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REALIZATIONS OF INFINITE PRODUCTS, RUELLE OPERATORS AND WAVELET FILTERS

DANIEL ALPAY, PALLE JORGENSEN, AND IZCHAK LEWKOWICZ

Contents

Abstract. Using the system theory notion of state-space realization of matrix-valued rational functions, we describe the Ruelle operator associated with wavelet filters. The resulting realization of infinite products of rational functions have the following four features: 1) It is defined in an infinite-dimensional complex domain. 2) Starting with a realization of a single rational matrix-function M, we show that a resulting infinite product realization obtained from M takes the form of an (infinite-dimensional) Toeplitz operator with the symbol that is a reflection of the initial realization for M. 3) Starting with a subclass of rational matrix functions, including scalar-valued ones corresponding to low-pass wavelet filters, we obtain the corresponding infinite products that realize the Fourier transforms of generators of $\mathbf{L}_2(\mathbb{R})$ wavelets. 4) We use both the realizations for M and the corresponding infinite product to obtain a matrix representation of the Ruelle-transfer operators used in wavelet theory. By "matrix representation" we refer to the slanted (and sparse) matrix which realizes the Ruelle-transfer operator under consideration.

1. Introduction

Among many applications of rational matrix-valued functions are their use as filters in signal processing, and in the construction of classes of wavelets. In the latter case, the matrix function to be considered is made up of a prescribed system of scalar valued functions of a single complex variable. If N is a scaling number for the wavelet under consideration, then there are associated systems of N scalar-valued functions representing each of the corresponding N frequency bands. Each such system produces a matrix-valued function. This particular approach to wavelet filters was considered in [9, 8, 5, 2, 11, 1]. The function corresponding to low-pass yields a father function for a wavelet when certain technical assumptions are imposed. Here we consider instead

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the matrix-valued approach: it has the advantage that it allows one to treat the combination of individual bands in a single analysis. However the issues involving infinite products in the matrix-valued case are more subtle, and we address them below. For example, to understand the infinite product formed from a rational matrix-valued function of a single complex variable, one must introduce an infinite number of complex variables. We show that, under suitable assumptions, the infinite product-function in turn then also has a realization as a function of one variable. While our motivation derived initially from the study of wavelet filters, we note that there is a host of other applications of infinite products of rational matrix functions. Indeed, the framework for our consideration of infinite products goes beyond that of wavelet filters. We shall consider these more general settings in the last section of our paper. The latter non-wavelet applications are derived from the theory of systems. Indeed the theory of realization of systems is also a key tool in our analysis of infinite products.

In [2, Section 4], we characterized wavelet filters as functions of the form

$$M(z)=QU(z^N)\Delta(z)V \tag{1.1}$$

where

$$V=\tfrac{1}{\sqrt{N}}\left(\epsilon_N^{-\ell j}\right)_{\ell,j=0,\ldots,N-1}\qquad\qquad \epsilon_N:=e^{i\frac{2\pi}{N}},$$

is (up to scaling) the usual discrete Fourier transform matrix,

$$\Delta(z):=\begin{pmatrix}1&&&\\&z^{-1}&&\\&&\ddots&\\&&&z^{-(N-1)}\end{pmatrix},$$

U is a rational $(N\times N)$-valued function which takes unitary values on the unit circle, with no poles outside the closed unit disk, and and Q is an arbitrary (constant) unitary matrix. One can explicitly write (1.1) as

$$M(z)=\tfrac{1}{\sqrt{N}}\begin{pmatrix}m_0(z)&m_0(\epsilon_N z)&\cdots&m_0(\epsilon_N^{N-1}z)\\ m_1(z)&m_1(\epsilon_N z)&\cdots&m_1(\epsilon_N^{N-1}z)\\ \vdots&&&\\ m_{N-1}(z)&m_{N-1}(\epsilon_N z)&\cdots&m_{N-1}(\epsilon_N^{N-1}z)\end{pmatrix}. \tag{1.2}$$

Note that $M(z)$ in (1.1) is unitary on $\mathbb{T}$.

An earlier relevant result on wavelet filter (1.1), (1.2) appeared in [11]. In [3] and [4], we recently further explored rectangular rational functions which are (co)isometric on the unit circle (with poles anywhere, but $\mathbb{T}$).

Following (1.2) one can write,

$$\begin{pmatrix}m_0(z)\\ m_1(z)\\ \vdots\\ m_{N-1}(z)\end{pmatrix}=QU(z^N)\begin{pmatrix}1\\ z^{-1}\\ \vdots\\ z^{-(N-1)}\end{pmatrix}. \tag{1.3}$$

In the sequel, by choosing in (1.1)

$$Q=\left(U(1)\Delta(1)V\right)^*=\left(U(1)V\right)^* \tag{1.4}$$

we shall normalize the filters so that in (1.1)

$$M(1) = I_N \ .$$

This normalization in particular forces that the upper left entry of M to satisfy

$$m_0(1) = 1.$$

This last condition is crucial to consider infinite products. For $m_0(z)$ in (1.2) we set:

$$m(z) := m_0(z). \tag{1.5}$$

The wavelet father function $\varphi(w)$ is given by its Fourier transform

$$\widehat{\varphi}(w) = \prod_{k=1}^{\infty} m(e^{\frac{2\pi i w}{N^k}}). \tag{1.6}$$

For details, see e.g. [8] and [9].

