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New Topological C-Algebras With Applications in Linear Systems Theory

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NEW TOPOLOGICAL $\mathbb{C}$-ALGEBRAS WITH APPLICATIONS IN LINEAR SYSTEMS THEORY

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Contents

Abstract. Motivated by the Schwartz space of tempered distributions $\mathscr{S}'$ and the Kondratiev space of stochastic distributions $\mathcal{S}_{-1}$ we define a wide family of nuclear spaces which are increasing unions of (duals of) Hilbert spaces $\mathscr{H}_p', p \in \mathbb{N}$, with decreasing norms $\|\cdot\|_p$. The elements of these spaces are functions on a free commutative monoid. We characterize those rings in this family which satisfy an inequality of the form $\|f*g\|_p \leq A(p-q)\|f\|_q\|g\|_p$ for all $p \geq q+d$, where $*$ denotes the convolution in the monoid, $A(p-q)$ is a strictly positive number and d is a fixed natural number (in this case we obtain commutative topological $\mathbb{C}$-algebras). Such an inequality holds in $\mathcal{S}_{-1}$, but not in $\mathscr{S}'$. We give an example of such a ring which contains $\mathscr{S}'$. We characterize invertible elements in these rings and present applications to linear system theory.

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1. Introduction

As is well known, the Schwartz space $\mathscr{S}'$ of complex tempered distributions can be viewed as the space of sequences of complex numbers $f = (f_n)_{n\in\mathbb{N}_0}$ subject to

$$\sum_{n\in\mathbb{N}_0} (n+1)^{-2p}|f_n|^2 < \infty \text{ for some } p \in \mathbb{N}.$$

Setting

$$\|f\|_p = \left(\sum_{n\in\mathbb{N}_0} (n+1)^{-2p}|f_n|^2\right)^{1/2} < \infty,$$

one can represent $\mathscr{S}'$ as a union of an increasing sequence of Hilbert spaces $\mathscr{H}_1', \mathscr{H}_2', \ldots$of complex sequences, with decreasing norms:

$$\mathscr{H}_p' = \left\{f = (f_n)_{n\in\mathbb{N}_0} : \|f\|_p < \infty\right\}. \tag{1.1}$$

In Hida's white noise space theory, a counterpart of $\mathscr{S}'$ was introduced by Kondratiev, see [21] and the references therein, in the following way. We begin with a definition:

Definition 1.1. *ℓ denotes the set of sequences of elements of $\mathbb{N}_0$, indexed by $\mathbb{N}$,*

$$(\alpha_1, \alpha_2, \ldots)$$

where $\alpha_j \neq 0$ for at most a finite number of indices.

The stochastic counterpart of $\mathscr{S}'$ is the space $\mathcal{S}_{-1}$ of families of complex numbers $f = (f_\alpha)_{\alpha\in\ell}$ indexed by ℓ and such that

$$\sum_{\alpha\in\ell} |f_\alpha|^2(2\mathbb{N})^{-\alpha p} < \infty \text{ for some } p \in \mathbb{N}.$$

In the above expression, one sets

$$(2\mathbb{N})^\alpha = 2^{\alpha_1}4^{\alpha_2}6^{\alpha_3}\cdots$$

We here set

$$\|f\|_p = \|(f_\alpha)_{\alpha\in\ell}\|_p = \left(\sum_{\alpha\in\ell} |f_\alpha|^2(2\mathbb{N})^{-\alpha p}\right)^{1/2}, \tag{1.2}$$

and denote

$$\mathscr{K}_p' = \left\{f = (f_\alpha)_{\alpha\in\ell} : \|f\|_p < \infty\right\}.$$

In a way similar to $\mathscr{S}'$, the space $\mathcal{S}_{-1}$ is the union of the increasing sequence of Hilbert spaces $\mathscr{K}_1', \mathscr{K}_2', \ldots$ with decreasing norms (1.2). The elements of $\mathcal{S}_{-1}$ are called stochastic distributions and play an important role in stochastic partial differential equations, see [21]. We also refer to the papers [2, 4, 5] where $\mathcal{S}_{-1}$ is used to develop a new approach to linear stochastic systems. Recall that the convolution of two elements of $\mathcal{S}_{-1}$, $f = (f_\alpha)_{\alpha\in\ell}$ and $g = (g_\alpha)_{\alpha\in\ell}$, (which is called the Wick product and denoted by $\diamond$) is defined by

$$f \diamond g = \left(\sum_{\beta\leq\alpha} f_\beta g_{\alpha-\beta}\right)_{\alpha\in\ell}$$

and satisfies the inequality

$$\|f \diamond g\|_p \leq A(p-q)\|f\|_q\|g\|_p, \quad \text{for all } p \geq q+2, \tag{1.3}$$

where

$$A(p-q) = \left(\sum_{\alpha\in\ell} (2\mathbb{N})^{-(p-q)\alpha}\right)^{\frac{1}{2}}$$

is finite. This inequality is due to Våge, see [32], [21]. It expresses in particular that the multiplication operator

$$g \mapsto f \diamond g$$

is bounded from the Hilbert space $\mathscr{K}'_p$ into itself where $f \in \mathscr{K}'_q$ and $p \geq q+2$. It plays a key role in the applications mentioned above.

In view of (1.3) it is a natural question to ask if a similar inequality holds in $\mathscr{S}'$. Here lies an important structural difference between $\mathscr{S}'$ and $\mathcal{S}_{-1}$. If $f = (f_n)_{n\in\mathbb{N}_0}$ and $g = (g_n)_{n\in\mathbb{N}_0}$ belong to $\mathscr{S}'$, their convolution which is defined by

$$f * g = \left(\sum_{m\leq n} f_m g_{n-m}\right)_{n\in\mathbb{N}_0}$$

also belongs to $\mathscr{S}'$. Nevertheless, as will be proved in the sequel (see Corollary 6.1), one cannot have an inequality of the kind (1.3), that is:

$$\|f * g\|_p \leq B(p-q)\|f\|_q\|g\|_p, \quad \text{for all } p \geq q+d, \tag{1.4}$$

where $d \in \mathbb{N}$ is preassigned, $B(p-q) > 0$ is a constant which depends only on $p-q$ in $\mathbb{N}$, and where f runs through $\mathscr{H}'_q$ and g runs through $\mathscr{H}'_p$. Since such an inequality does not hold, the origin of the present study was to find nuclear spaces containing $\mathscr{S}'$ such that an appropriate inequality of the type (1.3) holds for the convolution.

More generally, in the present paper we define a wide family of nuclear spaces in terms of positive functions over a commutative monoid, and give a characterization of those in which an inequality of the type (1.3) holds. Since such an inequality was first proved by Våge (in the setting of the Kondratiev space of stochastic distributions) we call these spaces

Våge spaces. We show that these spaces are in particular topological $\mathbb{C}$-algebras, and give a characterization of their invertible elements. We then consider the tensor product of two Våge spaces, and show that it is a Våge space too.

The Schwartz space of tempered distributions $\mathscr{S}'$ is not a Våge space. We define a Våge space, containing $\mathscr{S}'$. This space is the dual of a space which is included in the Schwartz space of test functions $\mathscr{S}$, consists of entire functions, and is invariant under the Fourier transform. One can thus define the Fourier transform on its dual, and study, as suggested to us by Palle Jorgensen, connections with the theory of hyperfunctions (see [11] for the latter). This will be done in another publication. We present some important properties of this space, and characterize it both in terms of sequences and in terms of entire functions.

The paper consists of seven sections besides the introduction, and we now describe its content. A family of spaces of functions over a commutative monoid which includes the space $\mathscr{S}'$, and which we call *regular admissible spaces*, is introduced in Section 2. In Section 3 we characterize regular admissible spaces in which convolution satisfies an inequality of the type (1.3). We also prove that these spaces are topological $\mathbb{C}$-algebras, which we call Våge spaces. Invertible elements in these rings are characterized in Section 4. In Section 5 we prove that the tensor product of two Våge spaces is a Våge space. The last three sections are devoted to examples and applications. In Section 6 we define a Våge space which contains $\mathscr{S}'$. Some results of E. Hille on Hermite series play an important role in the arguments. In Section 7 we consider the Kondratiev space. Finally, applications to linear system theory are outlined in Section 8.

2. A new family of nuclear spaces and Gelfand triples

In this section we introduce a family of nuclear spaces of functions over a commutative monoid which we use in the sequel. We begin with a definition and a preliminary result on such positive functions. Let A be a subset of $\mathbb{N}$. We denote

$$\ell_A = \mathbb{N}_0^{(A)} = \left\{\alpha \in \mathbb{N}_0^A : \text{supp}(\alpha) \text{ is finite}\right\} = \oplus_{n\in A}\mathbb{N}_0 e_n, \tag{2.1}$$

where, for $n \in \mathbb{N}$, we have denoted by e_n the sequence with all elements equal to 0, at exception of the n-th one, equal to 1. For two elements

$$\alpha = \sum_{n \in A} \alpha_n e_n \quad \text{and} \quad \beta = \sum_{n \in A} \beta_n e_n$$

in ℓ_A, we define $\alpha + \beta = \sum_{n \in A} (\alpha_n + \beta_n) e_n$. In other words, $(\ell_A, +, 0)$ is the free commutative monoid generated by the countable (or finite) set $\{e_n\}_{n \in A}$. Moreover, we consider the following partial order induced by the addition above: For $\alpha, \beta \in \ell_A$, we define $\alpha \leq \beta$ if there exists $\gamma \in \ell_A$ such that $\alpha + \gamma = \beta$.

Definition 2.1. *Let A be a subset of $\mathbb{N}$, and let ℓ_A be defined by (2.1). A positive function $a : \ell_a \to \mathbb{R}$ (that is $\alpha \mapsto a_\alpha$ where $a_\alpha > 0$ for any $\alpha \in \ell_A$) is called* admissible *if $a_0 = 1$ and $a_{e_n} > 1$ for all $n \in A$.*
Let $d \in \mathbb{N}$. The admissible function a is called d-regular *(or simply regular) if furthermore*

$$\sum_{n \in A} \frac{1}{a_{e_n}^d - 1} < \infty. \tag{2.2}$$

It is superexponential *(resp.* exponential*) if*

$$a_\alpha a_\beta \leq a_{\alpha+\beta} \quad (\text{resp. } a_\alpha a_\beta = a_{\alpha+\beta}) \quad \forall \alpha, \beta \in \ell_A. \tag{2.3}$$

Two examples of exponential regular admissible positive functions are as follows:

(a) The set A has cardinal one. Then, $\ell_A = \mathbb{N}_0$. We take $a_n = c^n$ with $c > 1$.

(b) We set $A = \mathbb{N}$. Then, $\ell_A = \ell$, where ℓ is as in Definition 1.1. We take

$$a_\alpha = (2\mathbb{N})^\alpha = 2^{\alpha_1} 4^{\alpha_2} 6^{\alpha_3} \cdots, \quad \alpha \in \ell. \tag{2.4}$$

As mentioned in the introduction, this last example occurs in Hida's white noise space theory, in the definition of spaces of stochastic distributions.

