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Kolmogorov's axioms for probabilities with values in hyperbolic numbers

Daniel Alpay, M. Elena Luna–Elizarrarás and Michael Shapiro

Abstract. We introduce the notion of a probabilistic measure which takes values in hyperbolic numbers and which satisfies the system of axioms generalizing directly Kolmogorov's system of axioms. We show that this new measure verifies the usual properties of a probability; in particular, we treat the conditional hyperbolic probability and we prove the hyperbolic analogues of the multiplication theorem, of the law of total probability and of Bayes' theorem. Our probability may take values which are zero-divisors and we discuss carefully this peculiarity.

1. Introduction

The hyperbolic numbers (called also split-complex, or perplex, or double numbers, etc.) are known since long ago but they are not as popular as complex numbers or quaternions. At the same time they possess many interesting properties; in particular, the ring \mathbb{D} of hyperbolic numbers admits a partial order \preceq which has a good compatibility with the other algebraic structures of \mathbb{D} . Consider the inequality $0 \preceq x \preceq 1$; it turns out that it has a well-defined set of solutions in \mathbb{D} and one can think of them as of the probabilities of some random events.

This was a motivation of the present work: to test how this conjecture operates. First of all, we give a review of hyperbolic numbers making a special emphasize on the properties of non-negative hyperbolic numbers. Next, we introduce direct generalizations of Kolmogorov's axioms where a probabilistic measure takes values in hyperbolic numbers. It is followed by a series of the immediate properties of such probabilistic measures. The last Section 5 "Conditional probability" introduces this notion, including the case of probabilities which are zero-divisors in \mathbb{D} , and presents the hyperbolic generalizations of several classic facts: multiplication theorem, independence of random events, law of total probability, Bayes' theorem.

Altogether, we have shown that the basic facts (elementary but underlying) of the classic probability theory extend onto the situation under consideration. Thus one can expect that the whole building can be constructed as well.

Our approach can be seen in different ways. First of all, since we replace the range \mathbb{R} of probabilistic measures by a hypercomplex system \mathbb{D} then the approach can be interpreted as an attempt to consider probabilistic measures with non–numerical (in the sense of non–real) values. The most renowned research line in this direction is that of quantum probability but the two are rather distant from each other, see, e.g., [1], [8], [7].

On the other hand, the hyperbolic numbers can be seen as a real two–dimensional algebra with the underlying linear space \mathbb{R}^2 . Hence, the hyperbolic probability has the following interpretation: one deals with a stochastic experiment which generates the necessity to endow the σ –algebra of the events with two probabilistic measures which are seen as \mathbb{R}^2 –valued measures; what is more, a rich multiplicative structure is introduced on the range of such measures.

Such situations may emerge in mathematical statistics in testing composite hypotheses.

Another example is provided by thermodynamics and statistical physics. Consider a physical system which has two (or more) minima of free energy. If the system is in an equilibrium then it can be in any of these states with certain probabilities but it cannot be known for sure in which of them; this is exactly the situation we are interested in.

We believe that our approach will be useful in treating such situations although in the present work we limit ourselves with considering the basics of purely mathematical theory.

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2. A review of hyperbolic numbers

Information about hyperbolic numbers is dispersed in many sources. We concentrate in this section some basic facts which can be found in more details in [2], [6].

The ring of hyperbolic numbers is the commutative ring \mathbb{D} defined as

$$\mathbb{D} := \{a + b\mathbf{k} \mid a, b \in \mathbb{R}; \mathbf{k}^2 = 1, \mathbf{k} \notin \mathbb{R}\}.$$

There is a conjugation, the \dagger –conjugation, on hyperbolic numbers:

$$\mathfrak{z}^\dagger := a - b\mathbf{k}.$$

This \dagger –conjugation is an additive, involutive and multiplicative operation on \mathbb{D} :

1. $(\mathfrak{z} + \mathfrak{w})^\dagger = \mathfrak{z}^\dagger + \mathfrak{w}^\dagger$;
2. $(\mathfrak{z}^\dagger)^\dagger = \mathfrak{z}$;
3. $(\mathfrak{z}\mathfrak{w})^\dagger = \mathfrak{z}^\dagger\mathfrak{w}^\dagger$.

Note that given $\mathfrak{z} = a + b\mathbf{k} \in \mathbb{D}$, then

$$\mathfrak{z}\mathfrak{z}^\dagger = a^2 - b^2 \in \mathbb{R},$$

from which it follows that any hyperbolic number \mathfrak{z} with $\mathfrak{z}\mathfrak{z}^\dagger \neq 0$ is invertible, and its inverse is given by

$$\mathfrak{z}^{-1} = \frac{\mathfrak{z}^\dagger}{\mathfrak{z}\mathfrak{z}^\dagger}.$$

If, on the other hand, $\mathfrak{z} \neq 0$ but $\mathfrak{z}\mathfrak{z}^\dagger = a^2 - b^2 = 0$ then \mathfrak{z} is a zero-divisor. In fact there are no other zero-divisors. We denote the set of zero-divisors by $\mathfrak{S}_{\mathbb{D}}$, thus

$$\mathfrak{S}_{\mathbb{D}} := \{\mathfrak{z} = a + b\mathbf{k} \mid \mathfrak{z} \neq 0, \mathfrak{z}\mathfrak{z}^\dagger = a^2 - b^2 = 0\}.$$

It turns out that there are two very special zero-divisors in \mathbb{D} . Set

$$\mathbf{e} := \frac{1}{2}(1 + \mathbf{k}),$$

then its \dagger -conjugate is

$$\mathbf{e}^\dagger := \frac{1}{2}(1 - \mathbf{k}).$$

It is immediate to check that \mathbf{e} and \mathbf{e}^\dagger are zero-divisors, and they are mutually complementary idempotent elements. Thus, the two sets

$$\mathbb{D}_{\mathbf{e}} := \mathbf{e} \cdot \mathbb{D} \quad \text{and} \quad \mathbb{D}_{\mathbf{e}^\dagger} := \mathbf{e}^\dagger \cdot \mathbb{D}$$

are (principal) ideals in the ring \mathbb{D} and they have the properties:

$$\mathbb{D}_{\mathbf{e}} \cap \mathbb{D}_{\mathbf{e}^\dagger} = \{0\}$$

and

$$\mathbb{D} = \mathbb{D}_{\mathbf{e}} + \mathbb{D}_{\mathbf{e}^\dagger}. \tag{2.1}$$

Formula (2.1) is called the idempotent decomposition of \mathbb{D} . Every hyperbolic number $\mathfrak{z} = a + b\mathbf{k}$ can be written as

$$\mathfrak{z} = a + b\mathbf{k} = (a + b)\mathbf{e} + (a - b)\mathbf{e}^\dagger =: \nu_1\mathbf{e} + \nu_2\mathbf{e}^\dagger. \tag{2.2}$$

Formula (2.2) is called the idempotent representation of a hyperbolic number. It has a remarkable feature: the algebraic operations of addition, multiplication, taking of inverse, etc. can be realized component-wise.

Observe that the sets $\mathbb{D}_{\mathbf{e}}$ and $\mathbb{D}_{\mathbf{e}^\dagger}$ can be written as

$$\mathbb{D}_{\mathbf{e}} = \{r\mathbf{e} \mid r \in \mathbb{R}\} = \mathbb{R}\mathbf{e}; \quad \mathbb{D}_{\mathbf{e}^\dagger} = \{t\mathbf{e}^\dagger \mid t \in \mathbb{R}\} = \mathbb{R}\mathbf{e}^\dagger.$$

Remark 2.1. *It will be useful to have in mind the following properties:*

- (a) $\mathfrak{z} \in \mathbb{D}_{\mathbf{e}}$ if and only if $\mathfrak{z}\mathbf{e} = \mathfrak{z}$;
- (b) $\mathfrak{z} \in \mathbb{D}_{\mathbf{e}^\dagger}$ if and only if $\mathfrak{z}\mathbf{e}^\dagger = \mathfrak{z}$.

