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Characterizations of Families of Rectangular, Finite Impulse Response, Para-Unitary Systems

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CHARACTERIZATIONS OF FAMILIES OF RECTANGULAR, FINITE IMPULSE RESPONSE, PARA-UNITARY SYSTEMS

DANIEL ALPAY, PALLE JORGENSEN, AND IZCHAK LEWKOWICZ

ABSTRACT. We here study Finite Impulse Response (FIR) rectangular, not necessarily causal, systems which are (para)-unitary on the unit circle (=the class $\mathcal{U}$). First, we offer three characterizations of these systems. Then, introduce a description of all FIRs in $\mathcal{U}$, as copies of a real polytope, parametrized by the dimensions and the McMillan degree of the FIRs.

Finally, we present six simple ways (along with their combinations) to construct, from any FIR, a large family of FIRs, of various dimensions and McMillan degrees, so that whenever the original system is in $\mathcal{U}$, so is the whole family.

A key role is played by Hankel matrices.

1. INTRODUCTION

This work is on the crossroads of Operator and Systems theory from the mathematical side and Control, Signal Processing and Communications theory from the engineering side. It addresses problems or employs tools from all these areas. Thus, it is meant to serve as a bridge between the corresponding communities. We start by formally laying out the set-up.

1.1. **Finite Impulse Response.** We here focus on $p \times m$-valued polynomials of a complex variable z, of the form

$$(1.1) \qquad F(z) = z^q \left(z^{-1}B_1 + \ldots + z^{-n}B_n \right),$$

where the natural n and the integer q are parameters. Hereafter, we relate to Laurent polynomials when the powers of z may be positive or negative (or both).

The polynomial $F(z)$ in (1.1) may be viewed as the (two-sided) Z-transform of $\Phi(t)$ a (discrete) time sequence,

$$(1.2) \quad \Phi(t) = \delta_K(t - q + 1)B_1 + \ldots + \delta_K(t - q + n)B_n \qquad t \quad \text{integral variable,}$$

where δ_K is the Kronecker delta,

$$\delta_K(\beta) = \begin{cases} 1 & \beta = 0, \\ 0 & \beta \neq 0, \end{cases} \qquad \beta \quad \text{integer.}$$

Hence, in Engineering terminology $\Phi(t)$ (and often $F(z)$) is referred to as Finite Impulse Response (i.e. the support of $\Phi(t)$ is finite).

Moreover $F(z)$ will be called *causal* whenever $1 \geq q$ (i.e. $\Phi(t) \equiv 0$ for all $0 > t$) and *strictly causal* if $0 \geq q$. Similarly, *(strictly) anti-causal* when $(q \geq n+1)$ $q \geq n$.

When anti-causal, $F(z)$ in (1.1) is a usual polynomial with non-negative powers of the variable z.

Finite Impulse Response functions (=Laurent polynomials) are of numerous applications in communications control and signal processing see e.g. [12], [13], [14], [15], [27], [28], [39], [42].

1.2. **Unitary symmetry.** Let $\mathbb{T}$ be the unit circle,

$$\mathbb{T} := \{ z \in \mathbb{C} \ : \ |z| = 1 \ \}.$$

We denote by $\mathcal{U}$ the class of $p \times m$-valued rational functions having unitary symmetry on the unit circle, i.e.

(1.3)

$$\mathcal{U} := \left\{ \ F(z) \ : \ \begin{cases} \left(F(z) \right)^* F(z) \equiv I_m & p \geq m & \text{isometry} \\ \\ F(z) \left(F(z) \right)^* \equiv I_p & m \geq p & \text{co} - \text{isometry} \end{cases} \quad \forall z \in \mathbb{T} \right\}.$$

In engineering terminology, if in addition all poles of $F(z)$ are within the open unit disk (=Schur stable), the function $F(z)$ is called *lossless*[1], see e.g. [17], [40, Section 14.2] or *all-pass*[2].

For a given $p \times m$-valued rational function $F(z)$ we here define the $m \times p$-valued *conjugate* rational function as,

$$F^{\#}(z) := \left(F \left(\frac{1}{z^*} \right) \right)^*.$$

Note that

$$F^{\#}(z)_{|_{z \in \mathbb{T}}} = \left(F(z)_{|_{z \in \mathbb{T}}} \right)^*.$$

It is well known, see e.g. [1, Eq. (3.1)], [23, Définition 35], that for rational functions condition (1.3) is equivalent to the following,

$$F \in \mathcal{U} \qquad \Longleftrightarrow \qquad \forall z \in \mathbb{C} \quad \begin{cases} F^{\#}(z) F(z) \equiv I_m & p \geq m & \text{isometry,} \\ \\ F(z) F^{\#}(z) \equiv I_p & m \geq p & \text{co} - \text{isometry.} \end{cases}$$

The interest in the class $\mathcal{U}$ stems from various aspects: For realization see e.g. [1], [2], [8] [17] and [41] for factorization see e.g. [33] and for some signal processing applications see e.g. [10].

1.3. **The current work.** The interest in Finite Impulse Response functions within $\mathcal{U}$ (=para-unitary, in signal processing "dialect") is vast, see e.g. the books [11], [25], [29, Section 7.3], [37, Section 5.2], [40, Section 6.5], the theses [23], [31] and the papers [4], [5], [7], [10], [16], [24], [26], [32], [34], [36] and [43].

For example, the classical spectral factorization of self adjoint matrices and the singular values decomposition of rectangular matrices (for constant matrices see e.g. [21, Theorems 2.5.6, 4.1.5] and [21, Theorem 7.3.5], respectively) have been generalized to matrix valued polynomials (where at least one of the factors is para-unitary). These extension have several signal processing applications and were studied in [30] and [38].

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[1]Passive electrical circuits are either dissipative or lossless.

[2]For example, in studying classical filters a "high-pass" could be viewed as an "all-pass" minus a "low-pass".

Similarly, the Q-R factorization of constant matrices has been generalized to matrix-valued polynomials (one of which is para-unitary). In [12], [13] this was studied along with applications to OFDM communications

This work focuses on characterizations of families of rectangular (not necessarily causal) Finite Impulse Response (FIR) functions within $\mathcal{U}$.

This work is aimed at three different communities: mathematicians interested in classical analysis, signal processing engineers and system and control engineers. Thus adopting the terminology familiar to one audience, may intimidate or even alienate the other. For example, as already mentioned, matrix-valued Laurent polynomials (powers of various signs) and not necessarily causal Finite Impulse Response systems, are virtually the same entity seen by a different community. Similarly, what is known to engineers as McMillan degree also arises in geometry of loop groups as an index.

Books like [11], [19], [37], and the theses [23], [31] have made an effort to be at least "bi-lingual". Lack of space prevents us from providing even a concise dictionary of relevant terms. Thus, we shall try to employ only basic concepts.

This work is organized as follows.

In Subsection 2.1 we construct the Hankel matrix $\mathbf{H}$ associated with a polynomial $F(z)$ in (1.1) and show how that McMillan degree can be extracted from it. In Subsection 2.2 we show how to construct from a single Laurent polynomial a whole family of Laurent polynomials of various powers and degrees. Moreover, whenever the original polynomial is in $\mathcal{U}$, so is the whole resulting family. This construction is based on the Hankel matrix $\mathbf{H}$. The details are relegated to the Appendix.

In subsection 3.1 we present a characterization of causal Schur stable rational functions in $\mathcal{U}$ through their minimal realization matrix. In subsection 3.2 we present the Blaschke-Potapov characterization of rational functions in $\mathcal{U}$. Here, the rational functions may have poles everywhere (including infinity) except the unit circle. In Theorem 3.5 we introduce three convenient formulations of this result, in the framework of FIRs.

As a by-product, this enables us to offer an easy-to-use description of all FIRs in $\mathcal{U}$ with McMillan degree and dimensions as parameters. In fact, for causal systems, it turns out to be a genuine real polytope and in general these are copies of this polytope. This is in particular convenient if one wishes to: (i) Design FIRs within $\mathcal{U}$ through optimization, see e.g. [34].
(ii) Iteratively apply para-unitary similarity, see e.g. [23, Section 3.3], [30], [36]. In signal processing literature, this is associated with with *channel equalization* and in communications literature with *decorrelation of signals* or (iii) Iteratively apply Q-R factorization in the framework of communications, see e.g. [12], [13].

In Subsection 4.1 we return to the Hankel $\mathbf{H}$ and show that more information is "encoded" in it: Out of $\mathbf{H}$ one can check whether or not $F(z)$ is in $\mathcal{U}$ (see Theorem 4.1). Moreover, just from the singular values of $\mathbf{H}$ one can deduce whether $F(z)$ is square or rectangular (see Proposition 4.2). It turns out that the conditions there are closely related to those in the Nehari problem where one approximates an anti-causal rational polynomial, by a causal one, see e.g. [19, Section 12.8].

A concluding remark is given at the end.

2. THE HANKEL MATRIX AND FIR

In this section we present our first use of Hankel matrices. They will be re-appear in Section 4 and the Appendix.

2.1. Realization of FIR and the Hankel matrix.
In this subsection we focus on the McMillan degree a $p \times m$-valued polynomial polynomial $F(z)$

$$(2.1) \qquad F(z) = z^q \left(z^{-1} B_1 + \ldots + z^{-n} B_n \right),$$

where the natural n and the integer q are parameters.

We here specialize textbook material on state space realization (of not necessarily causal systems) and the corresponding Hankel matrices, see e.g. [19, Subsection 12.8.1].

We start by recalling some classical facts concerning realization of $p \times m$-valued rational function $F(z)$. With a slight abuse of notation, an arbitrary $p \times m$-valued rational function $F(z)$ may be written as

$$(2.2) \qquad F(z) = F_l(z) + D + F_r(z)$$

where D is a constant matrix and

$$\lim_{z \to 0} F_l(z) = 0_{p \times m} \qquad\qquad \lim_{z \to \infty} F_r(z) = 0_{p \times m} ,$$

(the subscripts stand for "left" and "right"). Note that $F_l(z)$, $F_r(z)$ may be viewed as the (two-sided) Z-transform of *strictly anti-causal*, *strictly causal* (discrete) time sequences, respectively.

Recall that $\tilde{F}_r(z)$ a $p \times m$-valued rational function with no poles at infinity[3] is given by,

$$(2.3) \qquad \tilde{F}_r(z) := F_r(z) + D = C(zI_n - A)^{-1} B + D.$$

Sometimes it is convenient to present $\tilde{F}_r(z)$ in (2.3) by its $(\nu+p) \times (\nu+m)$ realization matrix R, i.e.

$$(2.4) \qquad R := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right).$$

A realization of $\tilde{F}_r(z)$ in (2.3), is called *minimal* if in (2.4), ν the dimension of A, is the smallest possible. This ν is called the *McMillan degree* of $\tilde{F}_r(z)$ in (2.3).

As already mentioned, in the polynomial framework in (2.1), $F(z)$ is causal when $1 \geq q$ and strictly causal when $0 \geq q$. A special attention will be devoted to the case where in (2.1) $q = 0$, i.e.

$$(2.5) \qquad F_o(z) := \tilde{F}_r(z)_{|_{q=0}} = F(z)_{|_{q=0}} = z^{-1} B_1 + \ldots , z^{-n} B_n ,$$

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[3]=Bounded at infinity and in engineering called *causal* or colloquially *proper*.

For future reference, we introduce the following notation[4]

$$(2.6) \qquad \mathbf{B}_\eta := \begin{pmatrix} 0_{\eta \cdot p \times m} \\ B_1 \\ \vdots \\ B_n \end{pmatrix} \qquad \eta = 0, \ 1, \ 2, \ \ldots$$

and by $\mathbf{J}_k$ the following $kp \times kp$ block-shift matrix,

$$\mathbf{J}_k = \begin{pmatrix} & I_p & & \\ & & \ddots & \\ & & & I_p \\ 0_{p \times p} & & & \end{pmatrix}.$$

With this notation, specializing the realization matrix (2.4) to the case of $F_o(z)$ in (2.5), yields

$$(2.7) \qquad R := \left(\begin{array}{cc|c} \mathbf{J}_n & & \mathbf{B}_0 \\ \hline I_p & 0_{p \times (n-1)p} & 0_{p \times m} \end{array} \right),$$

where $\mathbf{B}_0$ is as in (2.6) (with $\eta = 0$).

