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Addendum
Entropy and information causality in general probabilistic theories

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Abstract. In this addendum to our paper (2010 *New J. Phys.* **12** 033024), we point out that an elementary consequence of the strong subadditivity inequality allows us to strengthen one of the main conclusions of that paper.

In this addendum to our paper [1], we point out that an elementary consequence of the strong subadditivity inequality allows us to strengthen one of the main conclusions of that paper.

In [1], we defined notions of measurement and mixing entropy for a broad class of probabilistic theories (which includes finite-dimensional quantum theory). We defined a theory to be *monoentropic* if for every state of a system in the theory, the mixing entropy (denoted S) is equal to the measurement entropy (denoted H). This broad class of theories has a notion of composite system, and of the 'marginal state' of a subsystem given a state for the composite system, so that the strong subadditivity inequality may be formulated for real-valued functions of states, including the measurement and mixing entropies. We proved the following theorem.

Theorem 1 (theorem 4 of [1]). *Suppose that a theory has the following properties:*

(1) *it is monoentropic, meaning that measurement entropy equals mixing entropy for all systems,*
(2) *its measurement entropy is strongly subadditive and*
(3) *its measurement entropy satisfies the Holevo bound.*

Then the theory satisfies information causality. It follows that any theory satisfying these conditions cannot violate Tsirel'son's bound.

In [1] we noted that monoentropicity was only used to establish the technical condition of positivity of conditional information, $H(A|C)$, defined as $H(AC) - H(C)$, when A is classical. This condition does not hold for all theories in the broad class we consider. Shortly after submitting the final version of [1], but too late for inclusion therein, we noticed that it does, however, hold for all theories satisfying strong subadditivity, enabling us to strengthen our theorem as follows.

Theorem 2. *Suppose that a theory's measurement entropy is strongly subadditive and satisfies the Holevo bound. Then the theory satisfies information causality. It follows that any theory satisfying these conditions cannot violate Tsirel'son's bound.*

As noted in our earlier paper, one can replace the two premises of this theorem with the single premise that the measurement entropy satisfies the data processing inequality.

To see that $H(A|C) \geqslant 0$ for classical A follows from strong subadditivity, we recall (from e.g. lemma 1 of [1]), that strong subadditivity is equivalent to the positivity of conditional mutual information, $I(A:B|C) \geqslant 0$. Letting A and B be perfectly correlated classical systems in this inequality, we have

$$\begin{aligned} I(A:B|C) &= H(A|C) + H(B|C) - H(AB|C) \\ &= H(AC) - H(C) + H(BC) \\ &\quad - H(C) - H(ABC) + H(C) \\ &= H(AC) + H(BC) - H(ABC) - H(C). \end{aligned}$$

Since A and B are perfectly correlated classical systems, $H(AC) = H(BC) = H(ABC)$. Consequently, in this case $I(A:B|C) = H(AC) - H(C) \equiv H(A|C)$. But by strong subadditivity this is $\geqslant 0$.

The primary use of the full definition of the measurement entropy in the proof of theorem 4 of [1] (i.e. theorem 1 above) was to establish monoentropicity, and thence positivity of conditional entropy. So the present observations suggest that the analogue of theorem 2 might actually hold for any real-valued function of states satisfying a few other weak properties of the measurement entropy that are used in the proof. The most obvious of these are that it reduce to the classical entropy on classical systems, and that $H(A:B|C) = H(A|C)$ for perfectly correlated classical A and B. However, we defer rigorous investigation of this possibility to future work.

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