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PONTRYAGIN DE BRANGES ROVNYAK SPACES OF SLICE HYPERHOLOMORPHIC FUNCTIONS

DANIEL ALPAY, FABRIZIO COLOMBO, AND IRENE SABADINI

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ABSTRACT. We study reproducing kernel Hilbert and Pontryagin spaces of slice hyperholomorphic functions which are analogs of the Hilbert spaces of analytic functions introduced by de Branges and Rovnyak. In the first part of the paper we focus on the case of Hilbert spaces, and introduce in particular a version of the Hardy space. Then we define Blaschke factors and Blaschke products and we consider an interpolation problem. In the second part of the paper we turn to the case of Pontryagin spaces. We first prove some results from the theory of Pontryagin spaces in the quaternionic setting and, in particular, a theorem of Shmulyan on densely defined contractive linear relations. We then study realizations of generalized Schur functions and of generalized Carathéodory functions.

1. INTRODUCTION

Functions s analytic in the open unit disk \mathbb{D} and contractive there, or equivalently such that the kernel

$$\frac{1 - s(z)s(w)^*}{1 - zw^*}$$

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is positive definite in \mathbb{D} , play an important role in operator theory, and their study is a part of a field called Schur analysis. The present work is a continuation of [2], and deals with various aspects of Schur analysis in the case of slice hyperholomorphic functions. To review the classical case, and to present the outline of the paper, we first recall a definition: A signature matrix is a matrix J (say with complex entries; in the sequel quaternionic entries will be allowed) which is both self-adjoint and unitary. We denote by $\text{sq}_- J$ the multiplicity (possibly equal to 0) of the eigenvalue -1 . Let now J_1 and J_2 be two signature matrices, belonging to $\mathbb{C}^{N \times N}$ and $\mathbb{C}^{M \times M}$ respectively, and assume that

$$\text{sq}_- J_1 = \text{sq}_- J_2.$$

Functions Θ which are $\mathbb{C}^{M \times N}$ -valued and meromorphic in \mathbb{D} , and such that the kernel

$$(1.1) \quad K_\Theta(z, w) = \frac{J_2 - \Theta(z)J_1\Theta(w)^*}{1 - zw^*}$$

has a finite number of negative squares in \mathbb{D} are called generalized Schur functions, and have been studied by Krein and Langer in a long series of papers; see for instance [37, 38, 40, 39, 41]. These authors consider also the case of operator-valued functions and other classes, in particular, kernels of the form

$$(1.2) \quad k_\varphi(z, w) = \frac{\varphi(z)J + J\varphi(w)^*}{1 - zw^*},$$

where φ is $\mathbb{C}^{N \times N}$ valued and analytic in a neighborhood of the origin, and $J \in \mathbb{C}^{N \times N}$ is a signature matrix, and the counterparts of these kernels when the open unit disk is replaced by the open upper half-plane. Meromorphic functions Θ for which the kernel (1.1) has a finite number of negative squares are called generalized Schur functions, and meromorphic functions Θ for which the kernel (1.2) has a finite number of negative squares are called generalized Carathéodory functions. Associated problems (such as realization and interpolation questions) have been studied extensively.

As mentioned above, a study of Schur analysis in the setting of slice hyperholomorphic functions has been initiated recently in [2], and it is the purpose of the present paper to continue this study. The paper [2] was set in the Hilbert spaces framework, and presented in particular the notions and properties of Schur multipliers, de Branges Rovnyak space, and coisometric realizations in the slice hyperholomorphic setting. In the first part of this work we also focus on the Hilbert space case, while in the second part we consider the case of indefinite inner product spaces.

To set the present work into perspective we recall that the theory of slice hyperholomorphic functions represents a novelty with respect to other theories of hyperholomorphic functions that can be defined in the quaternionic setting since it allows the definition the quaternionic functional calculus and its associated S -resolvent operator. The importance of the S -resolvent operator, in the context of this paper, is the definition of the quaternionic version of the operator $(I - zA)^{-1}$ that appears in the realization function $s(z) = D + zC(I - zA)^{-1}B$. It turns out that when A is a quaternionic matrix and p is a quaternion then $(I - pA)^{-1}$ has to be replaced by

$$(I - pA)^{-*} = (I - \bar{p}A)(|p|^2 A^2 - 2\text{Re}(p)A + I)^{-1}$$

which is equal to $p^{-1}S_R^{-1}(p^{-1}, A)$ where $S_R^{-1}(p^{-1}, A)$ is the right S -resolvent operator associated to the quaternionic matrix A .

Slice hyperholomorphic functions have two main formulations according to the fact that the functions we consider are defined on quaternions and are quaternion-valued, in this case the functions are called slice regular, see [32, 15, 19] or the functions are defined on the Euclidean space \mathbb{R}^{N+1} and have values in the Clifford Algebra \mathbb{R}_N and are also called slice monogenic functions, see [25, 26]. We also point out that there exists a non constant coefficients differential operator whose kernel contains slice hyperholomorphic functions defined on suitable domains, see [17].

Slice hyperholomorphicity has applications in operator theory: specifically, in the case of quaternions, it allows the definition of a quaternionic functional calculus, see e.g. [16, 18, 21]; while slice monogenic functions admit a functional calculus for n -tuples of operators, see [24, 20, 22]. The book [27] collects some of the main results on the theory of slice hyperholomorphic functions and the related functional calculi.

Finally we mention the paper [10, 11, 9], where Schur multipliers were introduced and studied in the quaternionic setting using the Cauchy-Kovalevskaya product and series of Fueter polynomials, and the papers [33, 45, 44], which treat various aspects of a theory of linear systems in the quaternionic setting. Our approach is quite different from the methods used there.

The paper consists of nine sections besides the introduction, and its outline is as follows: In Sections 2 and 3 we review some basic definitions on slice hyperholomorphic functions. In Section 4 we discuss the notion of multipliers in the case of reproducing kernel Hilbert spaces of slice hyperholomorphic functions. In Section 5 we discuss the Hardy space in the present setting, and introduce Blaschke products. Interpolation in the Hardy space is studied in Section 6. Sections 7-11 are in the setting of indefinite metric spaces. A number of facts on quaternionic Pontryagin spaces as well as a proof of a theorem of Shmulyan on relations are proved in Section 7. Negative squares are discussed in Section 8, while Section 9 introduces generalized Schur functions and discusses their realizations. We also consider in this section the finite dimensional case. Finally, we briefly discuss in Section 10 the case of generalized Carathéodory functions.

2. SLICE HYPERHOLOMORPHIC FUNCTIONS

In the literature there are several notions of quaternion valued hyperholomorphic functions. In this paper we consider a notion which includes power series in the quaternionic variable, the so-called slice regular or slice hyperholomorphic functions, see [27]. In order to introduce the class of slice hyperholomorphic functions, we fix some preliminary notations. By \mathbb{H} we denote the algebra of real quaternions $p = x_0 + ix_1 + jx_2 + kx_3$. A quaternion can also be written as $p = \text{Re}(p) + \text{Im}(p)$ where $x_0 = \text{Re}(p)$ and $ix_1 + jx_2 + kx_3 = \text{Im}(p)$ but also as $q = \text{Re}(p) + I_p|\text{Im}(p)|$ where $I_p = \text{Im}(p)/|\text{Im}(p)|$, as long as p is non real. The element I belongs to the 2-sphere

$$\mathbb{S} = \{p = x_1i + x_2j + x_3k : x_1^2 + x_2^2 + x_3^2 = 1\}$$

of unit purely imaginary quaternions.