Proposition 2.2. *Let $a : \ell_A \to \mathbb{R}$ be a superexponential d-regular admissible positive function. Then,*

$$\sum_{\alpha \in \ell_A} a_\alpha^{-d} < \infty.$$

Furthermore, if a is exponential rather than superexponential then d-regularity is also necessary for the family $(a_\alpha^{-d})_{\alpha \in \ell_A}$ to be summable.

Proof. We first prove the theorem for the case $d = 1$. It follows from (2.3) that for every $\alpha \in \ell_A$,

$$\prod_{n \in A} a_{e_n}^{\alpha_n} \leq a_\alpha.$$

Therefore,

$$\begin{aligned}\sum_{\alpha \in \ell_A} a_\alpha^{-1} &\leq \sum_{\alpha \in \ell_A} \prod_{n \in A} a_{e_n}^{-\alpha_n} \\ &= \prod_{n \in A} \sum_{\alpha_n = 0}^{\infty} a_{e_n}^{-\alpha_n} \\ &= \prod_{n \in A} \frac{1}{1 - a_{e_n}^{-1}} \\ &= \prod_{n \in A} \left(1 + \frac{1}{a_{e_n} - 1}\right) < \infty\end{aligned}$$

If $a_\alpha a_\beta = a_{\alpha+\beta}$, then $\forall \alpha \in \ell_A$, $\prod_{n \in A} a_{e_n}^{\alpha_n} = a_\alpha$. Therefore,

$$\sum_{\alpha \in A} a_\alpha^{-1} = \prod_{n \in A} \left(1 + \frac{1}{a_{e_n} - 1}\right),$$

which converges if and only if $\sum_{n \in A} \frac{1}{a_{e_n} - 1} < \infty$. In case $d > 1$, we take a^d instead of a. □

When $A = \mathbb{N}$ and a_α is given by (2.4) we obtain as a corollary a result of Zhang, proved in 1992, see [34], [21]. Zhang's proof uses Abel's convergence test. The result itself is necessary in order to present some important properties of Kondratiev spaces of stochastic test functions and stochastic distributions.

Corollary 2.3 (Zhang [34])**.** *Let $d \in \mathbb{N}$. $\sum_{\alpha \in \ell} (2\mathbb{N})^{-d\alpha} < \infty$ if and only if $d > 1$.*

Proof. We take $\ell_A = \ell$. Thus

$$\sum_{\alpha \in \ell} (2\mathbb{N})^{-d\alpha} < \infty \quad \iff \quad \sum_{n \in \mathbb{N}} \frac{1}{(2n)^d - 1} < \infty,$$

which is true if and only if $d > 1$. □

The spaces $\mathscr{S}'$ and $\mathcal{S}_{-1}$ are strong dual of Fréchet spaces. Namely, $\mathscr{S}'$ is the strong dual of the Schwartz space $\mathscr{S}$ of Hermite series

$$f(x) = \sum_{n=0}^{\infty} f_n \xi_n(x), \tag{2.5}$$

where $(\xi_n)_{n\in\mathbb{N}_0}$ denote the Hermite functions, and the coefficients f_n are complex numbers, and such that

$$\sum_{n\in\mathbb{N}_0}(n+1)^{2p}|f_n|^2<\infty \text{ for all } p\in\mathbb{N},$$

and $\mathcal{S}_{-1}$ is the dual of the Kondratiev space $\mathcal{S}_1$ of stochastic test functions which can be seen as families of complex numbers $(f_\alpha)_{\alpha\in\ell}$ indexed by ℓ and such that

$$\sum_{\alpha\in\ell}(\alpha!)^2(2\mathbb{N})^{\alpha p}|f_\alpha|^2<\infty \text{ for all } p\in\mathbb{N},$$

where

$$(2\mathbb{N})^\alpha=\prod_{k=1}^\infty(2k)^{\alpha_k}\quad\text{and}\quad\alpha!=\prod_{k=1}^\infty(\alpha_k!). \tag{2.6}$$

The triples $(\mathscr{S},\mathbf{L}_2(\mathbb{R},dx),\mathscr{S}')$ and $(\mathcal{S}_1,\mathcal{W},\mathcal{S}_{-1})$, where $\mathcal{W}$ denotes the white noise space, are Gelfand triples. We now define a wide family of nuclear topological vector spaces, which includes $\mathscr{S}'$ and $\mathcal{S}_{-1}$, and which is closed under tensor products, as dual of certain Fréchet spaces.

Let $a:\ell_A\to\mathbb{R}$ be a positive function, that is $\alpha\mapsto a_\alpha$ where $a_\alpha>0$ for any $\alpha\in\ell_A$. We denote the weighted Hilbert space with respect to a by

$$\ell_a^2=\left\{(\varphi_\alpha)_{\alpha\in\ell_A}:\sum_{\alpha\in\ell_A}|\varphi_\alpha|^2a_\alpha<\infty\right\}. \tag{2.7}$$

When a is the constant function 1 we simply denote $\ell^2=\ell_1^2$ (i.e $\ell^2=\ell^2(\ell_A)$). We define the countably normed space

$$\mathcal{F}_a=\left\{(\varphi_\alpha)_{\alpha\in\ell_A}:\sum_{\alpha\in\ell_A}|\varphi_\alpha|^2a_\alpha^p<\infty \text{ for all } p\in\mathbb{N}\right\}=\bigcap_{p\in\mathbb{N}}\ell_{a^p}. \tag{2.8}$$

Theorem 2.4. *Let $a:\ell_A\to\mathbb{R}$ be a positive function.*

(a) The space $\mathcal{F}_a$ endowed with the topology defined by the norms

$$\|\varphi\|_p^2=\sum_{\alpha\in\ell_A}|\varphi_\alpha|^2a_\alpha^p,\quad p=1,2,\dots$$

is a Fréchet space.

(b) If $a>1$ (that is, for any $\alpha\in\ell_A$, $a_\alpha>1$), then the space $\mathcal{F}_a$ is continuously included in ℓ^2.

(c) The space $\mathcal{F}_a$ is nuclear if and only if there exists $d \in \mathbb{N}$ such that

$$\sum_{\alpha \in \ell_A} a_\alpha^{-d} < \infty \tag{2.9}$$

Proof.

(a) By definition, $\mathcal{F}_a = \bigcap_{p \in \mathbb{N}} \ell_{a^p}$, and thus it is a Fréchet space. We first note that the norms are non decreasing and compatible (i.e. every sequence which is a Cauchy sequence with respect to the two norms and converges to zero with respect to one of them, converges to zero also with respect to the second). See [14, p. 17-18]

(b) We have

$$\|\varphi\|_{\ell^2}^2 = \sum_{\alpha \in \ell_A} |\varphi_\alpha|^2 \leq \sum_{\alpha \in \ell_A} |\varphi_\alpha|^2 a_\alpha^p = \|\varphi\|_{\ell_{a^p}}^2.$$

(c) Defining $\delta_\alpha = (\delta_{\alpha,\beta})_{\beta \in \ell_A}$ such that $\delta_{\alpha,\beta} = 0$ if $\alpha \neq \beta$ and $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$, it is clear that $(\delta_\alpha a_\alpha^{-p/2} b_\alpha)_{\alpha \in \ell_A}$ is an orthonormal base of ℓ_{a^p}. Now, $\mathcal{F}_a = \bigcap_{p \in \mathbb{N}} \ell_{a^p}$ is nuclear if and only if for every $p \in \mathbb{N}$ there exists $q > p$ such that the natural embedding $\iota : \ell_{a^q} \to \ell_{a^p}$ is Hilbert-Schmidt. The equality

$$\begin{aligned} \text{tr } (\iota^* \iota) &= \sum_{\alpha \in \ell_A} \langle \iota^* \iota (\delta_\alpha a_\alpha^{-q/2} b_\alpha), (\delta_\alpha a_\alpha^{-q/2} b_\alpha) \rangle_q \\ &= \sum_{\alpha \in \ell_A} \|\iota(\delta_\alpha a_\alpha^{-q/2} b_\alpha)\|_p^2 \\ &= \sum_{\alpha \in \ell_A} a_\alpha^{-(q-p)} \|\iota(\delta_\alpha a_\alpha^{-p/2} b_\alpha)\|_p^2 = \sum_{\alpha \in \ell_A} a_\alpha^{-(q-p)} \end{aligned}$$

yields that $\mathcal{F}_a$ is nuclear if and only if for any p there exists $q > p$ such that

$$\sum_{\alpha \in \ell_A} a_\alpha^{-(q-p)} < \infty. \tag{2.10}$$

If (2.9) holds for some $d \in \mathbb{N}$, then setting $q = p + d$ leads to the requested result. On the opposite direction, if $\mathcal{F}_a$ is nuclear, then for $p = 1$ there exists $q > p$ such that (2.10) holds. Setting $d = q - p = q - 1$ yields the requested result.

□

Definition 2.5. *Let a be a d-regular admissible positive function. The space $\mathcal{F}_a'$ is called a d-regular admissible space. The space is called regular admissible if it is d-regular admissible for some $d \in \mathbb{N}$.*

We denote by $(\ell^2_{a^p})'$ the dual of $\ell^2_{a^p}$. Then

$$(\ell^2_{a^1})' \subseteq (\ell^2_{a^2})' \subseteq \cdots \subseteq (\ell^2_{a^p})' \subseteq \cdots \subseteq \mathcal{F}'_a,$$

and the dual space $\mathcal{F}'_a$ is the union of the increasing sequence of the spaces $(\ell^2_{a^p})'$, i.e.,

$$\mathcal{F}'_a = \bigcup_{p\in\mathbb{N}} (\ell^2_{a^p})'.$$

See [14, p. 35-36]. Since a Fréchet space is nuclear if and only if its strong dual is nuclear, $\mathcal{F}'_a$ is also nuclear.