It should be pointed out that in some engineering circles functions of the form of (1.1) are referred to as (multi-resolution) *Filter Bank*, whose applications transcend wavelets, see e.g. [18], [19] and even go beyond signal processing, see e.g. [16].

Here we limit our discussion to discrete time systems, where the variables, input, output, and state, are time series, that is, functions on $\mathbb{Z}$. A linear time-invariant system in this model will then be specified by a transfer matrix $M(z)$, also called a transfer function. It is a rational matrix-valued function of a single complex variable. Moreover the complex variable z is dual to time, and so it represents frequency. If $M(z)$ has no pole at infinity (following engineering literature), a corresponding state space realization is any quadruple of matrices A, B, C, and D of appropriate size, such that

$$M(z) = D + C(zI - A)^{-1}B$$

holds.

We assume that M in (1.1), (1.2) is a matrix-valued rational function analytic at infinity, while for $m(z)$, its upper left entry, we introduce a state space realization

$$m(z) = D + C(zI - A)^{-1}B \tag{1.7}$$

where we can assume that the realization is minimal (that is, the size of A is the smallest possible one), and that in particular A has no spectrum on the unit circle since M is analytic on the unit circle. Our use of the term realization conforms to its common use in the theory of systems from the study of dynamical systems and filters in engineering, and pioneered by Kalman and others; see [14, 15, 17]. Aspects of realization of filter bank as in (1.1), (1.2) were already addressed in [11], [18] and [19].

The paper consists of five sections besides the introduction, and its outline is as follows. In Section 2 we introduce state-space realization formulas of finite products of rational functions, each of a different variable. Infinite products are considered in Section 3. As we will see in that section an important role is played by the Toeplitz operator with the related symbol equal to

$$A + zB(I - zD)^{-1}C. \tag{1.8}$$

In Section 4, we compute the Markov parameters associated with $|m(z)|^2$ in terms of the given realization of m. In Section 5, we study the Ruelle operator and connections with rational wavelet filters are studied in Section 6.

2. Finite products

As is well known, see e.g [6], [14, Section 6.4], [15], [17, Section 6.5], every $(p\times q)$-valued rational function $R(z)$ analytic at infinity can be written as

$$R(z) = D + C(zI - A)^{-1}B, \tag{2.1}$$

for matrices A, B, C and D of appropriate sizes. Equation (2.1) is called a realization of R and we shall sometimes use the abbreviated form

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right). \tag{2.2}$$

In general, the realization is highly non unique. When the dimension of the state space, (i.e. the upper left block A is say $d \times d$), is *minimal*, the realization is unique up to a similarity matrix, meaning that the only freedom in the choice of the realization is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} T & 0 \\ 0 & I_p \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} T^{-1} & 0 \\ 0 & I_q \end{pmatrix}, \tag{2.3}$$

where $T \in \mathbb{C}^{d\times d}$ is an arbitrary invertible matrix and where D is assumed to belong to $\mathbb{C}^{p\times q}$. An important formula for the realization of the product is given in the next lemma:

Lemma 2.1. *Let R_1 and R_2 be two matrix-valued rational functions analytic at infinity, and with realizations*

$$R_1(z) = D_1 + C_1(zI_{n_1} - A_1)^{-1}B_1 \quad \textit{and} \quad R_2(z) = D_2 + C_2(zI_{n_2} - A_2)^{-1}B_2.$$

Assume that the product R_1R_2 makes sense. Then a realization of $R_1(z)R_2(z)$ is compactly given by:

$$\left(\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right) \tag{2.4}$$

An important tool in our argument is the counterpart of (2.4) when each function depends on a *different variable*. See Lemma 2.3.

Factorization of rational matrix-valued functions of one variable is classical. In contrast, factorization theory of rational functions of several complex variables $z_1, \ldots, z_u$ is not well developed. However, here we consider matrix-valued rational functions of the form

$$M(z) = M_1(z_1)M_2(z_2)\cdots M_u(z_u) \tag{2.5}$$

where $M_1, \ldots, M_u$ are matrix-valued rational functions of appropriate sizes and analytic at infinity.

For future reference we mention the following result, whose proof is a direct verification, and will be omitted.