Definition 2.1. Let $\Omega \subseteq \mathbb{H}$ be an open set and let $f : \Omega \rightarrow \mathbb{H}$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ passing through 1 and I and denote by $x + Iy$ an element in \mathbb{C}_I .

- (1) We say that f is a left slice regular function (or slice regular or slice hyperholomorphic) if, for every $I \in \mathbb{S}$, we have:

$$\frac{1}{2} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + Iy) = 0.$$

- (2) We say that f is right slice regular function (or right slice hyperholomorphic) if, for every $I \in \mathbb{S}$, we have

$$\frac{1}{2} \left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy) I \right) = 0.$$

Definition 2.2. The set of all elements of the form $\operatorname{Re}(p) + J|\operatorname{Im}(p)|$ when J varies in \mathbb{S} is denoted by $[p]$ and is called the 2-sphere defined by p .

The most important feature of slice hyperholomorphic functions is that, on a suitable class of open sets described below, they can be reconstructed by knowing their values on a complex plane \mathbb{C}_I by the so-called Representation Formula.

Definition 2.3. Let Ω be a domain in \mathbb{H} . We say that Ω is a slice domain (*s-domain* for short) if $\Omega \cap \mathbb{R}$ is non empty and if $\Omega \cap \mathbb{C}_I$ is a domain in \mathbb{C}_I for all $I \in \mathbb{S}$. We say that Ω is axially symmetric if, for all $p \in \Omega$, the 2-sphere $[p]$ is contained in Ω .

Theorem 2.4 (Representation Formula). Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric s-domain. Let f be a left slice regular function on $\Omega \subseteq \mathbb{H}$. Then the following equality holds for all $p = x + I_p y \in \Omega$:

$$(2.3) \quad f(p) = f(x + I_p y) = \frac{1}{2} [f(z) + f(\bar{z})] + \frac{1}{2} I_p I [f(\bar{z}) - f(z)],$$

where $z := x + Iy$, $\bar{z} := x - Iy \in \Omega \cap \mathbb{C}_I$. Let f be a right slice regular function on $\Omega \subseteq \mathbb{H}$. Then the following equality holds for all $p = x + I_p y \in \Omega$:

$$(2.4) \quad f(x + I_p y) = \frac{1}{2} [f(z) + f(\bar{z})] + \frac{1}{2} [f(\bar{z}) - f(z)] I I_p.$$

The Representation Formula allows to extend any function $f : \Omega \subseteq \mathbb{C}_I \rightarrow \mathbb{H}$ defined on an axially symmetric s-domain Ω and in the kernel of the corresponding Cauchy-Riemann operator to a function $f : \tilde{\Omega} \subseteq \mathbb{H} \rightarrow \mathbb{H}$ slice hyperholomorphic where $\tilde{\Omega}$ is the smallest axially symmetric open set in \mathbb{H} containing Ω . Using the above notations, the extension is obtained by means of the *extension operator*

$$(2.5) \quad \operatorname{ext}(f)(p) := \frac{1}{2} [f(z) + f(\bar{z})] + \frac{1}{2} I_p I [f(\bar{z}) - f(z)], \quad z, \bar{z} \in \Omega \cap \mathbb{C}_I, \quad p \in \tilde{\Omega}.$$

For example, in the case of the kernel associated to the Hardy space, the extension operator applied to the function $\sum_{n=0}^{\infty} z^n \bar{w}^n$ gives (see Proposition 5.3 in [2]):

Proposition 2.5. Let p and q be quaternionic variables. The sum of the series $\sum_{n=0}^{+\infty} p^n \bar{q}^n$ is the function $k(p, q)$ given by

$$(2.6) \quad k(p, q) = (1 - 2\operatorname{Re}(q)p + |q|^2 p^2)^{-1} (1 - pq) = (1 - \bar{p}\bar{q})(1 - 2\operatorname{Re}(p)\bar{q} + |p|^2 \bar{q}^2)^{-1}.$$

The kernel $k(p, q)$ is defined for all p outside the 2-sphere defined by $[q^{-1}]$ (or, equivalently, for all q outside the 2-sphere $[p^{-1}]$). Moreover:

- a) $k(p, q)$ is slice hyperholomorphic in p and right slice hyperholomorphic in \bar{q} ;
- b) $\overline{k(p, q)} = k(q, p)$.

The function $k(p, q)$ in the preceding proposition is positive definite, and is the reproducing kernel of the slice hyperholomorphic counterpart of the Hardy space $\mathbf{H}_2(\mathbb{B})$ of functions analytic in the open unit ball \mathbb{B} , see [2] and Section 5 below.

Remark 2.6. The two possible expressions for $k(p, q)$ given in (2.6) correspond to the left slice regular reciprocal of $1 - p\bar{q}$ in the variable p and to the right slice regular reciprocal in the variable \bar{q} , (see the discussion in [2, Proposition 5.3]) and these two reciprocals coincide. Thus, in the sequel, we will often write $(1 - p\bar{q})^{-\star}$ instead of $k(p, q)$.

Remark 2.7. Note that whenever a function $k(p, q)$ is slice hyperholomorphic in p and is Hermitian, then $k(p, q)$ is right slice hyperholomorphic in \bar{q} .

3. SLICE HYPERHOLOMORPHIC MULTIPLICATION

We recall that, given two left slice hyperholomorphic functions f, g , it is possible to introduce a binary operation called the \star -product, such that $f \star g$ is a slice hyperholomorphic function. Similarly, given two right slice hyperholomorphic functions, we can define their \star -product. When considering in same formula both the products, it may be useful to distinguish between them and in this case we will write \star_l or \star_r according to the fact that we are using the left or the right slice regular product. When there is no subscript, we will mean that we are considering the left \star -product.

Let $f, g : \Omega \subseteq \mathbb{H}$ be slice regular functions such that their restrictions to the complex plane \mathbb{C}_I can be written as $f_I(z) = F(z) + G(z)J$, $g_I(z) = H(z) + L(z)J$ where $J \in \mathbb{S}$, $J \perp I$. The functions F, G, H, L are holomorphic functions of the variable $z \in \Omega \cap \mathbb{C}_I$ and they exist by the splitting lemma, see [27], p. 117. The \star_l -product of f and g is defined as the unique left slice hyperholomorphic function whose restriction to the complex plane \mathbb{C}_I is given by

$$(3.1) \quad (F(z) + G(z)J) \star_l (H(z) + L(z)J) := (F(z)H(z) - G(z)\overline{L(\bar{z})}) + (G(z)\overline{H(\bar{z})} + F(z)L(z))J.$$

Pointwise multiplication and slice multiplication are different, but they can be related as in the following result, [27, Proposition 4.3.22]:

Proposition 3.1. *Let $U \subseteq \mathbb{H}$ be an axially symmetric s -domain, $f, g : U \rightarrow \mathbb{H}$ be slice hyperholomorphic functions and let us assume that $f(p) \neq 0$. Then*

$$(3.2) \quad (f \star g)(p) = f(p)g(f(p)^{-1}pf(p)),$$

for all $p \in U$.

Remark 3.2. The transformation $p \rightarrow f(p)^{-1}pf(p)$ is clearly a rotation in \mathbb{H} , since $|p| = |f(p)^{-1}pf(p)|$ and allows to rewrite the \star -product as a pointwise product. Note also that if $f \star g(p) = 0$ then either $f(p) = 0$ or $g(f(p)^{-1}pf(p)) = 0$.

