Stochastic Processes Induced By Singular Operators

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Abstract. In this paper we study a general family of multivariable Gaussian stochastic processes. Each process is prescribed by a fixed Borel measure σ on $\mathbb{R}^n$. The case when σ is assumed absolutely continuous with respect to Lebesgue measure was studied earlier in the literature, when $n = 1$. Our focus here is on showing how different equivalence classes (defined from relative absolute continuity for pairs of measures) translate into concrete spectral decompositions of the corresponding stochastic processes under study. The measures σ we consider are typically purely singular. Our proofs rely on the theory of (singular) unbounded operators in Hilbert space, and their spectral theory.

Contents

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1. Introduction

We study Gaussian stochastic processes with stationary mean-square-increments. These are used in for example modeling of time-dependent phenomena in signal processing, in information theory, in telecommunication, and in a host of applications. A process $\{X(t)\}$ indexed by $t \in \mathbb{R}^n$ on a probability space is said to have stationary mean-square increments if the mean-square of each of the increments $X(t) - X(s)$ depends only on the difference $t - s$. Note that we are not restricting the nature of the statistical distribution of the increments, only the expectation of the mean-square. In particular, we are not assuming independent increments. Nonetheless, this mean-square property allows us to analyze decompositions of the processes and stochastic integrals with the use of spectral theory and tools from the theory of operators in Hilbert space.

There are several reasons for the importance of stationary mean-square increment processes. First they include a rich family of processes from applications; secondly, these are precisely the processes that admit a canonical action of a unitary group of time-transformations; the unitary operators act on $\mathbf{L}_2(\Omega, \mathcal{A}, P)$ where $(\Omega, \mathcal{A}, P)$ is the underlying probability space. For $n = 1$, it will be a one-parameter group of unitary operators; and in general, $n > 1$, it will be a unitary representation of the additive group $\mathbb{R}^n$. In the case of $n = 1$, and Brownian motion, Wiener and Kakutani proved that this one-parameter group is a group of point transformations acting ergodically; see [21]. The third reason is purely mathematical: the class of stationary mean-square increments processes admits a classification. We turn to this in Theorem 5.2 below. In Section 8 we introduce the unitary representations of $\mathbb{R}^n$.

It is possible to characterize the processes mentioned in the previous paragraphs by the family $\mathcal{C}$ of regular positive Borel measures σ on $\mathbb{R}^n$ subject to

$$\int_{\mathbb{R}^n} \frac{d\sigma(u)}{(1 + |u|^2)^p} < \infty \tag{1.1}$$

for some $p \in \mathbb{N}_0$. Such a measure σ is the spectral function of a homogeneous generalized stochastic field in the sense of Gelfand. See [9, p. 283]. Here we study this correspondence in reverse. Earlier work so far was restricted to the case when the spectral density is assumed to be absolutely continuous and $n = 1$, see [1]. The case of singular measure was considered in [2]. Here, we are making no restriction on

the spectral type. We associate to $\sigma \in \mathcal{C}$ four natural objects:

(a) The quadratic form

$$q_\sigma(\psi) = \int_{\mathbb{R}^n} |\widehat{\psi}(u)|^2 d\sigma(u), \tag{1.2}$$

where ψ belongs to the Schwartz space $\mathcal{S}_{\mathbb{R}^n}$, and where

$$\widehat{\psi}(u) = \int_{\mathbb{R}} e^{-iux}\psi(x)dx$$

denotes the Fourier transform of ψ.

(b) A linear operator Q_σ such that

$$q_\sigma(\psi) = \|Q_\sigma\psi\|^2_{\mathbf{L}_2(\mathbb{R}^n,dx)}, \quad \psi \in \mathcal{S}_{\mathbb{R}^n}.$$

(c) A generalized stochastic process $\{X_\sigma(\psi), \psi \in \mathcal{S}_{\mathbb{R}^n}\}$ such that

$$E[X_\sigma(\psi_1)\overline{X_\sigma(\psi_2)}] = \int_{\mathbb{R}^n} \widehat{\psi_1}(u)\overline{\widehat{\psi_2}(u)}d\sigma(u),$$

for $\psi_1, \psi_2 \in \mathcal{S}_{\mathbb{R}^n}$.

(d) Let $\mathcal{S}_{\mathbb{R}^n}(\mathbb{R})$ the Schwartz space of *real-valued* Schwartz functions of n-variables, and

$$\Omega = \mathcal{S}'_{\mathbb{R}^n}(\mathbb{R}) \tag{1.3}$$

its dual, the space of all tempered distributions. The fourth object associated to σ is a probability measure $d\mu_\sigma$ on $\mathcal{S}'_{\mathbb{R}^n}(\mathbb{R})$ such that

$$e^{-\frac{\|\widehat{\psi}\|^2_{\mathbf{L}_2(d\sigma)}}{2}} = \int_\Omega e^{i\langle\omega,\psi\rangle}d\mu_\sigma(\omega), \quad \psi \in \mathcal{S}_{\mathbb{R}^n}(\mathbb{R}),$$

where $\langle\ ,\ \rangle$ denotes the duality between $\mathcal{S}_{\mathbb{R}^n}(\mathbb{R})$ and $\mathcal{S}'_{\mathbb{R}^n}(\mathbb{R})$. One purpose of this paper is to study various connections between these quantities. See in particular Theorem 5.2.

We note that the measures σ of interest in building stochastic processes include those for which $\mathbf{L}_2(\sigma)$ has a Fourier basis, i.e., an orthogonal and total family of complex exponentials, where the frequencies are made up from some discrete subset S in $\mathbb{R}^n$. When this happens, we say that (σ, S) is a spectral pair. They have been extensively studied in [5, 19, 20, 6, 4]. Aside from their application to the constructing highly fluctuating stochastic processes, the spectral pair measures have additional uses: The measures σ that are part of a spectral pair have the following features:

1. They are different from those usually considered in the theory of stochastic processes, and yet they produce explicit models with easy rules for computation.
2. They possess intriguing multiscale properties, and selfsimilarity.

We also note that one can extend, as in our previous paper [2], the generalized process X_σ to be defined for $t \in \mathbb{R}^n$. When $n = 1$, this is done by replacing the variable $\phi \in \mathcal{S}$ by the functions $\xi_t(u) = \frac{e^{itu}-1}{u}$.

For $n = 1$, the quadratic form (1.2) appears in particular in the book of Gelfand and Vilenkin; see [9, Théorème 1′, p. 258]. Still for $n = 1$, the papers [1] and [2] can be seen as the beginning of such a study, for certain absolutely continuous measures, and for certain singular measures respectively. For other treatments of the theory of Gaussian processes, we refer to the following books, [16], a classic, and [13] of a more recent vintage.

Remark 1.1. *We note that in the paper both the spaces of real-valued Schwartz functions $\mathcal{S}_{\mathbb{R}^n}(\mathbb{R})$ and complex-valued Schwartz functions $\mathcal{S}_{\mathbb{R}^n}$, and their duals, come into play. The real-valued case appears mostly when we make use of the Bochner-Minlos theorem. (See for example equations* (2.1) *and* (2.3) *below.) And the corresponding complex spaces are used in computation of adjoint operators, with reference to some given Hilbert space inner product. When needed, it is simple to insert a complexification.*

The outline of the paper is as follows. The paper consists of eleven sections besides the introduction. Section 2 is a preliminary section where we review Hida's construction of the white noise space. Given a positive measure σ on $\mathbb{R}^n$, satisfying (1.1), we study in Sections 3 and 4 the associated quadratic form q ($:= q_\sigma$) defined by (1.2). In Theorem 4.2 we prove that q is closable if and only if σ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$. These results have implications on the nature of our stochastic processes X_σ (indexed by the Schwartz space $\mathcal{S}_{\mathbb{R}^n}$), which is built in Section 5. See Theorem 5.2 where we construct the associated process X_σ, and its path-space measure μ_σ on the Schwartz space $\mathcal{S}'_{\mathbb{R}^n}(\mathbb{R})$ of real-valued tempered distributions (see (1.3)). In Theorem 6.1, we realize $\mathbf{L}_2(\mathcal{S}'_{\mathbb{R}^n}(\mathbb{R}), \mathcal{B}(\mathcal{S}'_{\mathbb{R}^n}(\mathbb{R})), \mu_\sigma)$ in an explicit manner as a reproducing kernel Hilbert space. A second generalized stochastic process associated to σ is constructed in Section 7. Ergodicity is studied in Section 8. In section 9, we prove that two processes constructed from measures σ_1 and σ_2 from $\mathcal{C}$ are orthogonal in the white noise space $\mathcal{W}$ if and only σ_1 and σ_2 are mutually singular

measures. More generally, we introduce a Hilbert space $\mathscr{H}$ of sigma-functions such that each measure σ from $\mathcal{C}$ generates a closed subspace $\mathscr{H}(\sigma)$ in $\mathscr{H}$, with pairs of mutually singular measures generating orthogonal subspaces in $\mathscr{H}$. In Theorem 9.1, we show that our stochastic processes X_σ are indexed via an infinite-dimensional Fourier transform by equivalence classes of measures σ, with equivalence meaning mutually absolute continuity. As a result we note that the case of closable q in Theorem 4.2 only represents a single equivalence class of processes. Therefore our study of the possibilities for measures σ not absolutely continuous with respect to Lebesgue measure, introduces a multitude of new processes not studied earlier. An example is worked out in Section 12; see also [2] for related examples. A generalized Karhunen-Loève expansion is studied in Section 10. Our result, Theorem 10.2, offers a direct integral decomposition of the most general stochastic process X_σ. Its conclusion yields a formula for separation of variables (i.e., time and sample-point) in the sense of Karhunen-Loève; decomposing X_σ as a direct integral of standard i.i.d. $N(0,1)$ random variables. We prove in Section 11 that the space $\mathcal{C}$ of measures satisfying (1.1) for some p is a module. As mentioned above, the last section is devoted to an example.