Lemma 2.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and let $a : \mathcal{H}_1 \to \mathcal{H}_1$, $b : \mathcal{H}_2 \to \mathcal{H}_2$, and $c : \mathcal{H}_2 \to \mathcal{H}_1$ be bounded linear operators, with a and b invertible. Then*

$$\begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & a^{-1}cb^{-1} \\ 0 & b^{-1} \end{pmatrix}.$$

The following very simple lemma is the key to the formulas we develop:

Lemma 2.3.

$$\begin{aligned} \left(D_1 + C_1(z_1 I_{n_1} - A_1)^{-1}B_1\right)\left(D_2 + C_2(z_2 I_{n_2} - A_2)^{-1}B_2\right) = \\ = D + C(\Lambda(z) - A)^{-1}B \end{aligned} \tag{2.6}$$

where,

$$\Lambda(z) := \begin{pmatrix} z_1 I_{n_1} & \\ & z_2 I_{n_2} \end{pmatrix} \tag{2.7}$$

and the realization array (2.2) *takes the form,*

$$\left(\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right) \tag{2.8}$$

Proof. We first note that by Lemma 2.2 we have:

$$\begin{aligned} (\Lambda(z) - A)^{-1} &= \begin{pmatrix} z_1 I_{n_1} - A_1 & -B_1C_2 \\ 0 & z_2 I_{n_2} - A_2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} (z_1 I_{n_1} - A_1)^{-1} & (z_1 I_{n_1} - A_1)^{-1}B_1C_2(z_2 I_{n_2} - A_2)^{-1} \\ 0 & (z_2 I_{n_2} - z_2 A_2)^{-1} \end{pmatrix}. \end{aligned}$$

Therefore we have

$$\begin{aligned} &D + C(\Lambda(z) - A)^{-1}B = D_1D_2 + \\ &+ \begin{pmatrix} C_1 & D_1C_2 \end{pmatrix} \begin{pmatrix} (z_1 I_{n_1} - z_1 A_1)^{-1} & (z_1 I_{n_1} - A_1)^{-1}B_1C_2(z_2 I_{n_2} - A_2)^{-1} \\ 0 & (z_2 I_{n_2} - A_2)^{-1} \end{pmatrix} \begin{pmatrix} B_1D_2 \\ B_2 \end{pmatrix} \end{aligned}$$

which is exactly the left side of (2.6). □

This formula can now be iterated to obtain a realization for a product (2.5). With

$$M_j(z_j) = D_j + C_j(z_j I_{n_j} - A_j)^{-1}B_j, \quad j = 1, \ldots u,$$

and $M(z) = M_1(z_1)M_2(z_2)\cdots M_u(z_u)$ we have

$$M(z) = D + C(\Lambda(z) - A)^{-1}B,$$

where

$$\Lambda(z) := \begin{pmatrix} z_1 I_{n_1} & & & \\ & z_2 I_{n_2} & & \\ & & \ddots & \\ & & & z_u I_{n_u} \end{pmatrix}$$

where $M(\infty)=\lim_{z\to\infty}M(z)$ and the realization array (2.2) takes the form,

$$\left(\begin{array}{cccccc|c} A_1 & B_1C_2 & B_1D_2C_3 & \cdots & B_1D_2\cdots D_{u-2}C_{u-1} & B_1D_2\cdots D_{u-1}C_u & B_1D_2\cdots D_u \\ & A_2 & B_2C_3 & \cdots & B_2D_3\cdots D_{u-2}C_{u-1} & B_2D_3\cdots D_{u-1}C_u & B_2D_3\cdots D_u \\ & & & \ddots & & & \vdots \\ & & & & A_{u-1} & B_{u-1}C_u & B_{u-1}D_u \\ & & & & & A_u & B_u \\ \hline C_1 & D_1C_2 & D_1D_2C_3 & \cdots & D_1\cdots D_{u-2}C_{u-1} & D_1\cdots D_{u-1}C_u) & D_1\cdots D_u \end{array}\right) \tag{2.9}$$

We note that the case where all the functions vanish at infinity, i.e. where $\lim_{z\to\infty}M(z)=0$, leads to very simple formulas, which we gather in the following lemma.

Lemma 2.4. *It holds that*

$$C_1(z_1I_{n_1}-A_1)^{-1}B_1C_2(z_2I_{n_2}-A_2)^{-1}B_2\cdots C_u(z_uI_{n_u}-A_u)^{-1}B_u=C(\Lambda(z)-A)^{-1}B,$$

where $\Lambda(z)$ *is as in* (2.7) *and the realization array* (2.2) *takes the form,*

$$\left(\begin{array}{cccccc|c} A_1 & B_1C_2 & 0 & \cdots & & 0 & 0 \\ & A_2 & B_2C_3 & 0 & \cdots & 0 & 0 \\ & & & \ddots & & & \vdots \\ & & & & A_{u-1} & B_{u-1}C_u & 0 \\ & & & & & A_u & B_u \\ \hline C_1 & 0 & & \cdots & & 0 & 0 \end{array}\right).$$

The significance of the realizations in (2.9) and in Lemma 2.4, goes beyond the scope of this work. In the sequel, we actually exploit a special case of it. See also Remark 6.4

3. Infinite products

While the framework of realizations is typically formulated for finite matrices, (as we point out below) a number of the results make sense for infinite matrices, hence for linear operators in Hilbert space. A case in point is the realizations we obtain now for our infinite products. We now wish to let $u\to\infty$ in (2.9) when all the functions R_j coincide:

$$R_1(z)=R_2(z)=\cdots=M(z),$$

where $M(z)$ is a matrix-valued rational function, analytic at infinity, with realization

$$M(z)=D+C(zI-A)^{-1}B. \tag{3.1}$$

We assume that $1\geq\|M(z)\|$ for all $z\in\mathbb{T}$.

