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ON ALGEBRAS WHICH ARE INDUCTIVE LIMITS OF BANACH SPACES

DANIEL ALPAY AND GUY SALOMON

Contents

Abstract. We introduce algebras which are inductive limits of Banach spaces and carry inequalities which are counterparts of the inequality for the norm in a Banach algebra. We then define an associated Wiener algebra, and prove the corresponding version of the well-known Wiener theorem. Finally, we consider factorization theory in these algebra, and in particular, in the associated Wiener algebra.

1. Introduction

The purpose of this paper is to establish a framework for algebras which are inductive limits of Banach spaces and carry inequalities which are counterparts of the inequality satisfied by the norm in a Banach-algebra. More precisely, let $\mathcal{A}$ be an algebra which is also the inductive limit of a family of Banach spaces $\{X_\alpha : \alpha \in A\}$ directed under inclusion. We call $\mathcal{A}$ a *strong algebra* if for any $\alpha \in A$ there exists $h(\alpha) \in A$ such that for any $\beta \geq h(\alpha)$ there is a positive constant $A_{\beta,\alpha}$ for which

$$\|ab\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta, \quad \text{and} \quad \|ba\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta. \tag{1.1}$$

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for every $a \in X_\alpha$ and $b \in X_\beta$. The case of a Banach algebra corresponds to the case where the set of indices A is a singleton.

The strong algebras are topological algebras in the sense of Naimark, i.e. they are locally convex, and the multiplication is separately continuous (this follows from the universal property of inductive limits; see Proposition 3.2). If furthermore, any bounded set in a strong algebra is bounded in some of the X_α, then the multiplication is jointly continuous. The inequalities (1.1) express the fact that for any $\alpha \in A$, each of the spaces $\{X_\beta : \beta \geq h(\alpha)\}$ "absorbs" X_α from both sides. Due to this property, one may evaluate (with elements of $\mathcal{A}$) power series, and therefore, consider invertible elements; see for example Proposition 4.2 and Theorem 4.4.

In [3] (see also [1]), we studied a special family of such algebras, which are inductive limits of $\mathbf{L}^2$ spaces of measurable functions over a locally compact group. Examples include the algebra of germs of holomorphic functions at the origin, the Kondratiev space of Gaussian stochastic distributions (see [10] for the latter), the algebra of functions $f : [1, \infty) \to \mathbb{C}$ for which $f(x)/x^p$ belongs to $\mathbf{L}^2([1, \infty))$ (this gives relations with the theory of Dirichlet series), and a new space of non-commutative stochastic distributions; see [2] for the latter.

In this paper, we first develop the general theory of strong algebras. Then, we associate to every strong algebra a Wiener algebra of functions in the following sense: Let $\mathcal{A} = \varinjlim X_\alpha$ be a strong algebra, and define Y_α to be the space of periodic functions

$$a(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}, \quad a_n \in X_\alpha,$$

on $-\pi \leq t < \pi$ to $\mathcal{A}$, with

$$\|a\|_\alpha = \sum_{n \in \mathbb{Z}} \|a_n\|_\alpha < \infty.$$

We call the inductive limit $\varinjlim Y_\alpha$ of the Banach spaces Y_α, *the Wiener algebra associated to* $\mathcal{A}$. This family extends the case of Wiener algebras of functions with values in a Banach algebra. See [9] for the latter.

After showing that a Wiener algebra associated to a strong algebra is a strong algebra itself, we prove a strong algebra counterpart of the well known theorem of Wiener, namely, we show that an element is left/right/two-sided invertible in the Wiener algebra associated to $\mathcal{A}$,

if and only if its evaluation in any point of the circle $t \in [-\pi, \pi)$ is left/right/two-sided invertible in $\mathcal{A}$.

Finally, we consider factorization theory in strong algebras (and in particular in the associated Wiener algebra). More precisely, we show that if $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$, where $\mathcal{A}^+, \mathcal{A}^-$ are closed subalgebras, then any element a, which is "close enough" (in an appropriate sense) to 1, admits a factorization

$$a = a_- a_+,$$

where a_-, a_+ are invertible and satisfying $a_+ - 1, a_+^{-1} - 1 \in \mathcal{A}^+$, $a_- - 1, a_-^{-1} - 1 \in \mathcal{A}^-$.

The paper consists of five sections besides the introduction, and we now describe its content. A review on inductive limits, bornological spaces, and barreled spaces, and the definition of a strong algebra, are given in Section 2. Some topological results are given in Section 3. In Section 4 we study the underlying functional calculus. In Section 5 we study the Wiener algebra associated to a strong algebra, and prove the strong algebra version of the well-known Wiener theorem. The factorization theory is given in Section 6.

2. Topological review and the definition of a strong algebra

To introduce strong algebras, we first recall the definition of an inductive limit of normed spaces. This definition can be extended to an inductive limit of locally convex spaces; see [5, II.27, Proposition 4; II.29, Example II].

Definition 2.1. *Let $\{X_\alpha : \alpha \in A\}$ be a family of subspaces of a vector space X such that $X_\alpha \neq X_\beta$ for $\alpha \neq \beta$, directed under inclusion, satisfying $X = \bigcup_\alpha X_\alpha$, where A is directed under $\alpha \leq \beta$ if $X_\alpha \subseteq X_\beta$. Moreover, on each X_α ($\alpha \in A$), a norm $\|\cdot\|_\alpha$ is given, such that whenever $\alpha \leq \beta$, the topology induced by $\|\cdot\|_\beta$ on X_α is coarser than the topology induced by $\|\cdot\|_\alpha$. Then X, topologized with the inductive limit topology is called* the inductive limit of the normed spaces *$\{X_\alpha : \alpha \in A\}$.*

The inductive limit has the following universal property. Given any locally convex space Y, a linear map f from X to Y is continuous if and only if each of the restrictions $f|_{X_\alpha}$ is continuous with respect to the topology of X_α; see [5, II.27, Proposition 4]. This property allows

to take full advantage of the inequalities (2.1) in the definition of a strong algebra given now.