Proposition 2.6. *$\mathcal{F}'_a$ can be viewed as*

$$\left\{(f_\alpha)_{\alpha\in\ell_A} : \sum_{\alpha\in\ell_A} |f_\alpha|^2 a_\alpha^{-p} < \infty \text{ for some } p\in\mathbb{N}\right\}.$$

Proof. Let $f \in (\ell^2_{a^p})'$. It follows from Riesz's representation theorem that there exists $\psi = (\psi_\alpha) \in \ell^2_{a^p}$ with $\|\psi\|_{\ell^2_{a^p}} = \|f\|_{(\ell^2_{a^p})'}$ and such that

$$f(\cdot) = \langle \cdot, \psi\rangle_{\ell^2_{a^p}}.$$

Thus, for any $\varphi = (\varphi_\alpha) \in \ell^2_{a^p}$

$$f(\varphi) = \langle \psi, \varphi\rangle_{\ell^2_{a^p}} = \sum_{\alpha\in\ell_A} \varphi_\alpha \overline{\psi_\alpha} a^p_\alpha.$$

Setting $f_\alpha = \psi_\alpha a^p_\alpha$, we have

$$\sum_{\alpha\in\ell_A} |f_\alpha|^2 a_\alpha^{-p} = \sum_{\alpha\in\ell_A} |\psi_\alpha|^2 a^p_\alpha = \|\psi\|^2_{\ell^2_{a^p}} = \|f\|^2_{(\ell^2_{a^p})'}.$$

Moreover, for any (f_α) subjects to $\sum_{\alpha\in\ell_A} |f_\alpha|^2 a_\alpha^{-p} < \infty$

$$(f_\alpha) \mapsto \left((\varphi_\alpha) \mapsto \sum_{\alpha\in\ell_A} \varphi_\alpha \overline{f_\alpha}\right)$$

maps (f_α) to a continuous linear functional over $\ell^2_{a^p}$, and any composition of this mapping with $f \mapsto (f_\alpha)$, which was described before, yields the appropriate identity. Hence,

$$(\ell^2_{a^p})' \cong \ell^2_{a^{-p}} = \left\{(f_\alpha)_{\alpha\in\ell_A} : \sum_{\alpha\in\ell_A} |f_\alpha|^2 a_\alpha^{-p} < \infty\right\}. \tag{2.11}$$

Thus, $\mathcal{F}'_a$ can be viewed as

$$\bigcup_{p\in\mathbb{N}} \ell^2_{a^{-p}} = \left\{(f_\alpha)_{\alpha\in\ell_A} : \sum_{\alpha\in\ell_A} |f_\alpha|^2 a_\alpha^{-p} < \infty \text{ for some } p\in\mathbb{N}\right\}.$$

□

We note that the inner product $\langle\cdot,\cdot\rangle_{\ell^2}$ coincides with the antilinear duality $\langle\cdot,\cdot\rangle_{\mathcal{F}'_a,\mathcal{F}_a}$, whenever it makes sense. Considering the inclusion of dual spaces $(\ell^2)'$ in $\mathcal{F}'_a$, using Riesz theorem we have that,

$$\mathcal{F}_a \subseteq \ell^2 \subseteq \mathcal{F}'_a.$$

It is clear that the second inclusion is also continuous, since

$$\|f\|^2_{(\ell^2_{a^p})'} = \sum_{\alpha\in\ell_A} |f_\alpha|^2 a_\alpha^{-p} \leq \sum_{\alpha\in\ell_A} |f_\alpha|^2 = \|f\|^2_{\ell^2}.$$

Proposition 2.7. *$(\mathcal{F}_a, \ell^2(\ell_A), \mathcal{F}'_a)$ is a Gelfand triple.*

3. Våge spaces - a new family of topological $\mathbb{C}$-algebras

Considering the monoid ℓ_A for $A \subseteq \mathbb{N}$, it has the property that for any $\gamma \in \ell_A$

$$\{(\alpha,\beta)\in\ell_A^2 : \alpha+\beta=\gamma\} \text{ is finite.}$$

Thus it gives rise to the total $\mathbb{C}$-algebra of the monoid ℓ_A, namely $(\mathbb{C}^{\ell_A}, +, *)$, where $*$ denotes the convolution multiplication defined by

$$f * g = \left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta\right)_{\gamma\in\ell_A} \quad \text{for all } f,g\in\mathbb{C}^{\ell_A}. \tag{3.1}$$

We usually omit the $*$, and simply write fg instead of $f*g$. This algebra is an integral domain and a topological ring with respect to the product topology of $\mathbb{C}^{\ell}_A$. Denoting $x_n = (\delta_{e_n,\beta})_{\beta\in\ell_A} \in \mathbb{C}^{\ell_A}$, $x = (x_n)_{n\in A}$ and $x^\alpha = \prod_{n\in A} x_n^{\alpha_n}$, we obtain $x^\alpha = (\delta_{\alpha,\beta})_{\beta\in\ell_A} = \delta_\alpha$. Therefore

$$(f_\alpha)_{\alpha\in\ell_A} = \sum_{\alpha\in\ell_A} f_\alpha x^\alpha.$$

Thus, we obtain $\mathbb{C}^{\ell_A} = \mathbb{C}[[x]]$ and the convolution can be considered simply as formal series multiplication. For more details about total algebras of monoids see Bourbaki [9, p. 454].

The proof of the following proposition is straightforward and will be omitted.

Proposition 3.1. *Let f,g and h be in $\mathbb{C}^{\ell_A}$. Then, it holds that:*
(a) $fg = gf$.
(b) $(fg)h = f(gh)$.
(c) $f(g+h) = fg + fh$.

Recall that we introduced regular admissible spaces in Definition 2.5. When it does not lead to any confusion, we simply denote by $\|\cdot\|_p$ instead of $\|\cdot\|_{(\ell^2_{a^p})'} = \|\cdot\|_{\ell^2_{a^{-p}}}$.

Definition 3.2. *A regular admissible space $\mathcal{F}'_a = \bigcup_{p=1}^{\infty} \ell^2_{a^{-p}}$ is called a Våge space if there is $e \in \mathbb{N}$ such that for every $p \in \mathbb{N}$ and for every $p \geq q + e$*

$$\|fg\|_p \leq A(p-q)\|f\|_q\|g\|_p$$

for all $f \in \ell^2_{a^{-p}}$ and $g \in \ell^2_{a^{-p}}$, where $A(p-q)$ is a finite positive number. If $\mathcal{F}'_a$ is a Våge space, we call the minimal e with this property the index *of the space.*

We note that in particular, a Våge space $\mathcal{F}'_a$ is a subalgebra of $(\mathbb{C}^{\ell_A}, +, *)$, i.e., closed under convolution (and clearly under addition).

Theorem 3.3. *A d-regular admissible space $\mathcal{F}'_a$ is a Våge space if and only if*

$$a_\alpha a_\beta \leq a_{\alpha+\beta}, \quad \forall \alpha, \beta \in \ell_A,$$

i.e., $a : \ell_A \to \mathbb{R}$ is superexponential. Its index is then less or equal to d.

Proof. We follow the argument in [21, p. 118]. First we assume that for all $\alpha, \beta \in \ell_A$, $a_\alpha a_\beta \leq a_{\alpha+\beta}$. Since $\mathcal{F}'_a$ is a regular admissible space, $a_0 = 1$, $a_{e_n} > 1$ for all $n \in A$, and $\sum_{n\in A} \frac{1}{a^d_{e_n} - 1} < \infty$ for some $d \in \mathbb{N}$. Therefore by Proposition 2.2 we obtain for any $p - q \geq d$ $\sum_{\alpha\in\ell_A} a_\alpha^{-(p-q)} \leq \sum_{\alpha\in\ell_A} a_\alpha^{-d} < \infty$. We denote

$$A(p-q) = \left(\sum_{\alpha\in\ell_A} a_\alpha^{-(p-q)}\right)^{\frac{1}{2}}.$$

Now, supposed that $f \in \ell^2_{a^{-q}}$ and $g \in \ell^2_{a^{-p}}$, for some $p \geq q+d$. Then,

$$\begin{aligned}
\|fg\|_p^2 &= \sum_{\gamma\in\ell_A}\left|\sum_{\alpha\leq\gamma} f_\alpha g_{\gamma-\alpha} a_\gamma^{-p/2}\right|^2 \\
&\leq \sum_{\gamma\in\ell_A}\left(\sum_{\alpha\leq\gamma}|f_\alpha|a_\alpha^{-p/2}|g_{\gamma-\alpha}|a_{\gamma-\alpha}^{-p/2}\right)^2 \\
&\leq \sum_{\gamma\in\ell_A}\left(\sum_{\alpha,\alpha'\leq\gamma}|f_\alpha|a_\alpha^{-p/2}|f_{\alpha'}|a_{\alpha'}^{-p/2}|g_{\gamma-\alpha}|a_{\gamma-\alpha}^{-p/2}|g_{\gamma-\alpha'}|a_{\gamma-\alpha'}^{-p/2}\right) \\
&\leq \sum_{\alpha,\alpha'\in\ell_A}\left(|f_\alpha|a_\alpha^{-p/2}|f_{\alpha'}|a_{\alpha'}^{-p/2}\sum_{\gamma\geq\alpha,\alpha'}|g_{\gamma-\alpha}|a_{\gamma-\alpha}^{-p/2}|g_{\gamma-\alpha'}|a_{\gamma-\alpha'}^{-p/2}\right) \\
&\leq \left(\sum_{\beta\in\ell_A}|f_\beta|a_\beta^{-p/2}\right)^2\left(\sum_{\beta\in\ell_A}|g_\beta|^2a_\beta^{-p}\right)^{\frac{1}{2}}\left(\sum_{\beta\in\ell_A}|g_\beta|^2a_\beta^{-p}\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{\beta\in\ell_A}a_\beta^{-(p-q)}\right)\left(\sum_{\beta\in\ell_A}|f_\beta|^2a_\beta^{-q}\right)\left(\sum_{\beta\in\ell_A}|g_\beta|^2a_\beta^{-p}\right) \\
&= \left(A(p-q)\right)^2\|f\|_q^2\|g\|_p^2.
\end{aligned}$$

Thus, $\mathcal{F}_a$ is a Våge space of index which is less than or equal to d. On the other direction, assuming that $\mathcal{F}'_a$ is a Våge space of index e. Let q be a natural number and $p = q+e$. Then,

$$\|x^{\alpha+\beta}\|_p = \|x^\alpha x^\beta\|_p \leq A(p-q)\|x^\alpha\|_q\|x^\beta\|_p.$$

Therefore, $a_{\alpha+\beta}^{-p} \leq A(e)a_\alpha^{-p}a_\beta^{-q}$. Then, $A(e)^{-1/p}a_\alpha a_\beta^{q/p} \leq a_{\alpha+\beta}$. Thus, we obtain $a_\alpha a_\beta \leq a_{\alpha+\beta}$ as p goes to infinity. □

Corollary 3.4. *A Våge space is nuclear.*

Proof. If $\mathcal{F}'_a$ is a Våge space then $\sum_{\alpha\in\ell_A} a_\alpha^{-d} < \infty$. Applying Theorem 2.4, $\mathcal{F}_a$ is nuclear. However, since it is a Fréchet space, $\mathcal{F}'_a$ is also nuclear. □

We now show that the convolution is continuous in the strong topology of a Våge space. Before that, we need the following proposition.