The set of non-negative hyperbolic numbers is

$$\mathbb{D}^+ := \{\nu_1 \mathbf{e} + \nu_2 \mathbf{e}^\dagger \mid \nu_1, \nu_2 \geq 0\}.$$

We will need two more sets:

$$\mathbb{D}_\mathbf{e}^+ := \mathbb{D}_\mathbf{e} \cap \mathbb{D}^+ \setminus \{0\} \quad \text{and} \quad \mathbb{D}_{\mathbf{e}^\dagger}^+ := \mathbb{D}_{\mathbf{e}^\dagger} \cap \mathbb{D}^+ \setminus \{0\}.$$

Let us now define on \mathbb{D} the next relation: given $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{D}$ we write $\mathfrak{z}_1 \preceq \mathfrak{z}_2$ whenever $\mathfrak{z}_2 - \mathfrak{z}_1 \in \mathbb{D}^+$; this relation is reflexive, transitive and antisymmetric and therefore it defines a partial order on \mathbb{D} . Also, if we take $\alpha, \beta \in \mathbb{R}$ then $\alpha \preceq \beta$ if and only if $\alpha \leq \beta$, thus \preceq is an extension of the total order \leq on \mathbb{R} .

The next properties of the order \preceq will be useful in subsequent computations (for more details see [2]). Let $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}, \mathfrak{w} \in \mathbb{D}$.

1. If $\mathfrak{x} \preceq \mathfrak{y}$ and $\mathfrak{z} \in \mathbb{D}^+$, then $\mathfrak{z}\mathfrak{x} \preceq \mathfrak{z}\mathfrak{y}$.
2. If $\mathfrak{x} \preceq \mathfrak{y}$ and $\mathfrak{z} \preceq \mathfrak{w}$, then $\mathfrak{x} + \mathfrak{z} \preceq \mathfrak{y} + \mathfrak{w}$.
3. If $\mathfrak{x} \preceq \mathfrak{y}$, then $-\mathfrak{y} \preceq -\mathfrak{x}$.

Thanks to the good properties of the partial order \preceq , one defines the hyperbolic-valued modulus on \mathbb{D} by

$$|\mathfrak{z}|_{\mathbf{k}} = |\nu_1 \mathbf{e} + \nu_2 \mathbf{e}^\dagger|_{\mathbf{k}} := |\nu_1| \mathbf{e} + |\nu_2| \mathbf{e}^\dagger \in \mathbb{D}^+, \quad (2.3)$$

where $|\nu_1|, |\nu_2|$ denote the usual modulus of real numbers. The subindex \mathbf{k} is used to emphasize that this modulus is linked to the hyperbolic numbers with the imaginary unit \mathbf{k} . Moreover, the name “hyperbolic-valued modulus” for (2.3) is justified by the following properties (see [2], [6]):

- (I) $|\mathfrak{z}|_{\mathbf{k}} = 0$ if and only if $\mathfrak{z} = 0$.
- (II) $|\mathfrak{w}\mathfrak{z}|_{\mathbf{k}} = |\mathfrak{w}|_{\mathbf{k}} \cdot |\mathfrak{z}|_{\mathbf{k}}$.
- (III) $|\mathfrak{w} + \mathfrak{z}|_{\mathbf{k}} \preceq |\mathfrak{w}|_{\mathbf{k}} + |\mathfrak{z}|_{\mathbf{k}}$ for any $\mathfrak{z}, \mathfrak{w} \in \mathbb{D}$.

In particular, one may talk about the supremum $\sup_{\mathbb{D}}$ of bounded sets in \mathbb{D} with respect to this hyperbolic-valued modulus. Indeed, let $A \subset \mathbb{D}$, if there exists $M \in \mathbb{D}^+$ such that $|x|_{\mathbf{k}} \preceq M$ for any $x \in A$, we say that A is a \mathbb{D} -bounded set. Introduce

$$\begin{aligned} A_1 &:= \{x \in \mathbb{R} \mid \exists y \in \mathbb{R}, x\mathbf{e} + y\mathbf{e}^\dagger \in A\}, \\ A_2 &:= \{y \in \mathbb{R} \mid \exists x \in \mathbb{R}, x\mathbf{e} + y\mathbf{e}^\dagger \in A\}; \end{aligned}$$

if A is a \mathbb{D} -bounded set then A_1 and A_2 are bounded, and the $\sup_{\mathbb{D}} A$ can be computed as

$$\sup_{\mathbb{D}} A = \sup A_1 \mathbf{e} + \sup A_2 \mathbf{e}^\dagger.$$

It is worth noting that some hyperbolic modules can be endowed with a hyperbolic-valued norm. These norms have the expected properties, that is, if a hyperbolic module W has a hyperbolic-valued norm $\|\cdot\|_{\mathbb{D}}$, then the latte satisfies:

1. $\|x\|_{\mathbb{D}} \succ 0$ for all $x \in W$ and $\|x\|_{\mathbb{D}} = 0$ if and only if $x = 0 \in W$.
2. $\|\mathfrak{z}x\|_{\mathbb{D}} = |\mathfrak{z}|_{\mathbf{k}} \|x\|_{\mathbb{D}}$ for all $\mathfrak{z} \in \mathbb{D}$ and for all $x \in W$.
3. $\|x + w\|_{\mathbb{D}} \preceq \|x\|_{\mathbb{D}} + \|w\|_{\mathbb{D}}$ for all $x, w \in W$.

The strength of hyperbolic-valued norms defined on hyperbolic modules has been exploited in [2] and [3]. In the latter a version of Hahn–Banach Theorem for hyperbolic modules has been proved.

3. \mathbb{D} -valued probability

Definition 3.1. Let (Ω, Σ) be a measurable space, a function

$$P_{\mathbb{D}} : A \in \Sigma \mapsto P_{\mathbb{D}}(A) \in \mathbb{D}$$

with the properties:

(I) $P_{\mathbb{D}}(A) \succeq 0 \quad \forall A \in \Sigma;$

(II) $P_{\mathbb{D}}(\Omega) = \mathfrak{p}$, where \mathfrak{p} takes one of the three possible values $1, \mathbf{e}, \mathbf{e}^\dagger;$

(III) given a sequence $\{A_n\} \subset \Sigma$ of pairwise disjoint events, then

$$P_{\mathbb{D}} \left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} P_{\mathbb{D}}(A_n),$$

is called a \mathbb{D} -valued probabilistic measure, or a \mathbb{D} -valued probability, on the σ -algebra of events Σ . The triplet $(\Omega, \Sigma, P_{\mathbb{D}})$ is called a \mathbb{D} -probabilistic space.

Every \mathbb{D} -valued probabilistic measure can be written as

$$P_{\mathbb{D}}(A) = p_1(A) + p_2(A)\mathbf{k} = P_1(A)\mathbf{e} + P_2(A)\mathbf{e}^\dagger \quad (3.1)$$

with $P_1(A) = p_1(A) + p_2(A); P_2(A) = p_1(A) - p_2(A)$. The property (I) of $P_{\mathbb{D}}$ implies that

$$P_1(A) \geq 0, \quad P_2(A) \geq 0 \quad \forall A \in \Sigma.$$

The property (II) gives:

$$P_{\mathbb{D}}(\Omega) = \mathfrak{p} = P_1(\Omega)\mathbf{e} + P_2(\Omega)\mathbf{e}^\dagger,$$

that is:

(1) If $\mathfrak{p} = 1$ then $P_1(\Omega) = 1, P_2(\Omega) = 1$.