This classical approach has two limitations:
(i) Often the realization in (2.7) is not minimal. Moreover, the actual McMillan degree (bounded from above by np) is not apparent from (2.7).

(ii) Strictly speaking (2.4) and its special case in (2.7) are realization around zero, suitable for causal systems. Realizations for the anti-causal case (around $z = \infty$), are fairly common as well. Although known, it is more challenging to write down realizations of polynomials $F(z)$ containing powers of mixed signs, i.e. when in (2.1) $q \in [2, \ n-1]$.

We next show that representing realizations of FIR systems through Hankel matrices[5], circumvent both limitations:

It is suitable for all q in (2.1) and the McMillan degree is apparent.

In Subsection 4.1 below these Hankel matrices will be used to introduce a characterization of Finite Impulse Response functions within $\mathcal{U}$.

To study the Hankel matrix representation of $F(z)$ in (2.1),

$$F(z) = z^q \left(z^{-1} B_1 + \ \ldots \ + z^{-n} B_n \right), \qquad q \quad \text{integral parameter,}$$

we find it convenient to separately consider two extremes possibilities, and an intermediate case:

(i) $q \geq n + 1$ so $F(z)$ is strictly anti-causal and

(ii) $0 \geq q$ so $F(z)$ is strictly causal,

(iii) $q \in [1, \ n]$ so $F(z)$ is a genuine Laurent polynomial.

(i) Assume now that $F(z)$ in (2.1) is *strictly causal*, $-q := \eta \geq 0$. Here (2.1) takes the form

$$F(z) = F_r(z) = z^{-(1+\eta)} B_1 + \ \ldots \ + z^{-(n+\eta)} B_n \qquad \eta \geq 0.$$

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[4]In the sequel, boldface characters will stand for block-structured matrices.

[5]In general, the Hankel operator is infinite, but since we here focus on $F(z)$ in (2.2) with a Finite Impulse Response, the corresponding Hankel matrix is finite and no truncation is needed.

We shall denote by $\mathbf{H}_\eta$ (recall, block-structured matrices are represented by bold-face characters) the associated $p(n+\eta) \times m(n+\eta)$ Hankel matrix,

$$(2.8) \qquad \mathbf{H}_\eta = \begin{pmatrix} & & B_1 & \cdots & B_n & \\ & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & \\ B_1 & \cdots & B_n & & & \\ \vdots & \cdot^{\cdot^{\cdot}} & & & \\ B_n & & & & \end{pmatrix}.$$

The $p \times m$ (block) elements of $\mathbf{H}_\eta$ are known as the Markov parameters of $F_r(z)$ and in particular, the first (block) row of $\mathbf{H}_\eta$ is the impulse response of $F_r(z)$.

For completeness we add that the Hankel matrix can be obtained as a product of the observability and controllability matrices here it takes the form

$$\mathbf{H}_\eta = \begin{pmatrix} \mathbf{J}^0 & \mathbf{J}^1 & \cdots & \mathbf{J}^{n+\eta-1} \end{pmatrix} \begin{pmatrix} \mathbf{B}_\eta & & \\ & \ddots & \\ & & \mathbf{B}_\eta \end{pmatrix},$$

where $\mathbf{B}_\eta$ is as in (2.6), $\mathbf{J} = \mathbf{J}_{n+\eta}$ and $\mathbf{J}^0 = I_{(n+\eta)p}$.

Associating the Hankel matrix $\mathbf{H}_\eta$ in (2.8) with the polynomial $F_r(z)$ is fairly classical and goes back at least to [18, Eq. (7)].

(ii) The other extreme is where $F(z)$ is *strictly anti-causal*, i.e. in (2.1) $q \geq n+1$ so $F(z)$ is a genuine polynomial with $\eta := q - n - 1 \geq 0$,

$$F(z) = F_l(z) = z^{n+\eta}B_1 + \ldots + z^{1+\eta}B_n \qquad \eta \geq 0.$$

Then, the corresponding $p(n+\eta) \times m(n+\eta)$ Hankel matrix $\hat{\mathbf{H}}_\eta$ (hat for left polynomial) takes the form of

$$(2.9) \qquad \hat{\mathbf{H}}_\eta = \begin{pmatrix} & & B_n & \cdots & B_1 & \\ & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & \\ B_n & \cdots & B_1 & & & \\ \vdots & \cdot^{\cdot^{\cdot}} & & & \\ B_1 & & & & \end{pmatrix}.$$

(iii) In the intermediate case, where $q \in [1, \ n]$, $F(z)$ in (2.1) is a genuine Laurent polynomial and we shall write it in the form of (2.2) (with $B_q = D$) as

$$(2.10) \qquad \begin{aligned} F(z) &= F_l(z) + D + F_r(z) \\ F_r(z) &= z^{-1}B_{q+1} + z^{-2}B_{q+2} + \ldots + z^{q-n}B_n \qquad q \in [1, \ n]. \\ F_l(z) &= z^{q-1}B_1 + z^{q-2}B_2 + \ldots + z^1 B_{q-1} \ . \end{aligned}$$

The corresponding Hankel matrices for $F_r(z)$ and $F_l(z)$, respectively are

$$(2.11) \quad \mathbf{H} = \begin{pmatrix} B_{q+1} & B_{q+2} & \cdots & B_{n-1} & B_n \\ B_{q+2} & B_{q+3} & \cdots & B_n & \\ \vdots & & \cdot^{\cdot^{\cdot}} & & \\ B_{n-1} & B_n & & & \\ B_n & & & & \end{pmatrix} \qquad \hat{\mathbf{H}} = \begin{pmatrix} B_{q-1} & B_{q-2} & \cdots & B_2 & B_1 \\ B_{q-2} & B_{q-3} & \cdots & B_1 & \\ \vdots & & \cdot^{\cdot^{\cdot}} & & \\ B_2 & B_1 & & & \\ B_1 & & & & \end{pmatrix}.$$

It is well known, see e.g. [9, Theorem 4.5], that the McMillan degree of $F(z)$ in (2.2) and in (2.10) is equal to the sum of the ranks of $\hat{\mathbf{H}}$ and $\mathbf{H}$, the associated Hankel matrices. Thus we can now state the main result of this subsection.

Observation 2.1. *Let us denote by d the McMillan degree of the $p \times m$-valued (possibly Laurent) polynomial $F(z)$ in (2.1),*

$$F(z) = z^q \left(z^{-1}B_1 + \ldots + z^{-n}B_n \right).$$

Let $\mathbf{H}$, $\hat{\mathbf{H}}$ *be the Hankel matrices associated with* $F_r(z)$, $F_l(z)$, *respectively. Then,*

$$d = \begin{cases} \mathrm{rank}(\mathbf{H}) & 0 \geq q & \text{Eq.} \quad (2.8) \\ \mathrm{rank}(\hat{\mathbf{H}}) & q \geq n+1 & \text{Eq.} \quad (2.9) \\ \mathrm{rank}(\mathbf{H}) + \mathrm{rank}(\hat{\mathbf{H}}) & q \in [1, \ n] & \text{Eq.} \quad (2.11). \end{cases}$$

So far, we have used the Hankel matrix $\mathbf{H}$ to obtain the McMillan degree of a given FIR $F(z)$. In Subsection 4.1 below we show that this $F(z)$ is in $\mathcal{U}$ if and only if $\mathbf{H}^*\mathbf{H}$ (or $\mathbf{H}\mathbf{H}^*$) have a certain invariant subspace.

2.2. Families of FIR systems in $\mathcal{U}$. In this subsection we show how to produce, out of a given $p \times m$-valued Laurent polynomial,

$$(2.12) \qquad F(z) = z^q \left(z^{-1}B_1 + \ \ldots \ + z^{-n}B_n \right),$$

a whole family of Laurent polynomials of *various dimensions and powers*. Moreover, whenever the original one is in $\mathcal{U}$, then so is all the resulting family of polynomials.

Clearly, when $F(z)$ in (2.12) is in $\mathcal{U}$, so are $F(z)U_m$ and $U_pF(z)$, where U_m and U_p are arbitrary $m \times m$ and $p \times p$ constant unitary matrices.

Note also that if for some value of the parameter q, $F(z)$ in (2.12) is in $\mathcal{U}$, then this is the case for all q. Thus, without loss of generality, we can take all polynomials to be causal, i.e. in (2.12) $0 \geq q$.

We here illustrate six versions of the newly generated polynomials. Obviously, to further enrich the variety, they may be combined.

I. The reverse polynomial,

$$F_{\mathrm{rev}}(z) \quad := \quad z^{-1}B_n + z^{-2}B_{n-1} + \ \ldots \ + z^{1-n}B_2 + z^{-n}B_1 \ .$$

II. Preserving the McMillan degree.
For $-1 \geq q$ and $j = 1, \ \ldots \ , 1 - q$, one can generate $jp \times jm$-valued polynomials.
For example, taking in (2.12) $n = 4$ one may obtain:
a. For $q = -1$

$$F(z) \quad = \quad z^{-1}(z^{-1}B_1 + z^{-2}B_2 + z^{-2}B_3 + z^{-4}B_4) \qquad p \times m- \text{ valued}$$

$$F(z) \quad = \quad z^{-1} \begin{pmatrix} 0 & B_1 \\ B_1 & B_2 \end{pmatrix} + z^{-2} \begin{pmatrix} B_2 & B_3 \\ B_3 & B_4 \end{pmatrix} + z^{-3} \begin{pmatrix} B_4 & 0 \\ 0 & 0 \end{pmatrix} \qquad 2p \times 2m- \text{ valued},$$

b. For $q = -2$

$$F(z) \quad = \quad z^{-2}(z^{-1}B_1 + z^{-2}B_2 + z^{-2}B_3 + z^{-4}B_4) \qquad p \times m- \text{ valued}$$

$$F(z) \quad = \quad z^{-1} \begin{pmatrix} 0 & 0 \\ 0 & B_1 \end{pmatrix} + z^{-2} \begin{pmatrix} B_1 & B_2 \\ B_2 & B_3 \end{pmatrix} + z^{-3} \begin{pmatrix} B_3 & B_4 \\ B_4 & 0 \end{pmatrix} \qquad 2p \times 2m- \text{ valued}$$

$$F(z) \quad = \quad z^{-1} \begin{pmatrix} 0 & 0 & B_1 \\ 0 & B_1 & B_2 \\ B_1 & B_2 & B_3 \end{pmatrix} + z^{-2} \begin{pmatrix} B_2 & B_3 & B_4 \\ B_3 & B_4 & 0 \\ B_4 & 0 & 0 \end{pmatrix} \qquad 3p \times 3m- \text{ valued}.$$

We shall see that both polynomials in **a** share the same McMillan degree. A similar statement holds for the three polynomials in **b**.

III. Doubling the powers.
For a pair of parameters a integer and γ natural,

$$(2.13) \qquad F(z) = z^a \left(z^{-1\cdot\gamma}B_1 + z^{-2\cdot\gamma}B_2 + \ \ldots \ + z^{-n\cdot\gamma}B_n \right).$$

In particular, for $a = 0$, $F(z)$ in (2.13) may be written as $F(z^\gamma)$. Rational functions within $\mathcal{U}$, of this structure, suit what is known in signal processing as *filter banks*, see e.g. the books [37], [40] and the papers [4], [6], [14] [32], [20], [34], [42], [43].