Definition 2.2. *Let $\{X_\alpha : \alpha \in A\}$ be a family of Banach spaces directed under inclusions, and let $\mathcal{A} = \bigcup X_\alpha$ be its inductive limit. We call $\mathcal{A}$ a strong algebra if it is an algebra satisfying the property that for any $\alpha \in A$ there exists $h(\alpha) \in A$ such that for any $\beta \geq h(\alpha)$ there is a positive constant $A_{\beta,\alpha}$ for which*

$$\|ab\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta, \qquad \text{and} \qquad \|ba\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta. \tag{2.1}$$

for every $a \in X_\alpha$ and $b \in X_\beta$.

Since a strong algebra is an inductive limit of Banach spaces, it inherits two special structures of locally convex spaces, namely being bornological and barrelled. We recall that a locally convex space X is called bornological if every balanced, convex subset $U \subseteq X$ that absorbs every bounded set in X is a neighborhood of 0. Equivalently, a bornological space is a locally convex space on which each semi-norm that is bounded on bounded sets, is continuous. We also recall that a topological vector space is said to be barrelled if each convex, balanced, closed and absorbent set is a neighborhood of zero. Equivalently, a barreled space is a locally convex space on which each semi-norm that is semi-continuous from below, is continuous. With these definitions at hand, we can now state:

Proposition 2.3. *An inductive limit of Banach spaces (and in particular a strong algebra) is bornological and barrelled.*

Proof. A Banach space is clearly barrelled and bornological, and these two properties are kept under inductive limits; see [5, III.25, Corollary1, Corollary 3; III.12 Examples 1,3]. □

3. Topological results

The term "topological algebra" is sometimes refer to topological vector space together with a (jointly) continuous multiplication $(a, b) \mapsto ab$. However, in his book [12], M.A. Naimark gives the following definition for a topological algebra.

Definition 3.1 (M.A Naimark). *$\mathcal{A}$ is called topological algebra if:*

(a) $\mathcal{A}$ is an algebra;
(b) $\mathcal{A}$ is a locally convex topological linear space;
(c) the product ab is a continuous function of each of the factors a, b provided the other factor is fixed.

We will show that a strong algebra is a topological algebra in the sense of Naimark.

Proposition 3.2. *Let $\mathcal{A} = \bigcup_\alpha X_\alpha$ be a strong algebra and let $a \in \mathcal{A}$. Then the linear mappings $L_a : x \mapsto ax$, $R_a : x \mapsto xa$ are continuous. Thus, it is topological algebra in the sense of Naimark.*

Proof. Suppose that $a \in X_\alpha$, and let $L_a|_{X_\gamma} : X_\beta \to \mathcal{A}$ be the restriction of the map L_a to X_γ. If B is a bounded set of X_γ then in particular we may choose $\beta \geq h(\alpha)$ such that $\beta \geq \gamma$, so $B \subseteq \{x \in X_\beta : \|x\|_\beta < \lambda\}$. Thus, for any $x \in B$

$$\|L_a|_{X_\gamma}(x)\|_\beta \leq A_{\beta,\alpha}\lambda\|a\|_\alpha.$$

Hence, $L_a|_{X_\gamma}(B)$ is bounded in X_β and hence in $\mathcal{A}$. Thus, for any γ, $L_a|_{X_\gamma} : X_\gamma \to \mathcal{A}$ is bounded and hence continuous, so by the universal property of the inductive limits, L_a is continuous. The proof for R_a is similar. □

The boundedness assumption in the next theorem occurs in many natural cases, which are discussed after the proof of the theorem.

Theorem 3.3. *If in a strong algebra $\mathcal{A} = \bigcup X_\alpha$, any set is bounded if and only if it is bounded in some of the X_α, then the multiplication is jointly continuous.*

Proof of Theorem 3.3. Let $(a, b) \in \mathcal{A} \times \mathcal{A}$. Since

$$xy = (x - a)(y - b) + xb + ay - ab,$$

and in view of Proposition 3.2, $(x, y) \mapsto xy$ is continuous in (a, b) if and only if it is continuous at the origin. So it remains to show that the multiplication is continuous at the origin.

Since a product of bornological spaces is bornological, $\mathcal{A}\times\mathcal{A}$ is bornological. We will show that for every bounded set $B \subseteq \mathcal{A} \times \mathcal{A}$ and every convex, circled neighborhood V of 0 in $\mathcal{A}$, $m^{-1}(V)$ absorbs B (where $m(x, y) = xy$). Hence, by the definition of a bornological space (see Section 2) $m^{-1}(V)$ is a neighborhood of the origin.