Proposition 3.5. *Let $\mathcal{F}'_a$ be a regular admissible space and let (f_λ) be a net in $\mathcal{F}'_a$. Then $f_\lambda \to f$ in the strong topology if and only if there exists $p \in \mathbb{N}$ such that $f_\lambda, f \in \ell^2_{a^{-p}}$ and $f_\lambda \to f$ in the strong topology of $\ell^2_{a^{-p}}$.*

Proof. Suppose $f_\lambda \to f$ in the strong topology of $\mathcal{F}_a'$. In particular, $\{f_\lambda\}_{\lambda\in\Lambda} \cup \{f\}$ is strongly bounded. Therefore, there exists $p \in \mathbb{N}$ such that

$$\{f_\lambda\}_{\lambda\in\Lambda} \cup \{f\} \subseteq \ell^2_{a^{-p}}$$

(see [14, §5.3 p. 45]). Let B be a bounded set in $\ell^2_{a^p}$, then $B \cap \mathcal{F}_a$ is a dense subset of B. Therefore,

$$\sup_{\varphi\in B} |f_\lambda(\varphi) - f(\varphi)| = \sup_{\varphi\in B\cap\mathcal{F}_a} |f_\lambda(\varphi) - f(\varphi)| \to 0.$$

Thus, $f_\lambda \to f$ in the strong topology of $\ell^2_{a^{-p}}$. The opposite direction is clear. □

Theorem 3.6. *Let $\mathcal{F}_a'$ be a Våge space. Then the convolution is a continuous function $\mathcal{F}_a' \times \mathcal{F}_a' \to \mathcal{F}_a'$ in the strong topology. Hence $(\mathcal{F}_a', +, *)$ is a topological ℂ-algebra.*

Proof. Assuming $((f_\lambda, g_\lambda))_{\lambda\in\Lambda}$ is a net which converges to (f, g) in the strong topology of $\mathcal{F}_a' \times \mathcal{F}_a'$, then in particular, $f_\lambda \to f$ and $g_\lambda \to g$ in the strong topology of $\mathcal{F}_a'$. According to Proposition 3.5, there exist $p, q \in \mathbb{N}$ such that $f_\lambda, f \in \ell^2_{a^{-q}}$ and $g_\lambda, g \in \ell^2_{a^{-q}}$ where $f_\lambda \to f$ in the strong topology of $\ell^2_{a^{-p}}$ and $g_\lambda \to g$ in the strong topology of $\ell^2_{a^{-q}}$. We may assume that $p \geq q + d$. Since $\mathcal{F}_a'$ is a Våge space, $fg_\lambda = f * g_\lambda, fg = f * g \in \ell^2_{a^{-p}}$, and $* : \ell^2_{a^{-q}} \times \ell^2_{a^{-p}} \to \ell^2_{a^{-p}}$ is continuous. Since $(f_\lambda, g_\lambda) \to (f, g)$ in the strong topology of $\ell^2_{a^{-q}} \times \ell^2_{a^{-p}}$, $f_\lambda g_\lambda \to fg$ in the strong topology of $\ell^2_{a^{-p}}$. Again, using Proposition 3.5, we have that $f_\lambda g_\lambda \to fg$ in the strong topology of $\mathcal{F}_a'$. Thus the convolution is strongly continuous. □

We do not know if the convolution is continuous in the weak topology. We end this section with the weak topology analogue of Proposition 3.5.

Proposition 3.7. *Let (f_λ) be a net in $\mathcal{F}_a'$. Then $f_\lambda \to f$ in the weak topology if and only if there exists $p \in \mathbb{N}$ such that $f_\lambda, f \in \ell^2_{a^{-p}}$ and $f_\lambda \to f$ in the weak topology of $\ell^2_{a^{-p}}$.*

Proof. Suppose $f_\lambda \to f$ in the weak topology .In particular, $\{f_\lambda\} \cup \{f\}$ is weakly bounded, and thus strongly bounded (see [14, p. 48]). Therefore, there exists $p \in \mathbb{N}$ such that $\{f_\lambda\} \cup \{f\} \subseteq \ell^2_{a^{-p}}$. Moreover, $f_\lambda \to f$ pointwise on a dense subset of $\ell^2_{a^p}$, that is $\mathcal{F}_a$. Let $\epsilon > 0$ and $\varphi \in \ell^2_{a^p}$. We may choose $\psi \in \mathcal{F}_a$ such that $\|\varphi - \psi\|_p < \frac{\epsilon}{2(\|f\|+\sup_\lambda \|f_\lambda\|)}$, and $\lambda_0 \in \Lambda$ such that for all $\lambda \geq \lambda_0$, $|f_\lambda(\psi) - f(\psi)| < \frac{\epsilon}{2}$. Therefore,

$$\begin{aligned}|f_\lambda(\varphi) - f(\varphi)| &\leq |f_\lambda(\varphi) - f_\lambda(\psi)| + |f_\lambda(\psi) - f(\psi)| + |f(\psi) - f(\varphi)| \\ &\leq (\|f_\lambda\| + \|f\|)\|\varphi - \psi\| + |f_\lambda(\psi) - f(\psi)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Thus $f_\lambda \to f$ in the weak topology of $\ell^2_{a^{-p}}$. The opposite direction is clear. □

4. Invertible elements and power series

The $\mathbb{C}$-algebra $(\mathcal{F}'_a, +, *)$ is in particular a (unit) ring. We denote it by $\mathcal{R}$.

Definition 4.1. *Let $f = (f_\alpha)_{\alpha \in \ell_A} \in \mathcal{R}$. Then, $f_0 \in \mathbb{C}$ is called the generalized expectation of f and is denoted by $E[f]$.*

From this definition we have that

$$E[fg] = E[f]E[g], \quad \forall f, g \in \mathcal{R}.$$

We note that $E : \mathcal{R} \to \mathbb{C}$ is a homomorphism which maps $1_{\mathcal{R}}$ to $1_{\mathbb{C}}$. In the sequel, we will see it is the only homomorphism with this property (see Proposition 4.7).

Proposition 4.2. *Let M be a positive number. Then, for any $f \in \mathcal{R}$ such that $E[f] = 0$ there is $q \in \mathbb{N}$ such that $\|f\|_q < M$.*

Proof. Let $f = (f_\alpha) \in \ell^2_{a^{-p}}$ with $f_0 = 0$. Since $a_{e_n} > 1$ for $n = 1, 2, \ldots$, we have

$$a_\alpha = \prod_{n \in A} a_{e_n}^{\alpha_n} > 1 \quad \text{for all } \alpha \neq (0, 0, \ldots).$$

Therefore, for all $\alpha \in \ell_A$ $\lim_{q \to \infty} |f_\alpha|^2 a_\alpha^{-q} = 0$ (recall $f_0 = 0$) and for all $q > p$, $|f_\alpha|^2 a_\alpha^{-q} \leq |f_\alpha|^2 a_\alpha^{-p}$, whereas $\sum_{\alpha \in \ell_A} |f_\alpha|^2 a_\alpha^{-p} = \|f\|_p^2 < \infty$. Thus, the dominated convergence theorem implies

$$\lim_{q \to \infty} \|f\|_q^2 = \lim_{q \to \infty} \sum_{\alpha \in \ell_A} |f_\alpha|^2 a_\alpha^{-q} = \sum_{\alpha \in \ell_A} \lim_{q \to \infty} |f_\alpha|^2 a_\alpha^{-q} = 0.$$

□

Definition 4.3. *The n-th convolution power $f^n = f^{*n}$ of f is defined inductively as follows:*

$$f^n = \begin{cases} 1 & \text{if } n = 0 \\ f f^{(n-1)} & \text{if } n > 0 \end{cases}$$

Proposition 4.4. *Let $\mathcal{R}$ be a Våge space of index d and let f be in $\ell^2_{a^{-p}}$. Then $\forall n \in \mathbb{N}$, $f^n \in \ell^2_{a^{-(p+d)}}$. Moreover, $\|f^n\|_{p+d} \leq A(d)^n \|f\|_p^n$.*

Proof. Obviously, $f^0 = 1 \in \ell^2_{a^{-(p+d)}}$, and $\|f^0\|_{p+d} = A(d)^0 \|f\|_p^0$.
By induction, if we assume $f^n \in \mathcal{R}$, we get $f^{(n+1)} = ff^n \in \mathcal{R}$, and

$$\begin{aligned} \|f^{(n+1)}\|_{p+d} &= \|ff^n\|_{p+d} \\ &\leq A(d)\|f\|_p \|f^n\|_{p+d} \\ &\leq A(d)^n \|f\|_p^{n+1} < \infty \end{aligned}$$

□

More generally, given a polynomial $p(z) = \sum_{n=0}^N p_n z^n$ ($p_n \in \mathbb{C}$), we define its convolution version $p : \mathcal{R} \to \mathcal{R}$ by

$$p(f) = \sum_{n=0}^N p_n f^n$$

By Proposition 4.4, we have that $p(f) \in \mathcal{R}$ for $f \in \mathcal{R}$. The following proposition considers the case of power series.

Proposition 4.5. *Let $\phi(z) = \sum_{n\in\mathbb{N}} \phi_n z^n$ be a power series (with complex coefficients) which converges absolutely in the open disk with radius R. Then for any $f \in \mathcal{R}$ such that $|E[f]| < \frac{R}{A(d)}$ it holds that*

$$\phi(f) = \sum_{n\in\mathbb{N}} \phi_n f^n \in \mathcal{R}.$$

Proof. Applying Proposition 4.2, there exists q such that $\|f - E(f)\|_q < \frac{R}{A(d)} - |E[f]|$. Therefore,

$$\|f\|_q \leq \|f - E(f)1_{\mathcal{R}}\|_q + |E(f)| < \frac{R}{A(d)}.$$

Then by Proposition 4.4, for all $p \geq q + d$,

$$\begin{aligned} \sum_{n\in\mathbb{N}} |\phi_n| \|f^n\|_p &\leq \sum_{n\in\mathbb{N}} |\phi_n| A(d)^n \|f\|_q^n \\ &= \sum_{n\in\mathbb{N}} |\phi_n| (A(d)\|f\|_q)^n \\ &< \infty. \end{aligned}$$

Since $\ell^2_{a^{-p}}$ is a Hilbert space, $\phi(f) = \sum_{n\in\mathbb{N}} \phi_n f^n \in \ell^2_{a^{-p}}$. Thus, $\phi(f) \in \mathcal{R}$. □

Proposition 4.6. *The element $f \in \mathcal{R}$ is invertible if and only if $E[f]$ is invertible.*

Proof. If $E[f] \neq 0$, we can assume that $E[f] = 1$. By the Proposition 4.5 we have that $\sum_{n\in\mathbb{N}}(1-f)^n \in \mathcal{R}$. Furthermore,

$$f\left(\sum_{n\in\mathbb{N}}(1-f)^n\right) = 1.$$

Conversely, assume f invertible. Then there exists $f^{-1} \in \mathcal{R}$ such that $ff^{-1} = 1$. Hence, $E[f]E[f^{-1}] = E[ff^{-1}] = 1$. □

Proposition 4.7. *Let $\mathcal{R}$ be a Våge space. Then the following properties hold:*

(a) $GL(\mathcal{R})$ is open.
(b) The spectrum of $f \in \mathcal{R}$, $\sigma(f) = \{\lambda \in \mathbb{C} : f - \lambda 1_{\mathcal{R}}$ is not invertible $\}$ is the singleton $\{E[f]\}$.
(c) E is unique as a homomorphism $\mathcal{R} \to \mathbb{C}$ mapping $1_{\mathcal{R}}$ to $1_{\mathbb{C}}$.