(2) If $\mathfrak{p} = \mathbf{e}$ then $P_1(\Omega) = 1, P_2(\Omega) = 0$.

(3) If $\mathfrak{p} = \mathbf{e}^\dagger$ then $P_1(\Omega) = 0, P_2(\Omega) = 1$.

The property (III) leads to

$$P_i \left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} P_i(A_n) \quad \text{for } i = 1 \text{ and } 2.$$

Hence, to define a \mathbb{D} -valued probabilistic measure is equivalent to consider, on the same measurable space, a pair of unrelated, in general, usual \mathbb{R} -valued measures. In the case (1) both P_1 and P_2 are probabilistic measures; in case

(2) P_1 is a probabilistic measure and P_2 is a trivial one; in case (3) P_1 is a trivial measure and P_2 is a probabilistic measure. The cases (2) and (3) can be seen as two options of embedding the classic real–valued probabilistic measures into our new concept of \mathbb{D} –valued probabilistic measures: we identify such real–valued measures with \mathbb{D} –probabilistic measures which takes as its values only zero–divisors.

4. Properties of \mathbb{D} –valued probabilistic measures

- (I) Given $A \in \Sigma$, then $P_{\mathbb{D}}(A) + P_{\mathbb{D}}(A^C) = \mathfrak{p}$ where $A^C \in \Sigma$ is the complement of A .

Proof. $A \cup A^C = \Omega$, $A \cap A^C = \emptyset$, hence

$$P_{\mathbb{D}}(A) + P_{\mathbb{D}}(A^C) = P_{\mathbb{D}}(\Omega) = \mathfrak{p}.$$

□

- (II) $P_{\mathbb{D}}(\emptyset) = 0$.

Proof. $P_{\mathbb{D}}(\emptyset) = P_{\mathbb{D}}(\Omega^C) = \mathfrak{p} - P_{\mathbb{D}}(\Omega) = 0$.

□

- (III) If $A, B \in \Sigma$ with $A \subset B$ then $P_{\mathbb{D}}(A)$ and $P_{\mathbb{D}}(B)$ are comparable with respect to the partial order \preceq , what is more,

$$P_{\mathbb{D}}(A) \preceq P_{\mathbb{D}}(B).$$

Proof.

$$\begin{aligned} B &= B \cap \Omega = B \cap (A \cup A^C) \\ &= (B \cap A) \cup (B \cap A^C) = A \cup (A^C \cap B), \end{aligned}$$

where $A \cap (A^C \cap B) = \emptyset$. Hence $P_{\mathbb{D}}(B) = P_{\mathbb{D}}(A) + P_{\mathbb{D}}(A^C \cap B)$, and since $P_{\mathbb{D}}(A^C \cap B) \succeq 0$ we can add $P_{\mathbb{D}}(A)$ to both sides, proving with this the statement. □

Corollary 4.1. *The \mathbb{D} –probability of any event is comparable with \mathfrak{p} and is \mathbb{D} –less or equal to \mathfrak{p} .*

Indeed, it is always true that given $A \in \Sigma$, $A \subseteq \Omega$, hence $P_{\mathbb{D}}(A)$ is comparable with $P_{\mathbb{D}}(\Omega)$; what is more, $P_{\mathbb{D}}(A) \preceq P_{\mathbb{D}}(\Omega) = \mathfrak{p}$.

Corollary 4.2. *If $P_{\mathbb{D}}(\Omega) = \mathbf{e}$ then for any random event A there holds that $P_{\mathbb{D}}(A)$ is of the form $\lambda \mathbf{e}$ with $\lambda \in [0, 1]$. If $P_{\mathbb{D}}(\Omega) = \mathbf{e}^\dagger$ then for any random event A there holds that $P_{\mathbb{D}}(A)$ is of the form $\mu \mathbf{e}^\dagger$ with $\mu \in [0, 1]$.*

(IV) The addition theorem. Given a collection of events A_1, \dots, A_n , there holds:

$$P_{\mathbb{D}}\left(\bigcup_{i=1}^n A_n\right) = \sum_{i=1}^n P_{\mathbb{D}}(A_i) - \sum_{1 \leq i < j \leq n} P_{\mathbb{D}}(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} P_{\mathbb{D}}(A_i \cap A_j \cap A_k) + \dots + (-1)^{n-1} P_{\mathbb{D}}(A_1 \cap \dots \cap A_n).$$

Proof. By induction. For $n = 2$ since $A_1 \cup A_2 = A_1 \cup (A_1^C \cap A_2)$ and also $A_2 = (A_1 \cap A_2) \cup (A_1^C \cap A_2)$, then

$$\begin{aligned} P_{\mathbb{D}}(A_1 \cup A_2) &= P_{\mathbb{D}}(A_1) + P_{\mathbb{D}}(A_1^C \cap A_2) \\ &= P_{\mathbb{D}}(A_1) + P_{\mathbb{D}}(A_2) - P_{\mathbb{D}}(A_1 \cap A_2). \end{aligned}$$

□

(V) Given two events A and B , $P_{\mathbb{D}}(A \cup B)$ is comparable with $P_{\mathbb{D}}(A) + P_{\mathbb{D}}(B)$ and

$$P_{\mathbb{D}}(A \cup B) \preceq P_{\mathbb{D}}(A) + P_{\mathbb{D}}(B).$$

More generally, given events A_1, \dots, A_n there follows:

$$P_{\mathbb{D}}\left(\bigcup_{i=1}^n A_i\right) \preceq \sum_{i=1}^n P_{\mathbb{D}}(A_i).$$

(VI) Theorem of continuity of the \mathbb{D} -probability.

If $A_1 \supset A_2 \supset \dots \supset A_n \supset \dots$ and $A := A_1 \cap A_2 \cap \dots \cap A_n \cap \dots$ then

$$\lim_{n \rightarrow \infty} P_{\mathbb{D}}(A_n) = P_{\mathbb{D}}(A) = P_{\mathbb{D}}\left(\bigcap_{n=1}^{\infty} A_i\right).$$

Proof.

$$A_n = A \cup \left(\bigcup_{k=n}^{\infty} A_k \cap A_{k+1}^C\right)$$

and the summands are pairwise disjoint, hence

$$P_{\mathbb{D}}(A_n) = P_{\mathbb{D}}(A) + \sum_{k=n}^{\infty} P_{\mathbb{D}}(A_k \cap A_{k+1}^C).$$

The series here converges for any n , in particular, for $n = 1$, hence

$$P_{\mathbb{D}}(A_1) = P_{\mathbb{D}}(A) + \sum_{k=1}^{\infty} P_{\mathbb{D}}(A_k \cap A_{k+1}^C),$$

thus, the following sums go to zero:

$$\lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} P_{\mathbb{D}}(A_k \cap A_{k+1}^C) = 0.$$

Finally, $\lim_{n \rightarrow \infty} P_{\mathbb{D}}(A_n) = P_{\mathbb{D}}(A)$. \square

Note that the convergence here is considered with respect to the hyperbolic-valued modulus $|\cdot|_{\mathbf{k}}$, see again [2] for the details.

