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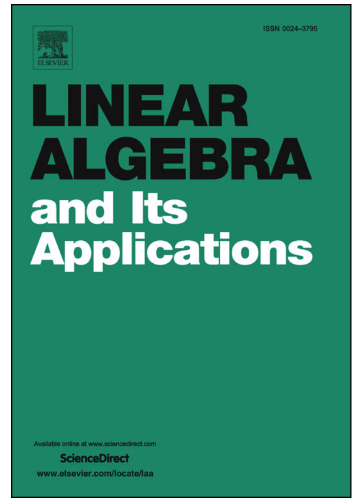

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"WRONG" SIDE INTERPOLATION BY POSITIVE REAL RATIONAL FUNCTIONS

DANIEL ALPAY AND IZCHAK LEWKOWICZ

Abstract. Using polynomial interpolation, along with structural properties of the family of rational positive real functions, we here show that a set of m nodes in the open *left* half of the complex plane, can always be mapped to anywhere in the complex plane by rational positive real functions whose degree is at most m. Moreover we introduce an *easy-to-find* parametrization in $\mathbb{R}^{2m+3}$ of a large subset of these interpolating functions.

1. Introduction

Problem Formulation

A framework for many classical interpolation problems is as follows. Given a set of distinct nodes

$$X = \{x_1, \ \dots\ , \ x_m\}$$

and image points

$$Y = \{y_1, \ \dots\ , \ y_m\}$$

(not necessarily distinct) and a family of functions $\mathcal{F}$, find whether there exist functions $f \in \mathcal{F}$ so that

$$y_j = f(x_j) \qquad j = 1, \ \dots\ , \ m. \tag{1.1}$$

If yes, parameterize all of them, preferably within a degree bound. There is a vast literature on the subject see e.g. [2]-[7], [10], [11], [13], [15]-[19]. To simplify the discussion, we here focus on scalar real rational functions. Thus degree simply means the maximum between the degree of the numerator and of the denominator polynomials. The polynomial (a.k.a. the Lagrange) interpolation is probably the best known problem in this framework, see e.g. [9, Section 2.10], [15] (in [16] it is attributed to [18]), for matrical version, see e.g. [12, Subsection 6.1.14]. For the case where $\mathcal{F}$ is the set of rational functions see [3] and if in addition all functions in $\mathcal{F}$ are analytic in a disk, centered at the

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origin of $\mathbb{C}$ and of a prescribed radius, the problem was addressed in [2].

We shall denote by $\mathbb{C}_r$ ($\overline{\mathbb{C}}_r$) the open (closed) right half plane (the subscript stands for "right"). The family of functions $\mathcal{F}$ we here focus on, is of *positive real*, i.e. analytically mapping the open right half plane to its closure. Namely, a real rational function $f(s)$ of a complex variable s is said to be positive if

$$\mathrm{Re}\,(f(s)) \geq 0 \qquad \forall s \in \mathbb{C}_r \ . \tag{1.2}$$

Interpolation problem with rational positive real functions can be further classified by the domain the set of nodes X belongs to.

If the nodes x_j are in $\mathbb{C}_r$, this amounts to the classical Nevanlinna-Pick interpolation problem, see e.g. [4, Theorem 18.1] and for real functions [19]. There, from the interpolation data one constructs the Pick matrix whose j,k element is given by

$$\frac{y_j^* + y_k}{x_j^* + x_k} \qquad j,\ k = 1,\ \dots\ ,\ m.$$

It is known that there exist interpolating functions if and only if the Pick matrix is positive semi-definite. Moreover, all interpolating functions may be parameterized through this Pick matrix. Recall that having the Pick matrix positive semi-definite implies that each 2-dimensional minor is non-negative, which in turn can be written as,

$$\frac{|x_j - x_k|^2}{\mathrm{Re}(x_j)\mathrm{Re}(x_k)} \geq \frac{|y_j - y_k|^2}{\mathrm{Re}(y_j)\mathrm{Re}(y_k)} \qquad m \geq j > k \geq 1. \tag{1.3}$$

This condition means that the map from the nodes $x_1\ ,\ \dots\ ,\ x_m$ to the image points $y_1\ ,\ \dots\ ,\ y_m$, is contractive in $\mathbb{C}_r$ in the sense of Eq. (1.3). This illustrates the fact that the interpolation problem at handealWrongSideInterpolation10aug2017:set.

If the nodes are confined to the imaginary axis, an interpolation scheme, elegant in its simplicity, appeared in [20].

If the nodes x_j are in $\overline{\mathbb{C}}_r$ (with possibly some of them on $i\mathbb{R}$) the problem is much harder, see e.g. [4, Chapter 21], [6], [7], [13] and [17].

If the interpolation data is in whole plane, provided that

$$\mathrm{Re}(x_j)\mathrm{Re}(y_j) > 0 \qquad j = 1,\ \dots\ ,\ m, \tag{1.4}$$

one can still resort to the classical Nevanlinna-Pick interpolation scheme: First, complete the data set so that if x, y is an interpolation pair, then so is $-x, -y$. Then, from this extended data, take the m nodes which are in $\mathbb{C}_r$, construct the the corresponding Pick matrix and proceed as usual. Finally, use the fact (see [19]) that whenever the Pick matrix is positive semi-definite, among the interpolation functions there exists some function with odd symmetry, i.e. $f(s) = -f(-s)$ (a.k.a. Foster or lossless functions, see e.g. [5], [8], [19]). If instead of the left and right half planes, $\mathbb{C}$ is partitioned to the unit disk and its exterior, a similar idea is presented in [2, Section 5].

In this work we focus on the case where the nodes $x_1, \ldots, x_m$ are all in $\mathbb{C}_l$ (the open left half plane). We parameterize a large subset of rational positive real interpolating functions whose degree is less or equal to m. In particular, it is shown that this set is never empty.

A key idea is the following: We construct two rational functions sharing the same denominator: (i) an interpolating function $p(s)$ (not necessarily positive real) and (ii) a strictly positive real rational function $\Delta(s)$ vanishing at the nodes. Thus, for all $r \in \mathbb{R}$ the parametric rational function

$$f(s) = p(s) + r\Delta(s), \tag{1.5}$$

is interpolating. Moreover, for r "sufficiently large", $f(s)$ turns to be positive real.

Interestingly, we can mention two ideas conceptually similar to those in the current work, which have appeared within completely different interpolation frameworks: (i) The fact that for interpolation by low degree rational functions, one should separately treat numerators and denominators, appeared in the context of Schur functions in [11, Theorems 1, 2]. (ii) In [1] we have used Eq. (1.5) for interpolation by structured matrix-valued polynomials. For example, where $p(s)$ was an interpolating polynomial which on $i\mathbb{R}$, attained Hermitian values, while the polynomial $\Delta(s)$, vanishing at the nodes, was positive definite on $i\mathbb{R}$, see [1, Eq. (1.6)].

In Section II we present a four steps interpolation procedure:

In Step 1 we parameterize all candidates for denominator polynomials of the sought interpolating functions. Namely, all real polynomials, of degree of at most m, non-vanishing at the nodes.

In Step 2, to each of these denominator polynomials we match a numerator to obtain $p(s)$, a rational interpolating function (not necessarily positive real).

In Step 3, we construct $\Delta(s)$ strictly positive real rational functions, vanishing at the nodes. We now restrict the denominators of $p(s)$, the rational functions from Step 2 to the subset of the resulting denimina-tors of $\Delta(s)$.