Thus, let B be any bounded set in $\mathcal{A} \times \mathcal{A}$. Then $B \subseteq B_1 \times B_2$ where B_i $i = 1, 2$ are bounded sets in $\mathcal{A}$ (see [13, pp. 27, 5.5]). So there exists $\alpha, \beta' \in A$ such that B_1 and B_2 are bounded in X_α and $X_{\beta'}$ respectively. We may choose $\beta \geq \beta', h(\alpha)$, and so B is bounded in $X_\alpha \times X_\beta$. In particular, $\|x\|_\alpha\|y\|_\beta$ is bounded for all $(x, y) \in B$. Therefore, from inequality (2.1), $\|xy\|_\beta$ is bounded for all $(x, y) \in B$, so $m(B)$ is bounded in $\mathcal{A}$. Thus, for any convex circled neighborhood V of the origin, V absorbs $m(B)$, i.e. there exists $\lambda > 0$ such that $m(B) \subseteq \lambda V$. Thus $m(\sqrt{\lambda}^{-1}B) \subseteq V$ so $B \subseteq \sqrt{\lambda}m^{-1}(V)$. □

There are several natural cases in which the assumption of the previous theorem holds. Namely, in each of the following instances of inductive limit of Banach spaces, any bounded set of $\bigcup X_\alpha$ is bounded in some of the X_α. Thus, when a strong algebra $\mathcal{A}$ is of one of these forms, then in particular the multiplication is jointly continuous.

(i) The set of indices A is the singleton $\{0\}$, and hence $\bigcup X_\alpha = X_0$ is a Banach space.

(ii) The set of indices A is $\mathbb{N}$, and $\bigcup X_n$ is the *strict* inductive limit of the X_n (and is then called an LB-space), that is, for any $m \geq n$ the topology of X_n induced by X_m, is the initial topology of X_n. See [13, Theorem 6.5, pp. 59] and [5, III.5, Proposition 6].

(iii) The set of indices A is $\mathbb{N}$, and the embeddings $X_n \hookrightarrow X_{n+1}$ are compact. See [5, III.6, Proposition 7].

(iv) The set of indices A is $\mathbb{N}$, and the inductive limit is a dual of reflexive Fréchet space. More precisely, let $\Phi_1 \supseteq \Phi_2 \supseteq \ldots$ be a decreasing sequence of Banach spaces, and assume that the corresponding countably normed space $\bigcap \Phi_n$ is reflexive. Then, $\bigcup \Phi_n'$, the strong dual of $\bigcap \Phi_n$ is the same as the inductive limit of the spaces $\Phi_1' \subseteq \Phi_2' \subseteq \ldots$ (as a topological vector space). See [5, IV.23, proof of Proposition 4] and [8, §5.3, pp.45-46] .

In fact, in [5, IV.26, Theorem 2] of N. Bourbaki, the following theorem is proved.

Theorem 3.4. *Let E_1 and E_2 be two reflexive Fréchet spaces, and let G a locally convex Hausdorff space. For $i = 1, 2$, let F_i be the strong dual of E_i. Then every separately continuous bilinear mapping $u : F_1 \times F_2 \to G$ is continuous.*

This gives another proof for the continuity of the multiplication in case (iv).

There are some cases where the topology on an inductive limit (that is, the inductive topology) is the finest topology such that the mappings $X_\alpha \hookrightarrow \bigcup X_\alpha$ are continuous (instead of the finest locally convex topology such that they are continuous). One example is when X is the inductive limit of a sequence of Banach spaces $\{X_n : n \in \mathbb{N}\}$, and the embeddings $X_n \hookrightarrow X_{n+1}$ are compact (see [5, III.6, Proposition 7, Lemma 1] and case (iii) above). In this case, we have the following sufficient condition on mappings $X \to X$ to be continuous.

Theorem 3.5. *Let X be the inductive limit of the family X_α, where its topology is the finest topology such that the mappings $X_\alpha \hookrightarrow \bigcup X_\alpha$ are continuous. Then any map (not necessarily linear) f from an open*

set $W \subseteq X$ to X which satisfies the property that for any α there is β such that $f(W \cap X_\alpha) \subseteq X_\beta$ and $f|_{W \cap X_\alpha}$ is continuous with respect to the topologies of $W \cap X_\alpha$ at the domain and X_β at the range, is a continuous function $W \to X$.

Proof. Note that in this case, U is open in X if and only if for any $U \cap X_\alpha$ is open in X_α for every α. Let U be an open set of X, and let $\alpha \in A$. By the assumption, there is β such that $f(W \cap X_\alpha) \subseteq X_\beta$ and $f^{-1}(U) \cap X_\alpha = f|_{W \cap X_\alpha}^{-1}(U \cap X_\beta)$ is open in $W \cap X_\alpha$. In particular, $f^{-1}(U) \cap X_\alpha$ is open in X_α, so $f^{-1}(U)$ is open in X and hence in W. $\square$

As a corollary to Theorem 3.5 we will show in the sequel that whenever a strong algebra $\mathcal{A}$ satisfies the assumption of the theorem then the set of invertible elements is open and that $a \mapsto a^{-1}$ is continuous. See Theorem 4.4.

There is a "well behaved" family of linear maps from an inductive limit of Banach spaces into itself, which we call admissible operators. These maps are in particular continuous, and sometimes all continuous linear maps are of this form.

Definition 3.6. *Let X be the inductive limit of the Banach spaces X_α. Then a linear map $T : X \to X$ which satisfies the property that for any α there is β such that $T(X_\alpha) \subseteq X_\beta$ and $T|_{X_\alpha}$ is continuous with respect to the topologies of X_α for the domain and X_β for the range, will be called an admissible operator of X. For an admissible operator $T : X \to X$ we denote by $\|T\|_\beta^\alpha$ the norm of $T|_{X_\alpha}$ when the range is restricted to X_β, whenever it makes sense, and otherwise we set $\|T\|_\beta^\alpha = \infty$.*

By the universal property of inductive limits, we conclude:

Proposition 3.7. *Any admissible operator is continuous.*

Remark 3.8. Note that if any bounded set in X is bounded in some X_α, then any continuous linear map is admissible.