Corollary 4.3. *If $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ and $A := \bigcup_{n=1}^{\infty} A_n$, then*

$$\lim_{n \rightarrow \infty} P_{\mathbb{D}}(A_n) = P_{\mathbb{D}}(A) = P_{\mathbb{D}}\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Proof. Take $B_n := A_n^C$ and use property (VI). \square

5. Conditional probability

Definition 5.1. *Let $(\Omega, \Sigma, P_{\mathbb{D}})$ be a probabilistic space, let A and B be two events. The conditional probability $P_{\mathbb{D}}(A|B)$ of the event A under the condition that B has happened is defined as:*

$$(1) \quad P_{\mathbb{D}}(A|B) := \frac{P_{\mathbb{D}}(A \cap B)}{P_{\mathbb{D}}(B)} \text{ if } P_{\mathbb{D}}(B) \succ 0 \text{ and } P_{\mathbb{D}}(B) \notin \mathfrak{S}_{\mathbb{D}};$$

$$(2) \quad P_{\mathbb{D}}(A|B) := P_{\mathbb{D}}(A) \text{ if } P_{\mathbb{D}}(B) = 0;$$

$$(3) \quad P_{\mathbb{D}}(A|B) := \frac{P_{\mathbb{D}}(A \cap B)}{\lambda_1} \mathbf{e} + P_{\mathbb{D}}(A) \mathbf{e}^{\dagger} \text{ if } P_{\mathbb{D}}(B) = \lambda_1 \mathbf{e}, \lambda_1 > 0;$$

$$(4) \quad P_{\mathbb{D}}(A|B) := P_{\mathbb{D}}(A) \mathbf{e} + \frac{P_{\mathbb{D}}(A \cap B)}{\lambda_2} \mathbf{e}^{\dagger} \text{ if } P_{\mathbb{D}}(B) = \lambda_2 \mathbf{e}^{\dagger}, \lambda_2 > 0.$$

Let us show that items (3) and (4) are in a complete agreement with (1). Indeed, using (3.1) write for any event A : $P_{\mathbb{D}}(A) = P_1(A) \mathbf{e} + P_2(A) \mathbf{e}^{\dagger}$. Hence, item (1) in idempotent representation reads:

$$P_{\mathbb{D}}(A|B) = \frac{P_1(A \cap B)}{P_1(B)} \mathbf{e} + \frac{P_2(A \cap B)}{P_2(B)} \mathbf{e}^{\dagger} = P_1(A|B) \mathbf{e} + P_2(A|B) \mathbf{e}^{\dagger},$$

meanwhile items (3) and (4) read:

$$\begin{aligned} P_{\mathbb{D}}(A|B) &= \frac{P_{\mathbb{D}}(A \cap B)}{\lambda_1} \mathbf{e} + P_{\mathbb{D}}(A) \mathbf{e}^{\dagger} \\ &= \frac{P_1(A \cap B) \mathbf{e} + P_2(A \cap B) \mathbf{e}^{\dagger}}{P_1(B)} \mathbf{e} + (P_1(A) \mathbf{e} + P_2(A) \mathbf{e}^{\dagger}) \mathbf{e}^{\dagger} \\ &= \frac{P_1(A \cap B)}{P_1(B)} \mathbf{e} + P_2(A) \mathbf{e}^{\dagger} \\ &= P_1(A|B) \mathbf{e} + P_2(A|B) \mathbf{e}^{\dagger} \end{aligned}$$

and

$$\begin{aligned}
 P_{\mathbb{D}}(A|B) &= P_{\mathbb{D}}(A)\mathbf{e} + \frac{P_{\mathbb{D}}(A \cap B)}{\lambda_2}\mathbf{e}^\dagger \\
 &= (P_1(A)\mathbf{e} + P_2(A)\mathbf{e}^\dagger)\mathbf{e} + \frac{P_1(A \cap B)\mathbf{e} + P_2(A \cap B)\mathbf{e}^\dagger}{\lambda_2}\mathbf{e}^\dagger \\
 &= P_1(A)\mathbf{e} + \frac{P_2(A \cap B)}{P_2(B)}\mathbf{e}^\dagger = P_1(A|B)\mathbf{e} + P_2(A|B)\mathbf{e}^\dagger.
 \end{aligned}$$

Thus, we see a complete compatibility of the formula in item (1) and of its analogues in items (3) and (4).

Let us show that for a fixed B , with $P_{\mathbb{D}}(B) \neq 0$, the conditional probability verifies all the axioms of the \mathbb{D} -probability, that is, it defines a new \mathbb{D} -probabilistic measure on the measurable space (B, Σ_B) where Σ_B is the σ -algebra of the sets of the form $A \cap B$ with $A \in \Sigma$.

Indeed, clearly $P_{\mathbb{D}}(A|B) \succeq 0$. Next, let's see that $P_{\mathbb{D}}(B|B) = \mathbf{p}$. Indeed:

(1) If $P_{\mathbb{D}}(B) \notin \mathfrak{S}_{\mathbb{D}}$, then

$$P_{\mathbb{D}}(B|B) = \frac{P_{\mathbb{D}}(B \cap B)}{P_{\mathbb{D}}(B)} = \frac{P_{\mathbb{D}}(B)}{P_{\mathbb{D}}(B)} = 1.$$

(2) If $P_{\mathbb{D}}(B) = \lambda_1\mathbf{e}$, then

$$P_{\mathbb{D}}(B|B) = \frac{P_{\mathbb{D}}(B \cap B)}{\lambda_1}\mathbf{e} + P_{\mathbb{D}}(B)\mathbf{e}^\dagger = \frac{P_{\mathbb{D}}(B)}{\lambda_1}\mathbf{e} = \mathbf{e}.$$

(3) If $P_{\mathbb{D}}(B) = \lambda_2\mathbf{e}^\dagger$, then

$$P_{\mathbb{D}}(B|B) = P_{\mathbb{D}}(B)\mathbf{e} + \frac{P_{\mathbb{D}}(B \cap B)}{\lambda_2}\mathbf{e}^\dagger = \frac{P_{\mathbb{D}}(B)}{\lambda_2}\mathbf{e}^\dagger = \mathbf{e}^\dagger.$$

Finally, if $A = \bigcup_{k=1}^{\infty} A_k$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ then:

1. If $P_{\mathbb{D}}(B) \notin \mathfrak{S}_{\mathbb{D}}$, then

$$\begin{aligned}
 P_{\mathbb{D}}(A|B) &= \frac{P_{\mathbb{D}}(A \cap B)}{P_{\mathbb{D}}(B)} = \frac{P_{\mathbb{D}}(\bigcup_{k=1}^{\infty} A_k \cap B)}{P_{\mathbb{D}}(B)} \\
 &= \frac{\sum_{k=1}^{\infty} P_{\mathbb{D}}(A_k \cap B)}{P_{\mathbb{D}}(B)} = \sum_{k=1}^{\infty} \frac{P_{\mathbb{D}}(A_k \cap B)}{P_{\mathbb{D}}(B)} \\
 &= \sum_{k=1}^{\infty} P_{\mathbb{D}}(A_k|B).
 \end{aligned}$$