In Step 4 to each of the resulting interpolating function $p(s)$, we add $r\Delta(s)$, a weighted version of the strictly positive real rational functions, vanishing at the nodes (which shares the same denominar). Thus, $f(s)$ in Eq. (1.5) is an interpolating function, of degree of at most m for all $r \in \mathbb{R}$. Furthermore, for r "sufficiently large" it is positive real.

A closer scrutiny reveals that all interpolating, positive real rational functions obtained, are so that the degree of the denominator is larger or equal to the degree of the numerator.

We complete our the description of positive real interpolating functions as follows: We repeat the previous steps by constructing positive real interpolating functions from the original nodes x_j *but to*[1] $\frac{1}{y_j}$. Finally, as the sought solution, we take the reciprocal of these functions.

In Section III we illustrate the above procedure by detailed examples and add concluding remarks.

2. A Recipe

2.1. **Step 1: Constructing all real monic polynomials, of degrees m and m − 1, non-vanishing at the nodes.** We first recall several basic facts concerning polynomial (a.k.a. Lagrange) interpolation, see e.g. [9, Section 2.10], [12, Subsection 6.1.14], [15], [16], [18]. Given a set of distinct points in $\mathbb{C}$,

$$X = \{x_1, \ \dots \ , \ x_m\}.$$

We shall denote by $\eta(s)$ the monic polynomial of the smallest degree vanishing at X,

$$\eta(s) := \prod_{j=1}^{m}(s - x_j). \tag{2.1}$$

Next, denote by $\phi_1(s), \ \dots \ , \ \phi_m(s)$ the monic divisors of $\eta(s)$ of degree $m-1$, i.e.

$$\phi_j(s) = \tfrac{\eta(s)}{s-x_j} = \prod_{\substack{k=1\\k\neq j}}^{m}(s - x_k) \qquad j = 1, \ \dots \ , \ m. \tag{2.2}$$

Given a set of (not necessarily distinct) points in $\mathbb{C}$,

$$Y = \{y_1, \ \dots \ , \ y_m\}.$$

[1]Assuming that $y_j \neq 0$ for all $j = 1, \ \dots \ , \ m$.

The unique polynomial $\tilde{d}(s)$ of degree of at most $m-1$, interpolating between X and Y, is given by,

$$\tilde{d}(s)=\sum_{j=1}^{m}\frac{y_j}{\phi_j(x_j)}\phi_j(s).$$

As already mentioned, in the sequel, we shall call X, *interpolating nodes*, Y *image points*, and relate to the pair X,Y as *interpolation data*.

From the above it follows that $\{\phi_1,\ \ldots\ ,\ \phi_m\}$ in (2.2) form a basis to the space P_{m-1} of all complex polynomials of degree of at most $m-1$, while $\{\phi_1,\ \ldots\ ,\ \phi_m,\eta\}$ form a basis to the space P_m of all complex polynomials of degree of at most m.

We thus arrive at the following classical conclusion.

Proposition 2.1. *Any polynomial $\tilde{d}(s)$ of degree of at most m can be written as*

$$\tilde{d}(s)=b\eta(s)+\sum_{j=1}^{m}c_j\phi_j(s)\qquad b,c_1,\ \ldots\ ,\ c_m\in\mathbb{C}.$$

Moreover, $\tilde{d}(s)$ does not vanish on X, if and only if

$$c_j\neq 0\qquad 1\ \ldots\ ,\ m,$$

and $\tilde{d}\in P_{m-1}$, if and only if, $b=0$.

From now on we shall focus on real polynomials and real rational functions. To this end, if necessary, the original set of nodes X is complemented so it is closed under complex conjugation. Namely,

$$\mathrm{Im}(x_j)>0\implies x_{j+1}=x_j^*\ .\tag{2.3}$$

Note that then in Eq. (2.1) the resulting $\eta(s)$ is real. Now, to render $\tilde{d}(s)$ real we shall write it as,

$$\tilde{d}(s)=b\eta(s)+\sum_{j=1}^{m}c_j\phi_j(s)\qquad b\in\mathbb{R}\qquad \begin{smallmatrix}\mathrm{Im}(x_j)>0 & 0\neq\ c_{j+1}=c_j^*\\ \mathrm{Im}(x_j)=0 & 0\neq\quad c_j\in\mathbb{R}.\end{smallmatrix}\tag{2.4}$$

Note that in particular, $\tilde{d}(s)$ may have multiple roots.

Without loss of generality, we shall find it convenient to distinguish in Eq. (2.4) between the cases $b=0$, $b\neq 0$ (see Proposition 2.10 below). Furthermore to ease the distinction, we differently denote the coefficients[2] of the polynomials for $b = 0$ and $b \neq 0$, i.e.

$$\begin{aligned} \tilde{d}_o(s) &:= \tilde{d}(s)_{|_{b=0}} &&= \sum_{j=1}^{m} \gamma_j \phi_j(s) \\ \tilde{d}_1(s) &:= \tilde{d}(s)_{|_{b\neq 0}} &&= b\eta(s) + \sum_{j=1}^{m} c_j \phi_j(s). \end{aligned}$$

We now can state the following.

Proposition 2.2. *For distinct nodes $x_1, \ldots, x_m \in \mathbb{C}$, closed under complex conjugation, let $\eta(s)$ and $\phi_1(s), \ldots, \phi_m(s)$ be as in Eqs.* (2.1) *and* (2.2)*, respectively. Let also $\gamma_1, \ldots, \gamma_m$ and $c_1, \ldots, c_m$ along with $b \in \mathbb{R}$, be all non-zero parameters as in Eq.* (2.4)*.*

The set of all real polynomials of degree, of at most m, with no roots at these nodes, can be parametrized by two families,

$$\begin{aligned} \tilde{d}_o(s) &= \sum_{j=1}^{m} \gamma_j \phi_j(s) & \deg\left(\tilde{d}_o(s)\right) &= m-1 \\ \tilde{d}_1(s) &= b\eta(s) + \sum_{j=1}^{m} c_j \phi_j(s) & \deg\left(\tilde{d}_1(s)\right) &= m. \end{aligned} \tag{2.5}$$

Up to this point we have mostly cited textbook material.

2.2. **Step 2: (Not necessarily positive real) interpolating functions with prescribed denominator.** With each of the denominators in Eq. (2.5), $\tilde{d}_o(s)$ and $\tilde{d}_1(s)$, we here match a numerator, denoted by $\nu_o(s)$ and $\nu_1(s)$, respectively, to obtain a rational (not necessarily positive real) interpolating function (from x_j to y_j).

Proposition 2.3. *Let the interpolation data in Eq.* (1.1) *be closed under complex conjugation*[3]*, where the nodes $x_1, \ldots, x_m$ are distinct and in $\mathbb{C}_l$. Let also the (non-zero) coefficients $\gamma_1, \ldots, \gamma_m$ and $c_1, \ldots, c_m$ be as in Eq.* (2.5)*.*

Construct the polynomials (where $\phi_j(s)$ as in Eq. (2.2)*)*

$$\nu_o(s) = \sum_{j=1}^{m} y_j \gamma_j \phi_j(s) \qquad \nu_1(s) = \sum_{j=1}^{m} y_j c_j \phi_j(s) \; . \tag{2.6}$$

[2]Although as before, $\begin{smallmatrix} \mathrm{Im}(x_j)>0 & 0\neq & \gamma_{j+1}=\gamma_j^* \\ \mathrm{Im}(x_j)=0 & 0\neq & \gamma_j\in\mathbb{R}. \end{smallmatrix}$

[3]Namely, if $\mathrm{Im}(x)_j > 0$ then $x_{j+1} = x_j^*$ and $y_{j+1} = y_j^*$.