Theorem 3.1. *Given a square* $M(z)$ *in* (3.1).
(*i*) *Assume that*

$$\lim_{k\to\infty}\|D^k\|^{1/k}<1. \tag{3.2}$$

Then, the operators

$$\mathscr{A} = \begin{pmatrix} A & BC & BDC & BD^2C & \cdots \\ 0 & A & BC & BDC & \cdots \\ 0 & 0 & A & BC & \cdots \\ & & & & \end{pmatrix}, \quad \ell_2(\mathbb{N})\otimes\mathbb{C}^m \Longrightarrow \ell_2(\mathbb{N})\otimes\mathbb{C}^m \tag{3.3}$$

$$\mathscr{B} = \begin{pmatrix} \vdots \\ BD^2 \\ BD \\ B \end{pmatrix}, \qquad \mathbb{C}^m \longrightarrow \ell_2(\mathbb{N})\otimes\mathbb{C}^m, \tag{3.4}$$

$$\mathscr{C} = \begin{pmatrix} C & DC & D^2C & \cdots \end{pmatrix}, \qquad \ell_2(\mathbb{N})\otimes\mathbb{C}^m \longrightarrow \mathbb{C}^m, \tag{3.5}$$

are bounded.
(ii) $\mathscr{A}$ *in* (3.3) *is the block-Toeplitz operator with symbol*

$$A + zB(I - zD)^{-1}C.$$

Proof. We have

$$A + zB(I - zD)^{-1}C = A + zBC + z^2BDC + z^3BD^2C + \cdots,$$

and hence the function $\phi(z) = A + zB(I - zD)^{-1}C$ is the symbol of the block Toeplitz operator (3.3). We note that, in view of (3.2)

$$\|\mathscr{A}\| = \|A + zB(I - zD)^{-1}C\|_\infty < \infty, \tag{3.6}$$

and so the block Toepliz operator $\mathscr{A}$ is bounded. We use the fact that a block Toeplitz operator with symbol $\phi(z)$ has norm $\|\phi\|_\infty$; see [12]. □

Remark 3.2. The same result holds *mutatis mutandis* when instead of (3.1), the alternative realization $M(z) = D + zC(I - zA)^{-1}B$ is chosen.

Remark 3.3. Recall that

$$\mathscr{B} = \mathcal{O}(D, B) \quad \text{and} \quad \mathscr{C} = \mathcal{C}(D, C)$$

in (3.4) and (3.5) are the observability and controllability operators (see e.g. [14, Section 6.2], [17])

Theorem 3.4. *Assume that* $M(z)$ *in* (3.1) *has no singularity at the point* $z = 1$ *and that* $M(1) = I$*, and let* $(z_k)_{k\in\mathbb{N}_0}$ *be a sequence of complex numbers which are not poles of* M *and such that*

$$\sum_{k=0}^{\infty} |1 - z_k| < \infty. \tag{3.7}$$

Then it holds that

$$\prod_{k=1}^{\infty} M(z_k) = \mathscr{C}(\Lambda(z) - \mathscr{A})^{-1}\mathscr{B} \tag{3.8}$$

where $\mathscr{A}, \mathscr{B}$ *and* $\mathscr{C}$ *are defined by* (3.3)-(3.5)*, and where*

$$\Lambda(z) = \begin{pmatrix} z_1 I_n & & \\ & z_2 I_n & \\ & & \ddots \end{pmatrix}.$$

Proof. Since the realization of M is assumed minimal, 1 is not in the spectrum of A and we have $M(1) = D + C(I_n - A)^{-1}B$. Thus

$$\begin{aligned} M(z) - M(1) &= D + C(zI_n - A)^{-1}B - D - C(I_n - A)^{-1}B \\ &= C(zI_n - A)^{-1}B - C(I_n - A)^{-1}B \\ &= (1-z)C(zI - A)^{-1}(I - A)^{-1}B. \end{aligned}$$

Furthermore, (3.7) implies in particular that $\lim_{k\to\infty} z_k = 1$. Let

$$K_0 = \max_{z\in V} \left\{ \|C(zI_n - A)^{-1}(I_n - A)^{-1}B\| \right\},$$

where V is a closed neighborhood of 1 in which m has no pole, and let

$$K = \max_{\text{where the } z_u \notin V} \left\{ K_0, \|C(z_uI_n - A)^{-1}(I_n - A)^{-1}B\| \right\}$$

Then we have

$$\|M(z_k) - I\| \le K \cdot |1 - z_k|.$$

Therefore the series $\sum_{k=1}^\infty \|I - M(z_k)\|$ and hence the product $\prod_{k=1}^\infty M(z_k)$ are convergent. The equality (3.8) is now easy to verify. □

Corollary 3.5. *Assume that $M(1) = I$, and let (θ_k) be a sequence of numbers on the real line such that*

$$\sum_{k=0}^\infty |\theta_k| < \infty.$$

Then the infinite product

$$\prod_{k=0}^\infty M(e^{it\theta_k}), \quad t \in \mathbb{R},$$

converges for all real t.