2. If $P_{\mathbb{D}}(B) = \lambda_1 \mathbf{e} \in \mathfrak{S}_{\mathbb{D}}$, since $A_k \cap B \subset B$ for any k and since $A \cap B \subset B$, write $P_{\mathbb{D}}(A_k \cap B) = \nu_k \mathbf{e}$ and

$$\begin{aligned} P_{\mathbb{D}}(A \cap B) &= \nu \mathbf{e} = P_1(A \cap B) \mathbf{e} = P_1 \left(\bigcup_{k=1}^{\infty} A_k \cap B \right) \mathbf{e} \\ &= \sum_{n=1}^{\infty} P_1(A_n \cap B) \mathbf{e} = \sum_{n=1}^{\infty} \nu_n \mathbf{e}, \end{aligned}$$

hence:

$$\begin{aligned} P_{\mathbb{D}}(A|B) &= \frac{P_{\mathbb{D}}(A \cap B)}{\lambda_1} \mathbf{e} + P_{\mathbb{D}}(A) \mathbf{e}^{\dagger} = \frac{\nu}{\lambda_1} \mathbf{e} + P_2(A) \mathbf{e}^{\dagger} \\ &= \frac{1}{\lambda_1} \sum_{k=1}^{\infty} \nu_k \mathbf{e} + \sum_{k=1}^{\infty} P_2(A_k) \mathbf{e}^{\dagger} \\ &= \sum_{k=1}^{\infty} \left(\frac{\nu_k}{\lambda_1} \mathbf{e} + P_2(A_k) \mathbf{e}^{\dagger} \right) \\ &= \sum_{k=1}^{\infty} (P_1(A_k|B) \mathbf{e} + P_2(A_k) \mathbf{e}^{\dagger}) \\ &= \sum_{k=1}^{\infty} P_{\mathbb{D}}(A_k|B). \end{aligned}$$

3. Similarly if $P_{\mathbb{D}}(B) = \lambda_2 \mathbf{e}^{\dagger} \in \mathfrak{S}_{\mathbb{D}}$.

Hence, $(B, \Sigma_B, P_{\mathbb{D}}(\cdot|B))$ is a new probabilistic space.

Theorem 5.2. (*Multiplication Theorem*) *Let $(\Omega, \Sigma, P_{\mathbb{D}})$ be a probabilistic space; let A and B be two events. Then*

$$P_{\mathbb{D}}(A \cap B) = P_{\mathbb{D}}(B) P_{\mathbb{D}}(A|B). \quad (5.1)$$

Proof. It is necessary to consider the different cases that arise.

- (a) If $P_{\mathbb{D}}(B) \succ 0$ and $P_{\mathbb{D}}(B) \notin \mathfrak{S}_{\mathbb{D},0}$, then we know that

$$P_{\mathbb{D}}(A|B) = \frac{P_{\mathbb{D}}(A \cap B)}{P_{\mathbb{D}}(B)},$$

hence (5.1) follows.

- (b) If $P_{\mathbb{D}}(B) = 0$, since $A \cap B \subset B$ then $P_{\mathbb{D}}(A \cap B) = 0$ implying (5.1).

- (c) If $P_{\mathbb{D}}(B) = \lambda_1 \mathbf{e}$ with $\lambda_1 > 0$ then

$$P_{\mathbb{D}}(A|B) = \frac{P_{\mathbb{D}}(A \cap B)}{\lambda_1} \mathbf{e} + P_{\mathbb{D}}(A) \mathbf{e}^{\dagger},$$

hence

$$P_{\mathbb{D}}(B)P_{\mathbb{D}}(A|B) = \lambda_1 P_{\mathbb{D}}(A|B)\mathbf{e} = P_{\mathbb{D}}(A \cap B)\mathbf{e}. \quad (5.2)$$

Since $A \cap B \subset B$, then

$$P_{\mathbb{D}}(A \cap B) = P_1(A \cap B)\mathbf{e} + 0\mathbf{e}^\dagger = P_1(A \cap B)\mathbf{e}.$$

Rewriting (5.2) one gets:

$$P_{\mathbb{D}}(B)P_{\mathbb{D}}(A|B) = P_1(A \cap B)\mathbf{e} = P_{\mathbb{D}}(A \cap B).$$

(d) If $P_{\mathbb{D}}(B) = \lambda_2 \mathbf{e}^\dagger$ with $\lambda_2 > 0$, one proceeds as in (c).

□

This theorem has a generalization for n random events.

Theorem 5.3. (*Generalized multiplication theorem.*) *Let A_1, \dots, A_n be random events. If they satisfy any of the following conditions:*

- (1) $P_{\mathbb{D}}(A_1 \cap \dots \cap A_n)$ is not a zero-divisor.
- (2) (a) *There exists $k_0 \in \{1, \dots, n\}$ such that $P_{\mathbb{D}}(A_{k_0}) = \mu_{k_0} \mathbf{e}$, with $\mu_{k_0} > 0$, i.e., $P_{\mathbb{D}}(A_{k_0})$ is a zero-divisor in $\mathbb{D}_{\mathbf{e}^\dagger}^+$,*

and

$$(b) P_{\mathbb{D}}\left(\bigcap_{\ell=1}^n A_\ell\right) \text{ belongs to } \mathbb{D}_{\mathbf{e}^\dagger}^+ \text{ also.}$$

- (3) (a) *There exists $k_0 \in \{1, \dots, n\}$ such that $P_{\mathbb{D}}(A_{k_0}) = \lambda_{k_0} \mathbf{e}^\dagger$, with $\lambda_{k_0} > 0$, i.e., $P_{\mathbb{D}}(A_{k_0})$ is a zero-divisor in $\mathbb{D}_{\mathbf{e}^\dagger}^+$,*

and

$$(b) P_{\mathbb{D}}\left(\bigcap_{\ell=1}^n A_\ell\right) \text{ belongs to } \mathbb{D}_{\mathbf{e}^\dagger}^+ \text{ also;}$$

then

$$P_{\mathbb{D}}(A_1 \cap \dots \cap A_n) = P_{\mathbb{D}}(A_1)P_{\mathbb{D}}(A_2|A_1) \cdots P_{\mathbb{D}}(A_n|A_1 \cap \dots \cap A_{n-1}). \quad (5.3)$$

Proof. Assume that (1) occurs. Since

$$\bigcap_{i=1}^{n-1} A_i \subset \bigcap_{i=1}^{n-2} A_i \subset \dots \subset A_1 \quad (5.4)$$

the hypothesis $P_{\mathbb{D}}\left(\bigcap_{\ell=1}^n A_\ell\right) = P_1\left(\bigcap_{\ell=1}^n A_\ell\right)\mathbf{e} + P_2\left(\bigcap_{\ell=1}^n A_\ell\right)\mathbf{e}^\dagger \notin \mathfrak{S}_{\mathbb{D},0}$ is

equivalent to say that $P_1\left(\bigcap_{\ell=1}^n A_\ell\right) > 0$ and $P_2\left(\bigcap_{\ell=1}^n A_\ell\right) > 0$, and it implies

that $P_{\mathbb{D}}(A_\ell) \notin \mathfrak{S}_{\mathbb{D},0}$ for all $\ell \in \{1, \dots, n\}$ and also that $P_{\mathbb{D}}\left(\bigcap_{i=1}^{n-k} A_i\right)$ is a strictly positive hyperbolic number (that is positive and not zero-divisor)

for all $k \in \{1, \dots, n-1\}$ and thus all conditional probabilities of the form $P_{\mathbb{D}}\left(A_k \mid \bigcap_{i=1}^{k-1} A_i\right)$ for $k \in \{2, \dots, n\}$ are well-defined, implying that

$$\begin{aligned} P_{\mathbb{D}}\left(\bigcap_{\ell=1}^n A_{\ell}\right) &= P_1(A_1)P_1(A_2 \mid A_1)\cdots P_1\left(A_n \mid \bigcap_{\ell=1}^{n-1} A_{\ell}\right) \mathbf{e} + \\ &\quad + P_2(A_1)P_2(A_2 \mid A_1)\cdots P_2\left(A_n \mid \bigcap_{\ell=1}^{n-1} A_{\ell}\right) \mathbf{e}^{\dagger}, \end{aligned}$$

hence (5.3) follows.