Then, the rational functions,

$$\tilde{p}_o(s) := \frac{\nu_o(s)}{\tilde{d}_o(s)} = \frac{\sum_{j=1}^{m} y_j \gamma_j \phi_j(s)}{\sum_{j=1}^{m} \gamma_j \phi_j(s)} \qquad \tilde{p}_1(s) := \frac{\nu_1(s)}{\tilde{d}_1(s)} = \frac{\sum_{j=1}^{m} y_j c_j \phi_j(s)}{b\eta(s) + \sum_{j=1}^{m} c_j \phi_j(s)}, \tag{2.7}$$

(with $0 \neq b \in \mathbb{R}$) interpolate between x_j and y_j.

This result follows directly from the definition of $\phi_j(s)$ in Eq. (2.2).

We next construct additional interpolating rational functions of degree of at most m.

Lemma 2.4. *Let $\eta(s)$, $d_k(s)$, $\nu_k(s)$, and $p_k(s)$, (with $k = 0, 1$) from Eqs. (2.1), (2.5), (2.6), and (2.7) respectively.*
The set of all real rational functions of degree m, vanishing at the nodes is given by,

$$\begin{aligned} \tilde{\Delta}_o(s) &= \left(\frac{\tilde{d}_o(s)}{\eta(s)}\right)^{-1} = \left(\sum_{j=1}^{m} \frac{\gamma_j}{s - x_j}\right)^{-1} \\ \tilde{\Delta}_1(s) &= \left(\frac{\tilde{d}_1(s)}{\eta(s)}\right)^{-1} = \left(b + \sum_{j=1}^{m} \frac{c_j}{s - x_j}\right)^{-1} \end{aligned} \tag{2.8}$$

with $\gamma_1, \ldots, \gamma_m$, b, $c_1, \ldots, c_m$ all non-zero.

Using $\tilde{\Delta}_k(s)$, define the rational functions,

$$\begin{aligned} \tilde{f}_o(s) &:= \tilde{p}_o(s) + r_o \tilde{\Delta}_o(s) \\ &= \frac{\sum_{j=1}^{m} y_j \gamma_j \phi_j(s) + r_o \eta(s)}{\sum_{j=1}^{m} \gamma_j \phi_j(s)} \qquad r_o \in \mathbb{R} \text{ parameter,} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \tilde{f}_1(s) &:= \tilde{p}_1(s) + r_1 \tilde{\Delta}_1(s) \\ &= \frac{\sum_{j=1}^{m} y_j c_j \phi_j(s) + r_1 \eta(s)}{b\eta(s) + \sum_{j=1}^{m} c_j \phi_j(s)} \qquad r_1 \in \mathbb{R} \text{ parameter.} \end{aligned} \tag{2.10}$$

Then the following is true.

(i) *For arbitrary $r_o, r_1 \in \mathbb{R}$ the functions $\tilde{f}_o(s)$ and $\tilde{f}_1(s)$ in Eqs. (2.9) (2.10) are of degree of at most m.*

(ii) *For arbitrary $r_o, r_1 \in \mathbb{R}$ the functions $\tilde{f}_o(s)$ and $\tilde{f}_1(s)$ in Eqs. (2.9) (2.10) interpolate from $x_1, \ldots, x_m$ to $y_1, \ldots, y_m$.*

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Proof : For convenience, throughout the proof, we omit the dependence on $k = 0, 1$ and simply write $\tilde{\Delta}(s)$, $\nu(s)$, $d(s)$ and $f(s)$.

(i) Recall that by construction (see Eqs. (2.1), (2.5) and Proposition 2.2) the polynomials $\eta(s)$ and $d(s)$ are relatively prime. Recall also (see Proposition 2.3) that

$$m = \text{degree}\,(\eta) \geq \text{degree}\,(\nu) = m - 1.$$

We thus have the following,

$$\begin{array}{rcc} m & = & \text{degree}\,(\tilde{\Delta}) \\ & = & \text{degree}\,(\frac{\eta}{d}) \\ & = & \text{degree}\,(d) + \text{degree}\,(\eta) \\ & = & \text{degree}\,(d) + \max\,(\text{degree}\,(\eta)\,, \text{degree}\,(\nu)) \\ & \geq & \text{degree}\,(d) + \text{degree}\,(\nu + r\eta) \\ & \geq & \text{degree}\,(\frac{\nu + r\eta}{d}) \\ & = & \text{degree}\,(\tilde{f})\,. \end{array}$$

(ii) This is immediate from Proposition 2.3 along with the definitions of $\tilde{\Delta}(s)$ and of $\tilde{f}(s)$, see Eqs. (2.8), (2.9) and (2.10), respectively. $\square$

The rest of this section is depvoted to extracting out of the resulting interpolating functions in Eqs. (2.9) and (2.10) positive real functions. To this end we assume hereafter that nodes are in the open left half plane, i.e.

$$x_1, \ \ldots \ , \ x_m \in \mathbb{C}_l \ .$$

We next devise a scheme based on the structure of interpolating functions (see Corollary 2.6 below), of easily constructing large subsets of positive real interpolating functions. This simplicity, comes on the expense of guranteeing finding *all* positive real interpolating functions.

2.3. **Step 3: All positive real rational functions of degree m, with prescribed denominator, and vanishing at the nodes.** We first recall well-known facts, which are fundamental to our construction.

Theorem 2.5. *The following is true.*

(i) *For prescribed data, the family of interpolating rational functions is convex.*
(ii) *The set of rational positive real functions forms a convex cone.*

As a non-empty intersection of convex sets, is convex, one can conclude the following.

Corollary 2.6. *For prescribed data set, whenever not empty, the family of rational positive real interpolating functions, is convex.*

We now resort to the following, see e.g. [8, Section 4.3], [14, Definition 6.4].

Definition 2.7. A rational function $f(s)$ will be called *Strictly Positive Real* if $f(s-\epsilon)$ is positive real, for some $\epsilon > 0$.

The following well known properties will be useful in the sequel.

Theorem 2.8. *(i) If a rational function $f(s)$ is strictly positive real, then*

$$\mathrm{Re}\,(f(s)) > 0 \qquad \forall s \in \overline{\mathbb{C}}_r \ .$$

(ii) The set of positive real functions forms a Convex Invertible Cone.

Item (i) follows from Definition 2.7 and for item (ii) see [8, Proposition 4.1.1].

Whenever the functions in Eq. (2.8) are strictly positive real, the tilde will be omitted and they will be denoted by $\Delta_o(s)$ and $\Delta_1(s)$. This is addressed next.

Example 2.9. We next illustrate the fact that for any set of nodes in $\mathbb{C}_l$, the open left-half plane, one can choose the coefficients $\gamma_1, \ \ldots \ , \gamma_m$ and $c_1, \ \ldots \ , c_m$, so that in Eq. (2.8) one obtains *strictly positive real* functions, $\Delta_o(s)$ and $\Delta_1(s)$.

As the reasoning is identical, we show it only for $c_1, \ \ldots \ , c_m$.

Specifically, if $x_j \in \mathbb{R}_-$ then $\frac{c_j}{s-x_j}$ is strictly positive real for all $c_j \in \mathbb{R}_+$. If $x_j \in \{\mathbb{C}_l \smallsetminus \mathbb{R}_-\}$ (and from Eq. (2.3) $x_{j+1} = x_j^*$) then taking $c_{j+1} = c_j^*$ yields,

$$\frac{c_j}{s-x_j} + \frac{c_j^*}{s-x_j^*} = 2\mathrm{Re}(c_j)\left(s + \frac{\left(-\mathrm{Re}(x_j) + \frac{\mathrm{Im}(c_j)\mathrm{Im}(x_j)}{\mathrm{Re}(c_j)}\right)s + |x_j|^2}{s + \left(-\mathrm{Re}(x_j) - \frac{\mathrm{Im}(c_j)\mathrm{Im}(x_j)}{\mathrm{Re}(c_j)}\right)}\right)^{-1} . \tag{2.11}$$

Thus, choosing c_j so that,

$$\mathrm{Re}(c_j) > \frac{|\mathrm{Im}(c_j)\mathrm{Im}(x_j)|}{-\mathrm{Re}(x_j)} \geq 0,$$

is sufficient to guarantee that the function in Eq. (2.11) is strictly positive real. $\square$

Motivated by the above example we can state the following.