Proof.

(a) By Proposition 4.6, we have that $\{f \in \mathcal{R} : E[f] \neq 0\}$ is the set of all invertible elements in $\mathcal{R}$. In other words, $GL(\mathcal{R}) = E^{-1}(GL(\mathbb{C}))$. In particular, since E is continuous, $GL(\mathcal{R})$ is open.
(b) Clearly, $f - \lambda 1_{\mathcal{R}}$ does not have an inverse if and only if $\lambda = E(f)$.
(c) Let $\varphi : \mathcal{R} \to \mathbb{C}$ be a homomorphism mapping $1_{\mathcal{R}}$ to $1_{\mathbb{C}}$ and let $f \in \mathcal{R}$. Since $\varphi(f - \varphi(f)1_{\mathcal{R}}) = 0$, $\varphi(f) \in \sigma(f)$, that is $\varphi(f) = E[f]$.

□

The notion of rational functions plays a key role in the theory of linear systems over commutative rings. See Section 8 below for a discussion and references. Here, using Proposition 4.6 we define rational functions with coefficients in $\mathcal{R}$ as in [5] and [2, Section 3] as elements in the quotient ring of $\mathcal{R}[z]$. Other characterizations can be given in terms of finite dimensional backward shift invariant spaces and realizations. See [5]. We mention that in the non-commutative case similar characterisations occur. See for instance [3, 6].

Definition 4.8. *A rational function with coefficients in $\mathcal{R}^{n\times m}$ is an expression of the form*

$$R(z) = p(z)(q(z))^{-1} \tag{4.1}$$

where $p \in (\mathcal{R}[z])^{n\times m}$, and $0 \neq q \in \mathcal{R}[z]$.

Note that, for any $f \in \mathcal{R}$ such that $E(q(f)) \neq 0$, $R(f)$ is well defined as an element of $\mathcal{R}^{n\times m}$.

In Section 8 we discuss some problems in the theory of linear systems using rational functions with coefficients in $\mathcal{R}$.

To conclude this section, we remark that one may consider classical interpolation problems (of the kind developed in [8]) in the present setting. For instance one can consider the following Nevanlinna-Pick interpolation problem:

Problem 4.9. *Given $N \in \mathbb{N}$ and N pairs of points $(a_1, b_1), \ldots, (a_N, b_N)$ in $\mathcal{R}^2$, find all power series ϕ such that*

$$\phi(a_j) = b_j, \quad j = 1, \ldots, N,$$

and such that, moreover, the function $z \mapsto E(\phi(z))$ is analytic and contractive in the open unit disk.

This problem, as well as more general interpolation problems, have been studied in [2] when $\mathcal{R}$ is the Kondratiev space $\mathcal{S}_{-1}$. Two important tools used there, and which are still valid in the setting of Våge spaces are the permanence of algebraic identities (see [7, p. 456]) and the definition and properties of analytic functions with values in the dual of a countably normed Hilbert space.

5. Tensor product of Våge spaces

When one considers the tensor product $E \otimes F$ of two locally convex Hausdorff spaces E, F, some "natural" topologies may be considered. Two such topologies are the π-topology and the ϵ-topology (see [15], [30, Chapter 43, p. 434]). These topologized tensor products of E and F are denoted respectively by $E \otimes_\pi F$ and $E \otimes_\epsilon F$. The completions of the tensor product of E and F with respect to the π-topology and the ϵ-topology, will then be denoted by $E \hat{\otimes}_\pi F$ and $E \hat{\otimes}_\epsilon F$ respectively. However, when it comes to nuclear spaces, things are getting much easier.

Theorem 5.1. *Let E be a locally convex Hausdorff space. Then, E is nuclear if and only if for every locally convex Hausdorff space F, $E \hat{\otimes}_\pi F = E \hat{\otimes}_\epsilon F$.*

A proof can be found in [30, Theorem 50.1 p. 511]. Thanks to this last theorem, we simply denote

$$E \hat{\otimes} F \overset{\text{def.}}{=} E \hat{\otimes}_\pi F = E \hat{\otimes}_\epsilon F$$

when one of the spaces E or F is nuclear. We also denote the usual tensor product of two Hilbert spaces E and F by $E \otimes F$. We recall the following result of Grothendieck on tensor products, see [15]:

Proposition 5.2. *Let E be a complete locally convex space of functions defined on a set T, such that its topology is finer than the pointwise convergence topology, and assume E to be nuclear. Then for every complete locally convex space F, the tensor product $E\hat{\otimes}F$ can be interpreted as the space of all functions $f : T \to F$ such that for all $y' \in F'$, $t \mapsto \langle y', f(t)\rangle_{F',F}$ is a function of E.*

For $A, B \subseteq \mathbb{N}$, let $a : \ell_A \to \mathbb{R}$ and $b : \ell_B \to \mathbb{R}$ be two positive functions, such that the associated countably Hilbert spaces $\mathcal{F}_a = \bigcap_{p\in\mathbb{N}} \ell^2_{a^p}$ and $\mathcal{F}_b = \bigcap_{p\in\mathbb{N}} \ell^2_{b^p}$ are nuclear. Since $\mathcal{F}_a$ is a complete locally convex space of functions defined on the free commutative monoid ℓ_A, and since its topology is finer than the pointwise topology, that is since

$$|\varphi_{\alpha\lambda} - \varphi_\alpha|^2 a^p_\alpha \leq \sum_{\alpha\in\ell_A} |\varphi_{\alpha\lambda} - \varphi_\alpha|^2 a^p_\alpha,$$

$\mathcal{F}_a\hat{\otimes}\mathcal{F}_b$ can be interpreted as the space of all elements of the form $\psi_{\beta,\alpha}$ such that for all $f = (f_\beta)_{\beta\in\ell_B} \in \mathcal{F}'_b$, $(\langle f, \psi_{\beta,\alpha}\rangle_{\mathcal{F}'_b,\mathcal{F}_b})_{\alpha\in\ell_A} \in \mathcal{F}_a$.

Theorem 5.3. *It holds that*

$$\mathcal{F}_a\hat{\otimes}\mathcal{F}_b = \bigcap_{p,q} \ell^2_{a^p}\otimes\ell^2_{b^q}.$$

Proof. Let $\psi = (\psi_{\beta,\alpha})$ be an element of $\mathcal{F}_a\hat{\otimes}\mathcal{F}_b$ with $\psi_{\beta,\alpha} > 0$ for any $\alpha \in \ell_A, \beta \in \ell_B$. Then for all $f = (f_\beta)_{\beta\in\ell_B} \in \mathcal{F}'_b$, $(\langle f, \psi_{\beta,\alpha}\rangle_{\mathcal{F}'_b,\mathcal{F}_b})_{\alpha\in\ell_A} \in \mathcal{F}_a$. In particular, we may choose $f_\beta = b_\beta^{\frac{q}{2}}$ for any $q \in \mathbb{N}$. Therefore,

$$\begin{aligned}\sum_{\alpha\in\ell_A}\sum_{\beta\in\ell_B} |\psi_{\beta,\alpha}|^2 b^q_\beta a^p_\alpha &\leq \sum_{\alpha\in\ell_A}\left|\sum_{\beta\in\ell_B} b_\beta^{\frac{q}{2}}\psi_{\beta,\alpha}\right|^2 a^p_\alpha \\ &= \sum_{\alpha\in\ell_A}\left|\langle (b_\beta^{\frac{q}{2}}), (\psi_{\beta,\alpha})\rangle_{\mathcal{F}'_b,\mathcal{F}_b}\right|^2 a^p_\alpha < \infty\end{aligned}$$

Thus, $(\psi_{\beta,\alpha}) \in \bigcap_{p,q} \ell^2_{a^p}\otimes\ell^2_{b^q}$. In case $(\psi_{\beta,\alpha}) \in \mathcal{F}_a\hat{\otimes}\mathcal{F}_b$ is an arbitrary element, then applying the last inequality to its positive real part, negative real part, positive imaginary part and negative imaginary part yields the requested result.

To prove the opposite direction, take $(\psi_{\beta,\alpha}) \in \bigcap_{p,q} \ell^2_{a^p} \otimes \ell^2_{b^q}$. Then for all $f = (f_\beta)_{\beta \in \ell_B} \in \mathcal{F}'_b$ there exists q such that $f \in \ell^2_{b^{-q}}$. Thus,

$$\begin{aligned}
\left| \langle f, (\psi_{\beta,\alpha}) \rangle_{\mathcal{F}'_b, \mathcal{F}_b} \right|^2 &= \left| \sum_{\beta \in \ell_B} \overline{f_\beta} \psi_{\beta,\alpha} \right|^2 \\
&= \left| \sum_{\beta \in \ell_B} \overline{f_\beta} b_\beta^{-\frac{q}{2}} \psi_{\beta,\alpha} b_\beta^{\frac{q}{2}} \right|^2 \\
&\leq \left(\sum_{\beta \in \ell_B} |f_\beta|^2 b_\beta^{-q} \right) \left(\sum_{\beta \in \ell_B} |\psi_{\beta,\alpha}|^2 b_\beta^q \right) \\
&= \|f\|^2_{\ell^2_{b^{-q}}} \sum_{\beta \in \ell_B} |\psi_{\beta,\alpha}|^2 b_\beta^q
\end{aligned}$$

Hence for all $p \in \mathbb{N}$,

$$\sum_{\alpha \in \ell_A} \left| \langle f, (\psi_{\beta,\alpha}) \rangle_{\mathcal{F}'_b, \mathcal{F}_b} \right|^2 a_\alpha^p \leq \|f\|^2_{\ell^2_{b^{-q}}} \sum_{\alpha \in \ell_A} \sum_{\beta \in \ell_B} |\psi_{\beta,\alpha}|^2 b_\beta^q a_\alpha^p < \infty$$

and so $(\psi_{\beta,\alpha}) \in \mathcal{F}_a \hat{\otimes} \mathcal{F}_b$. □

Proposition 5.4. *It holds that*

$$\bigcap_{p,q} \ell^2_{a^p} \otimes \ell^2_{b^q} = \bigcap_{p} \ell^2_{a^p} \otimes \ell^2_{b^p}$$

Proof. One direction is clear. The other direction follows from the inclusion

$$\ell^2_{a^{\max\{p,q\}}} \otimes \ell^2_{b^{\max\{p,q\}}} \subseteq \ell^2_{a^p} \otimes \ell^2_{b^q}$$

□

Applying Theorem 5.3 and Proposition 5.4 ,we obtain:

$$\mathcal{F}_a \hat{\otimes} \mathcal{F}_b = \bigcap_{p \in \mathbb{N}} \left\{ (\psi_{\alpha,\beta}) : \sum_{(\alpha,\beta) \in \ell_A \times \ell_B} |\psi_{\beta,\alpha}|^2 (b_\beta a_\alpha)^p < \infty \right\}. \tag{5.1}$$

Now, we can concatenate the indices in an obvious way. We define $C = 2B \cup (2A - 1)$ (disjoint union), $P_B : C \to B$, $P_A : C \to A$ the appropriate projections, $c_\gamma = b_{P_B(\gamma)} a_{P_A(\gamma)}$, and $\psi_\gamma = \psi_{P_B(\gamma), P_A(\gamma)}$. Therefore, we may write

$$\mathcal{F}_a \hat{\otimes} \mathcal{F}_b = \bigcap_{p \in \mathbb{N}} \left\{ (\psi_\gamma)_{\gamma \in \ell_C} : \sum_{\gamma \in \ell_C} |\psi_\gamma|^2 c_\gamma^p < \infty \right\},$$

and

$$(\mathcal{F}_a \hat{\otimes} \mathcal{F}_b)' = \bigcup_{p \in \mathbb{N}} \left\{ (\psi_\gamma)_{\gamma \in \ell_C} : \sum_{\gamma \in \ell_C} |\psi_\gamma|^2 c_\gamma^{-p} < \infty \right\}.$$

Proposition 5.5. *Let $a : \ell_A \to \mathbb{R}$ and $B : \ell_B \to \mathbb{R}$ be two admissible positive functions. Then:*

(a) $c : C \to \mathbb{R}$ (where $C = 2B \cup (2A - 1)$ and $c_\gamma = c_{P_B(\gamma)} a_{P_A(\gamma)}$) is admissible.
(b) If a and b are both d-regular, then so is c.
(c) If a and b are both superexponential, then so is c.