As a consequence of Proposition 3.1 one has:

Corollary 3.3. *If $\lim_{r \rightarrow 1} |f(re^{I\theta})| = 1$, for all I fixed in \mathbb{S} , then*

$$\lim_{r \rightarrow 1} |f \star g(re^{I\theta})| = |g(e^{I'\theta})|,$$

where $\theta \in [0, 2\pi)$, and $I' \in \mathbb{S}$ depends on θ and f .

Proof. Set $b = f(re^{I\theta})$. We can write $b = Re^{J\alpha}$ for suitable R, J, α and, by hypothesis, we can assume that $b \neq 0$ when $r \rightarrow 1$, thus b^{-1} exists. We have

$$\begin{aligned} b^{-1}re^{I\theta}b &= e^{-J\alpha}(re^{I\theta})e^{J\alpha} = r(\cos \alpha - J \sin \alpha)(\cos \theta + I \sin \theta)(\cos \alpha + J \sin \alpha) \\ &= r(\cos \theta + I \cos^2 \alpha \sin \theta - JI \cos \alpha \sin \alpha \sin \theta + IJ \cos \alpha \sin \alpha \sin \theta - JIJ \sin^2 \alpha \sin \theta) \\ &= r(\cos \theta + \cos \alpha e^{-J\alpha} I \sin \theta + e^{-J\alpha} IJ \sin \alpha \sin \theta) \\ &= r(\cos \theta + e^{-J\alpha} I e^{J\alpha} \sin \theta) = r(\cos \theta + I' \sin \theta), \end{aligned}$$

where $I' = e^{-J\alpha} I e^{J\alpha}$. Then, the result immediately follows from the equalities:

$$\lim_{r \rightarrow 1} |f \star g(re^{I\theta})| = \lim_{r \rightarrow 1} |f(re^{I\theta})g(b^{-1}re^{I\theta}b)| = \lim_{r \rightarrow 1} |g(re^{I'\theta})| = |g(e^{I'\theta})|.$$

□

Given a left slice regular function f it is possible to construct its slice regular reciprocal, which is denoted by $f^{-\star}$. The general construction can be found in [27]. In this paper we will be in need of the reciprocal of a polynomial or a power series with center at the origin that can be described in the easier way illustrated below.

Definition 3.4. *Given $f(p) = \sum_{n=0}^{\infty} p^n a_n$, let us set*

$$f^c(p) = \sum_{n=0}^{\infty} p^n \bar{a}_n, \quad f^s(p) = (f^c \star f)(p) = \sum_{n=0}^{\infty} p^n c_n, \quad c_n = \sum_{r=0}^n a_r \bar{a}_{n-r},$$

where the series converge. The left slice hyperholomorphic reciprocal of f is then defined as

$$f^{-\star} := (f^s)^{-1} f^c.$$

In an analogous way one can define the right slice hyperholomorphic reciprocal $f^{-\star} := f^c (f^s)^{-1}$, of a right slice regular function $f(q) = \sum_n a_n q^n$. Note that the series f^s has real coefficients.

Remark 3.5. Let Ω be an axially symmetric open set. We recall that if f is left slice hyperholomorphic in $q \in \Omega$ then $\overline{f(q)}$ is right slice hyperholomorphic in \bar{q} . This fact follows immediately from $(\partial_x + I\partial_y)f_I(x + Iy) = 0$, since by conjugation we get $\overline{f_I(x + Iy)}(\partial_x - I\partial_y) = 0$ for all $I \in \mathbb{S}$.

Lemma 3.6. *Let Ω be an axially symmetric s -domain and let $f, g : \Omega \rightarrow \mathbb{H}$ be two left slice hyperholomorphic functions. Then*

$$\overline{f \star_l g} = \bar{g} \star_r \bar{f},$$

where \star_l, \star_r are the left and right \star -products with respect to q and \bar{q} , respectively.

Proof. Let $f_I(z) = F(z) + G(z)J$, $g_I(z) = H(z) + L(z)J$ be the restrictions of f and g to the complex plane \mathbb{C}_I , respectively. The functions F, G, H, L are holomorphic functions of the variable $z \in \Omega \cap \mathbb{C}_I$ which exist by the splitting lemma and J is an element in the sphere \mathbb{S} orthogonal to I . The \star_r -product of two right slice hyperholomorphic

functions \bar{g} and \bar{f} in the variable \bar{q} is defined as the unique right slice hyperholomorphic function whose restriction to a complex plane \mathbb{C}_I is given by

$$(\overline{H(z)} - J \overline{L(z)}) \star_r (\overline{F(z)} - J \overline{G(z)}) := (\overline{H(z)} \overline{F(z)} - L(\bar{z}) \overline{G(z)}) - J (\overline{L(z)} \overline{F(z)} + H(\bar{z}) \overline{G(z)}).$$

Thus, comparing with (3.1), it is clear that

$$\overline{f_I \star_l g_I} = \overline{g_I} \star_r \overline{f_I},$$

and the statement follows by taking the unique right slice hyperholomorphic extension. \square

Remark 3.7. For the sake of completeness, we adapt some of the previous definitions in the case we consider matrix valued functions. We will say that a real differentiable function $f : \Omega \subseteq \mathbb{H} \rightarrow \mathbb{H}^{N \times M}$ is left (resp. right) slice hyperholomorphic if and only if for any linear and continuous functional Λ acting on $\mathbb{H}^{N \times M}$, the function Λf is left (resp. right) slice hyperholomorphic in Ω . If, in particular, $\Omega = \mathbb{B}$, then it can be shown with standard techniques that f is left slice hyperholomorphic if and only if $f(p) = \sum_{n=0}^{\infty} p^n A_n$, where $A_n \in \mathbb{H}^{N \times M}$ and the series converges in \mathbb{B} . Let $f : \mathbb{B} \rightarrow \mathbb{H}^{N \times M}$, $g : \mathbb{B} \rightarrow \mathbb{H}^{M \times L}$ be left slice hyperholomorphic and let $f(p) = \sum_{n=0}^{\infty} p^n A_n$, $g(p) = \sum_{n=0}^{\infty} p^n B_n$. The \star -product of f and g is defined as $f \star g := \sum_{n=0}^{\infty} p^n C_n$ where $C_n = \sum_{r=0}^n A_r B_{n-r}$. Analogous definitions can be given in the case we consider right slice hyperholomorphic functions.

Remark 3.8. When considering the function $\sum_{n=0}^{\infty} p^n A^n$ where $A \in \mathbb{H}^{N \times N}$ and $|p| < 1/\|A\|$, or, more in general, A is a bounded right linear quaternionic operator from a quaternionic Hilbert space to itself, then $(I - pA)^{-\star}$ can be constructed using the functional calculus (see [2, Proposition 2.16]): it is sufficient to construct the right slice regular inverse of $1 - pq$ with respect to q and then substitute q by the operator A . Note that we write $(I - pA)^{-\star}$ using the symbol \star instead of \star_r for simplicity and the discussion in Remark 2.6 justifies this abuse of notation.

4. MULTIPLIERS IN REPRODUCING KERNEL HILBERT SPACES

In this section we study the multiplication operators and their adjoints, we show that positivity implies the slice hyperholomorphicity for a class of functions and, finally, we prove that if a kernel is positive and slice hyperholomorphic then the corresponding reproducing kernel Hilbert space consists of slice hyperholomorphic functions.