2. Preliminaries

There are generally two approaches to measures on function spaces (path-space). The first one builds the measure from prescribed transition probabilities and uses a limit construction, called the Kolmogorov consistency principle. The second approach is based on the notion of generating function (computed from a probability measure), essentially a Fourier transform, obtained from the Bochner-Minlos theorem. This theorem insures the existence of a measure $d\mu_W$ on $\Omega = \mathcal{S}'_{\mathbb{R}^n}(\mathbb{R})$ (as defined by (1.3)) such that

$$e^{-\frac{\|\psi\|^2_{\mathbf{L}_2(\mathbb{R}^n,dx)}}{2}} = \int_\Omega e^{i\langle\omega,\psi\rangle} d\mu_W(\omega), \quad \psi \in \mathscr{S}_{\mathbb{R}^n}(\mathbb{R}). \tag{2.1}$$

The white noise space is defined to be

$$\mathcal{W} = \mathbf{L}_2(\Omega, \mathcal{B}, d\mu_W), \tag{2.2}$$

where $\mathcal{B}$ denotes the sigma-algebra of Borel sets. Furthermore, (2.1) induces an isomorphism $f \mapsto \widetilde{f}$ from the space of square summable real-valued functions of $\mathbf{L}_2(\mathbb{R}^n, dx)$ into $\mathcal{W}$ via the formula

$$\widetilde{\psi}(\omega) = \langle\omega, \psi\rangle, \quad \omega \in \Omega, \tag{2.3}$$

first for $\psi \in \mathcal{S}_{\mathbb{R}^n}(\mathbb{R})$ and then by continuity for every real-valued $f \in \mathbf{L}_2(\mathbb{R}^n, dx)$. Throughout our paper, we will be using the Bochner-Minlos theorem for a variety of positive definite functions on $\mathcal{S}_{\mathbb{R}^n}(\mathbb{R})$. Each will be required to be continuous on $\mathcal{S}_{\mathbb{R}^n}(\mathbb{R})$ with respect to the Fréchet topology. The positive definite functions we consider will have a form similar to that in the expression on the left hand side in (2.1), but with a different quadratic form occurring in the exponent. For background references we recommend [10], [11], [14] and [25].

3. A sesquilinear form

In our construction of Gaussian measures on the space of real tempered distributions in a Gelfand-triple via the Bochner-Minlos theorem, we will be making use of families of sesquilinear forms on the Schwartz space $\mathcal{S}_{\mathbb{R}^n}(\mathbb{R})$. The properties of the measures in turn depend on the nature of the sesquilinear forms under consideration, and we now turn to these below. We first recall that a map from the space of complex-valued Schwartz functions $\mathcal{S}_{\mathbb{R}^n}$ into $\mathbb{R}_+$ is called a positive quadratic form if for every $\phi, \psi \in \mathcal{S}_{\mathbb{R}^n}$ and $c \in \mathbb{C}$, it holds that:

$$\frac{1}{2}\left(q(\phi+\psi)+q(\phi-\psi)\right)=q(\phi)+q(\psi), \tag{3.1}$$

$$q(c\psi)=|c|^2q(\psi), \tag{3.2}$$

$$q(\psi)\geq 0. \tag{3.3}$$

Let $u=(u_1,u_2,\ldots,u_n)$ and $v=(v_1,v_2,\ldots,v_n)$ be in $\mathbb{R}^n$. We denote by

$$u.v=\sum_{k=1}^n u_kv_k \quad \text{and} \quad |u|^2=\sum_{k=1}^n u_k^2$$

the inner product of u and v and the norm of u respectively.

Definition 3.1. *We denote by $\mathcal{C}$ the space of positive measures $d\sigma$ on $\mathbb{R}^n$ such that* (1.1) *holds for some $p \in \mathbb{N}$:*

$$\int_{\mathbb{R}^n}\frac{d\sigma(u)}{(1+|u|^2)^p}<\infty$$

We associate to $d\sigma$ the space $\mathcal{M}(\sigma)$ of measurable functions ψ such that

$$\int_{\mathbb{R}^n}|\widehat{\psi}(u)|^2d\sigma(u)<\infty.$$

In view of (1.1), the space $\mathcal{M}(\sigma)$ contains the Schwartz space $\mathcal{S}_{\mathbb{R}^n}$ of Schwartz functions of n real variables. The set of functions ψ such that

$q_\sigma(\psi) < \infty$ plays an important role in [29, Theorem 3.1, p. 258] for $d\sigma(u)$ corresponding to the fractional Brownian motion, that is

$$d\sigma(u) = \frac{H(1-2H)}{\Gamma(2-2H)\cos(\pi H)}\,|u|^{1-2H}\,du, \quad H \in (0,1).$$

Still for $n = 1$, this set was further used in [1] when $d\sigma(u)$ is absolutely continuous with respect to Lebesgue measure, with appropriate conditions on the Radon-Nikodym derivative. Certain singular measures $d\sigma$ have been considered in [2].

We consider the hermitian form

$$L_\sigma(\psi_1,\psi_2) = \int_{\mathbb{R}^n} \widehat{\psi_1}(u)\overline{\widehat{\psi_2}(u)}d\sigma(u) = \langle \widehat{\psi_1}, \widehat{\psi_2}\rangle_{\mathbf{L}_2(\sigma)},$$

and denote

$$q_\sigma(\psi) = L_\sigma(\psi,\psi)$$

the associated quadratic form. Such forms appear in particular in the theory of generalized stochastic processes and generalized stochastic fields. See [9, Théorème 1′, p. 258, p. 283]. We now give another characterization of $\mathcal{C}$ in Proposition 3.3 below. We first need a preliminary lemma. In the statement of the lemma, U_t denotes, for $t \in \mathbb{R}^n$ and $f \in \mathbf{L}_2(\mathbb{R}^n, dx)$, the translation operator

$$(U_t f)(x) = f(x-t). \tag{3.4}$$

Lemma 3.2. *A positive semi-definite quadratic form q continuous on $\mathcal{S}_{\mathbb{R}^n}$ is of the form* (1.2)

$$q(\psi) = q_\sigma(\psi) = \int_{\mathbb{R}^n} |\widehat{\psi}(u)|^2 d\sigma(u)$$

for some $\sigma \in \mathcal{C}$ if and only if

$$q(U_t\psi) = q(\psi), \quad \forall \psi \in \mathcal{S}_{\mathbb{R}^n}. \tag{3.5}$$

Proof: It is clear that q_σ is a positive quadratic form which satisfies (3.5) since

$$\widehat{U_t f\psi}(u) = e^{iut}\widehat{\psi}(u).$$

To prove the converse statement, let $(\psi_p)_{p\in\mathbb{N}_0}$ be a sequence of elements in $\mathcal{D}_{\mathbb{R}^n}$ which converge to some element $\psi \in \mathcal{D}_{\mathbb{R}^n}$. Then there is a compact set $K \subset \mathbb{R}^n$ such that all the ψ_p and ψ have support inside K and $(\psi_p)_{n\in\mathbb{N}_0}$ as well as all the sequences of partial derivative converge uniformly to ψ and to the corresponding partial derivative respectively. Since K is bounded, it follows that, for every α and β in $\mathbb{N}_0^n$, the sequence $(x^\alpha\psi_p^{(\beta)}(x))_{n\in\mathbb{N}_0}$ converge uniformly to $x^\alpha\psi^{(\beta)}(x)$ on K, and

hence on $\mathbb{R}^n$. Thus, $(\psi_p)_{n\in\mathbb{N}_0}$ converges to ψ in the topology of $\mathcal{S}_{\mathbb{R}^n}$. Since q is assumed continuous in the topology of $\mathcal{S}_{\mathbb{R}^n}$ we have

$$\lim_{p\to\infty} q(\psi_p) = q(\psi).$$

Hence, q is also continuous as a quadratic form on $\mathcal{D}_{\mathbb{R}^n}$ Using [9, Théorème 6, p. 160] we see that q restricted to $\mathcal{D}_{\mathbb{R}^n}$ is of the form q_σ for some $\sigma \in \mathcal{C}$. By continuity it is equal to q_σ on $\mathcal{S}_{\mathbb{R}^n}$. $\square$

Proposition 3.3. *Let σ be a positive measure on $\mathbb{R}^n$. Then, $\sigma \in \mathcal{C}$ if and only if the map*

$$\psi \mapsto q_\sigma(\psi) = \int_{\mathbb{R}^n} |\widehat{\psi}(u)|^2 d\sigma(u)$$

is continuous in the Fréchet topology of $\mathcal{S}_{\mathbb{R}^n}$.

