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A NEW REALIZATION OF RATIONAL FUNCTIONS, WITH APPLICATIONS TO LINEAR COMBINATION INTERPOLATION

DANIEL ALPAY, PALLE JORGENSEN, IZCHAK LEWKOWICZ, AND DAN VOLOK

Abstract. We introduce the following *linear combination interpolation* problem (LCI), which in case of simple nodes reads as follows: Given N distinct numbers $w_1, \ldots w_N$ and $N+1$ complex numbers $a_1, \ldots, a_N$ and c, find all functions $f(z)$ analytic in an open set (depending on f) containing the points $w_1, \ldots, w_N$ such that

$$\sum_{u=1}^{N} a_u f(w_u) = c.$$

To this end we prove a representation theorem for such functions f in terms of an associated polynomial $p(z)$. We give applications of this representation theorem to realization of rational functions and representations of positive definite kernels.

Contents

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1. Introduction

Any function f analytic in a neighborhood of the origin can be uniquely written as

$$f(z) = \sum_{n=0}^{N-1} z^n f_n(z^N), \tag{1.1}$$

where $f_0, \ldots, f_{N-1}$ are analytic at the origin. Furthermore, the maps

$$T_n f = f_n, \quad n = 0, \ldots, N-1 \tag{1.2}$$

satisfy, under appropriate hypothesis, the Cuntz relations (see also (3.7)). See for instance [4], where applications to wavelets are given. In the present paper, we extend these methods, and we derive new and explicit formulas for solutions to a class of multi-point interpolation problems; not amenable to tools from earlier investigations. Following common use, by Cuntz relations we refer here to a symbolic representation of a finite set (say N) of isometries having orthogonal ranges which add up to the identity (operator). When N is fixed the notation O_N is often used. By a representation of O_N in a fixed Hilbert space, we mean a realization of the N-Cuntz relations in a Hilbert space. Here we will be applying this to specific Hilbert spaces of analytic functions which are dictated by our multi-point interpolation setting. In general it is known that the problem of finding representations of O_N is subtle. (The literature on representations of O_N is vast.) For example, no complete classification of these representations is known, but nonetheless, specific representations can be found, and they are known to play a key role in several areas of mathematics and its applications; e.g., to multi-variable operator theory, and in applications to the study of multi-frequency bands; see the cited references below. Realizations as in (1.1) then results from representations of O_N; the particulars of these representations are then encoded in the operators from (1.2). Readers not familiar with O_N and its representations are referred to [11, 21, 20, 19], and to Remark 3.7 below.

The outline of the paper is as follows. In Section 2, we replace z^N by an arbitrary polynomial $p(z)$, and prove a counterpart of the decomposition (1.1), see Theorem 2.1. The rest of the paper is organized as follows: In sections 3 -4, we discuss uniqueness of solutions, and (motivated by applications from systems theory) we extend our result in three ways, first to that of Banach space valued functions (Theorem 3.4) and then we specialize to the case rational functions (Theorem 3.5). Thirdly we study multipoint interpolation when derivatives are specified. In section 5, we give an application to positive definite kernels.

More precisely, a first application of Theorem 2.1 is in giving a new realization formula for rational functions. To explain the result, recall that a matrix-valued rational function W analytic at the origin can always be written in the form

$$W(z) = D + zC(I - zA)^{-1}B,$$

where $D = W(0)$ and A, B, C are matrices of appropriate sizes. Such an expression is called a state space realization, and plays an important role in linear system theory and related topics; see for instance [8] and [10] for more information. We here prove that a rational function analytic at the points $w_1, \ldots, w_n$ can always be written in the form

$$W(z) = Z(z)\gamma(I - p(z)\alpha)^{-1}\beta,$$

where $p(z)$ is a polynomial vanishing at the points $w_1, \ldots, w_n$ and of degree $N \geq n$, and

$$Z(z) = \begin{pmatrix} 1 & z & \cdots & z^{N-1} \end{pmatrix}, \tag{1.3}$$

and α, β, γ are matrices of appropriate sizes. Finally we give an application to decompositions of positive definite kernels and the Cuntz relations.