Let us begin by recalling the following definition, see [8]:

Definition 4.1. A quaternionic Hilbert space \mathcal{H} of \mathbb{H}^N -valued functions defined on an open set $\Omega \subseteq \mathbb{H}$ is called a reproducing kernel quaternionic Hilbert space if there exists a $\mathbb{H}^{N \times N}$ -valued function defined on $\Omega \times \Omega$ such that:

- (1) For every $q \in \Omega$ and $a \in \mathbb{H}^N$ the function $p \mapsto K(p, q)a$ belongs to \mathcal{H} .
- (2) For every $f \in \mathcal{H}$, $q \in \Omega$ and $a \in \mathbb{H}^N$

$$\langle f, K(\cdot, q)a \rangle_{\mathcal{H}} = a^* f(q).$$

The function $K(p, q)$ is called the reproducing kernel of the space. As observed in [8], Definition 4.1, one may ask the weaker requirement that \mathcal{H} is a quaternionic pre-Hilbert space. However, the next result proven in [8], guarantees that a reproducing kernel quaternionic pre-Hilbert space has a unique completion as a reproducing kernel quaternionic Hilbert space, which will be denoted by $\mathcal{H}(K)$.

Theorem 4.2. *Given an $\mathbb{H}^{N \times N}$ -valued function $K(p, q)$ positive on a set $\Omega \subset \mathbb{H}$, there exists a uniquely defined reproducing kernel quaternionic Hilbert space of \mathbb{H}^N -valued function defined on Ω and with reproducing kernel $K(p, q)$.*

Let us recall that $\mathcal{H}(K)$ is the completion of the linear span $\mathcal{H}^\circ(K)$ of functions of the form

$$(4.1) \quad p \mapsto K(p, q)a, \quad q \in \Omega, \quad a \in \mathbb{H}^N,$$

with the inner product

$$(4.2) \quad \langle K(\cdot, q)a, K(\cdot, s)b \rangle_{\mathcal{H}^\circ(K)} := b^* K(s, q)a.$$

Proposition 4.3. *Let ϕ be a slice hyperholomorphic function defined on an axially symmetric s -domain Ω and with values in $\mathbb{H}^{N \times M}$, and let $K_1(p, q)$ and $K_2(p, q)$ be positive definite kernels in Ω , respectively $\mathbb{H}^{M \times M}$ - and $\mathbb{H}^{N \times N}$ -valued, and slice hyperholomorphic in the variable p . Moreover,*

(1) *Assume that the slice multiplication operator*

$$M_\phi : \mathcal{H}(K_1) \rightarrow \mathcal{H}(K_2)$$

given by

$$M_\phi : f \mapsto \phi \star f$$

is continuous. Then, the adjoint operator is given by the formula:

$$M_\phi^*(K_2(\cdot, q)d) = K_1(\cdot, q) \star_r \phi^*(q)d.$$

(2) *The multiplication operator M_ϕ is bounded and with norm less or equal to k if and only if the function*

$$(4.3) \quad K_2(p, q) - \frac{1}{k^2} \phi(p) \star_l K_1(p, q) \star_r \phi(q)^*$$

is positive on Ω .

Proof. We compute the adjoint of the multiplication operator $M_\phi : \mathcal{H}(K_1) \rightarrow \mathcal{H}(K_2)$:

$$\begin{aligned} c^*(M_\phi^*(K_2(\cdot, q)d))(p) &= \langle M_\phi^*(K_2(\cdot, q)d), K_1(\cdot, p)c \rangle_{\mathcal{H}(K_1)} \\ &= \langle K_2(\cdot, q)d, \phi \star_l K_1(\cdot, p)c \rangle_{\mathcal{H}(K_2)} \\ &= \langle \phi \star_l K_1(\cdot, p)c, K_2(\cdot, q)d \rangle_{\mathcal{H}(K_2)}^* \\ &= (d^*(\phi(q) \star_l K_1(q, p))c)^* \\ &= c^*(\phi(q) \star_l K_1(q, p))^* d. \end{aligned}$$

Now observe that by Lemma 3.6 we have $(\phi(q) \star_l K_1(q, p))^* = K_1(p, q) \star_r \phi^*(q)$ and so

$$M_\phi^*(K_2(\cdot, q)d) = K_1(\cdot, q) \star_r \phi^*(q)d.$$

The positivity of (4.3) follows from the positivity of the operator $k^2 - M_\phi M_\phi^*$. Conversely, if (4.3) is positive, the standard argument shows that $\|M_\phi\| \leq k$. \square

Example 4.4. *Let us consider the case in which the kernel K is of the form*

$$K(p, q) = \sum_{n=0}^{\infty} p^n \bar{q}^n \alpha_n, \quad \alpha_n \in \mathbb{R}, \quad \forall n \in \mathbb{N}.$$

Then we have

$$\phi(p) \star_l K(p, q) = \sum_{n=0}^{\infty} p^n \phi(p) \bar{q}^n \alpha_n$$

and

$$(\phi(p) \star_l K(p, q))^* = \sum_{n=0}^{\infty} q^n \phi(p)^* \bar{p}^n \alpha_n,$$

from which we obtain

$$\phi(q) \star_l (\phi(p) \star_l K(p, q))^* = \phi(q) \star_l \sum_{n=0}^{\infty} q^n \phi(p)^* \bar{p}^n \alpha_n = \phi(q) \star_l K(q, p) \star_r \phi(p)^*.$$

Recall that we defined in [2] a Schur function to be a function S with values in $\mathbb{H}^{N \times M}$, slice hyperholomorphic in \mathbb{B} and such that the kernel

$$(4.4) \quad k_S(p, q) = \sum_{n=0}^{\infty} p^n (I_N - S(p)S(q)^*) \bar{q}^n = (I_N - S(p)S(q)^*) \star (1 - p\bar{q})^{-*}$$

is positive on \mathbb{B} . We will show in Theorem 4.6 below that the converse, i.e. that positivity forces hyperholomorphicity, is true for a subclass of slice hyperholomorphic functions. This subclass is denoted by \mathcal{N} and corresponds to those functions f such that $f : \mathbb{B} \cap \mathbb{C}_I \rightarrow \mathbb{C}_I$ for any $I \in \mathbb{S}$. For these functions the pointwise multiplication of f with a monomial of the form p^n is well defined and commutative since f takes the complex plane \mathbb{C}_{I_p} to itself and thus it behaves, on each plane, like a complex valued function.

We will be in need of the following preliminary result, see [8, Proposition 9.3].

Proposition 4.5. *Let K_1 and K_2 be two positive functions on a set Ω with values in $\mathbb{H}^{N \times N}$ and $\mathbb{H}^{M \times M}$, respectively. Let ϕ be a function defined on Ω and with values in $\mathbb{H}^{N \times M}$. The pointwise multiplication operator by ϕ is bounded and with norm less or equal to k if and only if the function*

$$(4.5) \quad K_2(p, q) - \frac{1}{k^2} \phi(p) K_1(p, q) \phi(q)^*$$

is positive on Ω .