In a similar way, one can obtain richer structures, e.g. of the form

$$(2.14) \qquad F(z) = z^{-1}B_1 + z^{-2}B_2 + z^{-3}B_3 + z^{-9}B_4 + z^{-10}B_5 + z^{-11}B_6 \ .$$

IV. Rectangular polynomials. Let ρ be a parameter so that $\frac{n}{\rho}$ is natural[6],

$$(2.15) \qquad F(z) = z^{a\rho}\left(z^{-1\cdot\gamma\rho}\mathbf{B}_1 + z^{-2\cdot\gamma\rho}\mathbf{B}_2 + \ \ldots \ + z^{-n\cdot\gamma\rho}\mathbf{B}_{\frac{n}{\rho}}\right)$$

a. $F(z)$ is a $\rho p \times m$-valued polynomial with coefficients,

$$\mathbf{B}_1 = \begin{pmatrix} B_1 \\ \vdots \\ B_\rho \end{pmatrix} \qquad \mathbf{B}_2 = \begin{pmatrix} B_{\rho+1} \\ \vdots \\ B_{2\rho} \end{pmatrix} \qquad \cdots \qquad \mathbf{B}_{\frac{n}{\rho}} = \begin{pmatrix} B_{n+1-\rho} \\ \vdots \\ B_n \end{pmatrix}$$

b. $F(z)$ is a $p \times \rho m$-valued polynomial with coefficients,

$$\mathbf{B}_1 = \begin{pmatrix} B_1 & \cdots & B_\rho \end{pmatrix} \qquad \mathbf{B}_2 = \begin{pmatrix} B_{\rho+1} & \cdots & B_{2\rho} \end{pmatrix} \qquad \cdots \qquad \mathbf{B}_{\frac{n}{\rho}} = \begin{pmatrix} B_{n+1-\rho} & \cdots & B_n \end{pmatrix}.$$

V. Composition of polynomials.
Out of the pair of polynomials,

$$(2.16) \qquad \begin{aligned} F_b(z) &= z^{-1}B_1 + \ldots \ z^{-n}B_n & p_b \times m_b \\ F_c(z) &= z^{-1}C_1 + \ldots \ z^{-l}C_l & p_c \times m_c \end{aligned} \qquad\qquad n \geq l,$$

construct the following third polynomial

$$F_d(z) = z^{-1}D_1 + \ \ldots \ + z^{-n}D_n \ ,$$

in three different forms.

a. $F_d(z)$ is $(p_b + p_c) \times (m_b + m_c)$-valued polynomial with coefficients,

$$D_1 = \begin{pmatrix} B_1 & 0 \\ 0 & C_1 \end{pmatrix} \ \cdots \ D_l = \begin{pmatrix} B_l & 0 \\ 0 & C_l \end{pmatrix} \ D_{l+1} = \begin{pmatrix} B_{l+1} & 0 \\ 0 & 0_{p_c \times m_c} \end{pmatrix} \ \cdots \ D_n = \begin{pmatrix} B_n & 0 \\ 0 & 0_{p_c \times m_c} \end{pmatrix},$$

or

$$D_1 = \begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix} \ \cdots \ D_l = \begin{pmatrix} 0 & B_l \\ C_l & 0 \end{pmatrix} \ D_{l+1} = \begin{pmatrix} 0 & B_{l+1} \\ 0_{p_c \times m_c} & 0 \end{pmatrix} \ \cdots \ D_n = \begin{pmatrix} 0 & B_n \\ 0_{p_c \times m_c} & 0 \end{pmatrix}.$$

If both $F_b(z)$ and $F_c(z)$ are isometries on $\mathbb{T}$, then so is $F_d(z)$.

If both $F_b(z)$ and $F_c(z)$ are co-isometries on $\mathbb{T}$, then so is $F_d(z)$.

b. For $m_c \geq m_b$, $F_d(z)$ is $(p_b + p_c) \times m_c$-valued polynomial with coefficients,

$$D_1 = \begin{pmatrix} \sqrt{\alpha}B_1 & 0_{p_b \times (m_c-m_b)} \\ \sqrt{1-\alpha}C_1 \end{pmatrix} \quad \cdots \quad D_l = \begin{pmatrix} \sqrt{\alpha}B_l & 0_{p_b \times (m_c-m_b)} \\ \sqrt{1-\alpha}C_L \end{pmatrix}$$

$$D_{l+1} = \begin{pmatrix} \sqrt{\alpha}B_{l+1} & 0_{p_b \times (m_c-m_b)} \\ 0_{p_c \times m_c} \end{pmatrix} \quad \cdots \quad D_n = \begin{pmatrix} \sqrt{\alpha}B_n & 0_{p_b \times (m_c-m_b)} \\ 0_{p_c \times m_c} \end{pmatrix} \qquad \alpha \in [0,1] \quad \text{parameter.}$$

If both $F_b(z)$ and $F_c(z)$ are isometries on $\mathbb{T}$, then so is $F_d(z)$

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[6]For given n and ρ, one can always find $\zeta \in [0, \rho - 1]$ so that $\frac{n+\zeta}{\rho}$ is natural. Then, the last part of $\mathbf{B}_{\frac{n}{\rho}}$ in (2.15) is comprized of zeros.

c. For $p_b \geq p_c$, $F_d(z)$ is $p_b \times (m_b + m_c)$-valued polynomial with coefficients

$$D_1 = \begin{pmatrix} \sqrt{\alpha}B_1 & \sqrt{1-\alpha}C_1 \\ 0_{(p_b - p_c) \times m_b} & \end{pmatrix} \quad \cdots \quad D_l = \begin{pmatrix} \sqrt{\alpha}B_l & \sqrt{1-\alpha}C_l \\ 0_{(p_b - p_c) \times m_b} & \end{pmatrix}$$

$$D_{l+1} = \begin{pmatrix} \sqrt{\alpha}B_{l+1} & 0_{p_b \times m_c} \\ 0_{(p_b - p_c) \times m_b} & \end{pmatrix} \quad \cdots \quad D_n = \begin{pmatrix} \sqrt{\alpha}B_n & 0_{p_b \times m_c} \\ 0_{(p_b - p_c) \times m_b} & \end{pmatrix}$$

$\alpha \in [0,1] \quad$ parameter.

If both $F_b(z)$ and $F_c(z)$ are co-isometries on $\mathbb{T}$, then so is $F_d(z)$

VI. Product of the polynomials.

$$F_b(z) \quad = \quad z^{-1}B_1 + \ldots z^{-n}B_n \qquad p_b \times \rho$$

$$F_c(z) \quad = \quad z^{-1}C_1 + \ldots z^{-l}C_l \qquad \rho \times m_c$$

i.e. $F_d(z)$ is the following $p_b \times m_c$-valued polynomial

$$F_b(z)F_c(z) := F_d(z) = z^{-1}\left(z^{-1}D_1 + \ldots + z^{-(n+l-1)}D_{n+l-1}\right),$$

where a straightforward computation yields that the coefficients $D_1, \ldots, D_{n+l-1}$ are given by,

(2.17) $$\mathbf{D}_{1,n+l-1} = \mathbf{H}_l \mathbf{T}_{n+l, \, \rho} \mathbf{C}_{1,l}$$

with $\mathbf{C}_{1,l}$ and $\mathbf{D}_{1,n+l-1}$ in the spirit of (2.6), $\mathbf{H}_l$ is the $(n+l)p \times (n+l)m$ Hankel matrix as in (2.8) (with $\eta = l$) and

(2.18) $$\mathbf{T}_{n+l, \, \rho} := \begin{pmatrix} & & & I_\rho \\ & & I_\rho & \\ & \cdot^{\cdot^{\cdot}} & & \\ I_\rho & & & \end{pmatrix}.$$

In the Appendix below we show how each of the new polynomials, in the above items **I** through **VI**, is constructed. A key tool there will be the associated Hankel matrix.

Furthermore, as already pointed out, we shall see that if the original polynomial $F(z)$ in (2.12) is in $\mathcal{U}$, so are all the resulting polynomials.

To this end, in Subsections 3.1, 3.2 and 4.1 respectively, we present three characterizations of Laurent polynomials in $\mathcal{U}$.

3. MINIMAL REALIZATION OF (CO)-ISOMETRIC FIR

3.1. **Characterization through realization matrices of Schur stable systems.** In this subsection we characterize, through a corresponding realization matrix, rational functions, analytic outside the open unit disk (Schur stable), which are in $\mathcal{U}$.

There are several variants of characterizations of $F \in \mathcal{U}$ through its minimal realization matrix R. The square case $(m = p)$ was addressed in [17, Lemma 2 & Theorem 3]. Another version of it appeared in [1, Theorem 3.1] and [2, Theorem 2.1]. Below we cite and adapted form of [40, Theorem 14.5.1]. A more general case was studied in [8, Theorem 4.5]. In fact, in [1], [2], [8] and [17] they address the *indefinite* inner product case where $(F(z))^* J_p F(z) = J_m$ with J_p, J_m signature matrices, i.e. diagonal matrices satisfying $J_p^2 = I_p$ and $J_m^2 = I_m$.

Theorem 3.1. *Let $F(z)$ be a $p \times m$-valued rational function whose poles are within the open unit disk and let R be a corresponding $(\nu+p) \times (\nu+m)$ minimal realization matrix (2.4)*

$$R := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right).$$

I. Assume that $p \geq m$. $F(z)$ is in $\mathcal{U}$ if and only if,

$$(3.1) \qquad\qquad R^* \mathrm{diag}\{I_\nu \quad I_m\} R = \mathrm{diag}\{I_\nu \quad I_m\}.$$

II. Assume that $m \geq p$. $F(z)$ is in $\mathcal{U}$ if and only if,

$$(3.2) \qquad\qquad R\,\mathrm{diag}\{I_\nu \quad I_p\} R^* = \mathrm{diag}\{I_\nu \quad I_p\}.$$

We now recall the notion of Controllability and Observability Gramians, see e.g. [19, Eqns (12.8.17), (12.8.37)].

Assuming that the spectrum of A (the upper left block in R in (2.4)) is within the open unit disk (Schur stable), W_{cont}, W_{obs}, the associated $\nu \times \nu$ Controllability and Observability Gramians, respectively, are given by the solution to the respective Stein equations

$$(3.3) \qquad W_{\mathrm{cont}} - AW_{\mathrm{cont}}A^* = BB^* \qquad\qquad W_{\mathrm{obs}} - A^*W_{\mathrm{obs}}A = C^*C.$$

We can now state the following whose proof is given in [5].

Lemma 3.2. *Let $F(z)$ be a $p \times m$-valued rational function whose poles are within the open unit disk and denote by W_{cont}, W_{obs} the associated controllability and observability gramians, respectively.*

Assume that $F(z)$ is in $\mathcal{U}$.

I. If $p \geq m$, $F(z)$ admits a state space realization R in (3.1) so that

$$(I_\nu - W_{\mathrm{cont}}) \quad \text{positive semidefinite} \qquad\qquad W_{\mathrm{obs}} = I_\nu \ .$$

II. If $m \geq p$, $F(z)$ admits a state space realization R in (3.2) so that

$$W_{\mathrm{cont}} = I_\nu \qquad\qquad (I_\nu - W_{\mathrm{obs}}) \quad \text{positive semidefinite}.$$

III. If $p = m$, $F(z)$ admits a state space realization R in (3.1), (3.2) so that

$$W_{\mathrm{cont}} = I_\nu \qquad\qquad W_{\mathrm{obs}} = I_\nu \ .$$

This result will be used in the proof of Proposition 4.2 below.

We conclude this subsection by illustrating the results of Observation 2.1, Theorem 3.1 and of Lemma 3.2

Example 3.3. I. Consider the 2×2-valued polynomial $F(x)$ in (1.1) with $n = 3$ i.e.

$$F(z) = z^q(z^{-1}B_1 + z^{-2}B_2 + z^{-3}B_3),$$

where

$$(3.4) \qquad B_1 = \tfrac{1}{5}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \qquad B_2 = \tfrac{1}{5}\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \qquad B_3 = \tfrac{1}{5}\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

and q is a parameter assuming the values 2 and 1.

(i) For $q = 2$, $F(z)$ takes the form,

$$F(z) = zB_1 + B_2 + z^{-1}B_3.$$

Namely, as in (2.10), and here (2.11) takes the form of $\mathbf{H} = B_3$ and $\hat{\mathbf{H}} = B_1$ so the McMillan degree is $d = \mathrm{rank}(\hat{\mathbf{H}}) + \mathrm{rank}(\mathbf{H}) = 2$.