The multipoint interpolation problem, which in the case where p has simple zeros $w_1, \ldots, w_N$ consists in finding all functions $f(z)$ analytic in a simply connected set (depending on f) containing the points $w_1, \ldots, w_N$ and such that

$$\sum_{u=1}^{N} a_u f(w_u) = c. \tag{1.4}$$

This can be equivalently written as

$$(a_1, \ \ldots \ , \ a_N) \begin{pmatrix} f(w_1) \\ \vdots \\ f(w_N) \end{pmatrix} = c.$$

Namely the points $f(w_1), \ \ldots \ , \ f(w_N)$ lie on a hyper-plane, so roughly speaking, the points $w_1, \ \ldots \ , \ w_N$ lie on some manifold.

This type of problem seems to have been virtually neglected in the litterature. In [3] the case of two points was considered in the setting of the Hardy space of the open unit disk. The method there consisted in finding an involutive self-map of the open unit disk mapping one of the points to the second one, and thus reducing the given two-point interpolation problem to a one-point interpolation problem with an added symmetry. This method cannot be extended to more than two points, but in special cases. In [6] we considered the interpolation condition (1.4) in the Hardy space. Connections with the Cuntz relations played a key role in the arguments.

2. A decomposition of analytic functions

We set

$$p(z)=\prod_{j=1}^{n}(z-w_j)^{\mu_j},\quad \sum_{j=1}^{n}\mu_j=N,$$

and recall that $Z(z)$ is given by (1.3).

Theorem 2.1. *Let Ω be a (possibly disconnected) neighborhood of $\{w_1,\ldots,w_n\}$. Then there exists a neighborhood Ω_0 of the origin, such that $p^{-1}(\Omega_0)\subset\Omega$ and every function $f(z)$, analytic in Ω, can be represented in the form*

$$f(z)=Z(z)F(p(z)),\quad z\in p^{-1}(\Omega_0), \tag{2.1}$$

where $F(z)$ is a $\mathbb{C}^N$-valued function, analytic in Ω_0.

Proof. Choose n simple closed counterclockwise oriented contours $\gamma_1,\ldots,\gamma_n$ with the following properties:

1. The function $f(z)$ is analytic on each contour γ_j and in the simply connected domain D_j encircled by γ_j.
2. For $j=1,\ldots,n$ the domain D_j contains the point w_j.
3. The domains $D_1,\ldots,D_n$ are pairwise disjoint.

Denote

$$D:=\bigcup_{j=1}^{n}D_j,\quad \rho:=\min\left\{|p(s)|\;:\;s\in\bigcup_{j=1}^{n}\gamma_j\right\}.$$

Since all the zeros of $p(z)$ are contained in D, $\rho>0$ and, by the maximum modulus principle, $p^{-1}(\Omega_0)\subset D$, where Ω_0 is the open disk of radius ρ centered at the origin. Furthermore, for $z\in p^{-1}(\Omega_0)$ it holds that

$$\begin{aligned} f(z)&=\frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\gamma_j}\frac{f(s)}{p(s)-p(z)}\frac{p(s)-p(z)}{s-z}ds\\ &=\frac{Z(z)}{2\pi i}\sum_{j=1}^{n}\int_{\gamma_j}\frac{Q(s)f(s)}{p(s)-p(z)}ds=Z(z)F(p(z)),\end{aligned}$$

where $Q(s)$ is a $\mathbb{C}^N$-valued polynomial, such that

$$\frac{p(s)-p(z)}{s-z}=Z(z)Q(s),$$

and

$$F(z)=\frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\gamma_j}\frac{Q(s)f(s)}{p(s)-z}ds,\quad z\in\Omega_0, \tag{2.2}$$

is a $\mathbb{C}^N$-valued function, analytic in Ω_0. □

Corollary 2.2. *Assume that f is a polynomial (resp. rational). Then F given by* (2.2) *is also a polynomial (resp. rational).*