Proof: One direction is done as in the proof of [2, Theorem 5.2]: If $\sigma \in \mathcal{C}$ we can write:

$$\begin{aligned} q_\sigma(\psi) &= \int_{\mathbb{R}^n} |\widehat{\psi}(u)|^2 (1+|u|^2)^p \frac{d\sigma(u)}{(1+|u|^2)^p} \\ &\le C \max_{u\in\mathbb{R}^n} | \left(|\widehat{\psi}(u)|^2 (1+|u|^2)^p \right), \end{aligned}$$

with $C = \int_{\mathbb{R}^n} \frac{d\sigma(u)}{(1+|u|^2)^p}$. We have

$$\begin{aligned} |\widehat{\psi}(u)|^2 &= | \int_{\mathbb{R}^n} \psi(x) \star \psi^\sharp(x) e^{-iu.x} dx| \\ &\le \int_{\mathbb{R}^n} |\psi(x) \star \psi^\sharp(x)| dx \\ &\le \left(\int_{\mathbb{R}^n} |\psi|(x) dx \right) \left(\int_{\mathbb{R}^n} |\psi^\sharp|(x) dx \right) \\ &= \left(\int_{\mathbb{R}^n} |\psi|(x) dx \right)^2. \end{aligned}$$

The terms involving powers of the components u_j are treated in the same way.

Conversely, if q_σ is continuous in the Fréchet topology, then as in the preceding lemma, it is continuous as a map from $\mathcal{D}_{\mathbb{R}^n}$ into $\mathbb{C}$. So q_σ is a translation invariant continuous sesquilinear form from $\mathcal{D}_{\mathbb{R}^n}$ into $\mathbb{C}$. By the previous lemma, there is a measure $\sigma_1 \in \mathcal{C}$ such that

$$q_\sigma(\psi) = \int_{\mathbb{R}^n} |\widehat{\psi}(u)|^2 d\sigma_1(u), \quad \forall \psi \in \mathcal{D}_{\mathbb{R}^n},$$

and so $\sigma=\sigma_1$. $\square$

In Theorem 3.4 below we prove that there exists a bounded linear operator Q from $\mathcal{S}_{\mathbb{R}^n}$ into $\mathbf{L}_2(d\sigma)$ such that

$$\int_{\mathbb{R}^n}\widehat{\psi_1}(u)\overline{\widehat{\psi_2}}(u)d\sigma(u)=\langle Q\psi_1,Q\psi_2\rangle_{\mathbf{L}_2(d\sigma)},\quad \forall\psi_1,\psi_2\in\mathcal{S}_{\mathbb{R}^n}. \tag{3.6}$$

The analysis of the problem uses Schwartz' kernel theorem and a factorization result of Gorniak and Weron; see [8] and [7]. By Schwartz' kernel theorem there is a continuous linear positive operator from $\mathcal{S}_{\mathbb{R}^n}$ into $\mathcal{S}'_{\mathbb{R}^n}$ such that

$$\int_{\mathbb{R}^n}\widehat{\psi_1}(u)\overline{\widehat{\psi_2}(u)}d\sigma(u)=\langle T\psi_1,\overline{\psi_2}\rangle.$$

The form is positive, and so T is a positive operator from $\mathcal{S}_{\mathbb{R}^n}$ into $\mathcal{S}'_{\mathbb{R}^n}$. It is proved in [8] that the space $\mathcal{S}_{\mathbb{R}^n}$ has the factorization property, meaning that any continuous positive operator from $\mathcal{S}_{\mathbb{R}^n}$ into $\mathcal{S}'_{\mathbb{R}^n}$ can be factorized via a Hilbert space: There exists a Hilbert space $\mathcal{H}$ and a continuous operator Q from $\mathcal{S}_{\mathbb{R}^n}$ into $\mathcal{H}$ such that $T=Q^*Q$. To conclude, it suffices to take an isomorphism between $\mathcal{H}$ and $\mathbf{L}_2(\mathbb{R}^n,dx)$. Note that the operator Q will not, in general, be bounded from $\mathbf{L}_2(\mathbb{R}^n,dx)$ into $\mathbf{L}_2(d\sigma)$.

The argument below does not give an explicit construction. However, in the proof of Theorem 3.4 the operator Q (denoted there by Q_σ) is constructed in an explicit way.

Theorem 3.4. *Let σ be a positive measure on $\mathbb{R}^n$ subject to* (1.1)*, and assume that*

$$\dim\,\mathbf{L}_2(d\sigma)=\infty.$$

There exists a continuous linear operator Q_σ from $\mathcal{S}_{\mathbb{R}^n}$ into $\mathbf{L}_2(d\sigma)$ such that (3.6) *holds.*

Proof: We first note that the operators M_{u_k} of multiplication by the variable u_k in $\mathbf{L}_2(d\sigma)$ are self-adjoint and commute with each other. Fix an isometric isomorphism W from $\mathbf{L}_2(d\sigma)$ onto $\mathbf{L}_2(\mathbb{R}^n,dx)$, let $h\in\mathbf{L}_2(d\sigma)$ be defined by

$$h(u)=\frac{1}{(1+|u|^2)^{p/2}},$$

and introduce $T_k=WM_{u_k}W^*$. Define $T=(T_1,\ldots,T_n)$, and

$$Q_\sigma\psi=(I+\sum_{k=1}^n T_k^2)^{p/2}\widehat{\psi}(T)Wh. \tag{3.7}$$

Then, Q_σ satisfies (3.6), as is seen using the functional calculus for commuting normal operators. More precisely, with

$$W^*TW = (W^*T_1W, W^*T_2W, \ldots, W^*T_nW)$$

in the third line, and

$$M_u = (M_{u_1}, M_{u_2}, \ldots, M_{u_n})$$

in the fourth line, we have:

$$\begin{aligned}
\|Q_\sigma\psi\|^2_{\mathbf{L}_2(\mathbb{R},dx)} &= \|(I+\sum_{k=1}^n T_k^2)^{p/2}\widehat{\psi}(T)Wh\|^2_{\mathbf{L}_2(\mathbb{R},dx)} \\
&= \|W^*(I+\sum_{k=1}^n T_k^2)^{p/2}\widehat{\psi}(T)Wh\|^2_{\mathbf{L}_2(\sigma)} \\
&= \|(I+\sum_{k=1}^n (W^*T_kW)^2)^{p/2}\widehat{\psi}(W^*TW)h\|^2_{\mathbf{L}_2(\sigma)} \\
&= \|(I+\sum_{k=1}^n M_{u_k}^2)^{p/2}\widehat{\psi}(M_u)h\|^2_{\mathbf{L}_2(\sigma)} \\
&= \int_{\mathbb{R}^n} \left|(1+|u|^2)^{p/2}\widehat{\psi}(u)\frac{1}{(1+|u|^2)^{p/2}}\right|^2 d\sigma(u) \\
&= \int_{\mathbb{R}^n} |\widehat{\psi}(u)|^2 d\sigma(u).
\end{aligned}$$

$\square$

Lemma 3.5. *The space $\mathcal{S}_{\mathbb{R}^n}$ is dense in $\mathbf{L}_2(\sigma)$*

Proof: The space $\mathcal{S}_{\mathbb{R}^n}$ is closed under pointwise product, and under conjugation. Furthermore, $\mathcal{S}_{\mathbb{R}^n}$ separates points. Therefore, it is dense in $C_0(\dot{\mathbb{R}}^n)$, where $\dot{\mathbb{R}}^n$ denotes the one point compactification of $\mathbb{R}^n$, and where $C_0(\dot{\mathbb{R}}^n)$ denotes the space of uniformly continuous functions f defined on $\mathbb{R}^n$ and such that $\lim_{|x|\to\infty} f(x) = 0$. $\square$

Corollary 3.6. *The completion of $\mathcal{S}_{\mathbb{R}^n}$ in the norm $\left(\int_{\mathbb{R}^n} |\widehat{\phi}(u)|^2 d\sigma(u)\right)^{1/2}$ is $\mathbf{L}_2(\mathbb{R}^n, dx)$.*

Proof: We have

$$Q\phi = (I+T^2)^{p/2}\widehat{\phi}(T)W\left(\frac{1}{(1+|u|^2)^{p/2}}\right),$$

and so

$$\|Q\phi\|_{\mathbf{L}_2(\mathbb{R}^n,dx)} = \|\widehat{\phi}\|_{\mathbf{L}_2(\sigma)}.$$

By construction $W(\mathbf{L}_2(\sigma)) = \mathbf{L}_2(\mathbb{R}^n, dx)$, and this concludes the proof. $\square$