A minimal realization (i.e. of dimension 2) is

$$F(z) = z v_1 v_1^* + B_2 + \frac{1}{z} v_3 v_3^* \qquad \text{with} \qquad v_1 = \tfrac{\sqrt{2}}{\sqrt{5}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_3 = \tfrac{\sqrt{2}}{\sqrt{5}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(ii)   For $q = 1$, $F(z)$ is causal and takes the form,

$$F(z) = B_1 + z^{-1} B_2 + z^{-2} B_3.$$

A straightforward dimension 4 realization of the form (2.7) is

$$\tilde{R}_r = \left(\begin{array}{cc|c} 0 & I_2 & B_2 \\ 0 & 0 & B_3 \\ \hline I_2 & 0 & B_1 \end{array} \right).$$

A closer scrutiny reveals that here (2.11) takes the form of $\hat{\mathbf{H}}$ empty, and

$$\mathbf{H} = \begin{pmatrix} B_2 & B_3 \\ B_3 & 0 \end{pmatrix} = \tfrac{1}{5}\begin{pmatrix} 0 & 3 & 2 & -2 \\ -3 & 0 & -2 & 2 \\ 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \end{pmatrix}.$$

Thus the McMillan degree is $d = \mathrm{rank}(\hat{\mathbf{H}}) + \mathrm{rank}(\mathbf{H}) = 0 + 2 = 2$.

Indeed, a minimal realization is,

$$R_r = \tfrac{1}{5}\left(\begin{array}{cc|cc} -2 & -2 & 1 & -4 \\ 2 & 2 & 4 & -1 \\ \hline -4 & 1 & 2 & 2 \\ 1 & -4 & 2 & 2 \end{array} \right).$$

Recall that from both parts of Theorem 3.1 it follows that $R_r R_r^* = \mathrm{diag}\{I_2 , I_2\} = R_r^* R_r$.

II.   Consider the 1×2-valued polynomial

(3.5) $$F(z) = z^q \left(z^{-1} \begin{pmatrix} 0 & -\tfrac{3}{5} \end{pmatrix} + z^{-2} \begin{pmatrix} \tfrac{4}{5} & 0 \end{pmatrix} \right),$$

where the parameter q assume the values 0 and 1.

For $q = 0$, $F(z)$ is strictly causal. The associated Hankel matrix $\mathbf{H}$ in (2.8) is given by,

(3.6) $$\mathbf{H} = \tfrac{1}{5}\begin{pmatrix} 0 & -3 & 4 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}$$

and the Hankel singular values are 1 and 0.8. A minimal realization is

$$R = \left(\begin{array}{cc|cc} 0 & \tfrac{4}{5} & 0 & -\tfrac{3}{5} \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array} \right).$$

Indeed, from part II of Theorem 3.1 it follows that R is co-isometry, i.e. $RR^* = \mathrm{diag}\{I_2 , 1\}$ and from Lemma 3.2, we have that the Observability Gramian is $W_{\mathrm{obs}} = \mathrm{diag}\{1 \quad \tfrac{16}{25}\}$.

Substituting in (3.5) $q = 1$ yields the causal polynomial,

$$F(z) = \begin{pmatrix} 0 & -\tfrac{3}{5} \end{pmatrix} + z^{-1} \begin{pmatrix} \tfrac{4}{5} & 0 \end{pmatrix}.$$

A corresponding minimal realization is

$$R = \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline \tfrac{4}{5} & 0 & -\tfrac{3}{5} \end{array} \right).$$

Indeed, $RR^* = \mathrm{diag}\{1 \quad 1\}$ and $W_{\mathrm{obs}} = \tfrac{16}{25}$.   □

In this subsection we have assumed that the system is causal and in particular Schur stable. In the sequel, we remove this restriction.

3.2. **A characterization through the Blaschke-Potapov product.** We first need some preliminaries. We shall use the fact that a $k \times k$ rank one orthogonal projection,

$$P^* = P = P^2 \qquad \text{rank}(P) = 1,$$

can always be written as

$$(3.7) \qquad\qquad P = vv^* \qquad v^*v = 1 \qquad v \in \mathbb{C}^k.$$

Using (3.7), a rank $k-1$ orthogonal projection Q i.e.

$$Q^* = Q^2 = Q \qquad\qquad \text{rank}(Q) = k-1,$$

can always be written as

$$(3.8) \qquad\qquad Q := I_k - vv^*.$$

We can now cite the classical Blaschke-Potapov product result (as appeared in [5]).

Lemma 3.4. *A $p \times m$-valued rational function $F(z)$, of McMillan degree d, is in $\mathcal{U}$, (1.3), if and only if*
(3.9)

$$p \geq m \quad F(z) = \left(\prod_{j=1}^{d} \left(I_p + \left(\frac{1-\alpha_j^* z}{z-\alpha_j} - 1 \right) v_j v_j^* \right) \right) U_{\text{iso}} \qquad \begin{matrix} v_j \in \mathbb{C}^p & v_j^* v_j = 1 \\ U_{\text{iso}} \in \mathbb{C}^{p \times m} & U_{\text{iso}}^* U_{\text{iso}} = I_m \\ \alpha_j \in \{\infty \cup \mathbb{C}\} \smallsetminus \mathbb{T} \end{matrix}$$

$$m \geq p \quad F(z) = U_{\text{coiso}} \left(\prod_{j=1}^{d} \left(I_m + \left(\frac{1-\alpha_j^* z}{z-\alpha_j} - 1 \right) v_j v_j^* \right) \right) \qquad \begin{matrix} v_j \in \mathbb{C}^m & v_j^* v_j = 1 \\ U_{\text{coiso}} \in \mathbb{C}^{p \times m} & U_{\text{coiso}} U_{\text{coiso}}^* = I_p \end{matrix} \; .$$

Recall $\prod_{j=1}^{0} := I$

In Theorem 3.5 below, we specialize this result to the FIR framework. To this end, we recall that

$$(3.10) \quad \left(I + \left(\frac{1-\alpha^* z}{z-\alpha} - 1 \right) vv^* \right)^{-1} = \left(I + \left(\frac{z-\alpha}{1-\alpha^* z} - 1 \right) vv^* \right) \qquad \begin{matrix} v^* v = 1 \\ \alpha \in \{\infty \cup \mathbb{C}\} \smallsetminus \mathbb{T}. \end{matrix}$$

We can now formulate the main result of this subsection.

Theorem 3.5. *Let $F(z)$ be a $p \times m$-valued Finite Impulse Response function of McMillan degree d.*

I. Assuming $p \geq m$, $F(z)$ is isometrically in $\mathcal{U}$ if and only if it can be written in one, and hence in each, of these three equivalent forms,

$$(3.11) \quad F(z) = \begin{cases} \prod_{j=1}^{\gamma} \left(I_p + (z-1) v_j v_j^* \right) \prod_{j=\gamma+1}^{d} \left(I_p + (\frac{1}{z} - 1) v_j v_j^* \right) U_{\text{iso}} & U_{\text{iso}}^* U_{\text{iso}} = I_m \\ \prod_{j=1}^{\gamma} \left(I_p + (z-1) v_j v_j^* \right) \left(\prod_{j=d}^{\gamma+1} \left(I_p + (z-1) v_j v_j^* \right) \right)^{-1} U_{\text{iso}} & v_j \in \mathbb{C}^p \\ \left(\prod_{j=\gamma}^{1} \left(I_p + (\frac{1}{z} - 1) v_j v_j^* \right) \right)^{-1} \prod_{j=\gamma+1}^{d} \left(I_p + (\frac{1}{z} - 1) v_j v_j^* \right) U_{\text{iso}} & v_j^* v_j = 1 \\ & \gamma \in [0, \; d]. \end{cases}$$

II. For $m \geq p$, $F(z)$ is co-isometrically in $\mathcal{U}$ if and only if it can be written in one, and hence in each, of these three equivalent forms,

$$(3.12) \quad F(z) = \begin{cases} U_{\mathrm{coiso}} \prod\limits_{j=\gamma+1}^{d} \left(I_m + (\frac{1}{z}-1)v_j v_j^*\right) \prod\limits_{j=1}^{\gamma} \left(I_m + (z-1)v_j v_j^*\right) & U_{\mathrm{coiso}} U_{\mathrm{coiso}}^* = I_p \\[2ex] U_{\mathrm{coiso}} \left(\prod\limits_{j=d}^{\gamma+1}\left(I_m + (z-1)v_j v_j^*\right)\right)^{-1} \prod\limits_{j=1}^{\gamma}\left(I_m + (z-1)v_j v_j^*\right) & v_j \in \mathbb{C}^m \\ & v_j^* v_j = 1 \\[2ex] U_{\mathrm{coiso}} \prod\limits_{j=\gamma+1}^{d}\left(I_m+(\frac{1}{z}-1)v_j v_j^*\right)\left(\prod\limits_{j=\gamma}^{1}\left(I_m+(\frac{1}{z}-1)v_j v_j^*\right)\right)^{-1} & \gamma \in [0,\ d]. \end{cases}$$

In particular, the function $F(z)$ in (3.11), (3.12) is causal for $\gamma = 0$ and anti-causal for $\gamma = d$.

The first version in both (3.11) and (3.12) is a re-writing of (3.9) with $\alpha_1 = \ldots = \alpha_\gamma = \infty$ and $\alpha_{\gamma+1} = \ldots = \alpha_d = 0$. Using (3.10), the other two versions follow from the first.

Note that it is only for the causal case, $\gamma = 0$, or the anti-causal case, $\gamma = d$, that in (3.11) a product of the form,

$$(3.13) \qquad \prod_{j=1}^{\gamma}\left(I_p + (z-1)v_j v_j^*\right) \prod_{j=\gamma+1}^{d}\left(I_p + (\frac{1}{z}-1)v_j v_j^*\right) U_{\mathrm{iso}} \qquad \begin{matrix} U_{\mathrm{iso}}^* U_{\mathrm{iso}} = I_m \\ v_j \in \mathbb{C}^p \\ v_j^* v_j = 1 \\ \gamma \in [0,\ d], \end{matrix}$$

is of McMillan degree d for *any choice* of the projections v_1, $\ldots$, v_d. For $\gamma \in [1,\ d-1]$, the McMillan degree of the expression in (3.13) is *at most d*. For example, repetitive use of (3.10) reveals that substituting in (3.13)

$$v_j v_j^* = v_{d+1-j} v_{d+1-j}^* \qquad j = 1,\ \ldots,\gamma \qquad d = 2\gamma \qquad \gamma \quad \text{natural,}$$

yields the *zero* degree function I_p.

Theorem 3.5 asserts that whenever $F \in \mathcal{U}$ is of McMillan degree d, *there exist* rank one orthogonal projections $v_1 v_1^*$, $\ldots$, $v_d v_d^*$ satisfying (3.11) and (3.12).

Recall that employing optimization for design of FIRs is fairly common, see e.g. [14], [15], [20], [39] and [42] and for FIRs within $\mathcal{U}$ see e.g. [34]. This motivates a *convex* description of large families of FIRs. To this end, we next specialize [5, Observation 4.3].

Corollary 3.6. *All $p \times m$-valued FIR rational functions of McMillan degree d in $\mathcal{U}$ may be parametrized by,*

$$(3.14) \qquad \begin{aligned} (d+1) \cdot [0,\ 2\pi)^{(2p-m-1)(m+d)+d(m-1)+m} & \qquad p \geq m \\[1ex] (d+1) \cdot [0,\ 2\pi)^{(2m-p-1)(p+d)+d(p-1)+p} & \qquad m \geq p. \end{aligned}$$

The causal subset (here, =lossless subset), may be parametrized by,

$$(3.15) \qquad \begin{aligned} {}[0,\ 2\pi)^{(2p-m-1)(m+d)+d(m-1)+m} & \qquad p \geq m \\[1ex] [0,\ 2\pi)^{(2m-p-1)(p+d)+d(p-1)+p} & \qquad m \geq p. \end{aligned}$$

There are a few parametrizations of all FIRs in $\mathcal{U}$, see e.g. [23, Propriété 41], [40, Section 14.4] and the more detailed study in [16]. The above choice is advantageous as the set of parameters in (3.15) is not only *convex*, but in fact a *polytope*. This corresponds to having in (3.11) or in (3.12), $\gamma = 0$. As the integral parameter γ

attains values in $[0, \ d]$, the parameter set in (3.14) are $d+1$ copies of this *polytope*. This is in particular convenient if one wishes to:

(i) Design FIRs within $\mathcal{U}$ through optimization, see e.g. [34].