Proof. Since $|e^{i\theta} - 1| \le |\theta|$ for θ real, we have

$$\|M(e^{it\theta_k}) - I\| \le K_1 \cdot |\theta_k|$$

where now we can take

$$K_1 = \max_{\theta\in[0,2\pi]} \|C(e^{i\theta}I_n - A)^{-1}(I_n - A)^{-1}B\|,$$

and hence the result. □

Corollary 3.6. *In the notation and hypothesis of Corollary 3.5, the product*

$$\prod_{k=0}^\infty M(e^{i\frac{2\pi w}{N^k}})$$

converges for every $w \in \mathbb{R}$.

4. Markov parameters

Let M be analytic in the exterior and on the boundary of the unit disk, with minimal realization $M(z) = D + C(zI - A)^{-1}B$. In particular $\sigma(A) \subset \mathbb{D}$. Let $h_0 + \sum_{k=1}^{\infty} \frac{h_k}{z^k}$ be the Laurent expansion at infinity of M. The coefficients $h_0, h_1, \ldots$ are called the Markov parameters of M, see e.g. [14, Subsections 5.1.2, 6.2.1], [17, Section 6.5]. They are given by $h_0 = D$ and

$$h_k = CA^{k-1}B, \quad k = 1, 2, \ldots \tag{4.1}$$

We extend the sequence h_k by

$$h_u = 0, \quad u < 0.$$

Since the spectral radius of A is strictly less than 1, we can set

$$\Gamma = \sum_{u=0}^{\infty} A^{*u} C^* C A^u. \tag{4.2}$$

Note that Γ is called the observability Gramian, and is the unique solution of the Stein equation

$$\Gamma - A^* \Gamma A = C^* C.$$

We set

$$Y = D^* C + B^* \Gamma A. \tag{4.3}$$

In view of the next result we recall that a rational function r with no poles on the unit circle belongs to the Wiener algebra of the disk, that is, can be written as

$$r(z) = \sum_{n \in \mathbb{Z}} z^n r_n,$$

where $\sum_{n \in \mathbb{Z}} |r_n| < \infty$. See for instance [13, Corollary 3.2].

Theorem 4.1. *Let $(c_n)_{n \in \mathbb{Z}}$ be defined by*

$$\|M(z)\|^2 = \sum_{n \in \mathbb{Z}} c_n z^n, \quad z \in \mathbb{T}. \tag{4.4}$$

Then

$$c_n = \sum_{j \in \mathbb{Z}} h_j^* h_{j+n}, \quad n \in \mathbb{Z}.$$

Therefore,

$$c_n = \begin{cases} B^* A^{*(-n-1)} Y^*, & n < 0, \\ D^* D + B^* \Gamma B, & n = 0, \\ Y A^{n-1} B, & n > 0. \end{cases} \tag{4.5}$$

Proof. For $n = 0$ we have

$$\begin{aligned} c_0 &= D^* D + \sum_{k=1}^{\infty} B^* A^{*(k-1)} C^* C A^{k-1} B \\ &= D^* D + B^* \left(\sum_{k=1}^{\infty} A^{*(k-1)} C^* C A^{k-1} \right) B \\ &= D^* D + B^* \Gamma B, \end{aligned}$$

where Γ is the Gramian matrix from (4.2).

We now assume $n > 0$. Then,

$$\begin{aligned} c_n &= h_0^* h_n + \sum_{k=1}^{\infty} h_k^* h_{k+n} \\ &= D^* C A^{n-1} B + \sum_{k=1}^{\infty} B^* A^{k-1} C^* C A^{k+n-1} B \\ &= D^* C A^{n-1} B + B^* \Gamma A^n B. \end{aligned}$$

Finally, for $n < 0$, we have:

$$\begin{aligned} c_n &= \sum_{k=-n}^{\infty} h_k^* h_{k+n} \\ &= \sum_{u=0}^{\infty} h_{u-n}^* h_u \\ &= B^* A^{*(-n-1)} C^* D + \sum_{u=1}^{\infty} B^* A^{*(u-n-1)} C^* C A^{u-1} B \\ &= B^* A^{*(-n-1)} C^* D + B^* A^{*(-n)} \Gamma B. \end{aligned}$$

□

Corollary 4.2. *Let d be the size of A (that is, $A \in \mathbb{C}^{d\times d}$). Let Y be defined by* (4.3) *and*

$$\mathcal{C}(A,B) = \begin{pmatrix} B & AB & A^2B & \cdots & A^{d-1}B \end{pmatrix}.$$

Then

$$c_n = YA^{n-1}B, \quad n = 1, \ldots \tag{4.6}$$

and

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_d \end{pmatrix} = Y \cdot \mathcal{C}(A,B). \tag{4.7}$$