Proposition 2.10. *Out of the polynomials $\eta(s)$, $\tilde{d}_o(s)$ and $\tilde{d}_1(s)$ in Eqs.* (2.1), (2.5) *respectively, one can choose the coefficients $c_1 \ , \ \ldots \ , \ c_m$,* *b and* $\gamma_1\ ,\ \dots\ ,\ \gamma_m$ *to construct all monic*[4] *polynomials,*

$$\begin{array}{llll} d_o(s) = & \sum_{j=1}^m \gamma_j\phi_j(s) & \sum_{j=1}^m \gamma_j = 1 & \deg\left(d_o(s)\right) = \ m-1 \\ d_1(s) = & \eta(s) + \sum_{j=1}^m c_j\phi_j(s) & & \deg\left(d_1(s)\right) = \ m, \end{array} \tag{2.12}$$

so that the rational functions from Eq. (2.8) *are strictly positive real, i.e.*

$$\begin{array}{ll} \Delta_o(s) = \left(\frac{d_o(s)}{\eta(s)}\right)^{-1} = \left(\sum_{j=1}^m \frac{\gamma_j}{s-x_j}\right)^{-1} & \sum_{j=1}^m \gamma_j = 1 \\ \Delta_1(s) = \left(\frac{d_1(s)}{\eta(s)}\right)^{-1} = \left(1 + \sum_{j=1}^m \frac{c_j}{s-x_j}\right)^{-1}. & c_1\cdots c_m \neq 0. \end{array} \tag{2.13}$$

Moreover:
The coefficients of $d_o(s)$*:* $\gamma_1\ ,\ \dots\ ,\ \gamma_m$ *form a convex set (excluding* $\gamma_j = 0$*) within a hyperplane in* $\mathbb{C}^m$*.*
The coefficients of $d_1(s)$*:* $c_1,\ \dots\ ,\ c_m$ *form a convex subset of* $\mathbb{C}^m$ *(excluding* m *hyperplanes* $c_j = 0$*). This set is positively unbounded in the sense that if in Eqs.* (2.5), (2.13)

$$c_1,\ \dots\ ,\ c_m$$

is an admissible set of parameters, then so is[5]

$$c_1+\delta_1,\ \dots\ ,\ c_m+\delta_m \qquad \delta_j \geq 0 \quad j = 1,\ \dots\ ,\ m,$$

Furthermore, the sets $\gamma_1\ ,\ \dots\ ,\ \gamma_m$ *and* $c_1,\ \dots\ ,\ c_m$ *can be parameterized by a convex subset of* $\mathbb{R}^m$ *(excluding the axes).*

Proof From Example 2.9 it follows that for arbitrary set of nodes, this family of $\Delta(s)$ functions is not empty.

To simplify establishing structural properties, we begin by ignoring the condition that neither $\gamma_1,\ \dots\ ,\ \gamma_m$ nor $c_1,\ \dots\ ,\ c_m$ vanish.

By Corollary 2.6 the set of interpolating positive real rational functions is convex [6].

[4]As in Eqs. (2.9) (2.10) $\tilde{\Delta}(s)$ is scaled by r, without loss of generality one can take $d(s)$ to be moinc.

[5] provided that, to preserve complex conjugation, $c_{j+1} = c_j^*$ implies $\delta_{j+1} = \delta_j$.

[6]We actually exploit the *geometry* of the set of interpolating functions. *Toplogy* of interpolating functions is used in [10].

Indeed, if $c_1\ ,\ \ldots\ ,\ c_m$ and $\hat{c}_1\ ,\ \ldots\ ,\ \hat{c}_m$ are two admissible sets in Eqs. (2.5), (2.13) then so is $\theta c_1 + (1-\theta)\hat{c}_1\ ,\ \ldots\ ,\ \theta c_m + (1-\theta)\hat{c}_m$, for all $\theta \in [0,\ 1]$.

The fact that $\sum\limits_{j=1}^{m} \gamma_j = 1$ forms a hyper-plane in $\mathbb{C}^m$ is straightforward. Next, to show that the set $c_1,\ \ldots\ ,\ c_m$ is positively unbounded, one can resort again to the construction in Example 2.9.

Recall however that the restriction that $\prod\limits_{j=1}^{m} c_j \neq 0$, implies that the set of coefficients $c_1,\ \ldots\ ,\ c_m$ forms an almost convex cone, as it excludes m hyper-planes, $c_j = 0$.

Real coefficients: Recall that in (2.5) the coefficients c_j are real or come in complex conjugate pairs. Specifically if there are q coefficients in the upper half plane and $m-2q$ are real, they are described by a point in $\mathbb{R}^m$. $\square$

The above analysis suggests that in the coefficient space, it is enough to find the *boundary* of the (almost convex) sets of admissible $\gamma_1,\ \ldots\ ,\ \gamma_m$ and $c_1,\ \ldots\ ,\ c_m$.

In the next step, we combine Propsitions 2.3 and 2.10 to construct positive real interpolating functions of degree of at most m.

2.4. **Step 4: Positive real interpolating functions.** To extract positive real functions, out of the set of interpolating functions $\tilde{f}(s)$ in Eqs. (2.9) and (2.10), we focus on those whose deniminator is given by Theorem 2.10. This is formalized next.

Lemma 2.11. *Let* $\eta(s)$, $d_k(s)$, $\nu_k(s)$, $p_k(s)$ *and* $\Delta_k(s)$, *(with* $k = 0, 1$*) from Eqs.* (2.1), (2.5), (2.6), (2.7) *and* (2.8) *respectively.*
Define the rational functions,

$$
\begin{aligned}
f_o(s) \quad &:= \quad p_o(s) + r_o\Delta_o(s) = \frac{\nu_o(s)}{d_o(s)} + r_o\frac{\eta(s)}{d_o(s)} \\
&= \quad \frac{\sum\limits_{j=1}^{m} y_j\gamma_j\phi_j(s) + r_o\eta(s)}{\sum\limits_{j=1}^{m} \gamma_j\phi_j(s)} \qquad r_o \in \mathbb{R} \text{ parameter},
\end{aligned}
\tag{2.14}
$$

and

$$f_1(s) \quad := \quad p_1(s) + {\scriptstyle r_1 \Delta_1(s)} = \tfrac{\nu_1(s)}{d_1(s)} + {\scriptstyle r_1} \tfrac{\eta(s)}{d_1(s)}$$

$$(2.15) \qquad = \qquad \frac{\sum\limits_{j=1}^{m} y_j c_j \phi_j(s) + r_1 \eta(s)}{b\eta(s) + \sum\limits_{j=1}^{m} c_j \phi_j(s)} \qquad r_1 \in \mathbb{R} \ \text{parameter}.$$

Then, $f_k(s)$ *(and* $p_k(s)$*) are interpolating function with* $\Delta_k(s)$ *strictly positive real, vanishing at the nodes (all sharing the same denominator).*

Furthermore, the quantities,

$$-\inf_{s \in \mathbb{C}_r} \tfrac{\text{Re } p_o(s)}{\text{Re } \Delta_o(s)} \qquad\qquad -\inf_{s \in \mathbb{C}_r} \tfrac{\text{Re } p_1(s)}{\text{Re } \Delta_1(s)}$$

are well defined.