Theorem 4.6. *Let $S : \mathbb{B} \rightarrow \mathbb{H}^{N \times M}$ be a function such that $S : \mathbb{B} \cap \mathbb{C}_I \rightarrow \mathbb{C}_I^{N \times M}$ for every $I \in \mathbb{S}$. The following are equivalent:*

- (1) *The function $\sum_{n=0}^{\infty} p^n (I_N - S(p)S(q)^*) \bar{q}^n$ is positive on \mathbb{B} .*
- (2) *The operator M_S is a contraction from $\mathbf{H}_2^M(\mathbb{B})$ to $\mathbf{H}_2^N(\mathbb{B})$.*
- (3) *S is a Schur function belonging to $\mathcal{N}(\mathbb{B})$.*

Proof. The equivalence between (1) and (2) follows as in [1] Theorem 2.6.3, and its proof is based on Proposition 4.5. Indeed, let us set in (4.5)

$$K_1(p, q) = I_M(1 - p\bar{q})^{-*}, \quad K_2(p, q) = I_N(1 - p\bar{q})^{-*}.$$

We have:

$$I_N(1 - p\bar{q})^{-*} - \frac{1}{k^2} S(p)(1 - p\bar{q})^{-*} S(q)^*$$

$$(4.6) \quad = I_N \sum_{n=0}^{\infty} p^n \bar{q}^n - \frac{1}{k^2} S(p) \left(\sum_{n=0}^{\infty} p^n \bar{q}^n \right) S(q)^*;$$

now observe that, by hypothesis, $S(p)$ commutes with p^n since S takes the complex plane \mathbb{C}_{I_p} to itself; similarly, $S(q)^*$ commutes with \bar{q}^n . So we obtain that (4.6) is equal to:

$$\sum_{n=0}^{\infty} p^n \left(I_N - \frac{1}{k^2} S(p) S(q)^* \right) \bar{q}^n.$$

Thus, if (1) holds then by Proposition 4.5 we conclude that M_S is a contraction from $\mathbf{H}_2^M(\mathbb{B})$ to $\mathbf{H}_2^N(\mathbb{B})$. Conversely, if (2) holds, then again Proposition 4.5 allows to conclude that (1) holds.

The implication (3) \Rightarrow (2) follows from the fact that S is a Schur function. We show that (2) implies (3). The function S is slice hyperholomorphic since $Sc \in \mathbf{H}_2^N(\mathbb{B})$ for any $c \in \mathbb{H}^M$. Observe that the function S is contractive since M_S^* acts as

$$M_S^* \left((1 - p\bar{q})^{-*} d \right) = (1 - p\bar{q})^{-*} S(q)^* d$$

and it is a contraction. \square

Definition 4.7. A subset Ω of \mathbb{B} is called a set of uniqueness if every slice hyperholomorphic function on \mathbb{B} which vanishes on Ω is identically zero on \mathbb{B} .

Example 4.8. Any open subset Ω of $\mathbb{B} \cap \mathbb{C}_I$ is a set of uniqueness. More in general, any subset Ω of $\mathbb{B} \cap \mathbb{C}_I$ for $I \in \mathbb{S}$ having an accumulation point in \mathbb{C}_I is a set of uniqueness.

Theorem 4.9. Let Ω be a set of uniqueness in \mathbb{B} and let S be a function defined on Ω such that $S : \Omega \cap \mathbb{C}_I \rightarrow \mathbb{C}_I^{N \times M}$ for every $I \in \mathbb{S}$. Then S can be extended slice hyperholomorphically to a Schur function in $\mathcal{N}(\mathbb{B})$ if and only if the kernel

$$(4.7) \quad \sum_{n=0}^{\infty} p^n (I_N - S(p) S(q)^*) \bar{q}^n$$

is positive on Ω .

Proof. If S can be extended hyperholomorphically to a Schur function, then the kernel (4.7) is positive definite on Ω . We prove the converse. Define the right linear quaternionic operator T as

$$T \left((1 - p\bar{q})^{-*} d \right) = (1 - p\bar{q})^{-*} S(q)^* d$$

for $q \in \Omega$ and reason as in the proof of Theorem 4.6. By assumption the kernel $\sum_{n=0}^{\infty} p^n (I_N - S(p) S(q)^*) \bar{q}^n$ is positive thus T is well defined and contractive. Its domain is dense since Ω is a set of uniqueness. So T extends to a contraction from \mathbf{H}_2^M to \mathbf{H}_2^N . Its adjoint is a contraction and for any $q \in \Omega$ and $F \in \mathbf{H}_2^N$ we have

$$\begin{aligned} \langle T^* F, (1 - p\bar{q})^{-*} d \rangle &= \langle F, T \left((1 - p\bar{q})^{-*} d \right) \rangle \\ &= \langle F, (1 - p\bar{q})^{-*} S(q)^* d \rangle \\ &= d^* S(q) F(q). \end{aligned}$$

Since we obtained a function equal to $S(q)F(q)$ on Ω , the choice $F = 1$ shows that $S = T^*1$ is the restriction to Ω of a Schur function. \square

To conclude this section we show that if a kernel $K(p, q)$ is positive and slice hyperholomorphic in p , then its corresponding reproducing kernel Hilbert space consists of slice hyperholomorphic functions.

Theorem 4.10. *Given an $\mathbb{H}^{N \times N}$ -valued function $K(p, q)$ on an open set $\Omega \subset \mathbb{H}$ let $\mathcal{H}(K)$ be the associated reproducing kernel quaternionic Hilbert space. Assume that for all $q \in \Omega$ the function $p \mapsto K(p, q)$ is slice hyperholomorphic. Then the entries of the elements of $\mathcal{H}(K)$ are also slice hyperholomorphic.*

Proof. It is enough to consider the case of \mathbb{H} -valued function because the matrix case works similarly. For any $f \in \mathcal{H}(K)$, $p, q \in \Omega$ and $\varepsilon \in \mathbb{R} \setminus \{0\}$ sufficiently small, we have

$$\frac{1}{\varepsilon}(K(p, q + \varepsilon) - K(p, q)) = \frac{1}{\varepsilon} \overline{(K(q + \varepsilon, p) - K(q, p))}.$$

Let $(u + Iv, x + Iy) \in \mathbb{C}_I \times \mathbb{C}_I$. We have that

$$\frac{\partial K(p, q)}{\partial x} = \frac{\partial \overline{K(q, p)}}{\partial u}.$$

In an analogous way, we have:

$$\frac{1}{\varepsilon}(K(p, q + I\varepsilon) - K(p, q)) = \frac{1}{\varepsilon} \overline{(K(q + I\varepsilon, p) - K(q, p))},$$

from which we deduce

$$\frac{\partial K(p, q)}{\partial y} = \frac{\partial \overline{K(q, p)}}{\partial v}.$$

The two families

$$\left\{ \frac{1}{\varepsilon}(K(p, q + \varepsilon) - K(p, q)) \right\}_{\varepsilon \in \mathbb{R} \setminus \{0\}}, \quad \left\{ \frac{1}{\varepsilon}(K(p, q + I\varepsilon) - K(p, q)) \right\}_{\varepsilon \in \mathbb{R} \setminus \{0\}},$$

are uniformly bounded in the norm and therefore have weakly convergent subsequences which converge to $\frac{\partial K(p, q)}{\partial x}$ and $\frac{\partial K(p, q)}{\partial y}$, respectively. Moreover we have

$$\frac{1}{\varepsilon}(f(p + \varepsilon) - f(p)) = \langle f(\cdot), \frac{1}{\varepsilon}(K(\cdot, p + \varepsilon) - K(\cdot, p)) \rangle_{\mathcal{H}(K)}$$

and

$$\frac{1}{\varepsilon}(f(p + I\varepsilon) - f(p)) = \langle f(\cdot), \frac{1}{\varepsilon}(K(\cdot, p + I\varepsilon) - K(\cdot, p)) \rangle_{\mathcal{H}(K)}.$$