Assuming now that (2) occurs, the hypothesis that $P_{\mathbb{D}}\left(\bigcap_{\ell=1}^n A_{\ell}\right)$ is a positive zero-divisor, let's say $P_{\mathbb{D}}\left(\bigcap_{\ell=1}^n A_{\ell}\right) = \mu \mathbf{e}$, $\mu > 0$, implies that $P_1\left(\bigcap_{\ell=1}^n A_{\ell}\right) = \mu > 0$.

On the other hand, assume that k_0 is the minimum integer in $\{1, 2, \dots, n\}$ such that $P_{\mathbb{D}}(A_{k_0})$ is a zero-divisor. This implies that $P_{\mathbb{D}}\left(\bigcap_{\ell=1}^k A_{\ell}\right) \in \mathbb{D}_{\mathbf{e}}$ for all $k \geq k_0$; writing in this case $P_{\mathbb{D}}\left(\bigcap_{\ell=1}^k A_{\ell}\right) = \nu_k \mathbf{e}$, one has:

$$P_{\mathbb{D}}\left(A_{k+1} \mid \bigcap_{\ell=1}^k A_{\ell}\right) = \frac{P_{\mathbb{D}}\left(\bigcap_{\ell=1}^{k+1} A_{\ell}\right)}{\nu_k} \mathbf{e} + P_{\mathbb{D}}(A_{k+1}) \mathbf{e}^{\dagger} \quad \text{for } k \geq k_0.$$

We have also that

$$\begin{aligned}
 P_{\mathbb{D}} \left(A_{k_0} \mid \bigcap_{\ell=1}^{k_0-1} A_{\ell} \right) &= \frac{P_{\mathbb{D}} \left(\bigcap_{\ell=1}^{k_0} A_{\ell} \right)}{P_{\mathbb{D}} \left(\bigcap_{\ell=1}^{k_0-1} A_{\ell} \right)} = \\
 &= \frac{\nu_k \mathbf{e}}{P_1 \left(\bigcap_{\ell=1}^{k_0-1} A_{\ell} \right) \mathbf{e} + P_2 \left(\bigcap_{\ell=1}^{k_0-1} A_{\ell} \right) \mathbf{e}^{\dagger}} \\
 &= \frac{\nu_k}{P_1 \left(\bigcap_{\ell=1}^{k_0-1} A_{\ell} \right)} \mathbf{e}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 P_{\mathbb{D}}(A_1) P_{\mathbb{D}}(A_2|A_1) \cdots P_{\mathbb{D}} \left(A_{k_0} \mid \bigcap_{\ell=1}^{k_0-1} A_{\ell} \right) \cdots P_{\mathbb{D}} \left(A_n \mid \bigcap_{\ell=1}^{n-1} A_{\ell} \right) &= \\
 = P_1(A_1) P_1(A_2|A_1) \cdots P_1 \left(A_{k_0} \mid \bigcap_{\ell=1}^{k_0-1} A_{\ell} \right) \cdots P_1 \left(A_n \mid \bigcap_{\ell=1}^{n-1} A_{\ell} \right) \mathbf{e} &= \\
 = P_1(A_1 \cap \cdots \cap A_n) \mathbf{e} = P_{\mathbb{D}}(A_1 \cap \cdots \cap A_n). &
 \end{aligned}$$

The case (3) is proved analogously. \square

Definition 5.4. Let A and B be two random events.

1. A is called independent of B if

$$P_{\mathbb{D}}(A|B) = P_{\mathbb{D}}(A).$$

2. B is called independent of A if

$$P_{\mathbb{D}}(B|A) = P_{\mathbb{D}}(B).$$

3. A and B are called mutually independent if A is independent of B and B is independent of A .

Let us analyze all possible situations.

- (i) Assume that $P_{\mathbb{D}}(A) = P_{\mathbb{D}}(B) = 0$. By definition, in this case $P_{\mathbb{D}}(A|B) = P_{\mathbb{D}}(A)$ and $P_{\mathbb{D}}(B|A) = P_{\mathbb{D}}(B)$, thus A and B are mutually independent. Moreover, it is enough to assume that one of the two probabilities only, say $P_{\mathbb{D}}(A)$, equals zero. Indeed, if $P_{\mathbb{D}}(A) = 0$ then $P_{\mathbb{D}}(A \cap B) = 0$ and hence

$$P_{\mathbb{D}}(B|A) = P_{\mathbb{D}}(B) \quad \text{and} \quad P_{\mathbb{D}}(A|B) = 0 = P_{\mathbb{D}}(A),$$

i.e., A and B are mutually independent. And in any case there follows $P_{\mathbb{D}}(A \cap B) = P_{\mathbb{D}}(A)P_{\mathbb{D}}(B)$.

- (ii) Assume that both $P_{\mathbb{D}}(A)$ and $P_{\mathbb{D}}(B)$ are not in $\mathfrak{S}_{\mathbb{D},0}$. That A is independent of B is equivalent to

$$P_{\mathbb{D}}(A|B) = P_{\mathbb{D}}(A) = \frac{P_{\mathbb{D}}(A \cap B)}{P_{\mathbb{D}}(B)}$$

which, in turn, is equivalent to

$$P_{\mathbb{D}}(A \cap B) = P_{\mathbb{D}}(A) \cdot P_{\mathbb{D}}(B).$$

In the same way, if B is independent of A then this is equivalent to

$$P_{\mathbb{D}}(B|A) = P_{\mathbb{D}}(B) = \frac{P_{\mathbb{D}}(A \cap B)}{P_{\mathbb{D}}(A)},$$

i.e.,

$$P_{\mathbb{D}}(A \cap B) = P_{\mathbb{D}}(B) \cdot P_{\mathbb{D}}(A).$$

Finally, under the assumed hypotheses A is independent of B if and only if B is independent of A if and only if A and B are mutually independent.

- (iii) Assume that $P_{\mathbb{D}}(A)$ and $P_{\mathbb{D}}(B)$ are zero-divisors which both belong to $\mathbb{D}_{\mathbf{e}}$: $P_{\mathbb{D}}(A) = \lambda \mathbf{e}$ and $P_{\mathbb{D}}(B) = \mu \mathbf{e}$ with positive reals λ and μ . This means that $P_1(A) = \lambda$ and $P_1(B) = \mu$ meanwhile $P_2(A) = P_2(B) = 0$ implying that $0 \leq P_1(A \cap B) =: \nu$ and $P_2(A \cap B) = 0$ (hence $P_{\mathbb{D}}(A \cap B) = \nu \mathbf{e}$). Suppose that A is independent of B ; this is equivalent to

$$\begin{aligned} \lambda \mathbf{e} &= P_{\mathbb{D}}(A) = P_{\mathbb{D}}(A|B) = \frac{P_{\mathbb{D}}(A \cap B)}{\mu} \mathbf{e} + P_{\mathbb{D}}(A) \mathbf{e}^{\dagger} \\ &= \frac{P_1(A \cap B)}{\mu} \mathbf{e} = \frac{\nu}{\mu} \mathbf{e}; \end{aligned} \tag{5.5}$$

thus A is independent of B if and only if

$$\lambda = \frac{\nu}{\mu}.$$

Assume the equality $\lambda = \frac{\nu}{\mu}$ holds. Considering now $P_{\mathbb{D}}(B|A)$ we have:

$$\begin{aligned} P_{\mathbb{D}}(B|A) &= \frac{P_{\mathbb{D}}(A \cap B)}{\lambda} \mathbf{e} + P_{\mathbb{D}}(B) \mathbf{e}^{\dagger} = \frac{\nu}{\lambda} \mathbf{e} = \frac{\nu}{\nu/\mu} \mathbf{e} \\ &= \mu \mathbf{e} = P_{\mathbb{D}}(B) \end{aligned} \tag{5.6}$$

which means that B is independent of A , and thus A and B are mutually independent. Observe that using (5.5) or (5.6) one concludes that for independent A and B

$$P_{\mathbb{D}}(A \cap B) = P_{\mathbb{D}}(A \cap B) \mathbf{e} = \lambda \mu \mathbf{e} = (\lambda \mathbf{e})(\mu \mathbf{e}) = P_{\mathbb{D}}(A)P_{\mathbb{D}}(B).$$

In the same way the case of $P_{\mathbb{D}}(A)$ and $P_{\mathbb{D}}(B)$ being both in $\mathbb{D}_{\mathbf{e}^\dagger}$ is covered.