Proof : For convenience, throughout the proof, we omit the dependence on $k = 0, 1$ and simply write $\nu(s)$, $d(s)$ and $f(s)$.

The construction in Proposition 2.10 guarantees that in Eq. (2.13)

$$\Delta(s) = \frac{\eta(s)}{d(s)}$$

is *strictly positive real* and thus by item (i) of Theorem 2.8

$$\text{Re} \frac{\eta(s)}{d(s)} > 0 \qquad \forall s \in \mathbb{C}_r \ .$$

Recalling that (i) the numerator $\eta(s)$ vanishes only at m points in $\mathbb{C}_l$, see Eq. (2.1) and (ii) in addition degree(η) $\geq$ degree(d) see Eq. (2.5), in fact

$$\text{Re} \frac{\eta(s)}{d(s)} \geq \delta > 0 \qquad \forall s \in \mathbb{C}_r \ .$$

Next, exploiting again the fact that $\Delta(s)$ is strictly positive real, see Theorem 2.8, implies that $d(s)$ does not vanish in $\overline{\mathbb{C}}_r$. We can thus conclude that

$$\inf_{s \in \mathbb{C}_r} \frac{\text{Re } p(s)}{\text{Re } \Delta(s)}$$

is well defined, so the claim is established. $\square$

So far we have described interpolating rational functions $f(s)$ of degree of at most m whose denominator is so that $\Delta(s)$ is strictly positive real. To proceed with the construction, the idea is very simple, see Eq.

(1.5) :
With the same $\eta(s)$, $d(s)$ and $\nu(s)$ construct the rational functions

$$f_a(s) := \frac{\nu(s)}{d(s)} + r_a\frac{\eta(s)}{d(s)} \quad \& \quad f_b(s) := \frac{\nu(s)}{d(s)} + r_b\frac{\eta(s)}{d(s)}$$

where r_a and r_b are real parameters. On the one hand, from Lemma 2.11 it follows that $f_a(s)$ and $f_b(s)$ interpolate between with same data. On the other hand, Theorems 2.8 and 2.10 imply that,

$$r_a > r_b \quad \Longrightarrow \quad \mathrm{Re}\left(f_a(s)\right) > \mathrm{Re}\left(f_b(s)\right) \quad x_j \neq s \in \mathbb{C}_r \ .$$

Next, recall that by Eq. (1.2) $f(s)$ is positive real whenever,

$$\mathrm{Re}\left(f(s)\right) \geq 0 \qquad \forall s \in \mathbb{C}_r \ .$$

Thus, one can formally define $\forall s \in \mathbb{C}_r$,

$$\underline{r}_o := \arg \min_{r_o \in \mathbb{R}} \mathrm{Re}\left(f_o(s)\right) = \arg \min_{r_o \in \mathbb{R}} \mathrm{Re}\left(p_o(s) + r_o\Delta_o(s)\right) \geq 0$$

(2.16)

$$\underline{r}_1 := \arg \min_{r_1 \in \mathbb{R}} \mathrm{Re}\left(f_1(s)\right) = \arg \min_{r_1 \in \mathbb{R}} \mathrm{Re}\left(p_1(s) + r_1\Delta_1(s)\right) \geq 0.$$

We next combine the above definition of $\underline{r}$ along with Lemma 2.11.

Proposition 2.12. *Let the rational function* $f_k(s)$ *and the scalars* $\underline{r}_k$ *(with* $k = 0, 1$*) be as in Eqs.* (2.14), (2.15) *and* (2.16)*, respectively. Then,*

$$\underline{r}_o := -\inf_{s \in \mathbb{C}_r} \frac{\mathrm{Re}\ p_o(s)}{\mathrm{Re}\ \Delta_o(s)} \qquad \underline{r}_1 := -\inf_{s \in \mathbb{C}_r} \frac{\mathrm{Re}\ p_1(s)}{\mathrm{Re}\ \Delta_1(s)}$$

and $f_k(s)$ *is positive real if and only if* $r_k \geq \underline{r}_k$.

Proof : For simplicity, we omit both the dependence on s and the subscript k. Using Eqs. (2.14) (2.15) note that

$$\mathrm{Re}\left(f\right) = \mathrm{Re}\left(\frac{\nu}{d} + r\frac{\eta}{d}\right) = \mathrm{Re}\left(\frac{\nu}{d}\right) + r\mathrm{Re}\left(\frac{\eta}{d}\right).$$

Now, f is positive real if and only if

$$\mathrm{Re}\left(f\right) \geq 0 \qquad \forall s \in \mathbb{C}_r \ .$$

Namely,

$$r\mathrm{Re}\left(\frac{\eta}{d}\right) \geq -\mathrm{Re}\left(\frac{\nu}{d}\right) \qquad \forall s \in \mathbb{C}_r \ ,$$

in turn, using the fact that $\frac{\eta}{d}$ is strictly positive real, see Theorem 2.10, this means that

$$r \geq \frac{-\mathrm{Re}\ \frac{\nu}{d}}{\mathrm{Re}\ \frac{\eta}{d}} \qquad \forall s \in \mathbb{C}_r \ .$$

Hence, one can conclude that f in Eqs. (2.14), (2.15) is positive real, if and only if,

$$r \geq \sup_{s \in \mathbb{C}_r} \frac{-\mathrm{Re}\ \frac{\nu}{d}}{\mathrm{Re}\ \frac{\eta}{d}} = -\inf_{s \in \mathbb{C}_r} \frac{\mathrm{Re}\ \frac{\nu}{d}}{\mathrm{Re}\ \frac{\eta}{d}} \ ,$$

and by Eq. (2.16), the proof is complete. □

Noting that $\deg(\eta_k) = m - 1$, for $k = 0, 1$, while $\deg(\psi) = m$ together with the fact that Δ_k is strictly positive real, guarantees the following.

Observation 2.13. *In Proposition 2.12,*

$$\underline{r}_o \geq 0 \qquad \underline{r}_1 \geq 0.$$

Note that from Eqs. (2.14) and (2.15) it follows that for $r_k > 0$, with $k = 0, 1$ whenever there is no pole-zero cancelation, the degree of the numerator of $f_o(s)$ or of $f_1(s)$ is m. Thus, all positive real interpolating functions $f(s)$ we have constructed are of degree at most m, but *under the restriction that the degree of the numerator is greater or equal to the degree of the denominator.* We next address the complementary case where the degree of the denominator is greater or equal to the degree of the numerator.

The basic idea is the following. Taking the original data, if one considers a function, say $g(s)$, interpolating from $x_1, \ \ldots \ , \ x_m$ to[7] $\frac{1}{y_1} \ , \ \ldots \ , \ \frac{1}{y_m}$, then $\frac{1}{g(s)}$ solves the original problem, where we have relied on the fact that the inverse of a positive real function, is positive real, see item (ii) of Theorem 2.8.

We follow the previous steps (while adding hat to the respective functions)

Theorem 2.14. *Let the interpolation data*[8] *be as in Eq.* (1.1)*, the (non-zero) denominator coefficients* $\gamma_1, \ \ldots \ , \ \gamma_m$ *and* $c_1, \ \ldots \ , \ c_m$ *be as in Eq.* (2.5)*.*
Construct the rational functions

$$\hat{p}_o(s) = \frac{\sum\limits_{j=1}^{m} \frac{\gamma_j}{y_j} \phi_j(s)}{\sum\limits_{j=1}^{m} \gamma_j \phi_j(s)} \qquad \hat{p}_1(s) = \frac{\sum\limits_{j=1}^{m} \frac{\hat{c}_j}{y_j} \phi_j(s)}{\sum\limits_{j=1}^{m} c_j \phi_j(s)} \ ,$$

[7]Assuming $y_j \neq 0$
[8]Assuming $y_j \neq 0$.