Proof. We first consider the case of a polynomial. For z near the origin we have

$$F(z)=\sum_{u=0}^{\infty}z^uF_u\quad\text{with}\quad F_u=\sum_{j=1}^{n}\frac{1}{2\pi i}\int_{\gamma_j}\frac{Q(s)f(s)}{p(s)^{u+1}}du.$$

Note that $\frac{1}{2\pi i}\int_{\gamma_j}\frac{Q(s)f(s)}{p(s)^{u+1}}du$ is the residue of the rational function $\frac{Q(s)f(s)}{p(s)^{u+1}}$ at the point w_j. For u large enough the difference of the degrees of the denominator and the numerator of this rational function is at least two, and so the sum of its residues is equal to 0 (the so-called exactity relation; see [17, p. 173] and [2, Exercise 7.3.6, p. 326]). Thus $F_u=0$ for u large enough and we conclude by analytic continuation that F is a polynomial.

In the case of a rational function consider the partial fraction representation, which is the sum of a polynomial (which we just have treated) and of terms of the form $\frac{1}{(s-a)^M}$, where a is not a zero of p. We thus need to show that, for such a, a sum of the form

$$G(z)=\frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\gamma_j}\frac{Q(s)}{(s-a)^M(p(s)-z)}ds, \tag{2.3}$$

is rational. Chose the contours $\gamma_1,\ldots,\gamma_n$ such that no zeroes of the equation $p(s)=p(a)$ lie inside or on them. Using the polynomial case, the result follows from writing

$$\begin{aligned} G(z) &= \frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\gamma_j}\frac{Q(s)}{(s-a)^M(p(s)-z)}ds \\ &= \frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\gamma_j}\frac{Q(s)\left(\frac{p(s)-p(a)}{s-a}\right)^M}{(p(s)-p(a))^M(p(s)-z)}ds \\ &= \frac{1}{2\pi i}\sum_{j=1}^{n}\int_{\gamma_j}Q(s)\left(\frac{p(s)-p(a)}{s-a}\right)^M\left(\frac{c(z)}{p(s)-z}+\right. \\ &\qquad\qquad\left.+\sum_{u=1}^{M}\frac{c_u(z)}{(p(s)-p(a))^u}\right)ds, \end{aligned}$$

for some complex numbers $c(z), c_1(z), \ldots, c_M(z)$ corresponding to the partial fraction expansion of the function $\frac{1}{(\lambda-z)(\lambda-p(a))^M}$:

$$\frac{1}{(\lambda-z)(\lambda-p(a))^M} = \frac{c(z)}{\lambda-z} + \sum_{u=1}^{M}\frac{c_u(z)}{(\lambda-p(a))^u}.$$

These are readily seen to be rational functions of z, and hence the function G above is rational. □

3. A new realization of rational functions

Denote by V the generalized $N \times N$ Vandermonde matrix

$$V = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix}, \text{ where } V_j = \begin{pmatrix} Z(w_j) \\ Z'(w_j) \\ \vdots \\ Z^{(\mu_j-1)}(w_j) \end{pmatrix},$$

and by C_w the linear operator

$$f(z) \mapsto \begin{pmatrix} C_{w_1}f \\ \vdots \\ C_{w_n}f \end{pmatrix}, \text{ where } C_{w_j}f := \begin{pmatrix} f(w_j) \\ f'(w_j) \\ \vdots \\ f^{(\mu_j-1)}(w_j) \end{pmatrix}.$$

By rearanging the rows the matrix V is readily seen to be invertible.