Remark 3.9. Note that if $\|S\|_\beta^\alpha < \infty$ and $\|T\|_\gamma^\beta < \infty$, then

$$\|TS\|_\gamma^\alpha \leq \|T\|_\gamma^\beta \|S\|_\beta^\alpha.$$

Proposition 3.10. *Let $T : X \to X$ be a admissible operator such that there exists α for which $\|T\|_\alpha^\alpha < 1$ then $I - T$ is invertible, and*

$$\|(I-T)^{-1}\|_\alpha^\alpha \leq \frac{1}{1 - \|T\|_\alpha^\alpha}.$$

Proof. Not that,

$$\|I+T+T^2+\cdots\|_\alpha^\alpha \le \sum_{n=0}^{\infty}(\|T\|_\alpha^\alpha)^n = \frac{1}{1-\|T\|_\alpha^\alpha} < \infty.$$

Moreover, one can check that

$$(I+T+T^2+\cdots)(I-T) = (I-T)(I+T+T^2+\cdots) = 1.$$

□

4. Power series and invertible elements

Henceforward, we assume that $\mathcal{A}$ is a unital strong algebra.

Proposition 4.1. *Assuming $\sum_{n=0}^{\infty} c_n z^n$ converges in the open disk with radius R, then for any $a \in \mathcal{A}$ such that there exist α, β with $\beta \ge h(\alpha)$ and $A_{\beta,\alpha}\|a\|_\alpha < R$ it holds that*

$$\sum_{n=0}^{\infty} c_n a^n \in X_\beta \subseteq \mathcal{A}.$$

Proof. This follows from

$$\sum_{n=0}^{\infty} |c_n| \|a^n\|_\beta \le \sum_{n=0}^{\infty} |c_n| (A_{\beta,\alpha}\|a\|_\alpha)^n \|1\|_\beta < \infty.$$

□

Proposition 4.2. *Let $a \in \mathcal{A}$ be such that there exists α, β with $\beta \ge h(\alpha)$ and $A_{\beta,\alpha}\|a\|_\alpha < 1$ then $1-a$ is invertible (from both sides) and it holds that*

$$\|(1-a)^{-1}\|_\beta \le \frac{\|1\|_\beta}{1-A_{\beta,\alpha}\|a\|_\alpha}, \quad \|1-(1-a)^{-1}\|_\beta \le \frac{A_{\beta,\alpha}\|a\|_\alpha\|1\|_\beta}{1-A_{\beta,\alpha}\|a\|_\alpha},$$

where

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Proof. Due to Proposition 4.1 we have that

$$\sum_{n=0}^{\infty} a^n \in X_\beta \subseteq \mathcal{A}.$$

Moreover, clearly

$$(1-a)\left(\sum_{n=0}^{\infty} a^n\right) = \left(\sum_{n=0}^{\infty} a^n\right)(1-a) = 1,$$

and we have that

$$\|(1-a)^{-1}\|_\beta \leq \sum_{n=0}^{\infty} \|a^n\|_\beta \leq \sum_{n=0}^{\infty}(A_{\beta,\alpha}\|a\|_\alpha)^n\|1\|_\beta = \frac{\|1\|_\beta}{1-A_{\beta,\alpha}\|a\|_\alpha},$$

and

$$\|1-(1-a)^{-1}\|_\beta \leq \sum_{n=1}^{\infty} \|a^n\|_\beta \leq \sum_{n=1}^{\infty}(A_{\beta,\alpha}\|a\|_\alpha)^n\|1\|_\beta = \frac{A_{\beta,\alpha}\|a\|_\alpha\|1\|_\beta}{1-A_{\beta,\alpha}\|a\|_\alpha}.$$

□

Proposition 4.3. *If $a \in \mathcal{A}$ has a left inverse $a' \in X_\alpha \subseteq \mathcal{A}$ (i.e. $a'a = 1$), then for any $\beta \geq h(\alpha)$ and $b \in X_\beta$ such that there exists $\gamma \geq h(\beta)$ with $A_{\gamma,\beta}A_{\beta,\alpha}\|a'\|_\alpha\|b\|_\beta < 1$, it holds that $a-b$ has a left inverse $(a-b)' \in X_\gamma$, where*

$$(a-b)' = a'\sum_{n=0}^{\infty}(ba')^n.$$

and

$$\|(a-b)'-a'\|_\gamma \leq A_{\gamma,\alpha}\|a'\|_\alpha \frac{A_{\gamma,\beta}A_{\beta,\alpha}\|a'\|_\alpha\|b\|_\beta\|1\|_\gamma}{1-A_{\gamma,\beta}A_{\beta,\alpha}\|a'\|_\alpha\|b\|_\beta}$$

Proof. We note that

$$A_{\gamma,\beta}\|ba'\|_\beta \leq A_{\gamma,\beta}A_{\beta,\alpha}\|a'\|_\alpha\|b\|_\beta < 1.$$

Thus, $1-ba'$ is invertible, and

$$a'(1-ba')^{-1}(a-b) = a'(1-ba')^{-1}(1-ba')a = 1.$$

Now, note that

$$\begin{aligned}\|(a-b)'-a'\|_\gamma &= \|a'(1-ba')^{-1}-a'\|_\gamma \\ &\leq A_{\gamma,\alpha}\|a'\|_\alpha\|(1-ba')^{-1}-1\|_\gamma \\ &\leq A_{\gamma,\alpha}\|a'\|_\alpha \frac{A_{\gamma,\beta}\|ba'\|_\beta\|1\|_\gamma}{1-A_{\gamma,\beta}\|ba'\|_\beta} \\ &\leq A_{\gamma,\beta}\|a'\|_\alpha \frac{A_{\gamma,\beta}A_{\beta,\alpha}\|a'\|_\alpha\|b\|_\beta\|1\|_\gamma}{1-A_{\gamma,\beta}A_{\beta,\alpha}\|a'\|_\alpha\|b\|_\beta}.\end{aligned}$$

□

We now give a corollary to Theorem 3.5.