Theorem 3.7. *Let $d\sigma \in \mathcal{C}$ be with the following property: For every compact subset K of $\mathbb{R}^n$, the exponential functions $\{e_t(u) = e^{it\cdot u},\ t \in \mathbb{R}^n\}$ are dense in $\mathbf{L}_2(K, d\sigma)$. Then, the operator Q_σ defined by* (3.7) *has dense range in $\mathbf{L}_2(\mathbb{R}^n, dx)$.*

Proof: Let $g \in \mathbf{L}_2(\mathbb{R}^n, dx)$ be such that

$$\langle Q_\sigma\psi, g\rangle_{\mathbf{L}_2(\mathbb{R}^n,dx)} = 0, \quad \forall\psi \in \mathcal{S}_{\mathbb{R}^n}.$$

Thus

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \overline{(W^*g)(u)}\widehat{\psi}(u)d\sigma(u) \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} (W^*g)(u)e^{iu\cdot x}\psi(x)d\sigma(u)dx \end{aligned}$$

Take ψ which has support inside a fix compact K. It follows that

$$\int_K (W^*g)(u)e^{-iu\cdot x}d\sigma(u) = 0 \quad a.e.$$

From the hypothesis we get that $(W^*g)(u) = 0$ almost everywhere on K. Since K is arbitrary, we have that $W^*g = 0$, and hence $g = 0$. $\square$

4. Closability

As noted, our processes are indexed by a family $\mathcal{C}$ of positive Radon measures σ on $\mathbb{R}^n$, see Definition 3.1. The distinction between the two cases when σ in $\mathcal{C}$ is absolutely continuous with respect to Lebesgue measure or not is crucial in applications of our processes to stochastic integrals and Ito formulas. In this section we show that the absolute continuity condition is equivalent to closability of a certain quadratic form in $\mathbf{L}_2(\mathbb{R}^n)$.

There are a variety of uses of sesquilinear forms q in operator theory and mathematical physics, and we refer to the book [22] for details. One property for a sesquilinear form q is closability. This notion depends on the choice of the ambient Hilbert space $\mathcal{H}$. The notion of closability for sesquilinear forms plays a crucial role in numerous applications. In the discussion below, the ambient Hilbert space $\mathcal{H}$ will

be $\mathbf{L}_2(\mathbb{R}^n, dx)$.

We recall the following definition.

Definition 4.1. *The quadratic form q defined on the Schwartz space $\mathcal{S}_{\mathbb{R}^n}$ is closable if the following condition holds: Given any sequence $(s_k)_{k\in\mathbb{N}}$ of elements of $\mathcal{S}_{\mathbb{R}^n}$ for which*

$$\lim_{k\rightarrow\infty} \|s_k\|_{\mathbf{L}_2(\mathbb{R}^n,dx)} = 0 \quad \text{and} \quad \lim_{k,\ell\rightarrow\infty} q(s_k - s_\ell) = 0,$$

it holds that

$$\lim_{k\rightarrow\infty} q(s_k) = 0.$$

Theorem 4.2. *The quadratic form $q_\sigma(\psi)$ is closable if and only if $d\sigma$ is absolutely continuous with respect to the Lebesgue measure.*

Proof: We first assume that q is closable in the sense of Definition 4.1, and we show that $d\sigma$ is absolutely continuous with respect to the Lebesgue measure. We divide the arguments into a number of steps.

STEP 1: *There exists a positive selfadjoint linear operator Γ with domain containing $\mathcal{S}_{\mathbb{R}^n}$ and such that*

$$q_\sigma(\psi) = \|\Gamma^{1/2}\psi\|^2_{\mathbf{L}_2(\mathbb{R}^n,dx)}, \quad \forall \psi \in S_{\mathbb{R}^n}. \tag{4.1}$$

This follows from Kato's theorem in [17, 22, 23] for closable quadratic forms. For the next step, recall that, for $t \in \mathbb{R}^n$, the translation operator U_t has been defined in (3.4).

STEP 2: *The operator Γ in (4.1) commutes with the translation operators U_t in $\mathbf{L}_2(\mathbb{R}^n, dx)$.*

Indeed, we have for $\psi \in \mathcal{S}_{\mathbb{R}^n}$

$$\widehat{U_t\psi}(u) = e^{it\cdot u}\widehat{\psi}(u), \quad t \in \mathbb{R}^n,$$

and so

$$q_\sigma(\psi) = q_\sigma(U_t\psi), \quad \forall \psi \in \mathcal{S}_{\mathbb{R}^n},$$

that is

$$\|\Gamma^{1/2}\psi\|^2_{\mathbf{L}_2(\mathbb{R}^n,dx)} = \|\Gamma^{1/2}U_t\psi\|^2_{\mathbf{L}_2(\mathbb{R}^n,dx)}.$$

By the uniqueness of the positive operator Γ in (4.1) we obtain

$$U_t^*\Gamma^{1/2}U_t = \Gamma^{1/2},$$

and hence

$$U_t^*\Gamma U_t = \Gamma.$$

Hence the selfadjoint operator Γ is in the commutant of the unitary n-parameter group $\{U_t\}_{t\in\mathbb{R}^n}$ acting on $\mathbf{L}_2(\mathbb{R}^n, dx)$.

STEP 3: Γ *is a convolution operator, that is there is* $m \in \mathbf{L}_1^{\mathrm{loc}}(\mathbb{R}^n, dx)$ *such that*

$$\Gamma\psi = m \star \psi. \tag{4.2}$$

This is because the group $(U_t)_{t\in\mathbb{R}^n}$ is multiplicity-free, and thus its commutant consists of convolution operators. In our applications of closable quadratic forms we make use of Kato's theory as presented in [23, 24] . The thrust of Kato's theorem is that there is a precise way to associate a selfadjoint operator to a closable quadratic form defined on a dense subspace in a Hilbert space. We further make use of results regarding unbounded operators commuting with algebras of bounded operators. These results are in [17, 18], and [35].

We can now conclude the proof:

$$q_\sigma(\psi) = \int_{\mathbb{R}^n} (m \star \psi)(x)\overline{\psi(x)}dx.$$

Applying Parseval's equality on the right side of the above equality we get

$$q(\psi) = \int_{\mathbb{R}^n} \widehat{m}(u)|\widehat{\psi}(u)|^2 du.$$

We obtain thus that $\widehat{m}(u) \geq 0$ and $d\sigma(u) = \widehat{m}(u)du$. It remains to prove that the converse statement holds: if $d\sigma$ is absolutely continuous with respect to Lebesgue measure, then q_σ is closable.

$\square$

5. The generalized process X_σ

The focus of our paper is a family $\mathcal{C}$ of regular Borel measures σ on $\mathbb{R}^n$ (see Definition 3.1). We will be assigning a stationary-increment processes to every σ in the set $\mathcal{C}$ (see Theorem 5.2). The properties of these processes will be studied in the rest of the paper. In Section 11 we show that $\mathcal{C}$ is closed under convolution of measures.

Definition 5.1. *We denote by* $\mathbf{L}(\mathcal{S}_{\mathbb{R}^n}, \mathbf{L}_2(\mathbb{R}^n, dx))$ *the space of linear operators* Q *from* $\mathcal{S}_{\mathbb{R}^n}$ *to* $\mathbf{L}_2(\mathbb{R}^n, dx)$.

In the following statement, recall that we have set $\Omega = \mathcal{S}'_{\mathbb{R}^n}(\mathbb{R})$.

Theorem 5.2. *Let $\sigma \in \mathcal{C}$. Then, there exists a probability measure μ_σ on $\mathcal{S}'_{\mathbb{R}^n}(\mathbb{R})$, an element $Q_\sigma \in \mathbf{L}(\mathcal{S}_{\mathbb{R}^n}, \mathbf{L}_2(\mathbb{R}^n, dx))$ and a generalized Gaussian stochastic process $\{X_\sigma(\psi)\}_{\psi \in \mathcal{S}_{\mathbb{R}^n}}$ such that:*

$$X_\sigma(\psi) = (\widetilde{Q_\sigma(\psi)}), \quad \forall \psi \in \mathcal{S}_{\mathbb{R}^n} \tag{5.1}$$

and

$$\int_\Omega e^{i\langle \omega, X_\sigma(\psi)\rangle} d\mu_W(\omega) = \int_\Omega e^{i\langle \omega, \psi\rangle} d\mu_\sigma(\omega) = e^{-\frac{\|\widehat{\psi}\|^2_{\mathbf{L}_2(\sigma)}}{2}}. \tag{5.2}$$

Proof: Using Theorem 3.4, we take an operator $Q_\sigma \in \mathbf{L}(\mathcal{S}_{\mathbb{R}^n}, \mathbf{L}_2(\mathbb{R}^n, dx))$ such that (3.6) holds. We define a generalized stochastic process $(X_\sigma)_{\psi \in \mathcal{S}_{\mathbb{R}^n}}$ via the formula (5.1). By definition of $d\mu_W$ we have

$$\int_\Omega e^{i\langle \omega, X_\sigma(\psi)\rangle} d\mu_W(\omega) = e^{-\frac{\|Q_\sigma \psi\|^2_{\mathbf{L}_2(\mathbb{R}^n, dx)}}{2}}. \tag{5.3}$$