Theorem 4.3. *There exists complex numbers $a_0, \ldots, a_{d-1}$ such that*

$$a_0c_1 + a_1c_2 + \cdots + a_{d-1}c_d + c_{d+1} = 0, \tag{4.8}$$

and more generally, for any $p \geq 1$,

$$a_0c_p + a_1c_{p+1} + \cdots + a_{d-1}c_{d+p-1} + c_{d+p} = 0. \tag{4.9}$$

Proof. By the Cayley-Hamilton theorem there exists numbers $a_0, a_1, \ldots, a_{d-1}$ such that

$$a_0 + a_1A + a_2A^2 + \cdots + a_{d-1}A^{d-1} + A^d = 0.$$

It then follows from (4.6) that we have (4.8), and more generally (4.9). □

Formulas (4.5) take a simpler form in a number of cases, which we mention as remarks:

Remark 4.4. If A is nilpotent (and then m is a polynomial in $1/z$) we have

$$c_{d+1} = c_{d+2} = \cdots = 0.$$

Remark 4.5. Assuming that $D=0$ implies that in (4.3) $Y=D^*C$ and (4.5) becomes,

$$c_n = \begin{cases} B^*A^{*(-n)}\Gamma B, & n\le 0,\\ B^*\Gamma A^nB, & n>0.\end{cases} \tag{4.10}$$

Remark 4.6. We recall that the observability Gramian Γ, see (4.2), is invertible if and only if the pair (C,A) is observable, meaning that

$$\cap_{u=1}^{d-1}\ker A^u=\{0\}$$

Then one assume $\Gamma=I_d$ by taking $T=\Gamma^{1/2}$ as similarity matrix in (2.3). When furthermore $D=0$ we then have

$$c_n = \begin{cases} B^*A^{*(-n)}B, & n\le 0,\\ B^*A^nB, & n>0.\end{cases} \tag{4.11}$$

Proposition 4.7. *Assume M is rational. Then the coefficients c_n satisfy the estimates of the form*

$$|c_k|\le Ce^{-\alpha|k|},\quad k\in\mathbb{Z}. \tag{4.12}$$

for every $\alpha>\rho(A)$, where $\rho(A)$ is the spectral radius of A.

5. Ruelle operator

Using $m(z)$ from (1.5), (1.7), the associated Ruelle operator (or transfer operator) is defined by (see [8, §3.2])

$$(Rf)(z)=\tfrac{1}{N}\sum_{\substack{w\in\mathbb{T}\\ w^N=z}}|m(w)|^2f(w). \tag{5.1}$$

See [8, p. 156]. In terms of the coefficients (4.4) in Theorem 4.1 it is the operator between appropriate subspaces of $\ell_2(\mathbb{Z})$ and with matrix representation

$$r_{\ell,j}=\tfrac{1}{N}c_{N\ell-j},$$

that is

$$(Rf)_\ell=\tfrac{1}{N}\sum_{k\in\mathbb{Z}}c_{N\ell-k}f_k. \tag{5.2}$$

Example 5.1. When $N=2$, $D=0$ and $\Gamma=I_d$ the matrix representation of the Ruelle operator is

$$\frac{1}{2}\begin{pmatrix}\cdots & & & & & & & \cdots\\ \cdots & B^*B & B^*AB & B^*A^{*2}B & B^*A^{*3}B & B^*A^{*4}B & & \cdots\\ \cdots & B^*A^2B & B^*AB & \boxed{B^*B} & B^*A^*B & B^*A^{*2}B & B^*A^{*3}B & \cdots\\ \cdots & B^*A^4B & B^*A^3B & B^*A^2B & B^*AB & B^*B & B^*A^*B & \cdots\\ \cdots & & & & & & & \cdots\end{pmatrix}$$

where the box denotes the $(0,0)$ element (see (4.11)).

Let

$$\mathscr{E}_r=\left\{(f_n)_{n\in\mathbb{Z}}\,;\ \|f\|_{r,1}\overset{\text{def.}}{=}\sum_{n\in\mathbb{Z}}e^{r|n|}|f_n|<\infty\right\}$$

and

$$\mathscr{E}_r^{(2)} = \left\{ (f_n)_{n\in\mathbb{Z}}\,;\ \|f\|_{r,2}^2 \overset{\text{def.}}{=} \sum_{n\in\mathbb{Z}} e^{r|n|}|f_n|^2 < \infty \right\}$$

See [20], [8, p. 158] for these last spaces. We note that an element of $\mathscr{E}_r$ satisfies an estimate of the form

$$|f_n| \le Ke^{-r|n|}, \quad n \in \mathbb{Z}, \tag{5.3}$$

for some $K > 0$.

Theorem 5.2. *Assume that the coefficients c_n satisfy the estimate* (4.12)*. Then for every choice of β and β' such that*

$$\alpha < \beta \quad \textit{and} \quad \beta' < N\alpha, \tag{5.4}$$

the Ruelle operator is continuous from $\mathscr{E}_\beta$ into $\mathscr{E}_{\beta'}$ and from $\mathscr{E}_\beta^{(2)}$ into $\mathscr{E}_{\beta'}^{(2)}$.