Theorem 4.4. *In case where the topology on $\mathcal{A}$ is the finest topology such that the mappings $X_\alpha \hookrightarrow \mathcal{A}$ are continuous, the set of invertible elements $GL(\mathcal{A})$ is open, and $(\cdot)^{-1} : GL(\mathcal{A}) \to GL(\mathcal{A})$ is continuous.*

Proof. Let $a \in GL(\mathcal{A})$ and assume that $a^{-1} \in X_\alpha$. Let U_a be the set of all $b \in \mathcal{A}$ such that there exists $\beta \geq h(\alpha)$ for which

$$\|b\|_\beta < \frac{1}{A_{h(\beta),\beta} A_{\beta,\alpha} \|a^{-1}\|_\alpha}.$$

Clearly $U_a \cap X_\beta$ is open in X_β for any β, so U_a is open. Moreover, for any $b \in U_a$

$$A_{h(\beta),\beta}\|a^{-1}b\|_\beta \leq A_{h(\beta),\beta} A_{\beta,\alpha}\|a^{-1}\|_\alpha \|b\|_\beta < 1.$$

In view of Theorem 4.2, $1 - a^{-1}b$ is invertible, and therefore $a - b = a(1 - a^{-1}b)$ is invertible too. Thus, $a + U_a \subseteq GL(\mathcal{A})$, and so $GL(\mathcal{A})$ is open.

Now, we note that,

$$\begin{aligned}(a+b)^{-1} - a^{-1} &= \left(a(1+a^{-1}b)^{-1}\right) - a^{-1} \\ &= (1+a^{-1}b)^{-1}a^{-1} - a^{-1} \\ &= \left((1+a^{-1}b)^{-1} - 1\right) a^{-1}.\end{aligned}$$

Therefore, for any $b \in U_a$,

$$\begin{aligned}\|(a+b)^{-1} - a^{-1}\|_{h(\beta)} &\leq A_{h(\beta),\alpha}\|a^{-1}\|_\alpha \|(1+a^{-1}b)^{-1} - 1\|_{h(\beta)} \\ &\leq A_{h(\beta),\alpha}\|a^{-1}\|_\alpha \frac{A_{h(\beta),\beta}\|a^{-1}b\|_\beta \|1\|_{h(\beta)}}{1 - A_{h(\beta),\beta}\|a^{-1}b\|_\beta} \\ &\leq A_{h(\beta),\alpha}\|a^{-1}\|_\alpha \frac{A_{h(\beta),\beta}A_{\beta,\alpha}\|a^{-1}\|_\alpha\|b\|_\beta \|1\|_{h(\beta)}}{1 - A_{h(\beta),\beta}A_{\beta,\alpha}\|a^{-1}\|_\alpha\|b\|_\beta}\end{aligned}$$

Thus, the function

$$u : b \mapsto (a+b)^{-1} - a^{-1}$$

satisfies $u(U_a \cap X_\beta) \subseteq X_{h(\beta)}$, and $u|_{U_a \cap X_\beta}$ is continuous with respect to the topologies of $U_a \cap X_\beta$ at the domain and $X_{h(\beta)}$ at the range. So by Theorem 3.5 it is continuous $U_a \to \mathcal{A}$. Since a was arbitrary, $(\cdot)^{-1} : GL(\mathcal{A}) \to GL(\mathcal{A})$ is continuous. ∎

5. A Wiener algebra associated to a strong algebra and a strong algebra version of the Wiener theorem

Definition 5.1. *Let $\mathcal{A} = \bigcup X_\alpha$ be a strong algebra. Let Y_α be the space of periodic functions*

$$a(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}, \quad a_n \in X_\alpha,$$

on $-\pi \leq t < \pi$ to $\mathcal{A}$, with

$$\|a\|_\alpha = \sum_{n \in \mathbb{Z}} \|a_n\|_\alpha < \infty.$$

The inductive limit $\mathcal{U} = \bigcup Y_\alpha$ of the Banach spaces Y_α is called the Wiener algebra associated to $\mathcal{A}$.

Remark 5.2. Assuming

$$\Phi_1 \supseteq \Phi_2 \supseteq \cdots \supseteq \Phi_p \supseteq \cdots$$

is a decreasing sequence of reflexive Banach spaces, so that $\bigcap \Phi_p$ is a reflexive Fréchet space, and suppose that $\mathcal{A} = \bigcup \Phi'_p$, the inductive limit of the duals, is a strong algebra (in particular its inductive limit topology coincides with its strong dual topology). We may define

$$\Psi_p = c_0(\mathbb{Z}; \Phi_p),$$

i.e. the space of all Φ_p-valued sequences $(x_n)_{n\in\mathbb{Z}}$ which satisfy

$$\lim_{n\to-\infty} \|x_n\|_p = 0, \quad \lim_{n\to\infty} \|x_n\|_p = 0.$$

This space is a Banach space, and its dual is

$$\Psi'_p = \ell_1(\mathbb{Z}; \Phi'_p),$$

i.e. the space of all Φ'_p-valued sequences $(a_n)_{n\in\mathbb{Z}}$ which satisfy

$$\sum_{n\in\mathbb{Z}} \|a_n\|_p < \infty.$$

(For further reading on Banach-valued sequences spaces and their duals we refer to [7] and [11]). In this case, $\bigcup \Psi'_p$ can be identified as the Wiener algebra associated to $\mathcal{A}$, but instead of considering only the inductive limit topology on it, we may also consider its topology as a strong dual of $\bigcap \Psi_p$. We do not know when these two topologies coincide.