□

The operator Q_σ is continuous from $\mathcal{S}_{\mathbb{R}^n}$ into $\mathbf{L}_2(\mathbb{R}^n, dx)$. Its adjoint is a continuous operator from $\mathbf{L}_2(\mathbb{R}^n, dx)$ into $\mathcal{S}'_{\mathbb{R}^n}$. We have the diagram

$$\begin{tikzcd}
\mathcal{S}_{\mathbb{R}^n} \arrow[r, hook, "i"] \arrow[dr, "Q_\sigma"] & \mathbf{L}_2(\mathbb{R}^n, dx) \arrow[r, hook, "i^*"] & \mathcal{S}'_{\mathbb{R}^n} \\
\mathcal{S}_{\mathbb{R}^n} \arrow[r, hook, "i"] & \mathbf{L}_2(\mathbb{R}^n, dx) \arrow[r, hook, "i^*"] \arrow[ur, "Q_\sigma^*"] & \mathcal{S}'_{\mathbb{R}^n}
\end{tikzcd}.$$

Theorem 5.3. *The operator*

$$Q_\sigma^* Q_\sigma \ : \ \mathcal{S}_{\mathbb{R}^n} \longrightarrow \mathcal{S}'_{\mathbb{R}^n}$$

is given by the formula

$$(Q_\sigma^* Q_\sigma \psi)(x) = \widehat{\sigma}(x - \cdot)(\psi),$$

meaning that

$$\langle Q_\sigma^* Q_\sigma \psi, \phi\rangle = \int_{\mathbb{R}^n} d\sigma(u) \widehat{\psi}(u) \overline{\widehat{\phi}(u)} du. \tag{5.4}$$

Proof: By definition of $\widehat{\sigma}$ we have

$$\langle \widehat{\sigma}(t-s), v(t,s)\rangle = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-iu(t-s)} v(t,s) dt ds \right) d\sigma(u).$$

Taking $v(t,s) = \psi(t)\overline{\phi}(s)$ we obtain

$$\langle \widehat{\sigma}(t-s), \psi(t)\phi(s)\rangle = \int_{\mathbb{R}^n} \psi(u) \overline{\phi(u)} \sigma(u).$$

But, by definition of the adjoint we have

$$\langle Q_\sigma\psi, Q_\sigma\phi\rangle = \langle Q_\sigma^* Q_\sigma\psi, \phi\rangle$$

and hence the result. $\square$

For related computation when $n = p = 1$ see [1, Section 2].

6. Reproducing kernels

In this section we study the reproducing kernel Hilbert space $\mathscr{H}(C_\sigma)$ associated to the positive definite function

$$C_\sigma(\phi - \psi) = e^{-\frac{\|\widehat{\phi}-\widehat{\psi}\|^2_{\mathbf{L}_2(\sigma)}}{2}},$$

where ϕ and ψ vary through the space $\mathcal{S}_{\mathbb{R}^n}(\mathbb{R})$ of *real-valued* Schwartz functions. Let us already mention that the fact that the functions are real play a key role in the arguments. We set for $f \in \mathbf{L}_2(\mu_\sigma)$

$$(\mathscr{F}_\sigma f)(\psi) = \int_\Omega e^{-i\langle\omega,\psi\rangle} f(\omega) d\mu_\sigma(\omega). \tag{6.1}$$

Theorem 6.1. *The map $\mathscr{F}_\sigma$ is isometric from $\mathbf{L}_2(\mu_\sigma)$ onto $\mathscr{H}(C_\sigma)$.*

Proof: We set

$$e_\phi(\omega) = e^{i\langle\omega,\phi\rangle}.$$

Since we are in the real case e_ϕ has modulus 1 and hence belongs to $\mathbf{L}_2(\mu_\sigma)$. We note that

$$(\mathscr{F}_\sigma e_\phi)(\psi) = \int_\Omega e^{-i\langle\omega,\psi\rangle} e^{i\langle\omega,\phi\rangle} d\mu_\sigma(\omega) = C_\sigma(\phi - \psi).$$

It follows from this equality that, for f in the closed linear span of the functions e_ψ in $\mathbf{L}_2(\Omega, d\mu_\sigma)$ the function (6.1) belongs to $\mathscr{H}(C_\sigma)$ and that

$$\|\mathscr{F}_\sigma f\|_{\mathscr{H}(C_\sigma)} = \|f\|_{\mathbf{L}_2(\Omega,d\mu_\sigma)}.$$

To conclude the proof of the theorem we need to show that the above closed linear span is equal to $\mathbf{L}_2(\Omega, d\mu_\sigma)$. The argument is a bit long, and divided into a number of steps.

STEP 1: *The closed linear span of the e_ψ is dense in $\mathbf{L}_2(\Omega, d\mu_\sigma)$ if and only if the polynomials are dense in $\mathbf{L}_2(\Omega, d\mu_\sigma)$.*

Indeed, one direction is clear from the power series expansion of the exponential. To prove the converse, consider ϕ of the form

$$\phi = \sum_{m=1}^{M} \lambda_m \phi_m,$$

where $M \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_M \in \mathbb{R}$ and $\phi_1, \ldots, \phi_M \in \mathcal{S}_{\mathbb{R}^n}(\mathbb{R})$, and let $F \in \mathbf{L}_2(\Omega, \mu_\sigma)$ be orthogonal to all the e_ϕ.

$$\int_\Omega F(\omega)e_\phi(\omega)d\mu_\sigma(\omega) = \int_\Omega F(\omega)e^{\lambda_1\langle\omega,\phi_1\rangle}\cdots e^{\lambda_1\langle\omega,\phi_M\rangle}d\mu_\sigma(\omega).$$

Differentiating this expression with respect to λ_1 and setting $\lambda_m = 0$, $m = 1, \ldots M$ we get

$$\int_\Omega F(\omega)\langle\omega, \phi_1\rangle d\mu_\sigma(\omega).$$

More generally, differentiating with respect to all the variables, we get orthogonality with respect to all the polynomials.

We denote by $\mathcal{H}$ the closed linear span of the random variable $\omega \mapsto \langle\omega, \phi\rangle$ in $\mathbf{L}_2(\Omega, \mu_\sigma)$.

STEP 2: *The map $\mathscr{F}_\sigma$ which to the random variable $\omega \mapsto \langle\omega, \phi\rangle$ associates the function $\widehat{\phi}$ extends to a unitary from $\mathcal{H}$ onto $\mathbf{L}_2(\sigma)$.*

Isometry is is a direct consequence of (5.2). The fact that $\mathscr{F}_\sigma$ is onto follows from Lemma 3.5.

We note that $\Gamma(\mathcal{H}) = \mathbf{L}_2(\Omega, \mu_\sigma)$, where Γ denotes the second quantization functor. We denote by $\Gamma(\mathscr{F}_\sigma)$ the extension of this map from $\mathbf{L}_2(\Omega, \mu_\sigma)$ into the Fock space $\mathcal{F}(\mathbf{L}_2(\sigma))$.

In the next step, we denote by H_m and h_m the Hermite polynomials and the Hermite functions respectively, $m = 0, 1, \ldots$. Furthermore, we set ℓ to be the set of sequences

$$(\alpha_1, \alpha_2, \ldots), \tag{6.2}$$

indexed by $\mathbb{N}$ with values in $\mathbb{N}_0$, for which only a finite number of elements $\alpha_j \neq 0$.

STEP 3: *The functions $H_\alpha(\omega) = \prod_{m=1}^\infty H_{\alpha_m}(\langle\omega, h_m\rangle)$ form an orthogonal system in $\mathbf{L}_2(\Omega, \mu_\sigma)$ when α runs through ℓ.*

See for instance [14, Theorem 22.24, p. 26].

STEP 4: *The span is dense.*

It is enough to prove that the polynomials are dense in $\mathbf{L}_2(\Omega, \mu_\sigma)$. This is turn follows from the fact that the "exponentials"

$$\Gamma(\phi) = \sum_{m=1}^{\infty} \frac{1}{\sqrt{m!}} \phi \otimes \cdots \otimes \phi$$

are dense in $\Gamma(\mathbf{L}_2(\sigma))$. See for instance [15, Proposition 2.14, p. 28] for the latter.