Proof. Let $f = (f_n)_{n\in\mathbb{Z}}$ be an element of $\mathscr{E}_\beta$, with β as in (5.4). The ℓ component of the vector Rf is given by (5.2),

$$(Rf)_\ell = \tfrac{1}{N}\sum_{k\in\mathbb{Z}} c_{N\ell-k}f_k$$

and, using (4.12) and (5.3) can be estimated as:

$$\begin{aligned} N|(Rf)_\ell| &\le \sum_{k\in\mathbb{Z}} |c_{N\ell-k}|\cdot|f_k| \\ &\le \sum_{k\in\mathbb{Z}} Ce^{-\alpha|N\ell-k|}\cdot Ke^{-\beta|k|} \\ &= \sum_{k\in\mathbb{Z}} CKe^{-\alpha(|N\ell-k|+|k|)}\cdot e^{(\alpha-\beta)|k|} \\ &\le CKe^{-\alpha|N\ell|}\sum_{k\in\mathbb{Z}} e^{(\alpha-\beta)|k|}. \end{aligned}$$

Hence for any $\beta' < N\alpha$,

$$\sum_{\ell\in\mathbb{Z}} |(Rf)_\ell)|\cdot e^{\beta'|\ell|} < \infty,$$

and so $Rf \in \mathscr{E}_{\beta'}$.

Let now $\epsilon > 0$ and $f, g \in \mathscr{E}_\beta$ and such that $\|f-g\|_{\beta,1} < \epsilon$. Then in a way similar to the above argument we write

$$\begin{aligned} N\|f-g\|_{\beta',1} &= \sum_{\ell\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |c_{N\ell-k}| \cdot |f_k - g_k| \right) \\ &\le \sum_{\ell\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} Ce^{-\alpha|N\ell-k|} \cdot e^{-\alpha|k|} \underbrace{e^{(\alpha-\beta')|k|}}_{\text{is } <1} e^{\beta'|k|} |f_k - g_k| \right) \\ &\le \sum_{\ell\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} Ce^{-\alpha|N\ell|} \cdot e^{\beta'|k|} |f_k - g_k| \right) \\ &= C \left(\sum_{\ell\in\mathbb{Z}} e^{-\alpha|N\ell|} \right) \cdot \|f-g\|_{\beta',1}. \end{aligned}$$

and continuity of R follows.

The case of the spaces $\mathscr{E}_\beta^{(2)}$ and $\mathscr{E}_{\beta'}^{(2)}$ is proved in the same way. □

For a result related to the following theorem in the non rational case, see [8, p. 158].

The first item in the next theorem is taken from [8, p.156-159], [10].

Theorem 5.3. *Let $m(z)$ be as in* (1.5), (1.7).
(1) *The Ruelle operator has finite trace, and its trace is given by the formula*

$$\operatorname{Tr} R = \tfrac{1}{N} \sum_{k\in\mathbb{Z}} c_{(N-1)k} = \tfrac{1}{N} \sum_{\substack{w\in\mathbb{T} \\ w^N=1}} |m(w)|^2.$$

(2) *In the rational case, we have*

$$\operatorname{Tr} R = \tfrac{1}{N} \left(DD^* + B^*\Gamma B + Y(I-A)^{-1}B^* + B(I-A^*)^{-1}Y^* \right).$$

Proof. The second item is a direct consequence of formulas (4.5) . □

6. Wavelets and rational filters

In this section, we show that starting from a rational wavelet filter, the infinite product (1.6) is indeed in $\mathbf{L}_2(\mathbb{R}, dx)$. To that purpose it is enough to prove that

$$R1 \le 1$$

for the corresponding Ruelle operator, where, by definition of R,

$$(R1)(z) = \tfrac{1}{N} \sum_{\substack{w\in\mathbb{T} \\ w^N=z}} |m(w)|^2,$$

with m from (1.5), (1.7).
Let M be a rational wavelet filter, that is a function of the form (1.1). Recall that its first column is given by (1.3).

Proposition 6.1. *It holds that*

$$R1 = 1 \tag{6.1}$$

Proof. From (1.3) and (1.4) we have that

$$m(z) = \sum_{j=1}^{N} z^{1-j}[W(z^N)]_{1j},$$

where $[W]_{jk}$ denotes the jk entry in a matrix W, with

$$W(z^N) := (U(1)V)^* U(z^N).$$

Thus, for $z \in \mathbb{T}$,

$$\begin{aligned}
\frac{1}{N}\sum_{\substack{\omega\in\mathbb{T}\\ \omega^N=z}} |m(\omega)|^2 &= \frac{1}{N}\sum_{\substack{\omega\in\mathbb{T}\\ \omega^N=z}} \left|\sum_{j=1}^{N}\omega^{1-j}\left[W(\omega^N)\right]_{1j}\right|^2\\
&= \frac{1}{N}\sum_{\substack{\omega\in\mathbb{T}\\ \omega^N=z}}\sum_{j,k=1}^{N}\omega^{-j+k}\left[W(z)\right]_{1j}\overline{\left[W(z)\right]_{1k}} \qquad \text{(note that } \omega^N=z)\\
&= \sum_{j,k=1}^{N}\delta_{jk}\left[W(z)\right]_{1j}\overline{\left[W(z)\right]_{1k}} \qquad \Big(\text{since } \frac{1}{N}\sum_{\substack{\omega\in\mathbb{T}\\ \omega^N=z}}\omega^{-j+k}=\delta_{jk}\Big)\\
&= \sum_{j=1}^{N}\left[W(z)\right]_{1j}\overline{\left[W(z)\right]_{1j}} = 1 \qquad \text{(since } W \text{ takes unitary values on } \mathbb{T}).
\end{aligned}$$