(ii) Iteratively apply para-unitary similarity, see e.g. [23, Section 3.3], [30], [36]. In signal processing literature, this is associated with with *channel equalization* and in communications literature with *decorrelation of signals* or

(iii) Iteratively apply Q-R factorization in the framework of communications, see e.g. [12], [13].

We now establish a connection between Blaschke-Potapov description of $F(z)$ in (3.11), (3.12) (with $\gamma = 0$) and $B_1, \ \ldots, \ B_n$ the coefficients of the polynomial $F(z)$ in (2.1) with $q = 1$, i.e.

$$(3.16) \qquad F(z) = B_1 + z^{-1}B_2 + \ \ldots \ z^{1-n}Z_n \ .$$

Observation 3.7. *Assume that the (causal) $p \times m$-valued polynomial $F(z)$ in (3.16) is in $\mathcal{U}$ and of McMillan degree d.*

Then if $m = p$ up to multiplication by a unitary matrix from the left or from the right, the square coefficient matrices $B_1, \ \ldots, \ B_n$, are given by,

$$
\begin{aligned}
B_1 &= Q_1 \cdots Q_d \\
B_2 &= \sum_{j=1}^{d} Q_1 \cdots Q_{j-1} v_j v_j^* Q_{j+1} \cdots Q_d \\
B_3 &= \sum_{k=j+1}^{d} \sum_{j=1}^{d-1} Q_1 \cdots Q_{j-1} v_j v_j^* Q_{j+1} \cdots Q_{k-1} v_k v_k^* Q_{k+1} \cdots Q_d \\
B_4 &= \sum_{q=k+1}^{d} \sum_{k=j+1}^{d-1} \sum_{j=1}^{d-2} Q_1 \cdots Q_{j-1} v_j v_j^* Q_{j+1} \cdots Q_{k-1} v_k v_k^* Q_{k+1} \cdots Q_{q-1} v_q v_q^* Q_{q+1} \cdots Q_d \\
&\vdots \\
B_n &= v_1 v_1^* \cdots \ v_d v_d^*
\end{aligned}
$$

where $v_1 v_1^*, \ \ldots, \ v_d v_d^*$, are rank one orthogonal projections (3.7) and $Q_j := I - v_j v_j^*$, $j = 1, \ \ldots, \ d$, see (3.8).

If $p \geq m$, the above $p \times p$ coefficient matrices $B_1, \ \ldots, \ B_n$ are multiplied from the right by a $p \times m$ isometry U_{iso}, $(U_{\mathrm{iso}}^* U_{\mathrm{iso}} = I_m)$.

If $m \geq p$, the above $m \times m$ coefficient matrices $B_1, \ \ldots, \ B_n$ are multiplied from the left by a $p \times m$ co-isometry U_{coiso}, $(U_{\mathrm{coiso}} U_{\mathrm{coiso}}^* = I_p)$.

In particular this implies that

$$F(1) = B_1 \ + \ldots \ + B_n = \begin{cases} U_{\mathrm{iso}} & F \text{ isometry} \\ U_{\mathrm{coiso}} & F \text{ co-isometry} \end{cases}$$

We conclude this section by pointing out that so far we have focused on characterizations of Finite Impulse Response functions in $\mathcal{U}$ through their *minimal realization*. In the sequel this restriction is removed.

4. HANKEL MATRICES - REVISITED

4.1. **Isometric FIR - a Hankel matrix characterization.** Let $F(z)$, (1.1), be a Finite Impulse Response (possibly rectangular) rational function,

$$F(z) = z^q \left(z^{-1}B_1 + \ \ldots, \ z^{-n}B_n \right) \qquad q \quad \text{parameter.}$$

Clearly, the above $F(z)$ is in $\mathcal{U}$, if and only if $F_o(z)$ in (2.5),

$$F(z)_{|_{q=0}} = F_o(z) := z^{-1}B_1 + \ldots , z^{-n}B_n ,$$

is in $\mathcal{U}$.

Thus, in the sequel we find it convenient to focus on $F_o(z)$. Subsequently, substituting in (2.6) and in (2.8) $\eta = 0$ one obtains the matrices

$$\mathbf{B}_0 = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix} \qquad \mathbf{H}_0 = \begin{pmatrix} B_1 & \cdots & B_n \\ \vdots & \iddots & \\ B_n & & \end{pmatrix}.$$

We shall also find it convenient to write $F_o(z)$ in (2.5) in each of the two following forms

(4.1) $$F_o(z) = \mathbf{Z}\mathbf{B}_0 = \hat{\mathbf{B}}\hat{\mathbf{Z}},$$

where $\mathbf{B}_0$ is as before and

(4.2) $$\hat{\mathbf{B}} := (B_1 , \cdots , B_n),$$

and

$$\mathbf{Z} := (z^{-1} \cdots z^{-n}) \otimes I_p = (z^{-1}I_p \cdots z^{-n}I_p)$$

(4.3) $$\hat{\mathbf{Z}} := \begin{pmatrix} z^{-1} \\ \vdots \\ z^{-n} \end{pmatrix} \otimes I_m = \begin{pmatrix} z^{-1}I_m \\ \vdots \\ z^{-n}I_m \end{pmatrix}.$$

where $\otimes$ denotes the usual Kronecker (=tensor) product, see e.g. [22, Section 4.2].

We now state the main result of this subsection.

Theorem 4.1. *Let $F_0(z)$ be a $p \times m$ polynomial in (2.5)*

$$F_0(z) = z^{-1}B_1 + \ldots + z^{-n}B_n,$$

and let $\mathbf{H}_0$ be the corresponding Hankel matrix.

The polynomial $F_o(z)$ is in $\mathcal{U}$ if and only if

(4.4)
$$(I_{nm} - \mathbf{H}_0^*\mathbf{H}_0) \begin{pmatrix} I_m \\ 0_{m(n-1) \times m} \end{pmatrix} = 0_{pn \times m} \qquad p \geq m$$

$$\begin{pmatrix} I_p & 0_{p \times (n-1)p} \end{pmatrix} (I_{np} - \mathbf{H}_0\mathbf{H}_0^*) = 0_{p \times np} \qquad m \geq p.$$

Proof: A straightforward substitution of $F_o(z)$ in the definition of $\mathcal{U}$ (1.3) yields the following characterization,
(4.5)
$$\begin{pmatrix} \sum_{j=1}^{n} B_j^* B_j \\ \sum_{j=1}^{n-1} B_{1+j}^* B_j \\ \vdots \\ \sum_{j=1}^{2} B_{n-2+j}^* B_j \\ B_n^* B_1 \end{pmatrix} = \begin{pmatrix} I_m \\ 0_{m(n-1) \times m} \end{pmatrix} \qquad p \geq m$$

$$\left(\sum_{j=1}^{n} B_j B_j^* , \sum_{j=1}^{n-1} B_{1+j} B_j^* , \cdots , \sum_{j=1}^{2} B_{n-2+j} B_j^* , B_n B_1^* \right) = \left(I_p , 0_{p \times (n-1)p} \right) \qquad m \geq p.$$

For the square case, see e.g. [23, Propriété 37].

Using the matrices $\mathbf{H}_0$ and $\mathbf{B}_0$ this may be equivalently written as,

(4.6)
$$\mathbf{H}_0^* \mathbf{B}_0 = \begin{pmatrix} I_m \\ 0_{m(n-1) \times m} \end{pmatrix} \qquad p \geq m,$$

$$\hat{\mathbf{B}} \mathbf{H}_0^* = \begin{pmatrix} I_p & 0_{p \times (n-1)p} \end{pmatrix} \qquad m \geq p.$$

where $\hat{\mathbf{B}}$ is as in (4.2).

Now, since

$$\mathbf{B}_0 = \mathbf{H}_0 \begin{pmatrix} I_m \\ 0_{m(n-1) \times m} \end{pmatrix} \qquad \text{and} \qquad \hat{\mathbf{B}} = \begin{pmatrix} I_p & 0_{p \times (n-1)p} \end{pmatrix} \mathbf{H}_0$$

the relation in (4.6) is equivalent to

$$\mathbf{H}_0^* \mathbf{H}_0 \begin{pmatrix} I_m \\ 0_{m(n-1) \times m} \end{pmatrix} \quad = \quad \begin{pmatrix} I_m \\ 0_{m(n-1) \times m} \end{pmatrix}$$

$$\begin{pmatrix} I_p & 0_{p \times (n-1)p} \end{pmatrix} \mathbf{H}_0 \mathbf{H}_0^* \quad = \quad \begin{pmatrix} I_p & 0_{p \times (n-1)p} \end{pmatrix},$$

which in turn can be written as (4.4). □

An alternative proof of the same result is given in [5, Theorem 5.2].

Theorem 4.1 offers a characterization of a Laurent polynomial $F(z)$ in $\mathcal{U}$ through the existence of a certain invariant subspace of $\mathbf{H}_0^* \mathbf{H}_0$ (or $\mathbf{H}_0 \mathbf{H}_0^*$), see (4.4). A closer scrutiny reveals that whenever $F(z)$ is in $\mathcal{U}$, more can be said about these matrices.

Proposition 4.2. *Let $F_o(z)$ be a $p \times m$-valued polynomial in (2.5)*

$$F_o(z) = z^{-1} B_1 + \ \ldots \ + z^{-n} B_n,$$

and let $\mathbf{H}_0$ be the corresponding Hankel matrix.

Assume that $F(z)$ is in $\mathcal{U}$.

I. For $p \geq m$,

(4.7)
$$I_{pn} - \mathbf{H}_0^* \mathbf{H}_0 = \begin{pmatrix} 0_{m \times m} & 0 \\ 0 & \Delta_{pn-m} \end{pmatrix},$$

where Δ_{pn-m} is a $(pn - m) \times (pn - m)$ positive semi-definite weak contraction.

Moreover, if $p = m$ then Δ_{pn-m} is an orthogonal projection.

II. For $m \geq p$,

(4.8)
$$I_{mn} - \mathbf{H}_0 \mathbf{H}_0^* = \begin{pmatrix} 0_{p \times p} & 0 \\ 0 & \Delta_{mn-p} \end{pmatrix},$$

where Δ_{mn-p} is a $(mn - p) \times (mn - p)$ positive semi-definite weak contraction.

Moreover, if $p = m$ then Δ_{mn-p} is an orthogonal projection.

Proof : The structure of the right hand side of (4.7) and (4.8), is immediate from (4.4). All is left to verify is the spectrum of Δ.

Recall that the square of the Hankel singular values, are the positive eigenvalues of $\mathbf{H}_0^* \mathbf{H}_0$ (or of $\mathbf{H}_0 \mathbf{H}_0^*$) which in turn are equal to the positive eigenvalues of the product of the Gramians $(W_{\mathrm{cont}} W_{\mathrm{obs}})$, appeared in (3.3), see e.g. [19, Eq. (12.8.43)].

From Lemma 3.2 we know that in the rectangular case, the spectrum of $(W_{\mathrm{cont}}W_{\mathrm{obs}})$ lies in the interval $[0,\ 1]$ and in the square case the non-zero eigenvalues of $(W_{\mathrm{cont}}W_{\mathrm{obs}})$ are all ones.

Finally, note that the eigenvalues of the right hand side of (4.7) and (4.8) are just 1 minus the eigenvalues of $\mathbf{H}_0^*\mathbf{H}_0$ or of $\mathbf{H}_0\mathbf{H}_0^*$, respectively. $\square$

It is interesting to point out that the condition of having the quantities in (4.7) and (4.8) respectively,

$$I_{pn} - \mathbf{H}_0^*\mathbf{H}_0 \qquad\qquad I_{mn} - \mathbf{H}_0\mathbf{H}_0^*$$

positive semidefinite (as we indeed do), commonly appears in the context of Nehari's problem where one approximates an anti-causal polynomial (positive powers of z) by a causal rational function (no pole at infinity), see e.g. [19, Section 12.8].

The following example illustrates the above results.