$\square$

7. The associated stochastic process, second construction, and the fundamental isomorphism

To understand the stochastic processes governed by measures σ from $\mathcal{C}$, we will be making use of a fundamental isomorphism for the associated white noise spaces. We turn to the details below. We denote by $\mathcal{W}_\sigma$ the space

$$\mathcal{W}_\sigma = \mathbf{L}_2(\mathcal{S}'_{\mathbb{R}^n}(\mathbb{R}), \mathcal{B}(\mathscr{S}'_{\mathbb{R}^n}(\mathbb{R})), d\mu_\sigma).$$

We present another Gaussian process with generalized covariance $q_\sigma(\psi_1, \psi_2)$ and an isomorphism between $\mathcal{W}$ and a closed subspace of $\mathcal{W}_\sigma$. Equation (5.2) can be rewritten as

$$E[e^{iY_\sigma(\psi)}] = e^{-\frac{\|\widehat{\psi}\|^2_{\mathbf{L}_2(d\sigma)}}{2}}, \tag{7.1}$$

where $Y_\sigma(\psi)$ denotes the random variable defined by

$$Y_\sigma(\psi) = \langle \omega, \psi \rangle_{\mathcal{S}'_{\mathbb{R}^n}(\mathbb{R}), \mathcal{S}_{\mathbb{R}^n}(\mathbb{R})}.$$

We have now the generalization of the map (2.3) for $\sigma \in \mathcal{C}$. We will denote this map $\widetilde{\psi}^\sigma$, so that

$$\widetilde{\psi}^\sigma = Y_\sigma(\psi). \tag{7.2}$$

A first consequence of the fact that (7.2) is an isometry is:

Theorem 7.1. *Let $\psi, \psi_1, \psi_2 \in \mathcal{S}_{\mathbb{R}^n}(\mathbb{R})$. Then:*

$$E[Y_\sigma(\psi)] = 0, \tag{7.3}$$

$$E[Y_\sigma(\psi_1)\overline{Y_\sigma(\psi_1)}] = \int_{\mathbb{R}^n} \widehat{\psi_1}(u)\overline{\widehat{\psi_2}(u)} d\sigma(u). \tag{7.4}$$

Theorem 7.2. *The map $\psi \mapsto Y_\sigma(\psi)$ extends to an isometric map from $\mathbf{L}_2(\mathbb{R}^n, dx)$ into $\mathcal{W}_\sigma$.*

Proof: In (7.1) replace ψ by $\epsilon\psi$ with $\epsilon \in \mathbb{R}$. Developing along the powers of ϵ both sides of (7.1) leads then to the result. $\square$

8. Ergodicity

We start with Theorem 5.2. Apply (5.2) to the function ψ_t defined by

$$\psi_t(x) = \psi(t + x),$$

we obtain

$$\int_{\mathbb{R}^n} e^{i\langle\omega, X_\sigma(\psi_t)\rangle} = e^{-\frac{1}{2}\|\widehat{\psi_t}\|^2_{\mathbf{L}_2(\sigma)}} = e^{-\frac{1}{2}\|\widehat{\psi}\|^2_{\mathbf{L}_2(\sigma)}}$$

since

$$\widehat{\psi}_t(x) = e^{itx}\widehat{\psi}(x).$$

Therefore, we have that, for every $\psi \in \mathscr{S}_{\mathbb{R}^n}(\mathbb{R})$ and every $t \in \mathbb{R}$,

$$\int_{\mathbb{R}^n} e^{i\langle\omega, X_\sigma(\psi_t)\rangle} d\mu_W(\omega) = \int_{\mathbb{R}^n} e^{i\langle\omega, X_\sigma(\psi)\rangle} d\mu_W(\omega).$$

For every $t \in \mathbb{R}^n$ the generalized processes

$$\{X_\sigma(\psi)\}_{\psi\in\mathscr{S}_{\mathbb{R}^n}(\mathbb{R})} \quad \text{and} \quad \{X_\sigma(\psi_t)\}_{\psi\in\mathscr{S}_{\mathbb{R}^n}(\mathbb{R})}$$

have the same generating function. It follows that there is a unitary map U_t such that

$$U_t X(\psi) = X(\psi_t), \quad \psi \in \mathcal{S}_{\mathbb{R}^n}(\mathbb{R}).$$

The family $(U_t)_{t\in\mathbb{R}^n}$ clearly forms a group.

Lemma 8.1. *Let $\mathcal{H}$ be a Hilbert space and $t \mapsto X(t)$ a function from $\mathbb{R}^n$ into $\mathcal{H}$. Then, the following are equivalent:*
(1) *There exists a function F_X on $\mathbb{R}^n$ such that*

$$\|X(t) - X(s)\|^2_{\mathcal{H}} = F_X(t - s), \quad \forall t, s \in \mathbb{R}^n. \tag{8.1}$$

(2) *For every $t \in \mathbb{R}^n$, the map $U(t)$ defined by*

$$U(t)X(s) = X(t + s), \quad s \in \mathbb{R}^n, \tag{8.2}$$

is unitary and satisfies

$$U(t_1 + t_2) = U(t_1)U(t_2), \quad \forall t_1, t_2 \in \mathbb{R}^n. \tag{8.3}$$

Proof: Assume that (8.1) holds. Then define $U(t)$ on the span of the linear span of the elements $\{X(s)\ ;,\ \ s \in \mathbb{R}^n\}$ by (8.2). Using (8.1), one checks that $U(t)$ extends from the subspace span $\{X(s)\ ;,\ \ s \in \mathbb{R}^n\}$ to all of $\mathcal{H}$, and that the extension, still denoted by $U(t)$, satisfies (8.3).

Conversely, assume that (2) is in force. Then

$$\begin{aligned}\|X(t) - X(s)\|^2_{\mathcal{H}} &= \|U(t)X(0) - U(s)X(0)\|^2_{\mathcal{H}} \\ &= \|U(t - s)X(0) - X(0)\|^2_{\mathcal{H}},\end{aligned}$$

using (8.3) and the unitarity assumption, and thus (8.1) holds. $\square$

We denote by $U(\mathcal{H})$ the set of unitary operators from $\mathcal{H}$ into itself. The proof of the following lemma is easy, and will be omitted.

Lemma 8.2. *Let $(X(t))_{t\in\mathbb{R}^n}$ be a process as defined in the preceding lemma, and assume* (8.1). *Let*

$$U \ : \ \mathbb{R}^n \longrightarrow U(\mathcal{H})$$

denote the corresponding representation of the group $(\mathbb{R}^n, +)$ acting by unitary operators on $\mathcal{H}$. The, the following are equivalent:
(1) $(U(t))_{t\in\mathbb{R}^n}$ *is a strongly continuous representation.*
(2) *The function F_X defined by* (8.1) *is continuous from $\mathbb{R}^n$ into $\mathcal{H}$.*

We apply the preceding results to the special case $\mathcal{H} = \mathbf{L}_2(\Omega, \mathcal{B}_\Omega, \mu_\sigma)$ (recall that $\Omega = \mathcal{S}'_{\mathbb{R}^n}(\mathbb{R})$; see (1.3)) is the Wiener space corresponding to the measure μ_σ as in Section 7, and where Y_σ is the corresponding stationary-mean-square process. The conditions in Lemmas 8.1 and 8.2 are satisfied, and we say that $(U_\sigma(t))_{t\in\mathbb{R}^n}$ is the corresponding unitary representation of the group $(\mathbb{R}^n, +)$. Then:

(8.4) $(U_\sigma(t)f)(\omega) = f(\omega(\cdot - t)), \quad \forall \omega \in \mathcal{S}'_{\mathbb{R}^n}$ and $\forall f \in \mathbf{L}_2(\Omega, \mathcal{B}_\Omega, \mu_\sigma)$.

Indeed, by Lemma 8.1, it is enough to determine $U_\sigma(t)$ on the linear span of the elements $Y_\sigma(s)$ when s runs through $\mathbb{R}^n$. For a given $s \in \mathbb{R}^n$ and $f = Y_\sigma(s)$ we have:

$$U_\sigma(t)Y_\sigma(s) = Y(s+t), \tag{8.5}$$

so that

$$(U_\sigma(t)Y_\sigma(s))(\omega) = (Y_\sigma(s+t))(\omega). \tag{8.6}$$

We now compute (8.4) for $f = Y\sigma(s)$. We have

$$\begin{aligned}
(U_\sigma(t)Y_\sigma(s))(\omega) &= (Y_\sigma(s+t))(\omega) \quad \text{(where we have used (8.5))} \\
&= \omega(s+t) \\
&= Y_\sigma(\omega(\cdot - t)),
\end{aligned}$$

by (7.2), which is the desired conclusion.

We conclude this section with:

Theorem 8.3. *Given $\sigma \in \mathcal{C}$, let X_σ be the corresponding generalized stochastic process defined by* (5.1)*,, and let μ_σ denote the measure on $\Omega = \mathcal{S}'_{\mathbb{R}^n}(\mathbb{R})$ from Theorem 5.2. Then the following two conditions are equivalent:*