(where $\phi_j(s)$ as in Eq. (2.2)) Then, for all $\hat{r}_o, \hat{r}_1 \in \mathbb{R}$, the rational functions,

$$\hat{f}_o(s) \;:=\; (\hat{p}_o(s) + \hat{r}_o\Delta_o(s))^{-1} \;=\; \left(\frac{\sum_{j=1}^{m}\frac{\gamma_j}{y_j}\phi_j(s)+\hat{r}_o\eta(s)}{\sum_{j=1}^{m}\gamma_j\phi_j(s)}\right)^{-1} \tag{2.17}$$

and

$$\hat{f}_1(s) \;:=\; (\hat{p}_1(s) + \hat{r}_1\Delta_1(s))^{-1} \;=\; \left(\frac{\sum_{j=1}^{m}\frac{c_j}{y_j}\phi_j(s)+\hat{r}_1\eta(s)}{\eta(s)+\sum_{j=1}^{m}c_j\phi_j(s)}\right)^{-1} \tag{2.18}$$

interpolate between x_j and $\frac{1}{y_j}$ with $j = 1, \ldots, m$.

Furthermore, the rational function $\hat{f}_k(s)$ (with $k = 0, 1$) in Eqs. (2.17), (2.18) is positive real if and only if $\hat{r}_k \geq \underline{\hat{r}}_k$ with

$$\underline{\hat{r}}_o := -\inf_{s\in\mathbb{C}_r}\frac{\mathrm{Re}\ \hat{p}_o(s)}{\mathrm{Re}\ \Delta_o(s)} \qquad\qquad \underline{\hat{r}}_1 := -\inf_{s\in\mathbb{C}_r}\frac{\mathrm{Re}\ \hat{p}_1(s)}{\mathrm{Re}\ \Delta_1(s)}\ .$$

We have shown that $\hat{f}_o(s)$ and $\hat{f}_1(s)$ are positive real interpolating functions of degree at most m, where the degree of the numerator is larger or equal to the degree of the denominator.

Similar to the previos reasoning, one can conclude the following.

Observation 2.15. *In Theorem 2.14,*

$$\underline{\hat{r}}_o \geq 0 \qquad\qquad \underline{\hat{r}}_1 \geq 0.$$

3. Examples and Concluding remarks

The above recipe is illustrated through simple examples.

A. We start by illustrating the role of $f_o(s)$ vs. $\hat{f}_o(s)$ in Eqs. (2.14) and (2.17), respectively to obtain interpolating functions having at $s = \infty$ either pole or zero.

(i) Find a minimal degree positive real function $f(s)$ mapping $x_1, \ldots, x_m \in \mathbb{C}_l$ to $y_1 = x_1, \ldots, y_m = x_m$. Clearly the sought solution is

$$f(s) = s.$$

We now follow the above recipe and substitute in Eq. (2.14)

$$\begin{aligned}
f_o(s) &= \frac{\sum_{j=1}^{m} y_j\gamma_j\phi_j(s) + r_o\eta(s)}{\sum_{j=1}^{m}\gamma_j\phi_j(s)} \\
&= \frac{\sum_{j=1}^{m} x_j\gamma_j\phi_j(s) + r_o\eta(s)}{\sum_{j=1}^{m}\gamma_j\phi_j(s)} && \text{for } y_j = x_j \\
&= \frac{\sum_{j=1}^{m} x_j\gamma_j\phi_j(s) + \frac{r_o}{m}\sum_{j=1}^{m}(s-x_j)\phi_j(s)}{\sum_{j=1}^{m}\gamma_j\phi_j(s)} \\
&= \frac{\sum_{j=1}^{m}\left(s + x_j(m\gamma_j - 1)\right)\phi_j(s)}{m\sum_{j=1}^{m}\gamma_j\phi_j(s)} && \text{for } r_o = 1 \\
&= s && \text{for } \gamma_j \equiv \tfrac{1}{m}\ .
\end{aligned}$$

(ii) Find a minimal degree positive real function $f(s)$ mapping $x_1, \ \ldots \ , \ x_m \in \mathbb{C}_l$ to $y_1 = \frac{1}{x_1}, \ \ldots \ , \ y_m = \frac{1}{x_m}$. Clearly the sought solution is

$$f(s) = \frac{1}{s}\ .$$

We now follow the above recipe and substitute in Eq. (2.17)

$$\begin{aligned}
\hat{f}_o(s) &= \left(\frac{\sum\limits_{j=1}^{m}\frac{\gamma_j}{y_j}\phi_j(s)+\hat{r}_o\eta(s)}{\sum\limits_{j=1}^{m}\gamma_j\phi_j(s)}\right)^{-1} \\
&= \left(\frac{\sum\limits_{j=1}^{m}x_j\gamma_j\phi_j(s)+\hat{r}_o\eta(s)}{\sum\limits_{j=1}^{m}\gamma_j\phi_j(s)}\right)^{-1} && \text{for } y_j=\tfrac{1}{x_j} \\
&= \left(\frac{\sum\limits_{j=1}^{m}\left(s+x_j(m\gamma_j-1)\right)\phi_j(s)}{m\sum\limits_{j=1}^{m}\gamma_j\phi_j(s)}\right)^{-1} && \text{for } \hat{r}_o=1 \\
&= \frac{1}{s} && \text{for } \gamma_j\equiv\tfrac{1}{m}\ .
\end{aligned}$$

B. Parametrize all positive real rational functions, of degree of at most two, so that

$$f(-1)=y_1 \qquad\qquad f(-3)=y_2\ ,$$

where $y_1, y_2 \in \mathbb{R}$ are arbitrary.

First for reference, a direct computation reveals that all rational functions, of degree of at most one, are given by

$$f(s)=\frac{(a(3y_2-y_1)+b(y_1-y_2))s+3a(y_2-y_1)+b(3y_1-y_2)}{2(as+b)}\ . \tag{3.1}$$

These functions are positive real whenever,

$$\begin{aligned}
a &\geq 0 \\
b &\geq 0 \\
a(3y_2-y_1)+b(y_1-y_2) &\geq 0 \\
3a(y_2-y_1)+b(3y_1-y_2) &\geq 0.
\end{aligned} \tag{3.2}$$

The conditions in Eq. (3.2) may be satisfied for all $y_1,\ y_2 \in \mathbb{R}$ *umless*, $0 > y_1 = y_2$.

This implies that for $y_1 = y_2 \geq 0$ there is a zero degree positive real interpolating function, see item (iii) below. For $0 > y_1 = y_2$, the positive real interpolating functions are of degree of at least two, see item (vi) below. In all other cases, there exist positive real interpolating functions of degree one and above.