Example 4.3. Consider part II of Example 3.3.   As $\mathbf{H}$ in (3.6) satisfies condition (4.4), from Theorem 4.1 it follows that the polynomial in (3.5) is in $\mathcal{U}$. $\square$

The above results in Theorem 4.1 and in Proposition 4.2 were formulated in the language of $\mathbf{H}_0^*\mathbf{H}_0$ or $\mathbf{H}_0\mathbf{H}_0^*$, where $\mathbf{H}_0$ is a *Hankel* matrix. We now show that these results can be equivalently formulated in terms of *block-triangular Toeplitz* matrices. Indeed, using $\mathbf{T}$ from (2.18) note that

$$\mathbf{H}_0^*\mathbf{H}_0 \;=\; (\mathbf{T}_{n,p}\mathbf{H}_0)^*(\mathbf{T}_{n,p}\mathbf{H}_0) \;=\; \begin{pmatrix} B_n & & \\ \vdots & \ddots & \\ B_1 & \cdots & B_n \end{pmatrix}^* \begin{pmatrix} B_n & & \\ \vdots & \ddots & \\ B_1 & \cdots & B_n \end{pmatrix}$$

$$\mathbf{H}_0\mathbf{H}_0^* \;=\; (\mathbf{H}_0\mathbf{T}_{n,m})(\mathbf{H}_0\mathbf{T}_{n,m})^* \;=\; \begin{pmatrix} B_n & \cdots & B_1 \\ & \ddots & \vdots \\ & & B_n \end{pmatrix} \begin{pmatrix} B_n & \cdots & B_1 \\ & \ddots & \vdots \\ & & B_n \end{pmatrix}^*$$

Technically, this relation is well known. The problem of finding a factorization of (block) positive semi-definite matrix to (block) triangular Toeplitz is classical, see e.g. [35]. In Subsection 2.1 we saw that Toeplitz operator are better established in system theory.

5. A CONCLUDING REMARK

Although modest in size, the family of para-unitary Finite Impulse Response systems (=(co)-isometric Laurent polynomials) is of great interest in various fields.

In Theorems 3.1, 3.5 and 4.1 we have offered three characterizations of this family. In Corollary 3.6 we introduced an easy-to-use parameterization of this set.

Finally, in Subsection 2.2 we suggested six ways (along with their combinations) to construct from a given para-unitary FIR system, a whole family of systems, of various dimensions and powers, all with this property. This may raise the following open problem of going in the opposite direction: Given a "complicated" FIR in $\mathcal{U}$. How to "factorize" it, following items **I** to **VI**, to simpler building blocks.

The nature of this work suggests that the same set appears, under possibly different terminology, in additional fields and further connections may be established.

References

[1] D. Alpay and I. Gohberg. "Unitary rational matrix functions" In I. Gohberg, editor, *Topics in interpolation theory of rational matrix-valued functions*, *Operator Theory: Advances and Applications*, Vol. 33, pp. 175–222. Birkhäuser Verlag, Basel, 1988.

[2] D. Alpay and I. Gohberg. "On Orthogonal Matrix Polynomial", In I. Gohberg, editor, *Orthogonal Matrix-Valued Polynomials and Applications*, *Operator Theory: Advances and Applications*, Vol. 34, pp. 25–46. Birkhäuser Verlag, Basel, 1988.

[3] D. Alpay, P.E.T Jorgensen, and I. Lewkowicz "Extending Wavelet Filters. Infinite-dimensions, the Non-rational case and Indefinite-inner Product Spaces", *Excursions in Harmonic Analysis Book Series*, Volume 2, Chapter 5, pp. 71-113, Springer-Birkhäuser, 2012.

[4] D. Alpay, P.E.T Jorgensen, and I. Lewkowicz "Parameterization of all Wavelet Filters: Input-output and State Space", *Sampling Theory in Signal and Image Processing*, Vol. 12, pp. 159-188, 2013.

[5] D. Alpay, P.E.T Jorgensen, and I. Lewkowicz "Characterizations of Rectangular, (Para)-Unitary Rational Functions", a manuscript. Available at
http://arxiv.org/abs/1410.0283

[6] D. Alpay, P.E.T Jorgensen, and I. Lewkowicz "Filter Banks - an Interpolation Approach", a manuscript.

[7] D. Alpay, P. Jorgensen, I. Lewkowicz and I. Marziano. "Representation Formulas for Hardy space functions through the Cuntz relations and new interpolation problems", in the volume: *Multiscale Signal Analysis and Modeling* (editors Xiaoping Shen and Ahmed Zayed), in the series Lecture Notes in Electrical Engineering, pp. 161-182, Springer, 2013.

[8] D. Alpay and M. Rakowski, "Co-Isometrically Valued Matrix Functions" *Operator Theory: Advances & Application*, Vol. 80, pp. 1-20, Birkhäuser, 1995.

[9] H. Bart, I. Gohberg and M. Kaashoek, Minimal Factorization of Matrix and Operator and Functions, *Operator Theory: Advances and Applications*, Vol. 1, Birkhäuser, 1979.

[10] O. Bratteli and P.E.T. Jorgensen, "Wavelet filters and infinite-dimensional unitary groups", *Proceedings of the International Conference on Wavelet Analysis and Applications (Guangzhou, China 1999), AMS/IP Stud. Adv. Math.*, Amer. Math. Soc. Vol. 25, pp. 35-65, 2002.

[11] O. Bratteli and P.E.T. Jorgensen, *Wavelets through the Looking Glass*, Birkhäuser, 2002.

[12] D. Cescato and H. Bölcskei, "QR Decomposition of Laurent Polynomial Matrices Sampled on the Unit Circle", *IEEE Trans. Inf. Theory*, Vol. 56, pp. 4754-4761, 2010.

[13] D. Cescato and H. Bölcskei, "Algorithms for Interpolation-Based QR Decomposition in MIMO-OFDM Systems", *IEEE Trans. Signal Proc.*, Vol. 59, pp. 1719-1733, 2011.

[14] L. Chai, J. Zhang, C. Zhang and E. Mosca, "Optimal Noise Reduction in Oversampled PR Filter Banks" *IEEE Trans. Sig. Proc.*, Vol. 57, pp. 3844-3857, 2009.

[15] T.N. Davidson, "Enriching the Art of FIR Filter Design via Convex Optimization (efficient evaluation of inherent tradeoffs)", *IEEE Signal Proc. Magazine*, pp. 89-101, May 2010.

[16] X. Gao, T.Q. Nguyen and G. Strang, "On Factorization of M-Channel Paraunitary Filterbanks", *IEEE Trans. Signal Proc.*, Vol. 49, pp. 1433-1446, 2001.

[17] Y. Genin, P. Van Dooren and T. Kailath, J.M. Delosme and Martin Morf, "On $\Sigma-$Lossless Transfer Functions and related Questions", *Lin. Alg. & Appl.*, Vol. 50, pp. 251-275, 1983.

[18] I. Gohberg, P. Lancaster and L. Rodman, "Representations and Divisibility of Operator Polynomials", *Can. J. Math.*, Vol. 30, pp. 1045-1069, 1978.

[19] B. Hassibi, A.H. Sayed and T. Kailath, *Indefinite-Quadratic Estimation and Control*, A Unified Approach to H^2 and H^∞ Theories, SIAM, 1999.

[20] H.G. Hoang, H.D. Tuan and T.Q. Nguyen, "Frequency Selective KYP Lemma, IIR Filter and Filter Bank Design", IEEE Trans. Sign. Proc., Vol. 57, pp. 956-965, 2009.

[21] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[22] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[23] S. Icart, "Matrices polynomiales et égaliation de canal", Mémoire d'Habilitation à diriger des recherches, Polytech'Nice- Sophia Antipolis Département Électronique, 2013 (In French).

[24] P.E.T. Jorgensen, "Matrix Factorizations, Algorithms, Wavelets", *Notices of the American Mathematical Society*, Vol. 50, pp. 880-894, 2003.

[25] P.E.T. Jorgensen, *Analysis and probability: wavelets, signals, fractals*, Graduate Texts in Mathematics, Vol. 234, Springer, 2006.

[26] P.E.T. Jorgensen, "Unitary matrix functions, wavelet algorithms, and structural properties of wavelets", *Gabor and wavelet frames*, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., Vol. 10, pp. 107-166, World Sci. Publ., Hackensack, NJ, 2007.

[27] Y. Levinson and L. Mirkin, "FIR stabilization in discrete one-sided model matching problem", *Automatica*, Vol. 48, pp. 2990-2992, 2012.

[28] Y. Levinson and L. Mirkin, "L^2 Optimization in discrete FIR Estimation: Exploiting State-Space Structure", *SIAM J. Cont. & Opt.*, Vol. 51, pp. 419-441, 2013.

[29] S. Mallat, *A Wavelet Tour of Signal Processing*, 3rd Edition, Academic Press, 2009.

[30] J.G. McWhirter, P.D. Baxter, T. Cooper, S. Redif and J. Foster, "An EVD Algorithm for Para-Hermitian Polynomial Matrices", *IEEE Trans. Sig. Proc.*, Vol. 55, pp. 2158-2169, 2007.

[31] M. Olivi, *Parametrization of rational lossless matrices with applications to linear system theory*, Mémoire d'Habilitation à Diriger des Recherches, Université De Nice Sophia Antipolis, Mthématique 2010.

[32] S. Oraintara, T.D. Tran, P.N. Heller and T.Q. Nguyen, "Lattice structure for regular para-unitary linear-phase filter-banks and M-band orthogonal symmetric wavelets", *IEEE Trans. Sig. Proc.*, Vol. 49, pp. 2659-2672, 2001.

[33] V.P. Potapov. "Multiplicative structure of J-nonexpansive matrix functions. Trudy Mosk. Math. Ob. 4 (1955), pp. 125236 [Russian]; English Translation: AMS Translations, Series 2, 15, pp. 131-243, 1960.

[34] S. Redif, J.G. McWhirter and S. Weiss, "Design of FIR Paraunitary Filter Banks for Subband Coding Using Polynomial Eigenvalue Decomposition", *IEEE Trans. Sig. Proc.*, Vol. 59, pp. 5253-5264, 2011.

[35] J. Rissanen, "Algorithms for Triangular Decomposition of Block Hankel and Toeplitz Matrices with Applications to Factoring Positive Matrix Polynomials", *Math. Comp.*, Vol. 27, No. 121, pp. 147-154, 1973.

[36] M. Sørensen, L. De Lathauwer, S. Icart and L. Deneire, "On Jacobi-type methods for blind equalization of para-unitary channels", *Signal Processing*, Vol. 92, pp. 617-625, 2012.

[37] G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, 1996

[38] M. Tohidan, H. Amindavar and A.M. Reza, "A DFT-based approximate eigenvalue and singular value decompositiob of polynomial matrices", *EURASIP J. Advances in Signal Processing*, Vol. 93, pp. 1-16, 2013.

[39] J. Tuqan and P.P. Vaidyanathan, "A State Space Approach to the Design of Globally Optimal FIR Energy Compaction Filters", *IEEE Trans. Sig. Proc.*, Vol. 48, pp. 2822-2838, 2000.

[40] P.P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall, Signal Processing Series, 1993.

[41] G. Valli, "Interpolation theory, loop groups and instantons", *J. Reine Math.*, Vol. 446, pp. 137-163, 1994.

[42] H. Vikalo, B. Hassibi, A. Erdogan and T. Kailath, "On Robust Signal Reconstruction in Noisy Filter Banks", *Eurasip Signal Processing*, Vol. 85, pp. 1-14, 2005.

[43] G. Yang, and N. Zheng, "An Optimization Algorithm for biorthogonal Wavelet Filter Banks Design", *Int. J. Wavelets Multiresolut. Infor. Process.*, Vol. 6, pp. 51–63, 2008.

Appendix: Construction of families of FIRs

For each of the items **I** through **VI** from Subsection 2.2 we here fill-in the following details:

1. Construct new polynomials from the original one.

2. Show that whenever the original polynomial was in $\mathcal{U}$, so are the newly constructed polynomials.

I. Reverse polynomial
Using **T** from (2.18),

$$F_{\text{rev}}(z) \quad := \quad z^{-1}B_n + z^{-2}B_{n-1} + \ldots + z^{1-n}B_2 + z^{-n}B_1$$

$$= \qquad\qquad \mathbf{ZT}_{n,p}\mathbf{B}_0 = \hat{\mathbf{B}}\mathbf{T}_{n,m}\hat{\mathbf{Z}}$$

.

The realtion with the corresponding Hankel matrix is straightforward and thus omitted.