Proposition 5.3. $\mathcal{U} = \bigcup_\alpha Y_\alpha$ *is a strong algebra.*

Proof. For any $\alpha \in A$ and $\beta \geq h(\alpha)$ and for any $a \in Y_\alpha$ and $b \in Y_\beta$, it holds that

$$\begin{aligned}
\|ab\|_\beta &= \sum_{n\in\mathbb{Z}} \left\| \sum_{m\in\mathbb{Z}} a_m b_{n-m} \right\|_\beta \\
&\leq \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} A_{\beta,\alpha} \|a_m\|_\alpha \|b_{n-m}\|_\beta \\
&\leq A_{\beta,\alpha} \|a\|_\alpha \|b\|_\beta.
\end{aligned}$$

Similarly, $\|ba\|_\beta \leq A_{\beta,\alpha} \|a\|_\alpha \|b\|_\beta$. □

Our principal result is the following theorem:

Theorem 5.4. *For any $a \in \mathcal{U}$, a is left invertible if and only if $a(t)$ is left invertible for every t.*

To prove this theorem, we follow the strategy of the papers [4] of Bochner and Phillips and [14] of Wiener. The proofs are adapted to the case of strong algebras. We begin with some lemmas.

Lemma 5.5. *If $a \in Y_\alpha \subseteq \mathcal{U}$ and if $a(0)$ has a left inverse, then there exists an element*

$$b(t) = \sum_{n\in\mathbb{Z}} b_n e^{int}$$

in $\mathcal{U}$ with the following properties:

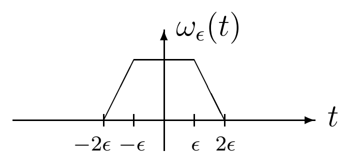
(i) the coefficient b_0 has a left inverse b_0', and there exists $\beta \geq h(\alpha)$ and $\gamma \geq h(\beta)$ such that

$$A_{\gamma,\beta}A_{\beta,\alpha}\sum_{n=1}^{\infty}(\|b_n\|_\alpha + \|b_{-n}\|_\alpha)\|b_0'\|_\beta < 1.$$

(ii) in some interval $t \in (-\epsilon, \epsilon)$, $b(t) = a(t)$.

Proof. We follow the proof of Wiener [14, pp. 12-14]. On the circle $[-\pi, \pi)$, we define the functions

$$\omega_\epsilon(t) = \begin{cases} 1, & |t| < \epsilon \\ 2 - \frac{|t|}{\epsilon}, & \epsilon \leq |t| < 2\epsilon \\ 0, & 2\epsilon \leq |t| \end{cases}$$



and

$$b_\epsilon(t) = \omega_\epsilon(t)a(t) + (1 - \omega_\epsilon(t))a(0) = \sum b_n(\epsilon)e^{int}.$$

Clearly, b_ϵ satisfies the property (ii) for any ϵ. As for property (i), by [14, (2.203), p. 13],

$$b_0(\epsilon) = a_0 + \sum_{n=1}^{\infty}(a_n + a_{-n})\left(1 + \frac{\cos\epsilon n - \cos 2\epsilon n}{\pi n^2\epsilon} - \frac{3\epsilon}{2\pi}\right).$$

Assuming $a \in Y_\alpha$, then there is $\alpha' \geq h(\alpha)$ such that $a(0)' \in X_{\alpha'}$. Since

$$\|b_0(\epsilon) - a(0)\|_\alpha \leq \sum_{n=1}^{\infty}\left(\frac{\cos\epsilon n - \cos 2\epsilon n}{\pi n^2\epsilon} - \frac{3\epsilon}{2\pi}\right)\|a_n + a_{-n}\|_\alpha \to 0,$$

we may choose $\beta \geq h(\alpha')$ and ϵ_1 such that for any $\epsilon \leq \epsilon_1$,

$$A_{\beta,\alpha'}A_{\alpha',\alpha}\|b_0(\epsilon) - a(0)\|_\alpha\|a(0)'\|_{\alpha'} < 1.$$

Thus, by Lemma 4.3, $b_0(\epsilon)$ has left inverse $b_0(\epsilon)' \in X_\beta$, and

$$\|b_0(\epsilon)' - a(0)'\|_\beta \leq A_{\beta,\alpha}\|a'(0)\|_\alpha \frac{A_{\beta,\alpha'}A_{\alpha',\alpha}\|b_0(\epsilon)-a(0)\|_\alpha\|a(0)'\|_{\alpha'}\|1\|_\beta}{1-A_{\beta,\alpha'}A_{\alpha',\alpha}\|b_0(\epsilon)-a(0)\|_\alpha\|a(0)'\|_{\alpha'}} \to 0.$$