Proof. c is admissible, since $c_0 = b_0 a_0 = 1$ and $c_{e_n} > 1$ for all $n \in C$. Moreover,

$$\sum_{n \in C} \frac{1}{c_{e_n}^d - 1} = \sum_{n \in A} \frac{1}{a_{e_n}^d - 1} + \sum_{n \in B} \frac{1}{b_{e_n}^d - 1},$$

and hence d-regularity of both a and b yields d-regularity of c. Finally, if both a and b are superexponential, clearly so is c. □

Finally, we give the following theorem.

Theorem 5.6. *Let E, F be two Fréchet spaces. If E is nuclear, we have the canonical isomorphism*

$$(E \hat{\otimes} F)' = E' \hat{\otimes} F'$$

A proof, in case both E and F are nuclear, is given in [30, (50.19), p. 525]. We can now state:

Theorem 5.7. *A tensor product of two Våge spaces is also a Våge space.*

Proof. Applying (5.1), Proposition 5.5 and Theorem 3.3, $(\mathcal{F}_a \hat{\otimes} \mathcal{F}_b)'$ is a Våge space. Theorem 5.6 yields the requested result. □

6. An extension of the space of tempered distributions

In this section we consider the special case $\ell_A = \mathbb{N}_0$ (i.e. $A = \{1\}$), and

$$a_n = (n + 1)^2.$$

The corresponding space $\mathcal{F}_a$ (defined by (2.8)) is identified with the Schwartz space $\mathscr{S}$ of rapidly decreasing smooth functions, and its dual is the space $\mathscr{S}'$ of tempered distributions. We will show (see Proposition 6.1 below) that $\mathscr{S}'$ is a regular admissible space, but it is not a Våge space. We will also construct a Våge space containing $\mathscr{S}'$.

We recall (see [26, Chapter IV, Section 2, p. 303], [29, p. 105]) that the *Hermite polynomials* $h_n(x)$ are defined by

$$h_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right), \quad n = 0, 1, \ldots, \tag{6.1}$$

Various notations and conventions are given for these polynomials. See the discussion on the end of page 105 of [29]. In particular, the multiplicative factor $(-1)^n$ (which does not appear in Sansone's book [26]) insures that the factor of x^n in h_n is positive. The *Hermite functions* $\xi_n(x)$ are defined by

$$\xi_n(x) = \pi^{-\frac{1}{4}}(2^n n!)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} h_n(x), \quad n = 0, 1, 2, \ldots. \tag{6.2}$$

The Hermite functions $(\xi_n)_{n\in\mathbb{N}_0}$ form an orthonormal basis of $\mathbf{L}_2(\mathbb{R}, dx)$. The *Schwartz space* $\mathscr{S}$ of smooth rapidly decreasing functions on $\mathbb{R}$ is defined by

$$\mathscr{S} = \left\{ f \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} \left| x^p f^{(q)}(x) \right| < \infty \text{ for all } p, q \in \mathbb{N}_0 \right\}$$

The Hermite functions are elements in the Schwartz space, and we have (see [24, Theorem V.13 p. 143]):

$$\mathscr{S} = \left\{ f = \sum_{n\in\mathbb{N}_0} f_n \xi_n : \sum_{n\in\mathbb{N}_0} |f_n|^2 (n+1)^{2p} < \infty \text{ for all } p \in \mathbb{N} \right\}.$$

Identifying $\sum_{n\in\mathbb{N}_0} f_n\xi_n$ with $(f_n)_{n\in\mathbb{N}_0}$ allows to identify $\mathbf{L}_2(\mathbb{R}, dx)$ with $\ell^2(\mathbb{N}_0)$ and $\mathscr{S}$ with $\mathcal{F}_a$.

Proposition 6.1. *$\mathscr{S}'$ is a regular admissible space which is nuclear, but is not a Våge space.*

Proof. First, we note that defining $a_n = (n+1)^2$ implies that $a : n \mapsto a_n$ is a 1-regular admissible function. It is indeed admissible since $a_0 = 1$ and $a_1 = 4 > 1$, and it is indeed 1-regular, since the sum in (2.2) is over the finite set $A = \{1\}$ and in particular does converge. Therefore, $\mathscr{S}'$ is a 1-regular admissible space. Since, $\sum_{n\in\mathbb{N}_0} \left((n+1)^2\right)^{-1} < \infty$, $\mathscr{S}$ is nuclear, and hence $\mathscr{S}'$ is also nuclear (of course, the nuclearity of $\mathscr{S}$ and $\mathscr{S}'$ is a standard result; see for instance [30]). Since a is not superexponential, that is

$$(n+1)^2(m+1)^2 \not\leq (n+m+1)^2,$$

$\mathscr{S}'$ is not a Våge space. $\square$

We define the following subspace of $\mathscr{S}$

$$\mathscr{G}=\left\{\sum_{n=0}^{\infty} f_n\xi_n : \sum_{n\in\mathbb{N}_0} |f_n|^2 2^{np}<\infty \text{ for all } p\in\mathbb{N}\right\}.$$

Proposition 6.2. *$\mathscr{G}'$ is a Våge space containing the Schwartz space $\mathscr{S}'$ of tempered distributions.*

Proof. Using the identification of $\sum_{n\in\mathbb{N}_0} f_n\xi_n$ with $(f_n)_{n\in\mathbb{N}_0}$, and defining $a_n=2^n$, we have that $\mathscr{G}$ is the corresponding countably Hilbert space $\mathcal{F}_a$ associated a (and as before, $\mathbf{L}_2(\mathbb{R},dx)$ is identified with $\ell^2(\mathbb{N}_0)$). Clearly, a is a 1-regular admissible function. It is indeed admissible since $a_0=1$ and $a_1=2>1$, and it is indeed 1-regular, since the sum in (2.2) is over the finite set $A=\{1\}$ and in particular does converge. Therefore, $\mathscr{G}'$ is a 1-regular admissible space. Since $a:n\mapsto 2^n$ is an exponential function, $\mathscr{G}'$ is a Våge space and is in particular nuclear. Moreover, the natural embeddings $\mathscr{G}\subseteq\mathscr{S}$ and $\mathscr{S}'\subseteq\mathscr{G}'$ are clearly continuous. Hence $\mathscr{G}$ is a closed subspace of $\mathscr{S}$, and $\mathscr{S}'$ is a closed subspace of $\mathscr{G}'$. ☐

Theorem 6.3. *$\mathscr{G}$ is the space of all entire functions $f(z)$ such that*

$$\iint_{\mathbb{C}} |f(z)|^2 e^{\frac{1-2^{-p}}{1+2^{-p}}x^2-\frac{1+2^{-p}}{1-2^{-p}}y^2}dxdy<\infty \quad \text{for all } p\in\mathbb{N}.$$

In the proof we make use of two results of Hille. The first result appear in [20, formula (1.3), p. 81] and [19, Theorem 2.2 p. 885]. For the second formula, see [20, formula (2.1) p. 82] and [18, p. 439-440]. In that last paper one can also find a history of the formula.

Theorem 6.4. *(Hille, [20]) The domain of absolute convergence of the series $\sum_{n=0}^{\infty} F_n\xi_n(z)$ is the strip $S_\tau=\{z\in\mathbb{C}:|\mathrm{Im}(z)|<\tau\}$, where*

$$\tau=-\limsup_{n\to\infty}(2n+1)^{-\frac{1}{2}}\log|F_n|.$$

Theorem 6.5. *(Hille, [18]) The series $\sum_{n=0}^{\infty}\xi_n(u)\xi_n(v)s^n$ converges for arbitrary complex values of u and v when $|s|<1$, and*

$$\sum_{n=0}^{\infty}\xi_n(u)\xi_n(v)s^n=\pi^{-\frac{1}{2}}(1-s^2)^{-\frac{1}{2}}e^{-\frac{(1+s^2)(u^2+v^2)-4svu}{2(1-s^2)}}.$$

Furthermore, we make use of the easy following proposition. See [25, §6, p. 61]. The space Γ_α bears various names, and in particular is called the Fock space.

Proposition 6.6. *For all* $0 < \alpha \leq 1$,

$$\Gamma_\alpha = \left\{ f \text{ is entire } : \frac{\alpha}{\pi} \iint_{\mathbb{C}} |f(z)|^2 e^{-\alpha|z|^2} dxdy < \infty \right\}$$

is a Hilbert space with a reproducing kernel $K_\alpha(z,w) = e^{\alpha\overline{w}z}$.

Proof of Theorem 6.3. Let $p \in \mathbb{N}$. For each $f \in \mathscr{G}$, $f = \sum_{n=0}^\infty f_n\xi_n$ whereas $\sum_{n=0}^\infty |f_n|^2 2^{np} < \infty$ for all $p \in \mathbb{N}$. In particular, $\lim_{n\to\infty} |f_n|^2 2^n = 0$. Therefore, for every n large enough, $\log|f_n| < -n\log\sqrt{2}$. Thus, in the notations of Theorem 6.4,

$$\tau = -\limsup_{n\to\infty} (2n+1)^{-\frac{1}{2}} \log|f_n| \geq \liminf_{n\to\infty} (2n+1)^{-\frac{1}{2}} n\log\sqrt{2} = \infty.$$

Therefore, denoting by $\mathscr{G}_p$ the space of all functions $f = \sum_{n=0}^\infty f_n\xi_n$ subject to $\sum_{n=0}^\infty |f_n|^2 2^{np} < \infty$ (i.e., $\mathscr{G}_p \cong \ell^2_{a^p}$), it is a Hilbert space of entire functions, and in particular, $\mathscr{G} = \bigcap_{p\in\mathbb{N}} \mathscr{G}_p$ is a countably Hilbert space of entire functions.