II. Preserving the McMillan degree.
For simplicity of exposition we consider as an illustrative example the polynomial $F(z)$ in (2.1) with $n = 4$ and the parameter q attaining the values -1 and -2.

a. For $q = -1$ the corresponding Hankel matrix, $\mathbf{H}_1$ from (2.8) can be, without loss of generality, extended with a row and a column of zeros so it is $6p \times 6m$. Now the resulting Hankel matrix may be partitioned in two forms,

$$\mathbf{H}_1 = \begin{pmatrix} 0 & B_1 & B_2 & B_3 & B_4 & 0 \\ B_1 & B_2 & B_3 & B_4 & 0 & 0 \\ B_2 & B_3 & B_4 & 0 & 0 & 0 \\ B_3 & B_4 & 0 & 0 & 0 & 0 \\ B_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \left(\begin{array}{cc|ccc|c} 0 & B_1 & B_2 & B_3 & B_4 & 0 \\ B_1 & B_2 & B_3 & B_4 & 0 & 0 \\ \hline B_2 & B_3 & B_4 & 0 & 0 & 0 \\ B_3 & B_4 & 0 & 0 & 0 & 0 \\ \hline B_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

which produce the two polynomials in item **II a**.

Following Theorem 4.1 note that

$$(I_{6m} - \mathbf{H}_1^*\mathbf{H}_1) \begin{pmatrix} I_{2m} \\ 0_{4m \times 2m} \end{pmatrix} \quad = \quad 0_{6m \times 2m} \qquad p \geq m$$

$$(I_{2p} \quad 0_{2p \times 4p})(I_{6p} - \mathbf{H}_1\mathbf{H}_1^*) \quad = \quad 0_{2p \times 6p} \qquad m \geq p,$$

so indeed the polynoial is in $\mathcal{U}$.

b. Substituting in $F(z)$ in (2.12) $q = -2$ yield a Hankel matrix $\mathbf{H}_2$ with three partitionings

$$\mathbf{H}_2 = \begin{pmatrix} 0 & 0 & B_1 & B_2 & B_3 & B_4 \\ 0 & B_1 & B_2 & B_3 & B_4 & 0 \\ B_1 & B_2 & B_3 & B_4 & 0 & 0 \\ B_2 & B_3 & B_4 & 0 & 0 & 0 \\ B_3 & B_4 & 0 & 0 & 0 & 0 \\ B_4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc|cc} 0 & 0 & B_1 & B_2 & B_3 & B_4 \\ 0 & B_1 & B_2 & B_3 & B_4 & 0 \\ \hline B_1 & B_2 & B_3 & B_4 & 0 & 0 \\ B_2 & B_3 & B_4 & 0 & 0 & 0 \\ \hline B_3 & B_4 & 0 & 0 & 0 & 0 \\ B_4 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \qquad\qquad \left(\begin{array}{ccc|ccc} 0 & 0 & B_1 & B_2 & B_3 & B_4 \\ 0 & B_1 & B_2 & B_3 & B_4 & 0 \\ B_1 & B_2 & B_3 & B_4 & 0 & 0 \\ \hline B_2 & B_3 & B_4 & 0 & 0 & 0 \\ B_3 & B_4 & 0 & 0 & 0 & 0 \\ B_4 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

so that the three polynomials in item **II b** are obtained.

These three polynomials are in $\mathcal{U}$, as following Theorem 4.1 one obtains,

$$(I_{6m} - \mathbf{H}_2^*\mathbf{H}_2) \begin{pmatrix} I_{3m} \\ 0_{3m \times 3m} \end{pmatrix} \quad = \quad 0_{6m \times 3m} \qquad p \geq m$$

$$(I_{3p} \quad 0_{3p \times 3p})(I_{6p} - \mathbf{H}_2\mathbf{H}_2^*) \quad = \quad 0_{3p \times 6p} \qquad m \geq p.$$

To make the last construction more realistic take for example, $m = p = 2$ and

$$B_1 = \tfrac{1}{10}\left(\begin{smallmatrix} 3 & 3 \\ 3 & 3 \end{smallmatrix}\right) \qquad B_2 = \tfrac{1}{5}\left(\begin{smallmatrix} 2 & -2 \\ 2 & -2 \end{smallmatrix}\right) \qquad B_3 = \tfrac{1}{5}\left(\begin{smallmatrix} -2 & -2 \\ 2 & 2 \end{smallmatrix}\right), \qquad B_4 = \tfrac{1}{10}\left(\begin{smallmatrix} 3 & -3 \\ -3 & 3 \end{smallmatrix}\right).$$

In items **III** through **VI** we produce more elaborate structures out of a given Hankel matrix. To this end, we find it convenient to introduce the following notation. Let $\alpha, \beta \geq 0$ and $\eta, \delta > 0$ be integers. One can construct the following $(\alpha + \beta + \eta)\delta \times \eta\delta$ isometry, i.e. $U_{\text{Iso}}^* U_{\text{Iso}} = I_{\eta\delta}$,

$$(5.1) \qquad U_{\text{Iso}}(\alpha, \beta, \eta, \delta) = I_\eta \otimes \begin{pmatrix} 0_{\beta \times \delta} \\ I_\delta \\ 0_{\alpha \times \delta} \end{pmatrix} = \begin{pmatrix} 0_{\beta \times \delta} & & & \\ I_\delta & & & \\ 0_{\alpha \times \delta} & & & \\ & 0_{\beta \times \delta} & & \\ & I_\delta & & \\ & 0_{\alpha \times \delta} & & \\ & & \ddots & \\ & & & 0_{\beta \times \delta} \\ & & & I_\delta \\ & & & 0_{\alpha \times \delta} \end{pmatrix}$$

Similarly, U_{Coiso} is the following $\eta\delta \times (\alpha + \beta + \eta)\delta$ co-isometry, i.e. $U_{\text{Coiso}} U_{\text{Coiso}}^* = I_{\eta\delta}$,

$$(5.2) \qquad \begin{aligned} U_{\text{Coiso}}(\alpha, \beta, \eta, \delta) &= I_\eta \otimes \begin{pmatrix} 0_{\delta \times \beta} & I_\delta & 0_{\delta \times \alpha\delta} \end{pmatrix} \\ &= \begin{pmatrix} 0_{\delta \times \beta} & I_\delta & 0_{\delta \times \alpha} & & & \\ & & 0_{\delta \times \beta} & I_\delta & 0_{\delta \times \alpha} & \\ & & & & \ddots & \\ & & & & 0_{\delta \times \beta} & I_\delta & 0_{\delta \times \alpha} \end{pmatrix}. \end{aligned}$$

Let $\rho \in \{1, 2, \ldots, n\}$ be so that $\frac{n}{\rho}$ is natural. Substitute in U_{Iso} and in U_{Coiso} see (5.1), (5.2) respectively: $\alpha = a\rho p$, $\beta = b\rho p$, with $a, b \geq 0$ integers, $\eta = \frac{n}{\rho}$, $\delta = \rho p$ and consider the pair of products,

$$(5.3) \qquad U_{\text{Iso}}\mathbf{B} = \begin{pmatrix} 0_{b\rho p \times m} \\ B_1 \\ \vdots \\ B_\rho \\ 0_{(a+b)\rho p \times m} \\ B_{\rho+1} \\ \vdots \\ B_{2\rho} \\ 0_{(a+b)\rho p \times m} \\ \vdots \\ 0_{(a+b)\rho p \times m} \\ B_{n+1-\rho} \\ \vdots \\ B_n \\ 0_{a\rho p \times m} \end{pmatrix},$$

and

$$(5.4) \qquad \begin{aligned} \hat{\mathbf{B}} U_{\text{Coiso}} &= (0_{p \times b\rho m}, \; B_1 \; \cdots \; B_\rho, \; 0_{p \times (a+b)\rho m} \\ &\qquad B_{\rho+1} \; \cdots \; B_{2\rho}, \; 0_{p \times (a+b)\rho m}, \; \cdots \qquad B_{n+1\rho} \; \cdots \; B_n, \; 0_{p \times a\rho m}). \end{aligned}$$

Both cases yield the same $(a + b + 1)np \times (a + b + 1)nm$ Hankel matrix, denoted by $\mathbf{H}(a, b, \rho)$. For example,

$$(5.5) \qquad \mathbf{H}(0,\ 2,\ 1) = \begin{pmatrix} & & \ddots & B_1 & & B_2 & \cdots & & B_n \\ & & & & \ddots & & \cdots & \iddots & \\ & B_1 & & \ddots & & \cdots & & \iddots & \\ & & & & \cdots & & \iddots & & \\ & B_2 & \cdots & \iddots & & \iddots & & & \\ & & \iddots & & \iddots & & & & \\ & B_n & & \iddots & & & & & \end{pmatrix},$$

or

$$(5.6) \qquad \mathbf{H}(2,\ 2,\ 2) = \begin{pmatrix} & & \ddots & B_1 & B_2 & & \cdots & & B_{n-1} & B_n \\ & & \ddots & \iddots & & & \cdots & \iddots & & \iddots \\ & B_1 & \iddots & & \cdots & & \iddots & \iddots & & \\ & B_2 & & \cdots & & \iddots & \iddots & & & \\ & & & & & \iddots & \iddots & & & \\ & & & \iddots & \iddots & & & & & \\ & B_{n-1} & \iddots & & & & & & & \\ & B_n & & & & & & & & \end{pmatrix},$$

or for $n = 6$

$$\mathbf{H}(0,\ 1,\ 3) = \begin{pmatrix} & & \ddots & B_1 & B_2 & B_3 & & & B_4 & B_5 & B_6 \\ & & \ddots & \iddots & & & & & & \\ & B_1 & \iddots & \iddots & & & & \iddots & & \\ & B_2 & \iddots & & & & \iddots & & & \\ & B_3 & & & & \iddots & & & & \\ & & & & \iddots & \iddots & & & \\ & & & \iddots & \iddots & & & & \\ & B_4 & \iddots & \iddots & & & & & \\ & B_5 & \iddots & & & & & & \\ & B_6 & & & & & & & \end{pmatrix}.$$

III.   Doubling the powers.
First note that $\mathbf{H}(a, b, 1)$ (i.e. when $\rho = 1$) corresponds to $p \times m$-valued polynomial in (2.13) with $\gamma := a + b + 1$.

As another example, the above Hankel matrix $\mathbf{H}(0,\ 1,\ 3)$ is associated with the polynomial $F(z)$ in (2.14).

IV.   Rectangular polynomials.
Another sample of a Hankel matrix associated with $U_{\mathrm{Iso}}\mathbf{B}_o$ in (5.3) (or $\hat{\mathbf{B}}_o U_{\mathrm{Coiso}}$ in (5.4)) is obtained when the parameters are $a = 2, b = 2 \ \rho = 2$, i.e. Now, multiplying $\mathbf{H}(a,\ b,\ \rho)$ from the *right* by U_{Iso} in (5.1) with the parameters $\alpha = (\rho - 1)m$, $\beta = 0$, $\eta = \frac{n}{\rho}(a + b + 1)$ and $\delta = m$ yields the following $(a + b + 1)np \times (a + b + 1)\frac{n}{\rho}m$

Hankel matrix (here $a = 2, b = 2, \rho = 2$)