Proposition 3.1. *Let $f(z)$ be a function, analytic in a neighborhood of $\{w_1, \ldots, w_n\}$ and let $F(z)$ be a $\mathbb{C}^N$ function, analytic in a neighborhood* *of the origin, which provides the decomposition* (2.1) *for the function* $f(z)$. *Then the Taylor expansion of* $F(z)$ *is given by*

$$F(z) = \sum_{k=0}^{\infty} z^k V^{-1} C_w (R_0^{(p)})^k f, \tag{3.1}$$

where $R_0^{(p)}$ *denotes the linear operator*

$$f(z) \mapsto \frac{f(z) - Z(z) V^{-1} C_w f}{p(z)}.$$

Remark 3.2. *A priori* the convergence in (3.1) is pointwise, and uniform on compact subsets of the origin where f is defined. When the underlying spaces are finite dimensional, or when some extra topological structure is given, one can rewrite (3.1) as

$$F(z) = V^{-1} C_w (I - z R_0^{(p)})^{-1} f.$$

Proof of Proposition 3.1. Since

$$p(w_j) = p'(w_j) = \cdots = p^{(\mu_j - 1)}(w_j) = 0, \quad j = 1, \ldots, n,$$

differentiate both sides of (2.1) at w_j to obtain

$$C_{w_j} f = V_j F(0), \quad j = 1, \ldots, n.$$

Hence, in vector notation,

$$C_w f = V F(0),$$

$$F(0) = V^{-1} C_w f, \tag{3.2}$$

and

$$(R_0^{(p)} f)(z) = Z(z)(R_0 F)(p(z)), \tag{3.3}$$

where

$$(R_w F)(z) = \frac{F(z) - F(w)}{z - w}$$

is the classical backward-shift operator. In particular, the function $R_0 F$ provides a decomposition (2.1) for the function $R_0^{(p)} f$. By induction, one may conclude that

$$((R_0^{(p)})^k f)(z) = Z(z) R_0^k F(p(z)), \quad k = 0, 1, 2, \ldots$$

hence, in view of (3.2),

$$(R_0^k F)(0) = V^{-1} C_w (R_0^{(p)})^k f, \quad k = 0, 1, 2, \ldots$$

□

Corollary 3.3. *Every function $f(z)$, analytic in a neighborhood of $\{w_1, \dots, w_n\}$, admits a unique decomposition* (2.1)*, in which (as follows form Corollary 2.2) F is a polynomial (resp. rational) when f is a polynomial (resp. rational).*

Theorem 2.1 has an analogue in the setting of analytic functions with values in a Banach space $\mathcal{B}$. In what follows, $\mathcal{B}^s$ denotes the product space

$$\mathcal{B}^s := \mathbb{C}^s \otimes \mathcal{B},$$

and the tensor product of a matrix $(a_{i,j}) \in \mathbb{C}^{r \times s}$ and a linear operator $A \in \mathcal{L}(B)$ is understood as the operator matrix

$$(a_{i,j}) \otimes A := (a_{i,j}A) \in \mathcal{L}(\mathcal{B}^s, \mathcal{B}^r).$$

Theorem 3.4. *Let $\mathcal{B}$ be a Banach space and let $f(z)$ be a $\mathcal{B}$-valued function, analytic in a neighborhood Ω of $\{w_1, \dots, w_n\}$. Then there exist a neighborhood Ω_0 of the origin, and a $\mathcal{B}^N$-valued function $F(z)$, analytic in Ω_0, such that*

$$f(z) = (Z(z) \otimes I_{\mathcal{B}})F(p(z)), \quad z \in p^{-1}(\Omega_0) \subset \Omega. \tag{3.4}$$

Furthermore, the Taylor expansion of $F(z)$ is given by

$$F(z) = \sum_{k=0}^{\infty} z^k V^{-1} C_w (R_0^{(p)})^k f, \tag{3.5}$$

where $R_0^{(p)}$ denotes the linear operator

$$f(z) \mapsto \frac{f(z) - ((Z(z)V^{-1}) \otimes I_{\mathcal{B}})C_w f}{p(z)}.$$