We now follow the recipe from the previous section.
From Step 1

$$\eta(s) = (s+1)(s+3) = s^2 + 4s + 3$$

and

$$\phi_1(s) = s+3 \qquad \phi_2(s) = s+1.$$

From Step 2, and using Eq. (2.5) yields

$$d_o(s) = s + \gamma \qquad \gamma \in [0,\ 4] \setminus \{1, 3\},$$

and

$$d_1(s) = s^2 + s(4 + c_1 + c_2) + 3 + 3c_1 + c_2$$

where c_1 and c_2 are such that $1 + \frac{c_1}{s+1} + \frac{c_2}{s+3}$ is strictly positive real. For $d_1(s)$ the set of admissible parameters is convex and positively unbounded[9] (excluding the axes $c_1 = 0$ and $c_2 = 0$), it is given by

$$c_2 > \begin{cases} -3(c_1+1) & \frac{1}{8} \geq c_1 \\ -\frac{1}{3}(\sqrt{c_1} + 2\sqrt{2})^2 & c_1 \geq \frac{1}{8}\ . \end{cases} \qquad c_1 c_2 \neq 0. \tag{3.3}$$

From Step 3

$$\frac{\nu_o(s)}{d_o(s)} = \frac{y_1(\gamma-1)}{2} + \frac{y_2(3-\gamma)}{2} + \frac{(\gamma-3)(\gamma-1)(y_2-y_1)}{2(s+\gamma)} \qquad \gamma \in [0,\ 4] \setminus \{1,3\}$$

$$\frac{\nu_1(s)}{d_1(s)} = \frac{(c_1y_1+c_2y_2)s+3c_1y_1+c_2y_2}{s^2+s(4+c_1+c_2)+3+3c_1+c_2} \qquad c_1,\ c_2 \quad \text{from Eq. (3.3).}$$

Now from Step 4

$$f_o(s) = r_o(s+4-\gamma) + \tfrac{y_1(\gamma-1)}{2} + \tfrac{y_2(3-\gamma)}{2} + \left(r_o + \tfrac{y_2-y_1}{2}\right)\frac{(\gamma-3)(\gamma-1)}{s+\gamma} \tag{3.4}$$

with $\gamma \in [0,\ 4] \setminus \{1,3\}$, and with $c_1,\ c_2$ from Eq. (3.3),

$$f_1(s) = \frac{r_1 s^2 + (4r_1 + c_1y_1 + c_2y_2)s + 3r_1 + 3c_1y_1 + c_2y_2}{s^2 + (4+c_1+c_2)s + 3 + 3c_1 + c_2}\ . \tag{3.5}$$

One can verify that taking r_o, r_1 "sufficiently large" renders $f_o(s)$, $f_1(s)$ positive real.

Next, assuming $y_1y_2 \neq 0$ and $\gamma \in [0,\ 4] \setminus \{1,3\}$,

$$\hat{f}_o(s) = \frac{2y_1y_2(s+\gamma)}{2y_1y_2\hat{r}_o s^2 + (8y_1y_2\hat{r}_o + (3-\gamma)y_1 + (\gamma-1)y_2)s + 6y_1y_2\hat{r}_o + (3-\gamma)y_1 + 3(\gamma-1)y_2} \tag{3.6}$$

[9]From Proposition 2.10 it follows that in particular it contains the whole first quadrant of the $\{c_2, c_1\}$ plane

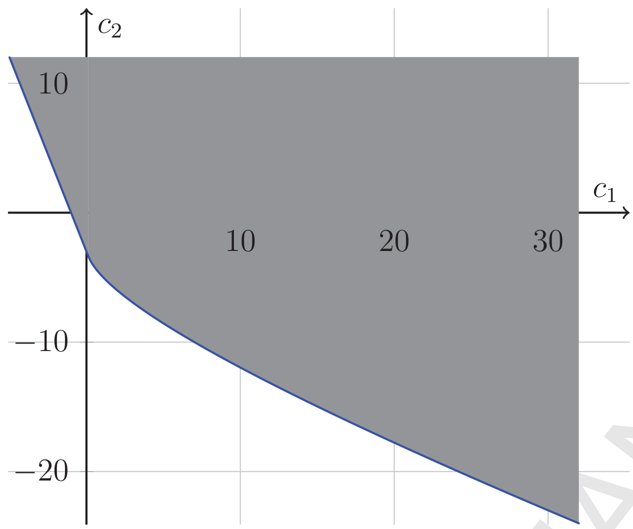

FIGURE 1. c_1, c_2 for $\frac{\eta}{d_1}$ strictly positive real, Eq. (3.3).

and with c_1, c_2 from Eq. (3.3),

$$\hat{f}_1(s) = \frac{s^2 + (4 + c_1 + c_2)s + 3 + 3c_1 + c_2}{\left(\frac{c_1}{y_1} + \frac{c_2}{y_2}\right)s + \frac{3c_1}{y_1} + \frac{c_2}{y_2} + \hat{r}_1(s+1)(s+3)} \tag{3.7}$$

Again, taking $\hat{r}_o$, $\hat{r}_1$ "sufficiently large" renders $\hat{f}_o(s)$, $\hat{f}_1(s)$ positive real.

Here are five particular cases.

(i) Recall that in the Introduction we pointed out that if $y_1, y_2 \in \mathbb{R}_-$, see Eq. (1.4), one can still try to resort to the classical Nevanlinna-Pick interpolation, seeking positive real odd fuctions so that

$$f(-1) = y_1 \qquad f(1) = -y_1 \qquad f(-3) = y_2 \qquad f(3) = -y_2 \ .$$

Now, the solvability condition in Eq. (1.3) reads,

$$\frac{y_2}{y_1} \in [\tfrac{1}{3}\ ,\ 3]\,, \tag{3.8}$$

and the resulting positive real odd interpolating functions (of degree of at most two) are

$$\begin{aligned} g_a(s) &= \tfrac{8y_1y_2s}{(y_2-3y_1)s^2+3(y_1-3y_2)} \\ g_b(s) &= \tfrac{(y_1-3y_2)s^2+3(y_2-3y_1)}{8s} . \end{aligned}$$

We now show, that these positive real odd functions, are special cases of the above recipe:
Indeed, assuming the condition in Eq. (3.8) is staisfied, from Eq. (3.5)

$$f_1(s)_{|_{c_1=\frac{4y_2}{3y_1-y_2} \quad c_2=\frac{12y_1}{y_2-3y_1} \quad r_1=0}} = g_a(s),$$

and from Eq. (3.7)

$$\hat{f}_1(s)_{|_{c_1=\frac{4y_1}{3y_2-y_1} \quad c_2=\frac{12y_2}{y_1-3y_2} \quad \hat{r}_1=0}} = g_b(s).$$

To further emphasize that our approach is different, in the four following special cases (ii), (iii) and (v), the condition in Eq. (3.8) is not satisfied, so the classical Nevanlinna-Pick interpolation is not applicable.

(ii) Take the special case where $y_1 = 1$ and $y_2 = 3$.
Clearly, $f(s) = -s$ is a real, anti-positive, minimal degree, interpolating function. We next seek minimal degree positive real interpolating functions.

Substituting these image points in $f_o(s)$ in Eq. (3.4) yields the following positive real interpolating functions,

$$f_o(s) = r_o s + (r_o + 1)\left(4 - \gamma + \frac{(\gamma-3)(\gamma-1)}{s+\gamma}\right) \qquad \gamma \in [0,\ 4] \smallsetminus \{1, 3\}.$$

To guarantee minimal degree, further substitute $\underline{r}_o = 0$, to obtain interpolating functions with zero at infinity,

$$f_o(s) = 4 - \gamma + \frac{(\gamma-3)(\gamma-1)}{s+\gamma} \qquad \gamma \in [0,\ 4] \smallsetminus \{1, 3\}.$$

Comparing with Eqs. (3.1) and (3.2) reveals that in this case our recipe yields all minimal degree (equals one) positive real interpolating functions.