Now, since $\left(\xi_n 2^{-\frac{np}{2}}\right)_{n\in\mathbb{N}_0}$ is an orthonormal basis for $\mathscr{G}_p$, and denoting $s = 2^{-p}$, the reproducing kernel of the Hilbert space $\mathscr{G}_p$ is given by

$$G(z,w) = \sum_{n=0}^\infty \xi_n(z)\overline{\xi_n(w)} 2^{-np} = \sum_{n=0}^\infty \xi_n(z)\xi_n(\overline{w}) s^n.$$

Applying Theorem 6.5, we have that

$$G(z,w) = \pi^{-\frac{1}{2}} (1-s^2)^{-\frac{1}{2}} e^{-\frac{(1+s^2)(z^2+\overline{w}^2)-4sz\overline{w}}{2(1-s^2)}}$$

Denoting $r(z) = \pi^{-\frac{1}{4}}(1-s^2)^{-\frac{1}{4}} e^{-\frac{1+s^2}{2(1-s^2)}z^2}$, considering the kernel $K_\alpha(z,w) = e^{\alpha\overline{w}z}$ with its associated Hilbert space Γ_α for $\alpha = \frac{2s}{1-s^2}$ (see Proposition 6.6), we have that

$$G(z,w) = r(z)K_\alpha(z,w)r(\overline{w}).$$

Therefore, the space $\mathscr{G}_p$ is equal to the space of functions of the form $f = rg$, with $g \in \Gamma_\alpha$ and norm

$$\|f\|_{\mathscr{G}_p} = \|g\|_{\Gamma_\alpha}.$$

We note that for $z = x + iy$,

$$\begin{aligned} 2 \cdot \frac{1+s^2}{2(1-s^2)} \mathrm{Re}(z^2) - \frac{2s}{1-s^2}|z|^2 &= \frac{(1+s^2)(x^2-y^2)}{1-s^2} - \frac{2s(x^2+y^2)}{1-s^2} \\ &= \frac{1-s}{1+s}x^2 - \frac{1+s}{1-s}y^2. \end{aligned}$$

Thus,

$$|r(z)|^{-2} = \sqrt{\pi(1-s^2)} e^{\frac{1-s}{1+s}x^2 - \frac{1+s}{1-s}y^2},$$

and, with $K_p = \frac{2^{1-p}}{\sqrt{\pi(1-2^{-2p})}}$,

$$\mathscr{G}_p = \left\{ f \text{ is entire } : \|f\|^2_{\mathscr{G}_p} = K_p \iint_{\mathbb{C}} |f(z)|^2 \, e^{\frac{1-2^{-p}}{1+2^{-p}}x^2 - \frac{1+2^{-p}}{1-2^{-p}}y^2} dxdy < \infty \right\},$$

and in particular $\mathscr{G} = \bigcap_{p\in\mathbb{N}} \mathscr{G}_p$ is the space of all entire functions $f(z)$ subject to

$$\iint_{\mathbb{C}} |f(z)|^2 \, e^{\frac{1-2^{-p}}{1+2^{-p}}x^2 - \frac{1+2^{-p}}{1-2^{-p}}y^2} dxdy < \infty \quad \text{for all } p \in \mathbb{N}.$$

□

We note that there are functions in the Schwartz space $\mathscr{S}$ which have no analytic continuation to an entire function but only to a holomorphic function on some strip. For example, one may consider the function

$$F(x) = \sum_{n\in\mathbb{N}_0} e^{-\sqrt{2n+1}} \xi_n(x) \in \mathscr{S}$$

It is indeed in the Schwartz space since $\sum_{n\in\mathbb{N}_0} e^{-2\sqrt{2n+1}}(n+1)^{2p} < \infty$ for all $p \in \mathbb{N}$. However,

$$\tau = -\limsup_{n\to\infty} (2n+1)^{-\frac{1}{2}} \log|F_n| = \liminf_{n\to\infty} (2n+1)^{-\frac{1}{2}} (2n+1)^{\frac{1}{2}} = 1.$$

Furthermore, clearly there are functions in the Schwartz space which have no analytic continuation to any holomorphic function, e.g., functions with compact support.

Remark 6.7. *As mentioned in the introduction, connections with the theory of hyperfunctions will be considered elsewhere.*

Remark 6.8. *Having now $\mathscr{G}'$ at hand, one can consider the tensor product $\mathscr{G}' \otimes \mathcal{S}_{-1}$, where $\mathcal{S}_{-1}$ is the Kondratiev space of stochastic distributions (see Section 7). By Section 5 it is a Våge space, and can be an appropriate setting to study stochastic linear systems. This will be done in a future publication.*

Remark 6.9. *If we define $a_0 = 1$ and for $n > 0$ $a_n = 2^{2^n}$, then a is again a 1-regular admissible function. The associated countably Hilbert space $\mathcal{F}_a$ is a space of entire functions. Moreover, since the function a is superexponential its dual is clearly a Våge space. We can define many other Våge spaces in a similar manner.*

7. The Kondratiev spaces

In this section we consider the special case $\ell_A = \ell$ (i.e. $A = \mathbb{N}$), and

$$a_\alpha = (2\mathbb{N})^\alpha.$$

Then the corresponding space $\mathcal{F}_a$ (defined by (2.8)) is identified with the Kondratiev space of Gaussian test functions $\mathcal{S}_1$, and its dual is the Kondratiev space of Gaussian stochastic distributions $\mathcal{S}_{-1}$. We will show (see Proposition 6.1 below) that $\mathcal{S}_{-1}$ is a Våge space. We also consider the Kondratiev space of Poissonian stochastic distributions, and show that it is also a Våge space.

We first need to recall a few definitions pertaining to the white noise space. The function $s \mapsto e^{-\frac{1}{2}\|s\|^2_{\mathbf{L}_2(\mathbb{R},dx)}}$ is positive definite on the Schwartz space of real-valued functions $\mathscr{S}_\mathbb{R}$, and continuous at the origin. The Bochner-Minlos theorem (see for instance [27, p. 10-11]) insures the existence of a probability measure $d\mu$ on the Borel σ-algebra of the dual space $\mathscr{S}'_\mathbb{R}$, such that

$$e^{-\frac{1}{2}\|s\|^2_{\mathbf{L}_2(\mathbb{R},dx)}} = \int_{\mathscr{S}'_\mathbb{R}} e^{i\langle s', s\rangle} d\mu(s') \quad \text{for all } s \in \mathscr{S}_\mathbb{R},$$

where the brackets denote the duality between $\mathscr{S}_\mathbb{R}$ and $\mathscr{S}'_\mathbb{R}$. This equality induces an isometric map

$$s \mapsto Q_s, \quad \text{where} \quad Q_s(s') = \langle s', s\rangle \quad (s \in \mathscr{S}_\mathbb{R}, s' \in \mathscr{S}'_\mathbb{R})$$

from $\mathscr{S}_\mathbb{R} \subset \mathbf{L}_2(\mathbb{R}, dx)$ into $\mathbf{L}_2(\mathscr{S}'_\mathbb{R}, \mathcal{B}, \mu)$.

The space $\mathcal{W} = \mathbf{L}_2(\mathscr{S}'_\mathbb{R}, \mathcal{B}, \mu)$ is called the Gaussian white noise space. We recall that the *Hermite polynomial functionals* $(H_\alpha)_{\alpha\in\ell} \subseteq \mathcal{W}$, which are defined by

$$H_\alpha(s') = \prod_{k=1}^{\infty} h_{\alpha_k}(Q_{\xi_{k-1}}(s')) \quad (s' \in \mathscr{S}'_\mathbb{R}),$$

form an orthogonal basis of $\mathcal{W}$, where (h_k) and (ξ_k) denote respectively the Hermite polynomials and the Hermite functions (see (6.1) and (6.2)). More precisely,

$$\mathcal{W} = \left\{ \sum_{\alpha\in\ell} f_\alpha H_\alpha : \sum_{\alpha\in\ell} |f_\alpha|^2 \alpha! < \infty \right\}, \tag{7.1}$$

where $\alpha! = \prod_{k=1}^{\infty}(\alpha_k)!$. The Kondratiev space of Gaussian test function $\mathcal{S}_1$ is defined by

$$\mathcal{S}_1 = \left\{ \sum_{\alpha\in\ell} f_\alpha H_\alpha : \sum_{\alpha\in\ell} |f_\alpha|^2 (2\mathbb{N})^{\alpha p} (\alpha!)^2 < \infty \text{ for all } p \in \mathbb{N} \right\}.$$

Clearly, $\mathcal{W}$ can be identified with ℓ^2 using the isometry

$$\sum_{\alpha\in\ell} f_\alpha H_\alpha \mapsto \left(f_\alpha (\alpha!)^{\frac{1}{2}} \right)_{\alpha\in\ell_A}.$$

In a similar way, defining

$$\mathcal{S}_{1,p} = \left\{ \sum_{\alpha\in\ell} f_\alpha H_\alpha : \sum_{\alpha\in\ell} |f_\alpha|^2 (2\mathbb{N})^{\alpha p} (\alpha!)^2 < \infty \right\},$$

it can be identified with $\ell^2(a^p)$ (for $a_\alpha = (2\mathbb{N})^\alpha$). Hence,

$$\mathcal{F}_a = \bigcap_{p\in\mathbb{N}} \ell^2(a^p) \cong \bigcap_{p\in\mathbb{N}} \mathcal{S}_{1,p} = \mathcal{S}_1.$$

The *Wick product* of two formal series $f = \sum_{\alpha\in\ell_A} f_\alpha H_\alpha$ and $g = \sum_{\alpha\in\ell_A} g_\alpha H_\alpha$, denoted by $\diamond$, was introduced by Hida and Ikeda, see [17]. It is defined by

$$f \diamond g = \sum_{\gamma\in\ell} \left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \right) H_\gamma. \tag{7.2}$$

The Kondratiev space of Gaussian stochastic distributions $\mathcal{S}_{-1}$, which is the dual of $\mathcal{S}_1$ and can be defined by

$$\mathcal{S}_{-1} = \left\{ \sum_{\alpha\in\ell} f_\alpha H_\alpha : \sum_{\alpha\in\ell} |f_\alpha|^2 (2\mathbb{N})^{-\alpha p} < \infty \text{ for some } p \in \mathbb{N} \right\} \cong \mathcal{F}'_a,$$

is not only closed under the Wick product, but as we present in the following proposition, also a Våge space. The following proposition is a result of Våge, proved in 1996, see [31] and [21, p. 118].