Proof. Let $\varphi \in \mathcal{B}^*$. Then, according to Theorem 2.1 and Proposition 3.1, the function $\varphi \circ f$ admits a unique decomposition (3.1) provided by the $\mathbb{C}^N$-valued function

$$F^{\varphi}(z) = \sum_{k=0}^{\infty} z^k V^{-1} C_w (R_0^{(p)})^k (\varphi \circ f).$$

Then

$$F^{\varphi}(z) = \sum_{k=0}^{\infty} z^k (I_N \otimes \varphi) F_k,$$

where

$$F_k := (V^{-1} \otimes I_{\mathcal{B}}) C_w (R_0^{(p)})^k f.$$

Since $F^{\varphi}(z)$ is analytic in an open disk

$$\Omega_0 = \{z \,:\, |z| < \rho\},$$

where ρ is independent of φ, the uniform boundedness principle implies that the $\mathcal{B}^N$-valued function

$$F(z) := \sum_{k=0}^{\infty} z^k F_k$$

is also analytic in Ω_0, and (3.4) follows from

$$F^\varphi = (I_N \otimes \varphi) \circ F, \quad \varphi \in \mathcal{B}^*.$$

□

The preceding analysis leads to a new kind of realization for rational functions.

Theorem 3.5. *Every rational $\mathbb{C}^{r\times s}$-valued function $f(z)$, which has no poles in $\{w_1, \dots, w_n\}$, can be written as*

$$f(z) = (Z(z) \otimes I_r)C(I - p(z)A)^{-1}B, \tag{3.6}$$

where A, B, C are constant matrices of appropriate sizes.

Proof. Write

$$\frac{1}{p(z)} = \sum_{j=1}^{n} \sum_{k=1}^{\mu_j} \frac{c_{j,k}}{(z - w_j)^k},$$

where $c_{j,k} \in \mathbb{C}$ are constants. Then the operator $R_0^{(p)}$ defined in (3.3) can be written as

$$R_0^{(p)} = \sum_{j=1}^{n} \sum_{k=1}^{\mu_j} c_{j,k} R_{w_j}^k.$$

Since $f(z)$ is a rational function, the space

$$\begin{aligned} \mathcal{L}(f) &:= \text{colspan}\{A^k f \,:\, k = 0, 1, 2, \dots\} \\ &\subset \text{colspan}\{R_{w_j}^k f \,:\, j = 1, \dots, n;\ k = 0, 1, 2, \dots\} \end{aligned}$$

is finite-dimensional. Choose a basis of this finite-dimensional space and let A, B, C be matrices representing the operators

$$\mathcal{L}(f) \ni fu \mapsto R_0^{(p)} fu \in \mathcal{L}(f), \quad u \in \mathbb{C}^s,$$
$$\mathbb{C}^s \ni u \mapsto fu \in \mathcal{L}(f),$$
$$\mathcal{L}(f) \ni fu \mapsto (V^{-1} \otimes I_r)C_w fu \in \mathbb{C}^{rN}, \quad u \in \mathbb{C}^s,$$

respectively. Then

$$f(z) = (Z(z) \otimes I_r) \sum_{k=0}^{\infty} p(z)^k C A^k B = (Z(z) \otimes I_r)C(I - p(z)A)^{-1}B.$$

□

We will call the realization (3.6) minimal if the size of the matrix A is minimal (for more on this notion, and equivalent characterizations, see for instance [9]). As a consequence of the uniqueness of the decomposition we also have:

Corollary 3.6. *When minimal, the realization* (3.6) *is unique up to a similarity matrix. Then, F is a polynomial if and only if A is nilpotent.*

Proof. It suffices to notice that the uniqueness of the decomposition (2.1) reduces the problem to the uniqueness of the minimality of the function $C(I-\lambda A)^{-1}B$ with $\lambda \in \mathbb{C}$. □