Similarly for $\hat{f}_o(s)$ in Eq. (3.6)

$$\hat{f}_o(s) = \frac{s+\gamma}{\left(\frac{\gamma-1}{2} + \frac{3-\gamma}{6}\right)s + \frac{3(\gamma-1)}{2} + \frac{3-\gamma}{6} + \hat{r}_o(s+1)(s+3)}$$

with $\gamma \in [0,\ 4] \setminus \{1, 3\}$. As before, to single out interpolating functions of degree one, we focus on cases where $\hat{\underline{r}}_o = 0$. However, then to guarantee positive realness, the range of the parameter γ needs to be further restricted, i.e.

$$\hat{f}_o(s) = 3\left(\gamma + \frac{(3-\gamma)(\gamma-1)}{s+\gamma}\right)^{-1} \qquad \gamma \in [\tfrac{3}{4},\ 4] \setminus \{1, 3\}.$$

Here, at infinity, the interpolating function has neither pole nor zero.

Finally note that comparison with Eqs. (3.1) and (3.2) reveals that in this case, the recipe produced *all* interpolating functions of degree one.

(iii) Take the special case where $y_1 = y_2 \geq 0$.
One can substitute in Eq. (3.4) $\underline{r}_o = 0$ to obtain the minimal (=zero) degree interpolating function $f_o(s) \equiv y_1$.
Similarly, one can substitute in Eq. (3.6) $\hat{\underline{r}}_o = 0$ to obtainn the minimal (=zero) degree interpolating function $\hat{f}_o(s) \equiv y_1$.

(iv) Take the special case where $0 > y_1 = y_2$.
Recall that from Eqs. (3.1) and (3.2) we know that there are no positive real interpolating function of degree less than two.

To obtain interpolating functions use the recipe and substitute in Eq. (3.4) to obtain,

$$f_o(s) = y_1 + r_o\frac{(s+3)(s+1)}{s+\gamma} \qquad \underline{r}_o = \begin{cases} -y_1\frac{\gamma}{3} & \gamma\in(1,\ 3) \\ -y_1\frac{1}{4-\gamma} & \gamma\in\{[0,\ 1)\cup(3,\ 4)\}. \end{cases}$$

Note that $\underline{r}_o$ turns to be unbounded, as γ approaches 4.

(v) Take the special case where $y_1 = 2$, $y_2 = 0$.
As before, substituting these image points in Eqs. (3.1) and (3.2) (with $\frac{b}{a} = \gamma$) reveals that *all* minimal degree (equals one) positive real interpolating functions are of the form[10]

$$f(s) = (\gamma - 1)\frac{s+3}{s+\gamma} \qquad 3 \neq \gamma > 1. \tag{3.9}$$

Next, address the case where the interpolating function is so that the degree of the denominator is strictly larger then the the degree of the numerator. Now, recall that since the set of image points contains zero, Theorem 2.14 cannot be used. Nevertheless, all required interpolating functions are obtained.

We start with a straightforward considerations: Since at $x = -1$, the numerator is non-zero, but it vanishes at $x = -3$, it must be (at least) of

[10]Substituting in Eq. (3.4) $r_o = 0$, yields the subset of the interpolating functions in Eq. (3.9), where $4 \geq \gamma$

degree one. Thus, the denominator is (at least) of degree two. Indeed, to obtain all minimal degree interpolating functions of the required nature, substitute in Eq. (3.5)

$$f_1(s)_{|r_1=0} = \frac{2c_1(s+3)}{(s+1)(s+3)+c_1(s+3)+c_2(s+1)} \ ,$$

where adapting Eq.(3.3),

$$0 \neq c_2 > \begin{cases} -3(c_1+1) & c_1 \in (0, \frac{1}{8}] \\ \\ -\frac{1}{3}(\sqrt{c_1}+2\sqrt{2})^2 & c_1 \geq \frac{1}{8} \ . \end{cases}$$

C. In the previous item the interpolation nodes were real. We here illustrate the fact that the recipe is identical for the non-real case, assuming the interpolation nodes are closed under complex conjugation.

Assume that the interpolation nodes are $x_1 = -\gamma + i\delta$ and $x_2 = -\gamma - i\delta$ where $\gamma > 0$ and $0 \neq \delta \in \mathbb{R}$. Hence,

$$\eta(s) = (s+\gamma)^2 + \delta^2$$

We now construct the denominator polynomials.

Following Proposition 2.10 a degree one numerator polynomial $d_o(s)$ is given by the condition that the following rational function is strictly positive real,

$$\begin{aligned} \frac{\eta(s)}{d_o(s)} &= \left(\frac{\frac{1}{2}+i\beta}{s+\gamma+i\delta} + \frac{\frac{1}{2}+i\beta}{s+\gamma+i\delta} \right)^{-1} \\ &= \frac{(s+\gamma)^2+\delta^2}{s+\gamma+2\beta\delta} \\ &= s+\gamma-2\beta\delta + \frac{\delta^2(1+4\beta^2)}{s+\gamma+2\beta\delta} \ , \end{aligned}$$

namely,

$$\gamma > 2|\beta\delta|.$$

Hence one arrives at the following parametrization,

$$d_o(s) = s + 2\gamma(1-\theta) \qquad \theta \in [0, \ 1).$$

$\square$

Concluding remarks

1. As already pointed out in Corollary 2.6, for arbitrary prescribed data set in $\mathbb{C}$, the family of all positive real interpolating functions is convex (whenever not empty).

In contrast, the set of rational functions of a degree of at most m is a cone, but highly non-convex. In fact, the degree of a sum of two rational functions is higher than the degree of each of the summands, unless one of the denominators divides the other.

When the interpolation nodes are in $\mathbb{C}_l$, the open left half plane, we here introduce an easy-to-compute parametrization of positive real interpolating functions as a subset of $\mathbb{R}^{2m+3}$, see item 2 for details.

2. For arbitrary interpolating data set in Eq. (1.1), closed under complex conjugation, with nodes in $\mathbb{C}_l$, a large subset of positive real interpolating functions of degree of at most m may be conveniently parametrized a union of convex subsets within $\mathbb{R}^{2m+3}$.

Indeed the coefficients in Eq. (2.12) are so that $c_1, \ \ldots \ , \ c_m$ form a positively unbounded convex subset of $\mathbb{R}^m$, which in particular contains $\mathbb{R}^m_+$, excluding the axes (see e.g. Figure 1). Next, $\gamma_1, \ \ldots \ , \ \gamma_m$ form a hyper-plane in $\mathbb{R}^{m-1}$. Finally, each of the four parameters $\underline{r}_o$, $\underline{r}_1$, $\underline{\hat{r}}_o$, $\underline{\hat{r}}_1$, lies in $\overline{\mathbb{R}}_+$.

3. Step 4 of the recipe relies on the fact that positive real rational functions form a convex cone and that the set of interpolating functions is convex. Steps 3 and 4 rely on the fact that the set of positive real rational functions is closed under inversion.

4. The parametrization through $f_o(s)$, $f_1(s)$, $\hat{f}_o(s)$, $\hat{f}_1(s)$ is motivated by *simplicity*. It is neither minimal, as the same interpolation function may be obtained in more than one way, see e.g. Example B(iii), nor is it comprehensive, as some of the minimal degree interpolating functions may be missing, see e.g. Example B(v).

5. While the parametrization through $f_o(s)$, $f_1(s)$, $\hat{f}_o(s)$, $\hat{f}_1(s)$ is convenient, focusing on *minimal degree* interpolating functions involves "fine tuning" of the parameters $\gamma_1, \ \ldots \ , \ \gamma_m, \ \ c_1, \ \ldots \ , \ c_m, \ \ r_o, \ \ r_1,$ $\hat{r}_o$ and $\hat{r}_1$, see Examples A, B.

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