Thus we can write

$$\frac{\partial f}{\partial u}(p) = \langle f(\cdot), \frac{\partial K(\cdot, p)}{\partial x} \rangle_{\mathcal{H}(K)},$$

and

$$\frac{\partial f}{\partial v}(p) = \langle f(\cdot), \frac{\partial K(\cdot, p)}{\partial y} \rangle_{\mathcal{H}(K)}.$$

To show that the function f is slice hyperholomorphic, we consider its restriction to any complex plane \mathbb{C}_I and we show that it is in the kernel of the corresponding Cauchy-Riemann operator:

$$\begin{aligned} \frac{\partial f}{\partial u} + I \frac{\partial f}{\partial v} &= \langle f, \frac{\partial K(\cdot, q)}{\partial x} \rangle_{\mathcal{H}(K)} + I \langle f(\cdot), \frac{\partial K(\cdot, q)}{\partial y} \rangle_{\mathcal{H}(K)} \\ &= \langle f, \frac{\partial K(\cdot, q)}{\partial x} - \frac{\partial K(\cdot, q)}{\partial y} I \rangle_{\mathcal{H}(K)} \\ &= \langle f, \overline{\frac{\partial K(q, \cdot)}{\partial u} + I \frac{\partial K(q, \cdot)}{\partial v}} \rangle_{\mathcal{H}(K)} = 0 \end{aligned}$$

since the kernel $K(q, p)$ is slice hyperholomorphic in the first variable q . \square

5. BLASCHKE PRODUCTS

The space $\mathbf{H}_2(\mathbb{B})$ was introduced in [2] as the space of power series $f(p) = \sum_{n=0}^{\infty} p^n f_n$, where the coefficients $f_n \in \mathbb{H}$ and are such that

$$(5.1) \quad \|f\|_{\mathbf{H}_2(\mathbb{B})} \stackrel{\text{def.}}{=} \sqrt{\sum_{n=0}^{\infty} |f_n|^2} < \infty.$$

$\mathbf{H}_2(\mathbb{B})$ endowed with the inner product

$$[f, g]_2 = \sum_{n=0}^{\infty} \overline{g_n} f_n, \quad \text{where} \quad g(p) = \sum_{n=0}^{\infty} p^n g_n$$

is the right quaternionic reproducing kernel Hilbert space with reproducing kernel

$$k(p, q) = \sum_{n=0}^{\infty} p^n \bar{q}^n = (1 - p\bar{q})^{-*}.$$

The norm (5.1) admits another expression.

Theorem 5.1. *The norm in $\mathbf{H}_2(\mathbb{B})$ can be written as*

$$\sup_{0 < r < 1, I \in \mathbb{S}} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{I\theta})|^2 d\theta \right]^{1/2} = \sup_{0 < r < 1} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{I\theta})|^2 d\theta \right]^{1/2}.$$

Proof. When one writes the power series expansion for f with center at 0, the equality is clear by the Parseval identity. Thus the norm can be defined as in the classical complex case by computing the integral on a chosen complex plane. \square

Let us prove some results associated to the Blaschke factors in the slice hyperholomorphic setting.

Definition 5.2. *Let $a \in \mathbb{H}$, $|a| < 1$. The function*

$$(5.2) \quad B_a(p) = (1 - p\bar{a})^{-*} \star (a - p) \frac{\bar{a}}{|a|}$$

is called Blaschke factor at a .

Lemma 5.3. *Let $a \in \mathbb{B}$. Then, $B_a(p)$ is a slice hyperholomorphic function in \mathbb{B} . Furthermore it holds that*

$$(5.3) \quad B_a(\bar{a})\bar{a} = \bar{a}B_a(\bar{a}).$$

Proof. Indeed $B_a(p)$ is slice hyperholomorphic by its definition, moreover we have

$$\begin{aligned}
 B_a(p) &= \left(\sum_{n=0}^{\infty} p^n \bar{a}^n \right) \star (a - p) \frac{\bar{a}}{|a|} \\
 (5.4) \quad &= \sum_{n=0}^{\infty} (p^n \bar{a}^n a - p^{n+1} \bar{a}^n) \frac{\bar{a}}{|a|} \\
 &= |a| + \sum_{n=0}^{\infty} p^{n+1} \bar{a}^{n+1} \left(|a| - \frac{1}{|a|} \right).
 \end{aligned}$$

Finally, (5.3) is a direct consequence of the last equality. \square

Remark 5.4. Set $\lambda(p) = 1 - p\bar{a}$. Then

$$(1 - p\bar{a})^{-\star} = (\lambda^c(p) \star \lambda(p))^{-1} \lambda^c(p).$$

Applying formula (3.2) to the products $\lambda^c(p) \star \lambda(p)$ and $\lambda^c(p) \star (a - p)$, we can rewrite (5.2) as

$$\begin{aligned}
 B_a(p) &= (\lambda^c(p) \star \lambda(p))^{-1} \lambda^c(p) \star (a - p) \frac{\bar{a}}{|a|} = (\lambda^c(p) \lambda(\tilde{p}))^{-1} \lambda^c(p) (a - \tilde{p}) \frac{\bar{a}}{|a|} \\
 (5.5) \quad &= \lambda(\tilde{p})^{-1} (a - \tilde{p}) \frac{\bar{a}}{|a|} = (1 - \tilde{p}\bar{a})^{-1} (a - \tilde{p}) \frac{\bar{a}}{|a|},
 \end{aligned}$$

where $\tilde{p} = \lambda^c(p)^{-1} p \lambda^c(p)$. Formula (5.5) represents the Blaschke factor $B_a(p)$ in terms of pointwise multiplication only.

Theorem 5.5. *Let $a \in \mathbb{H}$, $|a| < 1$. The Blaschke factor $B_a(q)$ has the following properties:*

- (1) *it takes the unit ball \mathbb{B} to itself;*
- (2) *it takes the boundary of the unit ball to itself;*
- (3) *it has a unique zero for $p = a$.*

Proof. By Remark 5.4 we write $B_a(p) = (1 - \tilde{p}\bar{a})^{-1} (a - \tilde{p}) \frac{\bar{a}}{|a|}$. Let us show that $|p| = |\tilde{p}| < 1$ implies $|B_a(p)|^2 < 1$. The latter inequality is equivalent to

$$|a - \tilde{p}|^2 < |1 - \tilde{p}\bar{a}|^2$$

which is also equivalent to

$$(5.6) \quad |a|^2 + |p|^2 < 1 + |a|^2 |p|^2.$$

The inequality (5.6) can be written as $(|p|^2 - 1)(1 - |a|^2) < 0$ and it holds when $|p| < 1$. When $|p| = 1$ we set $p = e^{I\theta}$, so that $\tilde{p} = e^{I'\theta}$ by the proof of Corollary 3.3; we have

$$|B_a(e^{I\theta})| = |1 - e^{I'\theta} \bar{a}|^{-1} |a - e^{I'\theta}| \frac{|\bar{a}|}{|a|} = |e^{-I'\theta} - \bar{a}|^{-1} |a - e^{I'\theta}| = 1.$$

Finally, from (5.5) it follows that $B_a(p)$ has only one zero that comes from the factor $a - \tilde{p}$. Moreover $B_a(a) = (1 - \tilde{a}\bar{a})^{-1} (a - \tilde{a}) \frac{\bar{a}}{|a|}$ where $\tilde{a} = (1 - a^2)^{-1} a(1 - a^2) = a$ and thus $B_a(a) = 0$. \square

Theorem 5.6. *Let $\{a_j\} \subset \mathbb{B}$, $j = 1, 2, \dots$ be a sequence of nonzero quaternions such that $[a_i] \neq [a_j]$ if $i \neq j$ and assume that $\sum_{j \geq 1} (1 - |a_j|) < \infty$. Then the function*

$$(5.7) \quad B(p) := \Pi_{j \geq 1}^* (1 - p \bar{a}_j)^{-*} \star (a_j - p) \frac{\bar{a}_j}{|a_j|},$$

where Π^* denotes the \star -product, converges uniformly on the compact subsets of \mathbb{B} .