Remark 3.7. The uniqueness allows us to give an interpretation on terms of generalized Cuntz relations. More precisely, define linear operators on analytic functions by $S_1, \ldots, S_N, T_1, \ldots, T_N$ by:

$$(S_j g)(z) = z^{j-1} g(p(z)) \quad \text{and} \quad T_j F = F_j, \quad j = 1, \ldots, N \tag{3.7}$$

where F is a $\mathbb{C}^N$-valued analytic function (see also (1.2) for the defintion of $T_1, \ldots, T_N$).Then the given decomposition (2.1) reads

$$T_i S_j = \delta_{ij} \quad \text{and} \quad \sum_{n=1}^{N} S_j T_j = I.$$

Remark 3.8. We note that in the case $\mu_1 = \cdots = \mu_n = 1$ the operator $R_0^{(p)}$ can be written as

$$R_0^{(p)} f(z) = \frac{f(z)}{p(z)} - \sum_{u=1}^{N} \frac{f(w_u)}{p'(w_u)(z-w_u)} \tag{3.8}$$

is reminiscent of a formula for a resolvent operator given in the setting of function theory on compact real Riemann surfaces. See [7, (4.1), p. 307]. This point is emphasized in the following proposition.

Proposition 3.9. *Let α and β be such that the roots $w_1(\alpha), \ldots, w_N(\alpha)$ and $w_1(\beta), \ldots, w_N(\beta)$ of the equations $p(z) = \alpha$ and $p(z) = \beta$ are all distinct ($w_u(\alpha) \neq w_v(\beta)$ for $u, v = 1, \ldots, N$). Then the resolvent equation*

$$R_\alpha^{(p)} - R_\beta^{(p)} = (\alpha - \beta) R_\alpha^{(p)} R_\beta^{(p)}$$

holds.

Proof. Indeed, on the one hand,

$$\left((R_\alpha^{(p)}-R_\beta^{(p)})(f)\right)(z)=$$

$$=\frac{f(z)}{p(z)-\alpha}-\sum_{u=1}^N\frac{f(w_u(\alpha))}{p'(w_u(\alpha))(z-w_u(\alpha))}-\frac{f(z)}{p(z)-\alpha}+\sum_{u=1}^N\frac{f(w_u(\alpha))}{p'(w_u(\alpha))(z-w_u(\alpha))}$$

$$=(\alpha-\beta)\frac{f(z)}{(p(z)-\alpha)(p(z)-\beta)}-\sum_{u=1}^N\frac{f(w_u(\alpha))}{p'(w_u(\alpha))(z-w_u(\alpha))}+\sum_{v=1}^N\frac{f(w_v(\beta))}{p'(w_v(\beta))(z-w_v(\beta))}.$$

On the other hand,

$$\begin{aligned}\left((R_\alpha^{(p)}\left(R_\beta^{(p)}(f)\right)\right)(z)&=\frac{\left(R_\beta^{(p)}(f)\right)(z)}{p(z)-\alpha}-\sum_{u=1}^N\frac{\left(R_\beta^{(p)}(f)\right)(w_u(\alpha))}{p'(w_u(\alpha))(z-w_u(\alpha))}\\&=\frac{f(z)}{(p(z)-\alpha)(p(z)-\beta)}-\sum_{v=1}^N\frac{f(w_v(\beta))}{p'(w_v(\beta))(z-w_v(\beta))(p(z)-\alpha)}-\\&\quad-\sum_{u=1}^N\frac{f(w_u(\alpha))}{(\underbrace{p(w_u(\alpha)}_{=\alpha}-\beta))p'(w_u(\alpha))(z-w_u(\alpha))}+\\&\quad+\sum_{u=1}^N\frac{\left(\sum_{v=1}^N\frac{f(w_v(\beta))}{p'(w_v(\beta))(w_u(\alpha)-w_v(\beta))}\right)}{p'(w_u(\alpha))(z-w_u(\alpha))}.\end{aligned}$$

