumulka, R.: Global Existence of Bell's Time-Inhomogeneous Jump Process for Lattice Quantum Field Theory. *Markov Proc. Rel. Fields* **11**: 1-18 (2005). math.PR/0312294 and mp_arc 04-11

[11] Georgii, H.-O., Tumulka, R.: Some Jump Processes in Quantum Field Theory. In J.-D. Deuschel, A. Greven (editors), *Interacting Stochastic Systems*, Berlin: Springer-Verlag (2004). math.PR/0312326

[12] Lopreore, C. L., Wyatt, R. E.: Quantum Wave Packet Dynamics with Trajectories. *Phys. Rev. Lett.* **82**: 5190-5193 (1999)

[13] Struyve, W., De Baere, W., De Neve, J., De Weirdt, S.: On Peres' Statement "Opposite Momenta Lead to Opposite Directions", Decaying Systems and Optical Imaging. *Found. Phys.* **34**: 963-985 (2004). quant-ph/0401095

[14] Sudbery, A.: Objective interpretations of quantum mechanics and the possibility of a deterministic limit. *J. Phys. A* **20**: 1743-1750 (1987)

[15] Valentini, A.: Subquantum Information and Computation. *Pramana - J. Phys.* **59**: 269-277 (2002). quant-ph/0203049

[16] Vink, J. C.: Quantum mechanics in terms of discrete beables. *Phys. Rev. A* **48**: 1808-1818 (1993)