- (iv) Assume that $P_{\mathbb{D}}(A)$ and $P_{\mathbb{D}}(B)$ are zero-divisors but now such that $P_{\mathbb{D}}(A) = \lambda \mathbf{e} \neq 0$ and $P_{\mathbb{D}}(B) = \mu \mathbf{e}^\dagger \neq 0$, or vice-versa. This gives, in particular, that since $A \cap B \subset A$ and $A \cap B \subset B$ then

$$P_1(A \cap B) = P_2(A \cap B) = 0,$$

that is $P_{\mathbb{D}}(A \cap B) = 0$. Consider $P_{\mathbb{D}}(A|B)$, one has:

$$P_{\mathbb{D}}(A|B) = \frac{P_{\mathbb{D}}(A \cap B)}{\mu} \mathbf{e}^\dagger + P_{\mathbb{D}}(A) \mathbf{e} = \lambda \mathbf{e} = P_{\mathbb{D}}(A),$$

which means that the hypotheses imply that A is independent of B . But

$$P_{\mathbb{D}}(B|A) = \frac{P_{\mathbb{D}}(A \cap B)}{\lambda} \mathbf{e} + P_{\mathbb{D}}(B) \mathbf{e}^\dagger = P_{\mathbb{D}}(B) \mathbf{e}^\dagger = \mu \mathbf{e}^\dagger = P_{\mathbb{D}}(B),$$

that is, B is independent of A .

Somewhat paradoxically, in this case A and B are always mutually independent. And there follows that

$$P_{\mathbb{D}}(A)P_{\mathbb{D}}(B) = (\lambda \mathbf{e})(\mu \mathbf{e}^\dagger) = 0 = P_{\mathbb{D}}(A \cap B).$$

- (v) The last case assumes that $P_{\mathbb{D}}(A)$ is zero-divisor and $P_{\mathbb{D}}(B)$ is an invertible hyperbolic number or vice-versa. Set

$$P_{\mathbb{D}}(A) = \lambda \mathbf{e} \neq 0, \quad P_{\mathbb{D}}(B) = \mu_1 \mathbf{e} + \mu_2 \mathbf{e}^\dagger \notin \mathfrak{S}_{\mathbb{D},0}.$$

In particular, this implies that

$$\nu := P_1(A \cap B) \geq 0 \quad \text{and} \quad P_2(A \cap B) = 0.$$

That A is independent of B is equivalent to

$$\begin{aligned} P_{\mathbb{D}}(A|B) &= \frac{P_{\mathbb{D}}(A \cap B)}{P_{\mathbb{D}}(B)} = \frac{\nu \mathbf{e}}{\mu_1 \mathbf{e} + \mu_2 \mathbf{e}^\dagger} = \frac{\nu}{\mu_1} \mathbf{e} \\ &= P_{\mathbb{D}}(A) = \lambda \mathbf{e}, \end{aligned}$$

i.e., $\lambda = \frac{\nu}{\mu_1}$.

On the other hand,

$$\begin{aligned} P_{\mathbb{D}}(B|A) &= \frac{P_{\mathbb{D}}(A \cap B)}{\lambda} \mathbf{e} + P_{\mathbb{D}}(B) \mathbf{e}^\dagger = \frac{P_1(A \cap B)}{\lambda} \mathbf{e} + P_2(B) \mathbf{e}^\dagger \\ &= \frac{\nu}{\lambda} \mathbf{e} + P_2(B) \mathbf{e}^\dagger = \mu_1 \mathbf{e} + \mu_2 \mathbf{e}^\dagger = P_{\mathbb{D}}(B), \end{aligned}$$

i.e., B is independent of A . Of course, the reasoning is reversible, thus A and B are mutually independent if and only if one of them is independent of the another one.

Finally note that from the above one has for independent events that $\nu = \lambda \mu_1$, hence:

$$P_{\mathbb{D}}(A \cap B) = \nu \mathbf{e} = (\lambda \mathbf{e})(\mu_1 \mathbf{e}) = P_{\mathbb{D}}(A)P_{\mathbb{D}}(B).$$

Resuming the just made analysis , one has the following

Corollary 5.1. *Given two random events A and B , then A is independent of B if and only if B is independent of A .*

Corollary 5.2. *If A and B are mutually independent events then the multiplication theorem becomes*

$$P_{\mathbb{D}}(A \cap B) = P_{\mathbb{D}}(A)P_{\mathbb{D}}(B). \quad (5.7)$$

Theorem 5.5. *If A and B are mutually independent events then so are A and B^C , A^C and B , A^C and B^C .*

Proof. It is enough to prove for A and B^C . Write $A = (A \cap B) \cup (A \cap B^C)$ with disjoint summands. Then $P_{\mathbb{D}}(A) = P_{\mathbb{D}}(A \cap B) + P_{\mathbb{D}}(A \cap B^C)$ from where

$$P_{\mathbb{D}}(A \cap B^C) = P_{\mathbb{D}}(A) - P_{\mathbb{D}}(A \cap B) = P_{\mathbb{D}}(A) - P_{\mathbb{D}}(A)P_{\mathbb{D}}(B). \quad (5.8)$$

Of course, we can factorize $P_{\mathbb{D}}(A)$ but the consequent computation depends on the value of $P_{\mathbb{D}}(\Omega)$. Thus we have to consider the following cases:

- (1) $P_{\mathbb{D}}(A) \notin \mathfrak{S}_{\mathbb{D},0}$, then $P_{\mathbb{D}}(\Omega) \notin \mathfrak{S}_{\mathbb{D},0}$ and in this case $\mathfrak{p} = 1$. Hence

$$P_{\mathbb{D}}(A \cap B) = P_{\mathbb{D}}(A)(1 - P_{\mathbb{D}}(B)) = P_{\mathbb{D}}(A)P_{\mathbb{D}}(B),$$

thus A and B^C are mutually independent.

- (2) If $P_{\mathbb{D}}(A) \in \mathfrak{S}_{\mathbb{D},0}$, let us say, $P_{\mathbb{D}}(A) = \lambda \mathbf{e}$, since $A \cap B^C \subset A$, then $P_{\mathbb{D}}(A \cap B^C) = \nu \mathbf{e}$ for some $\nu \geq 0$; write $P_{\mathbb{D}}(B) = \mu_1 \mathbf{e} + \mu_2 \mathbf{e}^\dagger$, and one has two more subcases:

- (a) If $P_{\mathbb{D}}(\Omega) = 1 \notin \mathfrak{S}_{\mathbb{D},0}$, then $P_{\mathbb{D}}(B^C) = 1 - P_{\mathbb{D}}(B) = (1 - \mu_1) \mathbf{e} + (1 - \mu_2) \mathbf{e}^\dagger$ and there follows:

$$\begin{aligned} \nu \mathbf{e} &= P_{\mathbb{D}}(A \cap B^C) \\ &= \lambda \mathbf{e} - \lambda \mathbf{e} (\mu_1 \mathbf{e} + \mu_2 \mathbf{e}^\dagger) = \lambda \mathbf{e} - \lambda \mathbf{e} (\mu_1 \mathbf{e}) \\ &= \lambda (1 - \mu_1) \mathbf{e} = (\lambda \mathbf{e}) ((1 - \mu_1) \mathbf{e}) = P_{\mathbb{D}}(A)P_{\mathbb{D}}(B^C), \end{aligned}$$

thus, A and B^C are mutually independent.