Proof. Let $\alpha_j(p) := B_{a_j}(p) - 1$. Using Remark 5.4 we have the chain of equalities:

$$\begin{aligned} \alpha_j(p) &= B_{a_j}(p) - 1 = (1 - \tilde{p} \bar{a}_j)^{-1} (a_j - \tilde{p}) \frac{\bar{a}_j}{|a_j|} - 1 \\ &= (1 - \tilde{p} \bar{a}_j)^{-1} \left[(a_j - \tilde{p}) \frac{\bar{a}_j}{|a_j|} - (1 - \tilde{p} \bar{a}_j) \right] \\ &= (1 - \tilde{p} \bar{a}_j)^{-1} \left[(|a_j| - 1) \left(1 + \tilde{p} \frac{\bar{a}_j}{|a_j|} \right) \right]. \end{aligned}$$

Thus, if $|p| < 1$ and recalling that $|\tilde{p}| = |p|$, we have

$$|\alpha_j(p)| \leq 2(1 - |p|)^{-1}(1 - |a_j|)$$

and since $\sum_{j=1}^{\infty} (1 - |a_j|) < \infty$ then $\sum_{j=1}^{\infty} |\alpha_j(p)|$ converges in \mathbb{B} and the statement follows. \square

Definition 5.7. *The function $B(p)$ defined in (5.7) is called Blaschke product.*

Remark 5.8. In the complex case the sequence of complex numbers $\{a_j\}$ turns out to be the sequence of zeroes of the Blaschke product. In the quaternionic case the situation is different and we shall discuss it in the next results. In order to illustrate the differences with the complex case, let us consider the simpler case in which we have a polynomial

$$P(p) = (p - a_1) \star \dots \star (p - a_n)$$

and assume that $[a_i] \neq [a_j]$ for all $i, j = 1, \dots, n$. Then, it can be verified that $p = a_1$ is a zero for the polynomial $P(p)$ while the other zeroes belong to the spheres $[a_j]$ defined by a_j for $j = 2, \dots, n$. Note that, in the case in which all the elements a_j belong to a same sphere for all $j = 1, \dots, n$, then the only zero of the polynomial is a_1 , see [43, Lemma 2.2.1] and it has multiplicity n . Moreover, whenever a polynomial and, more in general, a slice hyperholomorphic function f has two zeroes belonging to a same 2-sphere, then all the elements of the sphere are zeroes for f . Thus the zeroes of a slice hyperholomorphic function are either isolated points or isolated spheres, see [27].

Assume that the slice hyperholomorphic function f has zero set

$$Z = \{a_1, a_2, \dots\} \cup \{[c_1], [c_2], \dots\}.$$

Then it is possible to construct a suitable Blaschke product having Z_f as zero set. Let us begin with the case in which the zeros are isolated points. In the sequel, we will be in need of the following remark:

Remark 5.9. Direct computations show the following equality of polynomials:

$$(1 - pa) \star (a - p) \frac{\bar{a}}{|a|} = \left((a - p) \frac{\bar{a}}{|a|} \right) \star (1 - pa) = (a - p) \star (1 - pa) \frac{\bar{a}}{|a|}.$$

Proposition 5.10. *Let $Z = \{a_1, a_2, \dots\}$ be a sequence of elements in \mathbb{B} , $a_j \neq 0$ for all $j = 1, 2, \dots$ such that $[a_i] \neq [a_j]$ if $i \neq j$ and assume that $\sum_{j \geq 1} (1 - |a_j|) < \infty$. Then there exists a Blaschke product $B(p)$ having zero set at Z .*

Proof. Let us prove the statement by induction. By hypothesis the zero set of the required Blaschke product consists of isolated points, all of them belonging to different spheres. If $n = 1$, we have already proved that $B_1(p) := B_{a_1}(p)$ has a_1 as its unique zero. Let us assume that the statement holds for a_1, \dots, a_k , and so there exists a Blaschke product $B_k(p)$ vanishing at the given points and let us prove that we can construct a Blaschke product vanishing at a_1, \dots, a_k, a_{k+1} . Observe that it is possible to choose an element a'_{k+1} belonging to the sphere $[a_{k+1}]$ such that

$$B_k(p) \star B_{a'_{k+1}}(p)$$

has zeros a_1, \dots, a_{k+1} . In fact, consider the product

$$B_{k+1}(p) := B_k(p) \star (1 - p\bar{a}'_{k+1})^{-\star} \star (a'_{k+1} - p) \frac{\bar{a}'_{k+1}}{|a'_{k+1}|}$$

and rewrite it using Remark 5.9 in the form

$$B_{k+1}(p) := B_k(p) \star (a'_{k+1} - p) \star (1 - pa'_{k+1})(1 - 2\operatorname{Re}(\bar{a}'_{k+1})p + |\bar{a}'_{k+1}|^2 p^2)^{-1} \frac{\bar{a}'_{k+1}}{|a'_{k+1}|}.$$

We now observe that the zeros of $B_{k+1}(p)$ belonging to the ball \mathbb{B} come from the zeros of the product

$$\tilde{B}(p) := B_k(p) \star (a'_{k+1} - p).$$

Observe that

$$\tilde{B}(a_{k+1}) = B_k(a_{k+1})(a'_{k+1} - B_k(a_{k+1})^{-1}a_{k+1}B_k(a_{k+1}))$$

and in order that a_{k+1} is a zero of \tilde{B} , and so of B_{k+1} , it is sufficient to choose

$$a'_{k+1} = B_k(a_{k+1})^{-1}a_{k+1}B_k(a_{k+1}).$$

The convergence of the Blaschke product follows as in Theorem 5.6. \square

From now on, when we write $Z = \{(a, \mu)\}$ we mean that Z consists of the point a repeated μ times. Let us now prove the analog of Theorem 5.5 (3) in the case in which the point a has multiplicity μ .

Lemma 5.11. *Let $Z = \{(a, \mu)\}$ with $a \in \mathbb{B}$ and $a \neq 0$. The Blaschke product*

$$\begin{aligned} B(p) &:= \left((1 - p\bar{a})^{-\star} \star (a - p) \frac{\bar{a}}{|a|} \right)^{\star \mu} := \\ &= \underbrace{\left((1 - p\bar{a})^{-\star} \star (a - p) \frac{\bar{a}}{|a|} \quad \dots \quad (1 - p\bar{a})^{-\star} \star (a - p) \frac{\bar{a}}{|a|} \right)}_{\mu \text{ times}} \end{aligned}$$

has Z as zero set.