So we may choose $\epsilon_2 \leq \epsilon_1$, such that for any $\epsilon \leq \epsilon_2$,

$$\|b_0(\epsilon)'\|_\beta \leq \|a(0)'\|_\beta + 1.$$

Moreover, by [14, (2.205),(2.22) and (2.23) pp. 13-14],

$$\sum_{n=1}^{\infty}(\|b_n(\epsilon)\|_\alpha + \|b_{-n}(\epsilon)\|_\alpha) \leq \sum_{n\in\mathbb{Z}} \|a_n\|_\alpha A_n(\epsilon),$$

where for sufficiently small ϵ

$$A_n(\epsilon) \leq \min\left\{\sqrt{\epsilon}(2|n|c + \frac{9}{\pi}), \frac{15}{\pi}\right\}$$

(where c is some constant), so

$$\sum_{n=1}^{\infty}(\|b_n(\epsilon)\|_\alpha + \|b_{-n}(\epsilon)\|_\alpha) \leq \sum_{n\in\mathbb{Z}} \|a_n\|_\alpha A_n(\epsilon) \to 0.$$

Thus, we may choose $\epsilon_3 \leq \epsilon_2$ and $\gamma \geq h(\beta)$ such that for any $\epsilon \leq \epsilon_3$

$$A_{\gamma,\beta}A_{\beta,\alpha}\sum_{n=1}^{\infty}(\|b_n(\epsilon)\|_\alpha + \|b_{-n}(\epsilon)\|_\alpha)(\|a(0)'\|_\beta + 1) < 1.$$

Therefore

$$A_{\gamma,\beta}A_{\beta,\alpha}\sum_{n=1}^{\infty}(\|b_n(\epsilon)\|_\alpha + \|b_{-n}(\epsilon)\|_\alpha)\|b_0(\epsilon)'\|_\beta < 1.$$

Thus, we conclude that there exists ϵ small enough (i.e. ϵ_3) such that b_ϵ satisfies both properties (i) and (ii). □

Lemma 5.6. *If $b \in \mathcal{U}$ satisfies the property (i) of Lemma 5.5, then b has a left inverse in $\mathcal{U}$.*

Proof. Setting

$$c(t) = b(t) - b_0 = \sum_{n=1}^{\infty}\left(b_n e^{int} + b_{-n}e^{-int}\right),$$

we obtain that $b = c - (-b_0)$ has a left inverse b', and

$$b' = -b_0' \sum_{n=0}^{\infty}(-cb_0')^n.$$

□

We are now ready to prove the main theorem.

Proof of Theorem 5.4. If $a \in \mathcal{U}$, and if for any t, $a(t)$ is left invertible, then by Lemmas 5.5 and 5.6, there exists for each t a function $b_t \in \mathcal{U}$ and ϵ_t such that $(b_t a)|_{(t-\epsilon_t, t+\epsilon_t)} = 1|_{(t-\epsilon_t, t+\epsilon_t)}$. Since the circle is compact there exist $t_1, t_2, \cdots t_n$ such that the circle is covered by $\bigcup_{i=1}^n (t_i - \epsilon_{t_i}, t_i + \epsilon_{t_i})$. We can now piece the associated functions $(b_{t_i})_{i=1}^n$ to a function $a' \in \mathcal{U}$, such that $a'a = 1$. □

Corollary 5.7. *For any $a \in \mathcal{U}$, a is invertible (from both sided) if and only if $a(t)$ is invertible (from both sided) for every t.*

6. Canonical factorization in decomposing strong algebras

A decomposing strong algebra $\mathcal{A}$ is a unital strong algebra which is a direct sum

$$\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$$

of two closed subalgebras. The projection of $\mathcal{A}$ onto $\mathcal{A}^+$ parallel to $\mathcal{A}^-$ will be denoted by P and we set $Q = I - P$. An element $a \in GL(\mathcal{A})$ is said to admit a canonical factorization in case

$$a = a_- a_+,$$

where $a_-, a_+ \in GL(A)$ satisfy $a_+ - 1, a_+^{-1} - 1 \in \mathcal{A}^+$, $a_- - 1, a_-^{-1} - 1 \in \mathcal{A}^-$. We follow Clancey and Gohberg, who considered in [6] only the case where $\mathcal{A}$ is a Banach algebra.

Theorem 6.1. *Let $\mathcal{A}$ be a decomposing strong algebra $\mathcal{A}$ in which elements that have inverse on one side are invertible. The following statements about an element $a \in \mathcal{A}$ are equivalent:*

(a) The element $1 - a$ admits a canonical factorization.

(b) Each of the equations

$$x - P(ax) = 1, \quad y - Q(ya) = 1$$

is solvable in $\mathcal{A}$.

(c) For any pair of elements $f, g \in \mathcal{A}$, each of the equations

$$x - P(ax) = f, \quad y - Q(ya) = g$$

is uniquely solvable in $\mathcal{A}$.

This theorem and its proof is completely algebraically, so the proof given in [6] still holds for the case of strong algebras.

Let $\mathcal{A}$ be a decomposing strong algebra. For $a \in \mathcal{A}$ we define the operators T_a and R_a on $\mathcal{A}$ by

$$T_a(x) = P(ax) + Q(x), \quad R_a(x) = P(x) + Q(xa). \tag{6.1}$$

The result in Theorem 6.1 asserts that the element a admits a canonical factorization if and only if both T_a and R_a are invertible operators on $\mathcal{A}$.