□

Theorem 6.2. *Let m be the upper left entry of a wavelet filter, see* (1.5), (1.7). *Then the infinite product* (1.6) *converges to an* $\mathbf{L}_2(\mathbb{R})$ *function.*

Proof. We proceed in a number of steps.

STEP 1: *The infinite product* (1.6) *converges pointwise.*

This follows from Corollary 3.5 with $\theta_k = \frac{2\pi w}{N^k}$ since $M(1) = I_N$ and hence $m(1) = 1$.

We now set

$$f_k(w) = 1_{[-\frac{N^k}{2},\frac{N^k}{2}]}(w)\prod_{\ell=0}^{k} m\left(e^{i\frac{2\pi w}{N^\ell}}\right), \quad k = 0, 1, \dots \tag{6.2}$$

The key to the proof is to establish the identity

$$\int_{-\frac{N^k}{2}}^{\frac{N^k}{2}} |f_k(w)|^2 dw = 1. \tag{6.3}$$

STEP 2: (6.3) *holds:*

Indeed,

$$\begin{aligned}\int_{-\frac{N^k}{2}}^{\frac{N^k}{2}} |f_k(w)|^2 dw &= \int_{-\frac{N^k}{2}}^{\frac{N^k}{2}} \left(\prod_{\ell=0}^{k} |m\left(e^{i\frac{2\pi w}{N^\ell}}\right)|^2\right) dw \\ &= \int_{-\frac{N^{k-1}}{2}}^{\frac{N^{k-1}}{2}} \left(\prod_{\ell=0}^{k-1} |m\left(e^{i\frac{2\pi w}{N^\ell}}\right)|^2\right) \underbrace{\left(\sum_{p=0}^{N-1} |m\left(e^{2\pi i\frac{w+pN^{k-1}}{N^k}}\right)|^2\right)}_{\text{This is } R1} dw \\ &= \int_{-\frac{N^{k-1}}{2}}^{\frac{N^{k-1}}{2}} \left(\prod_{\ell=0}^{k-1} |m\left(e^{i\frac{2\pi w}{N^\ell}}\right)|^2\right) dw \quad (\text{since } R1 = 1) \\ &= \int_{-\frac{N^{k-1}}{2}}^{\frac{N^{k-1}}{2}} |f_{k-1}(w)|^2 dw \quad (\text{by definition of } f_{k-1}\text{; see (6.2)}).\end{aligned}$$

So we have

$$\int_{-\frac{N^k}{2}}^{\frac{N^k}{2}} |f_k(w)|^2 dw = \int_{-\frac{N^{k-1}}{2}}^{\frac{N^{k-1}}{2}} |f_{k-1}(w)|^2 dw.$$

Iterating this equality we get to

$$\begin{aligned}\int_{-\frac{N^k}{2}}^{\frac{N^k}{2}} |f_k(w)|^2 dw = \int_{-\frac{N^{k-1}}{2}}^{\frac{N^{k-1}}{2}} |f_{k-1}(w)|^2 dw = \cdots \\ \cdots = \int_{-\frac{1}{2}}^{\frac{1}{2}} |f_0(w)|^2 dw = \int_{-\frac{1}{2}}^{\frac{1}{2}} |m(e^{2\pi i w})|^2 dw < \infty\end{aligned}$$

STEP 3: *The pointwise limit function* $f(w) = \prod_{\ell=1}^{\infty} m\left(e^{2\pi i\frac{w}{N^\ell}}\right)$ *belongs to* $\mathbf{L}_2(\mathbb{R})$.

Indeed, by Fatou's lemma,

$$\begin{aligned}\int_{\mathbb{R}} |f(w)|^2 dw &= \int_{\mathbb{R}} \lim_{k\to\infty} |f_k(w)|^2 dw \\ &= \int_{\mathbb{R}} \liminf_{k\to\infty} |f_k(w)|^2 dw \\ &\le \liminf_{k\to\infty} \int_{\mathbb{R}} |f_k(w)|^2 dw < \infty\end{aligned}$$

in view of the previous step. □

Remark 6.3. The preceding arguments hold still in the case $R1 \le 1$. This covers important cases of rational filters for which the function U in (1.1) is only contractive as opposed to unitary.

Remark 6.4. In future work we shall use the multivariable state space techniques introduced in Section 2 to extend our results to the multivariable case, including the case of an infinite number of variables, and to study connections with infinite dimensional analysis (see [7] for the latter).

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