2012

Entropy and Information Causality in General Probabilistic Theories (Addendum)

Howard Barnum
University of New Mexico

Jonathan Barrett
University of Oxford

Lisa Orloff Clark
University of Otago

Matthew S. Leifer
Chapman University, leifer@chapman.edu

Robert Spekkens
Perimeter Institute for Theoretical Physics

See next page for additional authors

Follow this and additional works at: http://digitalcommons.chapman.edu/scs_articles

Part of the Quantum Physics Commons

Recommended Citation

This Article is brought to you for free and open access by the Science and Technology Faculty Articles and Research at Chapman University Digital Commons. It has been accepted for inclusion in Mathematics, Physics, and Computer Science Faculty Articles and Research by an authorized administrator of Chapman University Digital Commons. For more information, please contact laughtin@chapman.edu.
Entropic and Information Causality in General Probabilistic Theories (Addendum)

Comments
This article was originally published in New Journal of Physics, volume 14, in 2012. DOI: 10.1088/1367-2630/14/12/129401

Creative Commons License
This work is licensed under a Creative Commons Attribution-Noncommercial-Share Alike 3.0 License.

Copyright
IOP Publishing

Authors
Howard Barnum, Jonathan Barrett, Lisa Orloff Clark, Matthew S. Leifer, Robert Spekkens, Nicholas Stepanik, Alex Wilce, and Robin Wilke
Addendum

Entropy and information causality in general probabilistic theories

Howard Barnum\textsuperscript{1}, Jonathan Barrett\textsuperscript{2}, Lisa Orloff Clark\textsuperscript{3}, Matthew Leifer\textsuperscript{4,5}, Robert Spekkens\textsuperscript{4}, Nicholas Stepanik\textsuperscript{6}, Alex Wilce\textsuperscript{7} and Robin Wilke\textsuperscript{8}

\textsuperscript{1} Department of Physics and Astronomy, University of New Mexico, 1919 Lomas Boulevard NE, MSC07 4220, Albuquerque, NM 87131-0001, USA
\textsuperscript{2} Department of Computer Science, University of Oxford, Parks Road, Oxford OX1 3QD, UK
\textsuperscript{3} Department of Mathematics and Statistics, University of Otago, PO Box 56, Dunedin 9054, New Zealand
\textsuperscript{4} Perimeter Institute for Theoretical Physics, 31 Caroline St N, Waterloo, Ontario N2L 2Y5, Canada
\textsuperscript{5} Institute for Quantum Computing, University of Waterloo, 200 University Avenue W, Waterloo, Ontario N2L 3G1, Canada
\textsuperscript{6} Department of Mathematics, University of Connecticut, 198 Auditorium Road, Unit 3009, Storrs, CT 06269, USA
\textsuperscript{7} Department of Mathematical Sciences, Susquehanna University, Selinsgrove, PA 17870, USA
\textsuperscript{8} Department of Mathematics and Statistics, University of Vermont, Burlington, VT 05405, USA

E-mail: hbarnum@unm.edu, hnbarnum@aol.com, jonathan.barrett@cs.ox.ac.uk, lclark@maths.otago.ac.nz, matt@mattleifer.info, rspekkens@perimeterinstitute.ca, nicholas.stepanik@gmail.com, wilce@susqu.edu and rwilke@uvm.edu

New Journal of Physics 14 (2012) 129401 (3pp)
Received 22 August 2012
Published 5 December 2012
Online at http://www.njp.org/
doi:10.1088/1367-2630/14/12/129401
Abstract. In this addendum to our paper (2010 *New J. Phys.* 12 033024), we point out that an elementary consequence of the strong subadditivity inequality allows us to strengthen one of the main conclusions of that paper.

In this addendum to our paper [1], we point out that an elementary consequence of the strong subadditivity inequality allows us to strengthen one of the main conclusions of that paper.

In [1], we defined notions of measurement and mixing entropy for a broad class of probabilistic theories (which includes finite-dimensional quantum theory). We defined a theory to be *monoentropic* if for every state of a system in the theory, the mixing entropy (denoted $S$) is equal to the measurement entropy (denoted $H$). This broad class of theories has a notion of composite system, and of the ‘marginal state’ of a subsystem given a state for the composite system, so that the strong subadditivity inequality may be formulated for real-valued functions of states, including the measurement and mixing entropies. We proved the following theorem.

**Theorem 1 (theorem 4 of [1]).** Suppose that a theory has the following properties:

1. it is monoentropic, meaning that measurement entropy equals mixing entropy for all systems,
2. its measurement entropy is strongly subadditive and
3. its measurement entropy satisfies the Holevo bound.

Then the theory satisfies information causality. It follows that any theory satisfying these conditions cannot violate Tsirel’son’s bound.

In [1] we noted that monoentropicity was only used to establish the technical condition of positivity of conditional information, $H(A|C)$, defined as $H(AC) - H(C)$, when $A$ is classical. This condition does not hold for all theories in the broad class we consider. Shortly after submitting the final version of [1], but too late for inclusion therein, we noticed that it does, however, hold for all theories satisfying strong subadditivity, enabling us to strengthen our theorem as follows.

**Theorem 2.** Suppose that a theory’s measurement entropy is strongly subadditive and satisfies the Holevo bound. Then the theory satisfies information causality. It follows that any theory satisfying these conditions cannot violate Tsirel’son’s bound.

As noted in our earlier paper, one can replace the two premises of this theorem with the single premise that the measurement entropy satisfies the data processing inequality.

To see that $H(A|C) \geq 0$ for classical $A$ follows from strong subadditivity, we recall (from e.g. lemma 1 of [1]), that strong subadditivity is equivalent to the positivity of conditional mutual information, $I(A:B|C) \geq 0$. Letting $A$ and $B$ be perfectly correlated classical systems in this inequality, we have

$$I(A : B|C) = H(A|C) + H(B|C) - H(AB|C)$$

$$= H(AC) - H(C) + H(BC)$$

$$- H(C) - H(ABC) + H(C)$$

$$= H(AC) + H(BC) - H(ABC) - H(C).$$

*New Journal of Physics* 14 (2012) 129401 (http://www.njp.org/)
Since $A$ and $B$ are perfectly correlated classical systems, $H(AC) = H(BC) = H(ABC)$. Consequently, in this case $I(A : B|C) = H(AC) - H(C) \equiv H(A|C)$. But by strong subadditivity this is $\geq 0$.

The primary use of the full definition of the measurement entropy in the proof of theorem 4 of [1] (i.e. theorem 1 above) was to establish monoentropicity, and thence positivity of conditional entropy. So the present observations suggest that the analogue of theorem 2 might actually hold for any real-valued function of states satisfying a few other weak properties of the measurement entropy that are used in the proof. The most obvious of these are that it reduce to the classical entropy on classical systems, and that $H(A : B|C) = H(A|C)$ for perfectly correlated classical $A$ and $B$. However, we defer rigorous investigation of this possibility to future work.

**Acknowledgments**

This research was supported by the United States Government through grant OUR-0754069 from the National Science Foundation. It was also supported by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation. JB was supported by the EPSRC, the CHIST-ERA DIQIP project, and by grant FQXi-RFP3-1016A from the Foundational Questions Institute. At IQC, Matthew Leifer was supported in part by MITACS and ORDCF. At Perimeter Institute, Matthew Leifer was supported in part by grant RFP1-06-006 from The Foundational Questions Institute (fqxi.org).

**References**