Proving the resolvent identity amounts to showing that:

$$\begin{aligned}&\sum_{v=1}^N\frac{f(w_v(\beta))}{p'(w_v(\beta))(z-w_v(\beta))}=\\&=(\alpha-\beta)\left\{-\sum_{v=1}^N\frac{f(w_v(\beta))}{p'(w_v(\beta))(z-w_v(\beta))(p(z)-\alpha)}+\sum_{u=1}^N\frac{\left(\sum_{v=1}^N\frac{f(w_v(\beta))}{p'(w_v(\beta))(w_u(\alpha)-w_v(\beta))}\right)}{p'(w_u(\alpha))(z-w_u(\alpha))}\right\}\\&=(\alpha-\beta)\left\{-\sum_{v=1}^N\frac{f(w_v(\beta))}{p'(w_v(\beta))(z-w_v(\beta))(p(z)-\alpha)}+\right.\\&\quad+\sum_{v=1}^N\frac{f(w_v(\beta))}{p'(w_v(\beta))}\left\{\sum_{u=1}^N\frac{1}{p'(w_u(\alpha))(z-w_u(\alpha))(z-w_v(\beta))}+\right.\\&\qquad\left.\left.+\frac{1}{p'(w_u(\alpha))(w_u(\alpha)-w_v(\beta))(z-w_v(\beta))}\right\}\right\}.\end{aligned}\tag{3.9}$$

Taking into account the equality

$$\frac{1}{p(z)-\alpha}=\sum_{u=1}^{N}\frac{1}{p'(w_u(\alpha))(z-w_u(\alpha))}$$

we have

$$\sum_{u=1}^{N}\frac{1}{p'(w_u(\alpha))(w_u(\alpha)-w_v(\beta))}=\frac{1}{\alpha-p(w_v(\beta))}=\frac{1}{\alpha-\beta}.$$

(3.9) follows. □

Remark 3.10. Proposition 3.9 can be proved in an easier way using the classical resolvent identity by remarking that $(R_\alpha^{(p)}f)(z)=Z(z)(R_\alpha F)(p(z))$, where $f(z)=Z(z)F(p(z))$. The proof proposed here is more conducive to explicit links with the Riemann surface case.

We finally note that, at least in spirit, we used the theory of linear system in this section. See for instance [22, 10, 1] for more on this theory. In the sequel we resort to the theory of reproducing kernel Hilbert spaces. The reader may find the following references helpful: [27, 24, 1, 23, 26, 16, 28, 25]

4. Linear combination interpolation

In [6] we introduced a general problem of linear combination interpolation, and solved it in the setting of the Hardy space. Here, the preceding analysis enables us to solve a linear combination interpolation problem in the setting of functions analytic in the neighborhoods of given preassigned points.

Problem 4.1. *Given complex numbers* $a_{j,k}, j=1,\ldots,n; k=0,\ldots,\mu_j-1$; *and* c, *describe the set of all functions* f *analytic in a possibly disconnected neighborhood of the points* $w_1,\ldots,w_n$ *and such that*

$$\sum_{j=1}^{n}\sum_{k=0}^{\mu_j-1}a_{j,k}f^{(k)}(w_j)=c. \tag{4.1}$$

The idea is to use the decomposition (3.1) and to reduce the interpolation condition (4.1) to a *unique* interpolation condition for a *vector-valued* analytic function. Let

$$v=\begin{pmatrix}a_{1,0} & a_{1,1} & \cdots & a_{n,\mu_n-1}\end{pmatrix}.$$

Then (4.1) can be re-written as

$$vC_wf=c.$$

In view of Propositions 3.1, this last condition is equivalent to

$$vVF(0) = c,$$

which is a basic interpolation problem whose solution is given by

$$F(z) = \frac{V^*v^*}{vVV^*v^*}c + \left(I_N + (z-1)\frac{V^*v^*vV^*}{vVV^*v^*}G(z) \right),$$

where $G(z)$ is an arbitrary $\mathbb{C}^N$-valued function analytic in a neighborhood of the origin. Thus the solutions f are given by

$$\begin{aligned} f(z) &= Z(z)F(p(z)) \\ &= Z(z)\left(\frac{V^*v^*}{vVV^*v^*}c + \left(I_N + (p(z)-1)\frac{V^*v^*vV^*}{vVV^*v^*}G(p(z)) \right) \right). \end{aligned}$$

Furthermore, we obtain all the rational solutions of the interpolation when $G(z)$ is chosen rational.