(i) The representation $(U_\sigma(t))_{t\in\mathbb{R}^n}$ *in* (8.5) *has* 0 *as its only point spectrum, with a one-dimensional eigenspace.*
(ii) The action of $\mathbb{R}^n$ *on* Ω*:*

$$(t,\omega)\mapsto\omega(\cdot+t)$$

is ergodic with respect to the measure μ_σ

Proof: It suffices to combine Theorem 5.2 and Lemmas 8.1 and 8.2. $\square$

9. The Hilbert space of sigma-functions and applications

In Sections 3 through 7, we introduce a family of Gaussian stochastic processes as follows. Our construction goes beyond what has been done before in a number of ways, as we proceed to outline. The processes Y we consider are indexed by the Schwartz space $\mathcal{S}_{\mathbb{R}^n}$ (i.e., random variable-valued tempered distributions), and it is assumed that the expectation of the square of all the increments by Y is independent, not the increments themselves; i.e., that for fixed Y, the expectation of the squares of differences $Y(\psi)$ depends only on the differences of points ψ in the space of test functions $\mathcal{S}_{\mathbb{R}^n}$. This allows us then to determine a given process Y , via its covariance function, from a corresponding measure $d\sigma$ on $\mathbb{R}^n$. Conversely, we show that for every measure $d\sigma$ on $\mathbb{R}^n$, and falling in a suitably defined class $\mathcal{C}$, there is a uniquely defined process Y_σ determined by its two-point covariance functions. Given Y_σ, the formula (see Theorem 7.1) for its covariance is given by an integration with respect to $d\sigma$. In this section, we will be interested in the correlations computed from two such processes, each determined from a different measure $d\sigma$ in the class $\mathcal{C}$. But this then introduces a difficulty as there are two measures, and they might be relatively singular, or perhaps they may allow for a suitable comparison, for example with the use of a Radon-Nikodym derivative. So it is not at all clear how to compare and to compute the covariance for the different processes. In order to get around this we introduce a Hilbert space $\mathscr{H}$ of equivalence classes, each equivalence class determined by some measure $d\sigma$ from our class $\mathcal{C}$. More precisely, $\mathscr{H}$ consists of equivalence classes of pairs $(\psi, d\sigma)$ where ψ and $d\sigma$ are related, i.e., the function ψ is assumed in $\mathbf{L}_2(d\sigma)$. We are then able to write down an isometric transform for our processes which solves the comparison problem, i.e., comparing two processes computed from different measures $d\sigma$. Our transform is between the Hilbert space $\mathscr{H}$ and the $\mathbf{L}_2$ space of Wiener

white noise measure on the space of real tempered Schwartz distributions. We note that the Hilbert space $\mathscr{H}$, often called a Hilbert space of sigma-functions, has been used in the literature for a variety of different unrelated purpose, for example spectral theory [28], infinite products of measures [21], harmonic analysis [31] , probability theory [26], and more.

Let σ_1 and σ_2 be in $\mathcal{C}$, and let $f_i \in \mathbf{L}_2(d\sigma_i)$ for $i = 1, 2$. Following [28], we say that the two pairs (f_1, σ_1) and (f_2, σ_2) are equivalent if

$$f_1\sqrt{\frac{\mathrm{d}\sigma_1}{\mathrm{d}\lambda}} = f_2\sqrt{\frac{\mathrm{d}\sigma_2}{\mathrm{d}\lambda}}, \quad \lambda \ a.e., \tag{9.1}$$

where λ is a measure such that both σ_1 and σ_2 are absolutely continuous with respect to λ, and where, for instance, $\frac{\mathrm{d}\sigma_1}{\mathrm{d}\lambda}$ denotes the Radon-Nikodym derivative. It is enough to check (9.1) for $\lambda = \sigma_1 + \sigma_2$. The equality (9.1) defines indeed an equivalence relation. We denote by $f\sqrt{d\sigma}$ the equivalence class of (f, σ), and by $\mathscr{H}$ the space of all such equivalent classes. The form

$$\langle f_1\sqrt{d\sigma_1}, f_2\sqrt{d\sigma_2}\rangle_{\mathscr{H}} = \int_{\mathbb{R}^n} f_1(x)\overline{f_2(x)}\sqrt{\frac{\mathrm{d}\sigma_1}{\mathrm{d}\lambda}\frac{\mathrm{d}\sigma_2}{\mathrm{d}\lambda}}d\lambda$$

is a well defined inner product, and the space $\mathscr{H}$ endowed with this inner product is a Hilbert space. This space has been studied by numerous authors, and in particular by Kakutani, Lévy, Schwartz and Nelson. See [21], [30], [27].

We now define the generalization of the operator T_m in the present setting by

$$\widehat{R_\sigma f} = \widehat{f}\sqrt{\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}}. \tag{9.2}$$

With the use of the space $\mathscr{H}$ we now prove:

Theorem 9.1. *Let σ_1 and σ_2 be two measures in $\mathcal{C}$, and let X_{σ_1} and X_{σ_2} be the associated generalized processes given by* (5.3)*:*

$$\int_{\mathcal{W}} e^{i\langle X_{\sigma_k}(\psi),\omega\rangle} d\mu_W(\omega) = e^{-\frac{1}{2}\int_{\mathbb{R}^n} |\widehat{\psi}(u)|^2 d\sigma_k(u)}, \quad k = 1, 2.$$

Then, X_{σ_1} and X_{σ_2} are orthogonal in the white noise space $\mathcal{W}$ if and only if the measures σ_1 and σ_2 are mutually singular.

Proof: We denote W_σ the map which to $f \in \mathbf{L}_2(\mathrm{d}\sigma)$ associates the equivalence class $f\sqrt{\mathrm{d}\sigma}$, and by $\mathscr{H}(\sigma) = W_\sigma(\mathbf{L}_2(\mathrm{d}\sigma)) \subset \mathscr{H}$. Then the spaces $\mathscr{H}(\sigma_1)$ and $\mathscr{H}(\sigma_2)$ are orthogonal in $\mathscr{H}$ if and only if σ_1 and

σ_2 are mutually singular. Define operators R_1 and R_2 as in (9.2), that is:

$$\widehat{R_k f} = \widehat{f}\sqrt{\frac{\mathrm{d}\sigma_{\mathrm{k}}}{\mathrm{d}\sigma}}, \quad k = 1, 2,$$

with $\sigma = \sigma_1 + \sigma_2$. We have

$$\begin{aligned}
\langle \widehat{f}\sqrt{\mathrm{d}\sigma_1}, \widehat{f}\sqrt{\mathrm{d}\sigma_2}\rangle_{\mathscr{H}} &= \int_{\mathbb{R}^n} \widehat{f}\sqrt{\frac{\mathrm{d}\sigma_1}{\mathrm{d}\sigma}}\sqrt{\frac{\mathrm{d}\sigma_2}{\mathrm{d}\sigma}}\overline{\widehat{g}}d\sigma \\
&= \langle X_\sigma(R_1 f), X_\sigma(R_2 g)\rangle_{\mathcal{W}} \\
&= \langle X_{\sigma_1}(f), X_{\sigma_2}(g)\rangle_{\mathcal{W}}.
\end{aligned}$$

□

10. Generalized Karhunen-Loève expansion

Following [32] we define a random measure to be a countably additive function on a given sigma algebra $\mathcal{B}$, taking values in a space of random variables on a probability space. If moreover disjoint sets in $\mathcal{B}$ are mapped into independent random variables we say that the random measure is a Wiener process. Starting with one of our stochastic processes X as introduced in the first section, we show that it admits a direct integral decomposition along an essentially unique Wiener process Z_X, depending on X, and with Z_X Gaussian, i.e., Gaussian distributions in the fibers.

The classical Karhunen-Loève theorem (see e.g., [3]) applies to a restricted family of Gaussian processes, i.e., to a system of stochastic processes $\{X(t)\}$ indexed by $t \in \mathbb{R}$ (typically to represent time), and with specified joint distributions. When it applies, it offers a separation of variables, expanding the process $\{X(t)\}$ as a countable direct sum of a system of independent identically distributed (i.i.d.) $N(0, 1)$ random variables. Below (Theorem 10.2) we extend this to apply to the most general family of Gaussian processes (from our section 5) . A key step in our more general expansion is a systematic theory of direct integrals, taking the role of direct sum-expansions in the restricted setting.

The theme of this section is the study of normal fields and their role in direct integral decompositions of the stochastic processes we introduced in sect 5 above. Our normal fields (Definition 10.1) extend a notion of Wiener processes as introduced in [33]. In fact the study of Wiener processes was initiated in a special case in early papers by Ito.

A random measure over a fixed sigma-algebra $\mathcal{M}$ in a given measure space is a countably additive mapping from $\mathcal{M}$ into random variables of some probability space (Ω, P) , typically with P some fixed path-space measure. Alternatively, random measures are also known as stochastic processes indexed by a measure space. If a random measure takes stochastically independent values on disjoint sets in $\mathcal{M}$, it is called a Wiener process. Indeed, Wiener processes were extensively studied in [34], [33], see also [12]. Examples of Wiener processes are Poisson processes, normal distributions, and jump processes. A theorem in [33] states that every Wiener process naturally decomposes into a sum of three components, a Poisson, a normal, and jump process.

Definition 10.1. *Let $\mathcal{B}(\mathbb{R}^n)$ denote the sigma-algebra of Borel sets of $\mathbb{R}^n$. Fix a measure σ from $\mathcal{C}$. A normal field will be a function*

$$Z \ : \ \mathcal{B}(\mathbb{R}^n) \times \Omega \longrightarrow \mathbb{R},$$

with the following properties:

(1) *For every $\omega \in \Omega$, the function $Z(\cdot, \omega)$ is Borel-measurable on $\mathbb{R}^n$.*
(2) *For every Borel set A, $Z(A, \cdot) \in \mathcal{W}_\sigma$ and is Gaussian.*
(3) *$E_\sigma[Z(A, \cdot)] = 0$.*
(4) *For every $A_1, A_2 \in \mathcal{B}(\mathbb{R})$,*

$$E_\sigma[Z(A_1, \cdot)Z(A_2, \cdot)] = \int_{A_1 \cap A_2} \frac{d\sigma(u)}{(1+u^2)^p}.$$

The following theorem can be seen as a generalized Karhunen-Loève expansion for a special subfamily of elements of $\mathcal{C}$ (namely, $n = p = 1$).