- (b) If $P_{\mathbb{D}}(\Omega) = \mathbf{e}$ (note that the equality $P_{\mathbb{D}}(\Omega) = \mathbf{e}^\dagger$ is impossible) then necessarily $P_{\mathbb{D}}(B) = \mu_1 \mathbf{e}$ and $P_{\mathbb{D}}(B^C) = (1 - \mu_1) \mathbf{e}$, hence

$$\begin{aligned} P_{\mathbb{D}}(A \cap B^C) &= \lambda \mathbf{e} - \lambda \mathbf{e} \mu_1 \mathbf{e} \\ &= (\lambda \mathbf{e})(1 - \mu_1) \mathbf{e} = P_{\mathbb{D}}(A)P_{\mathbb{D}}(B^C), \end{aligned}$$

and A and B^C are mutually independent.

- (3) The case $P_{\mathbb{D}}(A) = \lambda_2 \mathbf{e}^\dagger$ is treated similarly.

□

Definition 5.6. *Random events A_1, \dots, A_n are called mutually (or jointly) independent if for any subset of indices i_1, \dots, i_r with $1 \leq i_1 < i_2 < \dots < i_r \leq n$ ($r \in \{2, \dots, n\}$) there holds:*

$$P_{\mathbb{D}}(A_{i_1} \cap \dots \cap A_{i_r}) = P_{\mathbb{D}}(A_{i_1}) \cdots P_{\mathbb{D}}(A_{i_r}).$$

If this holds for $r = 2$ only then the events are called pair-wise independent. Generally speaking, pair-wise independence and joint independence are different notions.

If A_1, \dots, A_n are mutually independent events then the general multiplication theorem holds in a simplified form:

$$P_{\mathbb{D}}(A_1 \cap \dots \cap A_n) = P_{\mathbb{D}}(A_1) \cdots P_{\mathbb{D}}(A_n).$$

Definition 5.7. *Let $(\Omega, \Sigma, P_{\mathbb{D}})$ be a \mathbb{D} -probabilistic space. Let H_1, \dots, H_n be pairwise disjoint random events with (not necessarily strictly) positive probabilities and such that $H_1 \cup \dots \cup H_n = \Omega$. Then the collection $\{H_1, \dots, H_n\}$ is called a fundamental (or complete) system of events (FSE).*

Theorem 5.8. *(hyperbolic law of total probability; complete hyperbolic probability formula). Given $(\Omega, \Sigma, P_{\mathbb{D}})$, A a random event; $\{H_1, \dots, H_n\}$ a FSE, then*

$$P_{\mathbb{D}}(A) = \sum_{i=1}^n P_{\mathbb{D}}(H_i) P_{\mathbb{D}}(A|H_i);$$

Proof. Since $A = A \cap \Omega = A \cap \left(\bigcup_{i=1}^n H_i \right) = \bigcup_{i=1}^n A \cap H_i$ and the events $A \cap H_i$ are pairwise disjoint then

$$P_{\mathbb{D}}(A) = \sum_{i=1}^n P_{\mathbb{D}}(A \cap H_i).$$

It is enough to apply now Theorem 5.2 (“Multiplication Theorem”). \square

Theorem 5.9. *(Bayes' theorem). Let $(\Omega, \Sigma, P_{\mathbb{D}})$, A and $\{H_1, \dots, H_n\}$ be as in the previous theorem, then:*

1) *if $P_{\mathbb{D}}(A)$ is an invertible hyperbolic number then*

$$P_{\mathbb{D}}(H_k|A) = \frac{P_{\mathbb{D}}(H_k) \cdot P_{\mathbb{D}}(A|H_k)}{\sum_{i=1}^n P_{\mathbb{D}}(H_i) \cdot P_{\mathbb{D}}(A|H_i)} = \frac{P_{\mathbb{D}}(H_k) \cdot P_{\mathbb{D}}(A|H_k)}{P_{\mathbb{D}}(A)}; \quad (5.9)$$

2) *if $P_{\mathbb{D}}(A) = \lambda \mathbf{e}$ with $\lambda > 0$ then*

$$\left(P_{\mathbb{D}}(H_k) \cdot P_{\mathbb{D}}(A|H_k) - P_{\mathbb{D}}(H_k|A) \cdot \sum_{i=1}^n P_{\mathbb{D}}(H_i) \cdot P_{\mathbb{D}}(A|H_i) \right) \mathbf{e} = 0. \quad (5.10)$$

3) If $P_{\mathbb{D}}(A) = \mu \mathbf{e}^\dagger$ with $\mu > 0$ then

$$\left(P_{\mathbb{D}}(H_k) \cdot P_{\mathbb{D}}(A|H_k) - P_{\mathbb{D}}(H_k|A) \cdot \sum_{i=1}^n P_{\mathbb{D}}(H_i) \cdot P_{\mathbb{D}}(A|H_i) \right) \mathbf{e}^\dagger = 0. \quad (5.11)$$

Proof. 1) Let $P_{\mathbb{D}}(A)$ be an invertible hyperbolic number then the multiplication theorem gives:

$$P_{\mathbb{D}}(A \cap H_k) = P_{\mathbb{D}}(H_k)P_{\mathbb{D}}(A|H_k) = P_{\mathbb{D}}(A)P_{\mathbb{D}}(H_k|A).$$

Thus, a part of formula (5.9) verifies. Using the hyperbolic law of total probability gives the rest of (5.9).

2) Let $P_{\mathbb{D}}(A) = \lambda \mathbf{e}$ with λ being a positive real number then the multiplication theorem leads to the equality:

$$P_{\mathbb{D}}(A)P_{\mathbb{D}}(H_k|A) = P_{\mathbb{D}}(H_k)P_{\mathbb{D}}(A|H_k). \quad (5.12)$$

Note that the left-hand side of (5.12) is an element of $\mathbb{D}_{\mathbf{e}}$, hence the right-hand side must be an element of $\mathbb{D}_{\mathbf{e}}$ also. But this is true indeed, since in the definition of $P_{\mathbb{D}}(A|H_k)$ the factor $P_{\mathbb{D}}(A \cap H_k)$ is involved and because $A \cap H_k \subset A$, hence $P_{\mathbb{D}}(A \cap H_k) \in \mathbb{D}_{\mathbf{e}}$. Now, recalling the property of Remark 2.1, equation (5.12) can be rewritten as

$$\begin{aligned} 0 &= P_{\mathbb{D}}(H_k)P_{\mathbb{D}}(A|H_k) - P_{\mathbb{D}}(H_k|A)P_{\mathbb{D}}(A) \\ &= (P_{\mathbb{D}}(H_k)P_{\mathbb{D}}(A|H_k) - P_{\mathbb{D}}(H_k|A)P_{\mathbb{D}}(A)) \mathbf{e}, \end{aligned}$$

finally using the hyperbolic law of total probability one obtains (5.10).

3) We proceed similarly to item 2). □

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