5. Representation in reproducing kernel Hilbert spaces

Here we focus on the case when $f(z)$ belongs to a reproducing kernel Hilbert space $\mathcal{H}(K)$ of analytic functions.

Proposition 5.1. *Let $K(z,w)$ be a positive definite function analytic in z an in $\overline{w}$ in an open set Ω which contains $w_1, \ldots, w_n$. There exists a neighborhood Ω_0 of the origin and a positive $\mathbb{C}^{N\times N}$-valued kernel $L(z,w)$, analytic in Ω_0, such that*

$$K(z,w) = Z(z)L(p(z),p(w))Z(w)^*, \quad z,w \in p^{-1}(\Omega_0) \subset \Omega.$$

Proof. Write $K(z,w) = C(z)C(w)^*$, where $C(z) : \mathcal{H}(K) \longrightarrow \mathbb{C}$ is the point evaluation functional:

$$C(z)f = f(z), \quad z \in \Omega.$$

Then $C(z)$ is a $\mathbf{L}(\mathcal{H}(K), \mathbb{C})$-valued function, analytic in Ω and, by Theorem 3.4, there exists a neighborhood Ω_0 of the origin, such that $p^{-1}(\Omega_0) \subset \Omega$ and

$$C(z) = Z(z)E(p(z)), \quad z \in \Omega_0,$$

where $E(z)$ is a $\mathbf{L}(\mathcal{H}(K), \mathbb{C}^{N\times 1})$-valued function, analytic in Ω_0. Now set

$$L(z,w) := E(z)E(w)^*$$

to compete the proof. □

Proposition 5.2. *Let $F \in \mathcal{H}(L)$. Then the function $Z(z)F(p(z))$, which is analytic a priori in $p^{-1}(\Omega_0)$, admits analytic continuation into Ω and is an element of the reproducing kernel Hilbert space $\mathcal{H}(K)$. Moreover, the operator $S : \mathcal{H}(L) \longrightarrow \mathcal{H}(K)$ determined by*

$$(SF)(z) = Z(z)F(p(z)), \quad F \in \mathcal{H}(L), z \in p^{-1}(\Omega_0), \tag{5.1}$$

is unitary.

Proof. Consider a linear relation in $\mathcal{H}(K) \times \mathcal{H}(L)$ spanned by

$$(K(\cdot, w), L(\cdot, p(w))Z(w)^*), \quad w \in p^{-1}(\Omega_0).$$

Since

$$\|K(\cdot, w)\|^2_{\mathcal{H}(K)} = K(w, w) = Z(w)L(p(w), p(w))Z(w)^* = \|L(\cdot, p(w))Z(w)^*\|^2_{\mathcal{H}(L)}$$

and since span$\{K_w : w \in p^{-1}(\Omega_0)\}$ is dense in $\mathcal{H}(K)$, the above relation is the graph of an isometry $T \in \mathbf{L}(\mathcal{H}(K), \mathcal{H}(L))$. The adjoint of T is the operator S. In view of Corollary 3.3, S is injective and hence unitary.

□

Remark 5.3. In the special case where the kernel L is block diagonal, $L = \text{diag}\ (L_1, \ldots, L_N)$, with $L_1, \ldots, L_N$ complex-valued positive definite kernels, we have the orthogonal decomposition

$$\mathcal{H}(L) = \oplus_{j=1}^N \mathcal{H}(L_j),$$

and, with S as in (5.1) we can define operators $S_1, \ldots, S_N$ via $S = \begin{pmatrix} S_1 & \cdots & S_N \end{pmatrix}$. These operators are given by (3.7) and satisfy the Cuntz relations. For the theory (and applications) of representations of Cuntz relations by operators in Hilbert space, see e.g. [13, 18, 12].

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