(5.7)

$$\mathbf{H}(2,\ 2,\ 2)U_{\text{Iso}} = \left(\begin{array}{c|c|c|c|c|c|c|c}
 & \begin{matrix}B_1\\B_2\end{matrix} & & & \begin{matrix}B_3\\B_4\end{matrix} & \begin{matrix}\ldots\\\ldots\end{matrix} & & \begin{matrix}B_{n-1}\\B_n\end{matrix} \\\hline
\begin{matrix}B_1\\B_2\end{matrix} & & & \begin{matrix}B_3\\B_4\end{matrix} & \begin{matrix}\ldots\\\ldots\end{matrix} & & \begin{matrix}B_{n-1}\\B_n\end{matrix} & \\\hline
 & & \begin{matrix}B_3\\B_4\end{matrix} & \begin{matrix}\ldots\\\ldots\end{matrix} & & \begin{matrix}B_{n-1}\\B_n\end{matrix} & & \\\hline
 & \begin{matrix}B_3\\B_4\end{matrix} & \begin{matrix}\ldots\\\ldots\end{matrix} & & \begin{matrix}B_{n-1}\\B_n\end{matrix} & & & \\\hline
\begin{matrix}B_3\\B_4\end{matrix} & \begin{matrix}\ldots\\\ldots\end{matrix} & & \begin{matrix}B_{n-1}\\B_n\end{matrix} & & & & \\\hline
\ldots & \vdots & \vdots & & & & & \\\hline
 & \begin{matrix}B_{n-1}\\B_n\end{matrix} & & & & & & \\\hline
\begin{matrix}B_{n-1}\\B_n\end{matrix} & & & & & & & \\\hline
 & & & & & & &
\end{array}\right),$$

which corresponds to the $p\rho \times m$-valued polynomial in **IV a.**

Similarly, multiplying $\mathbf{H}(a,\ b,\ \rho)$ from the *left* by U_{Coiso} in (5.2) with the parameters $\alpha = (\rho - 1)p$, $\beta = 0$, $\eta = \frac{n}{\rho}(a + b + 1)$ and $\delta = p$ yields the following $(a + b + 1)\frac{n}{\rho}p \times (a + b + 1)nm$ Hankel matrix (here $a = 2, b = 2, \rho = 2$)

(5.8)

$$U_{\text{Coiso}}\mathbf{H}(2,\ 2,\ 2) = \left(\begin{array}{c|c|c|c|c|c}
 & \begin{matrix}B_1 & B_2\end{matrix} & & & \begin{matrix}\ldots\end{matrix} & \begin{matrix}B_{n-1} & B_n\end{matrix} & \\\hline
\begin{matrix}B_1 & B_2\end{matrix} & & & & \begin{matrix}\ldots\end{matrix} & & \\\hline
 & & \begin{matrix}B_3 & B_4\end{matrix} & & \begin{matrix}\ldots\end{matrix} & & \\\hline
 & \begin{matrix}B_3 & B_4\end{matrix} & & & \begin{matrix}\ldots\end{matrix} & & \\\hline
\begin{matrix}B_3 & B_4\end{matrix} & & & & \begin{matrix}\ldots\end{matrix} & & \\\hline
\vdots & \vdots & \vdots & & \begin{matrix}\ldots\end{matrix} & & \\\hline
 & \begin{matrix}B_{n-1} & B_n\end{matrix} & & & & & \\\hline
\begin{matrix}B_{n-1} & B_n\end{matrix} & & & & & & \\\hline
 & & & & & &
\end{array}\right),$$

which corresponds to the $p \times \rho m$-valued polynomial in **IV b.**

It is easy to verify that if $\mathbf{H}_o$ satisfies the first line in (4.4) then so do all Hankel matrices of the form $\mathbf{H}_0 U_{\text{Iso}}$ (5.7).

Similarly, if $\mathbf{H}_0$ satisfies the second line in (4.4) then so do all Hankel matrices of the form $U_{\text{Coiso}}\mathbf{H}_0$ (5.8).

In the sequel, we shall adjust our previous notation in (2.8) of the Hankel matrix associated with

$$F(z) = z^{-(1+\eta)}B_1 + \ \ldots \ + z^{-(n+\eta)}B_n \qquad \eta \geq 0,$$

to $\mathbf{H}_{\mathbf{B},n,\eta}$ (in (2.8) the subscricts B and n were omitted, as so far they were evident from the context). For example, with the polynomial

$$z^{-(1+\eta)}C_1 + \ \ldots \ + z^{-(l+\eta)}C_l \qquad \eta = 0,\ 1,\ 2\ldots$$

one can associate the $(l + \eta)p_c \times (l + \eta)m_c$ Hankel matrix,

$$
\mathbf{H}_{\mathbf{C},l,\eta} =
\begin{pmatrix}
 & & & C_1 & \cdots & C_l & & \\
 & & \reflectbox{$\ddots$} & & & & \ddots & \\
 & C_1 & \cdots & C_l & & & & \reflectbox{$\ddots$} \\
 & \vdots & \reflectbox{$\ddots$} & & & & & \\
 & C_l & & & & & &
\end{pmatrix}.
$$

V. Composition of polynomials.

With the pair of poynomilas in (2.16) one can associated the Hankel matrices $\mathbf{H}_{\mathbf{B},n,0}$ and $\mathbf{H}_{\mathbf{C},l,0}$, which are of dimensions $np_b \times nm_b$ and $lp_c \times nm_c$, respectively.

Out of this pair, one can construct (at least) the *three* following Hankel matrices, all of the form $\mathbf{H}_{\mathbf{D},n,0}$:

a. A $n(p_b + p_c) \times n(m_b + m_c)$ Hankel matrix

or another $n(p_b + p_c) \times n(m_b + m_c)$ Hankel matrix

b. For $m_c \geq m_b$ a $n(p_b + p_c) \times nm_c$ Hankel matrix

$$
\left(
\begin{array}{c|c|c|c|c}
\begin{array}{c} \sqrt{\alpha}B_1 \; 0_{p_b \times (m_c - m_b)} \\ \sqrt{1-\alpha}C_1 \end{array} & & \begin{array}{cc} \sqrt{\alpha}B_l \; 0_{p_b \times (m_c - m_b)} \\ \sqrt{1-\alpha}C_l \end{array} & \begin{array}{c} \sqrt{\alpha}B_{l+1} \; 0_{p_b \times (m_c - m_b)} \\ 0_{p_c \times m_c} \end{array} & & \begin{array}{c} \sqrt{\alpha}B_n \; 0_{p_b \times (m_c - m_b)} \\ 0_{p_c \times m_c} \end{array} \\ \hline
\vdots & \ddots & & & \ddots \\ \hline
\begin{array}{c} \sqrt{\alpha}B_l \; 0_{p_b \times (m_c - m_b)} \\ \sqrt{1-\alpha}C_l \end{array} & & & \begin{array}{c} \sqrt{\alpha}B_n \; 0_{p_b \times (m_c - m_b)} \\ 0_{p_c \times m_c} \end{array} & \\
\begin{array}{c} \sqrt{\alpha}B_{l+1} \; 0_{p_b \times (m_c - m_b)} \\ 0_{p_c \times m_c} \end{array} & & \begin{array}{c} \sqrt{\alpha}B_n \; 0_{p_b \times (m_c - m_b)} \\ 0_{p_c \times m_c} \end{array} & & \\ \hline
\vdots & \ddots & & & \\ \hline
\begin{array}{c} \sqrt{\alpha}B_n \; 0_{p_b \times (m_c - m_b)} \\ 0_{p_c \times m_c} \end{array} & & & &
\end{array}
\right)
$$

c. For $p_b \geq p_c$ a $np_b \times (m_b + m_c)n$ Hankel matrix

$$
\left(
\begin{array}{cc|cc|cc|cc}
\begin{array}{c} \sqrt{\alpha}B_1 \\ 0_{(p_b - p_c) \times m_b} \end{array} & \sqrt{1-\alpha}C_1 & & \begin{array}{c} \sqrt{\alpha}B_l \\ 0_{(p_b - p_c) \times m_b} \end{array} & \sqrt{1-\alpha}C_l & \begin{array}{c} \sqrt{\alpha}B_{l+1} \\ 0_{(p_b - p_c) \times m_b} \end{array} & 0_{p_b \times m_c} & \begin{array}{c} \sqrt{\alpha}B_n \\ 0_{(p_b - p_c) \times m_b} \end{array} & 0_{p_b \times m_c} \\ \hline
\vdots & & \ddots & & & & & \ddots \\ \hline
\begin{array}{c} \sqrt{\alpha}B_l \\ 0_{(p_b - p_c) \times m_b} \end{array} & \sqrt{1-\alpha}C_l & & & & \begin{array}{c} \sqrt{\alpha}B_n \\ 0_{(p_b - p_c) \times m_b} \end{array} & 0_{p_b \times m_c} & \\
\begin{array}{c} \sqrt{\alpha}B_{l+1} \\ 0_{(p_b - p_c) \times m_b} \end{array} & 0_{p_b \times m_c} & & \begin{array}{c} \sqrt{\alpha}B_n \\ 0_{(p_b - p_c) \times m_b} \end{array} & 0_{p_b \times m_c} & & & \\ \hline
\vdots & \vdots & \ddots & & & & & \\ \hline
\begin{array}{c} \sqrt{\alpha}B_n \\ 0_{(p_b - p_c) \times m_b} \end{array} & 0_{p_b \times m_c} & & & & & &
\end{array}
\right)
$$

VI. Product of polynomials

Recall that out of

$$F_b(z) = z^{-1}B_1 + \dots z^{-n}B_n \qquad p_b \times \rho$$

$$F_c(z) = z^{-1}C_1 + \dots z^{-l}C_l \qquad \rho \times m_c$$

the following $p_b \times m_c$-valued polynomial was obtained

$$(5.9) \qquad F_d(z) := F_b(z)F_c(z) = z^{-1}\left(z^{-1}D_1 + \ \dots \ + z^{-(n+l-1)}D_{n+l-1}\right) \ .$$

where the coefficients $D_1 \, , \ \dots \, , \ D_{n+l-1}$ were explicitly given in (2.17).

Expressing, (2.17) in terms of corresponding Hankel matrices yields

$$(5.10) \qquad\qquad \mathbf{H}_{\mathbf{D},n+l-1,1} = \mathbf{H}_{\mathbf{B},n,l}\mathbf{T}_{m+l,\rho}\mathbf{H}_{\mathbf{C},l,n} \ ,$$

where the Hankel matrices $\mathbf{H}_{\mathbf{B},n,l}$, $\mathbf{H}_{\mathbf{C},l,n}$ and $\mathbf{H}_{\mathbf{D},n+l-1,1}$ are $(n+l)p_b \times (n+l)\rho$, $(n+l)\rho \times (n+l)m_c$ and $(n+l)p_b \times (n+l)m_c$ respectively, while $\mathbf{T}_{m+l,\rho}$ is the permutation matrix as in (2.18).

Next, to establish the fact that $F_d(z)$ is in $\mathcal{U}$, we go through the following steps.

First, from (5.10) note that

$$\mathbf{H}^*_{\mathbf{D},n+l-1,1}\mathbf{H}_{\mathbf{D},n+l-1,1} = \mathbf{H}^*_{\mathbf{C},l,n}\mathbf{T}_{m+l,\rho}\mathbf{H}^*_{\mathbf{B},n,l}\mathbf{H}_{\mathbf{B},n,l}\mathbf{T}_{m+l,\rho}\mathbf{H}_{\mathbf{C},l,n} \ .$$

Assuming now that $F_b \in \mathcal{U}$, it follows from (4.4) that

$$\mathbf{H}^*_{\mathbf{D},n+l-1,1}\mathbf{H}_{\mathbf{D},n+l-1,1} = \mathbf{H}^*_{\mathbf{C},l,n}\mathbf{T}_{m+l,\rho} \cdot \mathrm{diag}\{I_{(l+1)\rho} \quad \Delta_{(n-1)\rho}\} \cdot \mathbf{T}_{m+l,\rho}\mathbf{H}_{\mathbf{C},l,n}$$

$$= \mathbf{H}^*_{\mathbf{C},l,n} \cdot \mathrm{diag}\{\Delta_{(n-1)\rho} \quad I_{(l+1)\rho}\} \cdot \mathbf{H}_{\mathbf{C},l,n} \ ,$$

where $\Delta_{(n-1)\rho}$ is $(n-1)\rho \times (n-1)\rho$ positive semi-definite (weak) contraction.

Assuming now that also $F_c \in \mathcal{U}$, (carefully following the dimensions) it follows from (4.4) that

$$\mathbf{H}^*_{\mathbf{D},n+l-1,1}\mathbf{H}_{\mathbf{D},n+l-1,1} = \mathrm{diag}\{I_{2p_b} \quad \hat{\Delta}_{(n+l-2)p_b}\},$$

where $\hat{\Delta}_{(n+l-2)p_b}$ is a $(n+l-1)p_b \times (n+l-1)p_b$ positive semi-definite (weak) contraction, this part is estabilished and indeed $F_d \in \mathcal{U}$.

Showing the relation for $\mathbf{H}_{\mathbf{D},n+l-1,1}\mathbf{H}^*_{\mathbf{D},n+l-1,1}$ is quite similar and thus omitted.

$\square$

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