Proposition 7.1. *The Kondratiev space of Gaussian stochastic distributions $\mathcal{S}_{-1} = \mathcal{S}'_1$ is a Våge space.*

Proof. We note that $a : \alpha \mapsto (2\mathbb{N})^\alpha$ is a 2-regular admissible positive function. It is admissible since $a_0 = (2n)^0 = 1$ and $a_{e_n} = 2n > 1$, and it is 2-regular since $\sum_{n\in\mathbb{N}} \frac{1}{(2n)^2-1} < \infty$. Therefore,$\mathcal{S}_{-1} \cong \mathcal{F}'_a$ is a 2-regular admissible space. Since a is exponential, $\mathcal{S}_{-1}$ is a Våge space. □

Hida's theory can also be applied to Poisson processes. One then considers the function

$$\exp\left[\int_{\mathbb{R}}(e^{is(x)}-1)dx\right] \quad (s\in\mathscr{S}_{\mathbb{R}})$$

which is positive definite on $\mathscr{S}_{\mathbb{R}}$, and continuous at the origin. Here too, the Bochner-Minlos theorem insures the existence of a probability measure π on $\mathscr{S}_{\mathbb{R}}'$ and such that

$$\exp\left[\int_{\mathbb{R}}(e^{is(x)}-1)dx\right]=\int_{\mathscr{S}_{\mathbb{R}}'}e^{i\langle s',s\rangle}d\pi(s').$$

The Poissonian white noise space is $\mathcal{W}^\pi=\mathbf{L}_2(\mathscr{S}_{\mathbb{R}}',\mathcal{B},\pi)$, and admits a representation of the form (7.1), replacing the Hermite polynomial functionals $(H_\alpha)_{\alpha\in\ell}$ with the Charlier polynomial functionals $(C_\alpha)_{\alpha\in\ell}$, computed in terms of the Poisson-Charlier polynomials; see [21, p. 185]; we refer to [29, Chapter II, §2.81, p. 34-35] for the Poisson-Charlier polynomials. More precisely,

$$\mathcal{W}^\pi=\left\{\sum_{\alpha\in\ell}f_\alpha C_\alpha : \sum_{\alpha\in\ell}|f_\alpha|^2\alpha!<\infty\right\}.$$

The Kondratiev space of Poissonian test function $\mathcal{S}_1^\pi$ is defined by

$$\mathcal{S}_1^\pi=\left\{\sum_{\alpha\in\ell}f_\alpha C_\alpha : \sum_{\alpha\in\ell}|f_\alpha|^2(2\mathbb{N})^{\alpha p}(\alpha!)^2<\infty \text{ for all } p\in\mathbb{N}\right\}.$$

Since the associated positive function $a:\alpha\mapsto a_\alpha$ remains the same as before, we conclude the following proposition.

Proposition 7.2. *The Kondratiev space of Poissonian stochastic distributions is a Våge space.*

We refer to [22, Theorem 3.7 p. 192] for another example of a space where Våge inequality holds in the setting of white noise space analysis.

8. State space theory and Våge spaces

The results presented in [5, 2] for the case of the Kondratiev space $\mathcal{S}_{-1}$ of stochastic distributions extend to general Våge spaces. We refer to [12, 13, 23] for general background on the theory of linear systems when the coefficient space is $\mathbb{C}$ (we also refer to [1] for a survey), and to the papers [28, 16] and to the book [10] for more information on linear system on commutative rings, and in particular for the notions of controllable and observable pairs. These various notions are also reviewed in [5].

We begin this section with the following proposition. For completeness we give a short outline of the proof.

Proposition 8.1. *A matrix-valued rational function* $R(z) = p(z)(q(z))^{-1}$ *for which* $E(q(0)) \neq 0$ *can be written as*

$$R(z) = D + zC(I - zA)^{-1}B \tag{8.1}$$

where A*,* B*,* C*, and* D *are matrices of appropriate dimensions and with entries in the ring* $\mathcal{R}$*.*

Proof: We first note that a constant and the function z trivially have realizations of the form (8.1). Furthermore, if R_1 and R_2 are in the form (8.1),

$$R_j(z) = D_j + C_j(I_{N_j} - zA_j)^{-1}B_j, \quad j = 1, 2,$$

one has the realization formulas

$$R_1(z)R_2(z) = D + C(zI_N - A)^{-1}B, \tag{8.2}$$

where $N = N_1 + N_2$, $D = D_1D_2$ and

$$C = \begin{pmatrix} C_1 & D_1C_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1D_2 \\ B_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix},$$

and

$$R_1(z) + R_2(z) = D + C(I_N - zA)^{-1}B, \tag{8.3}$$

where $N = N_1 + N_2$, $D = D_1 + D_2$ and

$$C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

for the product and sum (provided the dimensions of R_1 and R_2 are such that these make sense). Next, if R is $\mathcal{R}^{p\times p}$ valued of the form (8.1) and D is invertible in $\mathcal{R}^{p\times p}$ then,

$$R^{-1}(z) = D^{-1} - D^{-1}C(I_N - z(A - BD^{-1}C)^{-1}BD^{-1},$$

which is also of the form (8.1). To conclude it remains to verify that realization is a property which stays under concatenation: If R_1 and R_2 are of the form (8.1) so are the functions

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} R_1 & R_2 \end{pmatrix},$$

provided the dimensions make sense. □

We note that, as was already mentioned after Definition 4.8, one can compute the value of a rational function of the form (4.1) at every point

$f \in \mathcal{R}$ such that $E(q(f)) \neq 0$. Let $q(z) = \sum_{m=0}^{M} q_m z^m$. Then, this last condition can be rewritten as

$$\sum_{m=0}^{M} E(q_m)(E(f))^m \neq 0.$$

Similarly, one can compute (8.1) at every $f \in \mathcal{R}$ such that $(I - E(fA))$ is invertible.

It is convenient to introduce the operators D_n, $n = 1, 2, \ldots$ defined by

$$D_n(x^\alpha) = \begin{cases} \alpha_n x^{\alpha - e_n} & \text{if } \alpha_n > 0 \\ 0 & \text{otherwise} \end{cases}$$

and by linear extension to any finite linear combination of such elements. We have in particular $D_n(XY) = D_n(X)Y + XD_n(Y)$.

As in [5] for $\mathcal{S}_{-1}$, given a Våge space $\mathcal{R}$ we define a rational function to be an expression of the form

$$D + zC(I - zA)^{-1}B,$$

where A, B, C and D are matrices of appropriate dimensions and with entries in $\mathcal{R}$. See Proposition 8.1 above. Before giving a sample result we recall that a pair $(C, A) \in \mathcal{R}^{p\times N} \times \mathcal{R}^{N\times N}$ is called *observable* if the map

$$f \mapsto \begin{pmatrix} Cf & CAf & CA^2 f & \cdots \end{pmatrix}$$

is injective from $\mathcal{R}^N$ into $(\mathcal{R}^p)^{\mathbb{N}}$. See [10, §2.2 p. 58]. In [5] it is proved for $\mathcal{S}_{-1}$ that an equivalent condition is:

$$C(I - zA)^{-1} f \equiv 0_{\mathcal{R}}^{p\times N} \Longrightarrow f = 0_{\mathcal{R}}^{N}.$$

The proof is the same for any Våge space.

Theorem 8.2. *Let $\hat{h}$ be a rational function with realization*

$$\hat{h}(z) = D + zC(I - zA)^{-1}B. \tag{8.4}$$

If the realization $E[\hat{h}](z) = E[D] + zE[C](I - zE[A])^{-1}E[B]$ is observable, then the realization (8.4) *is observable.*

Proof. We assume that $A \in \mathcal{R}^{N\times N}$. Let $f \in \mathcal{R}^N$ be such that $C(I - zA)^{-1} f \equiv 0$. We want to show that $f_\alpha = 0$ for all $\alpha \in \ell_A$. Since

$$E[\hat{h}](z) = E[D] + zE[C](I - zE[A])^{-1}E[B]$$

is an observable realization, we have that $E[C](I - zE[A])^{-1}E[f] \equiv 0$ implies $E[f] = 0$, and thus $f_0 = 0$. Now, since

$$\begin{aligned} D_n(C(I - zA)^{-1}f) &= D_n(C)(I - zA)^{-1}f + CD_n((I - zA)^{-1})f + \\ &\qquad + C(I - zA)^{-1}D_n(f) \\ &= 0, \end{aligned}$$

and since $E[f] = 0$, we have that $E[C](I - zE[A])^{-1}E[D_n(f)] = 0$, and thus $f_{e_n} = 0$. Furthermore, by a simple induction, since

$$D_n^m(C(I - zA)^{-1}f) = \sum_{k<m} U_k D_n^k(f) + C(I - zA)^{-1}D_n^m(f)$$

for some U_k, and since $D_n^m(C(I - zA)^{-1}f) = 0$ we have that

$$E[C](I - zE[A])^{-1}E[D_n^m(f)] = 0,$$

and therefore $f_{me_n} = 0$. Thus, $f_\alpha = 0$ for all $\alpha \in \ell_A$ such that $\alpha = (0, \dots, 0, \alpha_n, 0 \dots)$.
We may complete this proof as in [5]. □

In conclusion, Theorem 8.2 as well as Problem 4.9 (or more precisely, its solution presented in [2]) suggest that most of the classical linear system theory can be extended to our setting. This is important when one wants to take into account stochastic aspects of the theory. One such avenue consists of continuing the line of research initiated in [2, 4, 5]. One then considers input-output systems of the form

$$y_n = \sum_{k=0}^{n} h_k u_{n-k}, \quad n = 0, 1, \dots$$

where the input sequence $(u_n)_{n\in\mathbb{N}_0}$ and the impulse response $(h_n)_{n\in\mathbb{N}_0}$ are in some Våge space. The choice of the given Våge space is done to express for instance that the system is stochastic (then one chooses $\mathcal{S}_{-1}$). When the sequences consist of complex numbers, the product reduces to the product of complex numbers, and we are back in the classical theory.

As was already mentioned in Remark 6.8, another avenue is to define a stochastic linear system as a continuous mapping from the nuclear space $\mathscr{G} \otimes \mathcal{S}_1$ into its dual, to use Schwartz's kernel theorem and then to follow Zemanian's approach to linear systems. Test functions are now functions of the form (we write ω rather than s' for the variable in $\mathcal{S}_1$)

$$s(t,\omega)=\sum_{\substack{n\in\mathbb{N}_0\\ \alpha\in\ell}} H_\alpha(\omega)\xi_n(t)c_{n,\alpha} \tag{8.5}$$

where the coefficients $c_{n,\alpha}$ are in $\mathbb{C}$ and subject to

$$\|s\|_{p,q}^2=\sum_{n=0}^{\infty}\sum_{\alpha\in\ell}(\alpha!)^2|c_{n,\alpha}|^2(2\mathbb{N})^{q\alpha}2^{np}<\infty,\quad \forall p,q\in\mathbb{N}.$$

See [33] for the latter. Since the dual of $\mathscr{G}\otimes\mathcal{S}_1$ is a Våge space, one can get more precise results than the ones in [33].

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