Theorem 6.2. *Let $\mathcal{A}$ be a decomposing strong algebra $\mathcal{A}$ in which elements that have inverse on one side are invertible, and let $a \in \mathcal{A}$. If there is $\alpha \in A$ and $\beta \geq h(\alpha)$ such that*

$$A_{\beta,\alpha}\|1-a\|_\alpha < \Big(\max\{\|P\|_\beta^\beta, \|Q\|_\beta^\beta\}\Big)^{-1}$$

then a admits a canonical factorization $a = a_+a_-$, and the factors a_+, a_- may be chosen as $a_+ = x^{-1}$ and $a_- = y^{-1}$, where x,y are the solutions of

$$P(ax) + Q(x) = 1, \quad P(y) + Q(ya) = 1,$$

respectively.

Proof. Let T_a and R_a be the operators defined on $\mathcal{A}$ as in (6.1). Using the assumption, for any α such that $a \in X_\alpha$, and for any $\beta \geq h(\alpha)$, we obtain

$$\|(I-T_a)x\|_\beta = \|P((1-a)x)\|_\beta \leq \|P\|_\beta^\beta\|(1-a)x\|_\beta \leq \|P\|_\beta^\beta A_{\beta,\alpha}\|1-a\|_\alpha\|x\|_\beta,$$

and therefore,

$$\|I-T_a\|_\beta^\beta \leq A_{\beta,\alpha}\|1-a\|_\alpha\|P\|_\beta^\beta < 1.$$

Similarly

$$\|(I-R_a)x\|_\beta = \|Q(x(1-a))\|_\beta \leq \|Q\|_\beta^\beta\|x(1-a)\|_\beta \leq \|P\|_\beta^\beta A_{\beta,\alpha}\|1-a\|_\alpha\|x\|_\beta,$$

and therefore,

$$\|I-T_a\|_\beta^\beta \leq A_{\beta,\alpha}\|1-a\|_\alpha\|Q\|_\beta^\beta < 1.$$

Hence, by Proposition 3.10, T_a and R_a are invertible, so a admits a canonical factorization. By the proof of Theorem 6.1 (see [6]), it is obvious that the factors have the stated form. □

One principal result is the following corollary.

Corollary 6.3. *Let $\mathcal{A} = \bigcup X_\alpha$ be a strong algebra, and let*

$$\mathcal{U} = \left\{a(t) = \sum_{n=-\infty}^{\infty} a_n e^{int} : \sum_{n=-\infty}^{\infty} \|a_n\|_\alpha < \infty \text{ for some } \alpha\right\}$$

be the associated Wiener algebra. Denoting

$$\mathcal{U}^+ = \{a \in \mathcal{U} : a_n = 0, \forall n \geq 0\}, \quad \text{and} \quad \mathcal{U}_0^- = \{a \in \mathcal{U} : a_n = 0, \forall n < 0\}.$$

Then, any $a \in \mathcal{U}$, which is close enough to the identity, in the sense that there exists $\alpha \in A$ and $\beta \geq h(\alpha)$ such that

$$A_{\beta,\alpha}\|1 - a\|_\alpha < 1,$$

admits a canonical factorization with respect to $\mathcal{U}^+$ and $\mathcal{U}_0^-$.

Proof. Since the projections P of $\mathcal{U}$ onto $\mathcal{U}^+$, and Q of $\mathcal{U}$ onto $\mathcal{U}_0^-$, satisfy $\|P\|_\beta^\beta = \|Q\|_\beta^\beta = 1$, for all $\beta \in A$, the result follows from Theorem 6.2. $\square$

References

[1] D. Alpay and G. Salomon. New topological $\mathbb{C}$-algebras with applications in linear systems theory. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 15(2):1250011, 30, 2012.

[2] D. Alpay and G. Salomon. Non-commutative stochastic distributions and applications to linear systems theory. *Stochastic Process. Appl.*, 123(6):2303–2322, 2013.

[3] D. Alpay and G. Salomon. Topological convolution algebras. *J. Funct. Anal.*, 264(9):2224–2244, 2013.

[4] S. Bochner and R. S. Phillips. Absolutely convergent Fourier expansions for non-commutative normed rings. *Ann. of Math. (2)*, 43:409–418, 1942.

[5] N. Bourbaki. *Espaces vectoriels topologiques. Chapitres 1 à 5.* Masson, Paris, new edition, 1981. Éléments de mathématique. [Elements of mathematics].

[6] K. Clancey and I. Gohberg. *Factorization of matrix functions and singular integral operators*, volume 3 of *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel, 1981.

[7] N. Dinculeanu. *Vector measures.* Hochschulbücher für Mathematik, Band 64. VEB Deutscher Verlag der Wissenschaften, Berlin, 1966.

[8] I.M. Gelfand and G.E. Shilov. *Generalized functions. Volume 2.* Academic Press, 1968.

[9] I. Gohberg, S. Goldberg, and M.A. Kaashoek. *Classes of linear operators. Vol. II*, volume 63 of *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel, 1993.

[10] H. Holden, B. Øksendal, J. Ubøe, and T. Zhang. *Stochastic partial differential equations.* Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1996.

[11] I. E. Leonard. Banach sequence spaces. *J. Math. Anal. Appl.*, 54(1):245–265, 1976.

[12] M. A. Naĭmark. *Normed algebras.* Wolters-Noordhoff Publishing, Groningen, third edition, 1972. Translated from the second Russian edition by Leo F. Boron, Wolters-Noordhoff Series of Monographs and Textbooks on Pure and Applied Mathematics.

[13] H.H. Schaefer. *Topological vector spaces.* Springer, New York, 1986.

[14] N. Wiener. Tauberian theorems. *Ann. of Math.*, 33:1–100, 1932.

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