Theorem 10.2. *Let $d\sigma$ be a positive measure on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} \frac{d\sigma(u)}{u^2+1} < \infty.$$

Then there exists a normal field $Z(du, \cdot)$ such that

$$(X_\sigma(t))(\omega) = \int_{\mathbb{R}^n} \sqrt{1+|u|^2} \frac{e^{iu.t}-1}{u} Z(du, w), \quad \omega \in \Omega. \tag{10.1}$$

Proof: We proceed in a number of steps.

STEP 1: *Construction of a projection-valued measure.*

As in the proof of Theorem 3.4 we set M_u denote the operator of multiplication by the variable u in $\mathbf{L}_2(d\sigma)$ (recall that here $n = 1$) and

we fix an isometric isomorphism W from $\mathbf{L}_2(d\sigma)$ onto $\mathbf{L}_2(\mathbb{R}, dx)$, and let

$$h(u) = \frac{1}{\sqrt{1+u^2}}. \tag{10.2}$$

The spectral theorem applied to the selfadjoint operator $T = WM_uW^*$ leads to a projection-valued measure

$$P \quad : \quad \mathcal{B}(\mathbb{R}) \longrightarrow \text{Proj }(\mathbf{L}_2(\mathbb{R}, dx)),$$

such that

$$T = \int_{\mathbb{R}} \lambda P(d\lambda),$$

$$\int_{\mathbb{R}} \|P(d\lambda)f\|^2_{\mathbf{L}_2(\mathbb{R},dx)} = \|f\|^2_{\mathbf{L}_2(\mathbb{R},dx)}, \quad \forall f \in \mathbf{L}_2(\mathbb{R}, dx).$$

Recall that one defines $x(T) = \int_{\mathbb{R}} x(\lambda)P(d\lambda)$ for Borel functions.

STEP 2: *It holds that:*

$$P(A_1 \cap A_2) = P(A_1)P(A_2) \quad \forall A_1, A_2 \in \mathcal{B}(\mathbb{R}).$$

STEP 3: *Let h be defined by* (10.2) *and for $A \in \mathcal{B}(\mathbb{R})$ let*

$$Z(A, w) = \widetilde{P(A)Wh}, \tag{10.3}$$

where $\widetilde{\ }$ denotes the isomorphism $f \mapsto \widetilde{f}$ from $\mathbf{L}_2(\mathbb{R}^n, dx)$ into $\mathcal{W}$ defined by (2.3)*. Then Z is a normal field.*

Indeed, set $h_0 = Wh$. We have

$$\begin{aligned}
E[(Z(A_1, \cdot)Z(A_2, \cdot)] &= \langle P(A_1)h_0, P(A_2)h_0\rangle_{\mathbf{L}_2(\mathbb{R}^n,dx)} \\
&= \langle P(A_1 \cap A_2)h_0, h_0\rangle_{\mathbf{L}_2(\mathbb{R}^n,dx)} \\
&= \|P(A_1 \cap A_2)h_0\|^2_{\mathbf{L}_2(\mathbb{R}^n,dx)} \\
&= \|W^*P(A_1 \cap A_2)Wh_0\|^2_{\mathbf{L}_2(\sigma)} \\
&= \|1_{A_1 \cap A_2}h_0\|^2_{\mathbf{L}_2(\sigma)} \\
&= \int_{A_1 \cap A_2} \frac{d\sigma(u)}{1+u^2}.
\end{aligned}$$

STEP 4: (10.1) *holds.*

Indeed,

$$
\begin{aligned}
X_\sigma(t) &= \widetilde{Q_\sigma(1_{[0,t]})} \\
&= \widetilde{\sqrt{I+T^2}\widehat{1_{[0,t]}}}(T)h_0 \\
&= \int_{\mathbb{R}} \sqrt{1+u^2}\frac{e^{iut}-1}{u}\widetilde{P(du)h_0} \\
&= \int_{\mathbb{R}} \sqrt{1+u^2}\frac{e^{iut}-1}{u}Z(du,\cdot).
\end{aligned}
$$

$\square$

11. Convolution of measures $\mathcal{C}$

Recall that the family $\mathcal{C}$ was defined in Definition 3.1. The convolution is not stable in $\mathcal{C}$. To verify this, take $n=1$ and $d\sigma = d\lambda$ to be the Lebesgue measure. Clearly, $d\lambda \in \mathcal{C}$. On the other hand, we claim that $d\lambda \star d\lambda \notin \mathcal{C}$. Indeed, let $f \in C_c(\mathbb{R})$ (that is, continuous and with support compact), and such that, moreover

$$
\int_{\mathbb{R}} f(u)d\lambda(u) = K > 0.
$$

We have

$$
\int_{\mathbb{R}} f(u)d(\lambda\star\lambda)(u) = \iint_{\mathbb{R}\times\mathbb{R}} f(u+v)dudv = K\int_{\mathbb{R}} dv = \infty.
$$

It follows from Riesz' theorem that $d\lambda \star d\lambda$ is not a well defined Borel measure.

This example suggests to introduce the class $\mathcal{C}_b$ which consists of the positive Borel measures on $\mathbb{R}^n$ such that for every $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ and $C_{pq} > 0$ such that

$$
\int_{\mathbb{R}^n} \frac{d\sigma(u)}{(1+|u+v|^2)^q} \le \frac{C_{pq}}{(1+|v|^2)^p}. \tag{11.1}
$$

Theorem 11.1. *It holds that*

$$
\mathcal{C}\star\mathcal{C}_b \subset \mathcal{C}.
$$

Proof: For simplicity we consider the case $n=1$. Let $\sigma_1 \in \mathcal{C}$ and $\sigma_2 \in \mathcal{C}_b$. There exists $p \in \mathbb{N}$ such that

$$
\int_{\mathbb{R}} \frac{d\sigma_1(u)}{(1+u^2)^p} < \infty
$$

Since $\sigma_2 \in \mathcal{C}_b$, (11.1) is in force for some $q \in \mathbb{N}$ and $C_{pq} > 0$. Thus:

$$\begin{aligned}\int_{\mathbb{R}} \frac{d(\sigma_1 \star \sigma_2)(w)}{(1+w^2)} &= \iint_{\mathbb{R}^2} \frac{d\sigma_1(u) d\sigma_2(v)}{(1+(u+v)^2)^q} \\ &\le C_{pq} \int_{\mathbb{R}} \frac{d\sigma_1(u)}{(1+u^2)^p} < \infty.\end{aligned}$$

$\square$

12. An example: The Dirac comb

We take

$$\sigma(u) = \sum_{n \in \mathbb{Z}} \delta(n-u).$$

Then, $\mathbf{L}_2(d\sigma) = \ell_2(\mathbb{Z})$. Furthermore:

Proposition 12.1. *Let W be an isomorphism between $\ell_2(\mathbb{Z})$ onto $\mathbf{L}_2(\mathbb{R}, dx)$, and let Q be defined by*

$$Q\psi = W((\widehat{\psi}(n))_{n \in \mathbb{Z}}).$$

Then Q is a bounded operator from $\mathcal{S}_{\mathbb{R}}$ into $\mathbf{L}_2(\mathbb{R}, dx)$, and it holds that:

$$\int_{\mathbb{R}} |\widehat{\psi}(u)|^2 d\sigma(u) = \int_{\mathbb{R}} |Q\psi|(x)^2 dx \tag{12.1}$$

and Q^*Q is a bounded operator from $\mathcal{S}_{\mathbb{R}}$ into $\mathcal{S}'_{\mathbb{R}}$ defined by the periodization operator:

$$(Q^*Q\psi)(x) = \sum_{n \in \mathbb{Z}} \psi(x + 2\pi n). \tag{12.2}$$

Proof: For $\psi \in \mathcal{S}_{\mathbb{R}}$, integration by part shows that the sequence of Fourier coefficients $(\widehat{\psi}(n))_{n \in \mathbb{Z}}$ belongs to $\ell_2(\mathbb{Z})$. Therefore, equation (12.1) follows from the definition of σ and from the fact that W is an isomorphism from $\ell_2(\mathbb{Z})$ onto $\mathbf{L}_2(\mathbb{R}, dx)$. We now turn to (12.2). Let $\psi, \phi \in \mathcal{S}_{\mathbb{R}}$. We have

$$\begin{aligned}\int_{\mathbb{R}} (\sum_{n \in \mathbb{Z}} \psi(x+2\pi n))\overline{\phi(x)} dx &= \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \psi(x+2\pi n)\overline{\phi(x)} dx \\ &= \sum_{n \in \mathbb{Z}} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\psi}(u) e^{-2\pi n u i} \overline{\widehat{\phi}(u)} du \\ &= \sum_{n \in \mathbb{Z}} \widehat{\psi}(n) \overline{\widehat{\phi}(n)},\end{aligned}$$

where we have used Parseval equality for the second equality, and Poisson's formula for the third. □

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