Proof. We have:

$$(1 - p\bar{a})^{-\star} \star (a - p) \frac{\bar{a}}{|a|} = (1 - 2\operatorname{Re}(a)p + p^2|a|^2)^{-1} (1 - pa) \star (a - p) \frac{\bar{a}}{|a|}$$

thus, using the fact that $1 - 2\operatorname{Re}(a)p + p^2|a|^2$ has real coefficients, we can write

$$B(p) = (1 - 2p\operatorname{Re}(a) + p^2|a|^2)^{-\mu} \underbrace{\left((1 - pa) \star (a - p) \frac{\bar{a}}{|a|} \dots (1 - pa) \star (a - p) \frac{\bar{a}}{|a|} \right)}_{\mu \text{ times}}$$

and thanks to Remark 5.9 we obtain

$$B(p) = (1 - 2p\operatorname{Re}(a) + p^2|a|^2)^{-\mu} \left((a - p) \frac{\bar{a}}{|a|} \right)^\mu \star (1 - pa)^\mu.$$

Thus $B(p)$ has, in \mathbb{B} , a unique zero at $p = a$ of multiplicity μ . Note that the zero on the sphere $[1/a]$ which, as it can be proven, coincides with $1/a$ has to be excluded since $B(p)$ is not defined there, moreover $1/a \notin \mathbb{B}$. \square

Proposition 5.12. *Let $Z = \{(a_1, \mu_1), (a_2, \mu_2), \dots\}$ be a sequence of points $a_j \in \mathbb{B}$ with respective multiplicities $\mu_j \geq 1$, $a_j \neq 0$ for $j = 1, 2, \dots$. Let a'_j be such that $[a_i] \neq [a_j]$ if $i \neq j$ and $\sum_{j \geq 1} \mu_j(1 - |a_j|) < \infty$. Then there exists a Blaschke product of the form*

$$B(p) = \prod_{j \geq 1}^* (B_{a'_j}(p))^{\star \mu_j},$$

having zero set at Z , where $a'_1 = a_1$ and $a'_j \in [a_j]$ are suitably chosen elements, $j = 2, 3, \dots$

Proof. We prove the assertion by induction on the number of distinct zeros. If there is just one zero a_1 with multiplicity μ_1 , then the statement follows by Lemma 5.11. Let us assume that the statement holds in the case we have k different zeros a_i with respective multiplicities μ_i and let us prove that it holds for $k + 1$ different zeros. Let $B_k(p)$ be the Blaschke product having zeros at $Z = \{(a_1, \mu_1), \dots, (a_k, \mu_k)\}$ and let us consider

$$B_{k+1}(p) := B_k(p) \star (B_{a'_{k+1}}(p))^{\star \mu_k}$$

where a'_{k+1} is chosen such that $B_k(p) \star B_{a'_{k+1}}(p)$ has a zero at $p = a_{k+1}$. Then all the other zeros of B_{k+1} must belong to the sphere $[a_{k+1}]$. Moreover they must coincide with a_{k+1} otherwise the Blaschke product $(B_{a'_{k+1}}(p))^{\star \mu_k}$ vanishes at two different points on a same sphere, and thus it vanishes on the whole sphere. In particular, any two conjugate elements on the sphere are zeros of the product and so we would have:

$$\begin{aligned} B_a(p) \star B_{\bar{a}}(p) &= (1 - p\bar{a})^{-\star} \star (a - p) \frac{\bar{a}}{|a|} \star (1 - pa)^{-\star} \star (\bar{a} - p) \frac{a}{|a|} \\ &= (1 - 2\operatorname{Re}(a)p + p^2|a|^2)^{-1} (|a|^2 - 2\operatorname{Re}(a)p + p^2). \end{aligned}$$

However, it is immediate that the product $(B_{a'_{k+1}}(p))^{\star \mu_k}$ does not contain factors of the above form, thus all its zeros coincide with a_{k+1} as stated. The convergence of the Blaschke product follows as in Theorem 5.6. \square

If a Blaschke product of two factors has an entire sphere of zeros then, as discussed in the proof of the previous theorem, it has a specific form and we are led to the following definition:

Definition 5.13. *Let $a \in \mathbb{H}$, $|a| < 1$. The function*

$$(5.8) \quad B_{[a]}(p) = (1 - 2\operatorname{Re}(a)p + p^2|a|^2)^{-1} (|a|^2 - 2\operatorname{Re}(a)p + p^2)$$

is called Blaschke factor at the sphere $[a]$.

Remark 5.14. Note that the definition of $B_{[a]}(p)$ does not depend on the choice of the point a that identifies the 2-sphere. Indeed all the elements in the sphere $[a]$ have the same real part and module. It is easy to verify that Blaschke factor $B_{[a]}(p)$ vanishes on the sphere $[a]$.

The following result is immediate:

Proposition 5.15. *A Blaschke product having zeros at the set of spheres*

$$Z = \{([c_1], \nu_1), ([c_2], \nu_2), \dots\}$$

where $c_j \in \mathbb{B}$, the sphere $[c_j]$ is a zero of multiplicity ν_j , $j = 1, 2, \dots$ and $\sum_{j \geq 1} \nu_j(1 - |c_j|) < \infty$ is given by

$$\prod_{j \geq 1} (B_{[c_j]}(p))^{\nu_j}.$$

Proof. All the factors $B_{[c_j]}(p)$ have real coefficients and thus belong to the class \mathcal{N} (see Section 4), so we can use the pointwise product. The fact that the zeros are the given spheres follows by taking the zeros of each factor. The convergence of the infinite product follows as in Theorem 5.6. \square

Theorem 5.16. *A Blaschke product having zeros at the set*

$$Z = \{(a_1, \mu_1), (a_2, \mu_2), \dots, ([c_1], \nu_1), ([c_2], \nu_2), \dots\}$$

where $a_j \in \mathbb{B}$, a_j have respective multiplicities $\mu_j \geq 1$, $a_j \neq 0$ for $j = 1, 2, \dots$, $[a_i] \neq [a_j]$ if $i \neq j$, $c_i \in \mathbb{B}$, the spheres $[c_j]$ have respective multiplicities $\nu_j \geq 1$, $j = 1, 2, \dots$, $[c_i] \neq [c_j]$ if $i \neq j$ and

$$\sum_{i,j \geq 1} (\mu_i(1 - |a_i|) + \nu_j(1 - |c_j|)) < \infty$$

is given by

$$\prod_{i \geq 1} (B_{[c_i]}(p))^{\nu_i} \prod_{j \geq 1}^* (B_{a'_j}(p))^{\mu_j},$$

where $a'_1 = a_1$ and $a'_j \in [a_j]$ are suitably chosen elements, $j = 2, 3, \dots$

Proof. It follows from Propositions 5.10 and 5.12. \square

Theorem 5.17. *Let B_a be a Blaschke factor. The operator*

$$M_a : f \mapsto B_a \star f$$

is an isometry from $\mathbf{H}_2(\mathbb{B})$ onto itself.

Proof. We first consider $f(p) = p^u h$ and $g(p) = p^v k$ where $u, v \in \mathbb{N}_0$ and $h, k \in \mathbb{H}$, and show that

$$(5.9) \quad [B_a \star f, B_a \star g]_2 = \delta_{uv} \bar{k} h.$$

Using calculation (5.4), and with f and g as above, we have

$$(B_a \star f)(p) = p^u h |a| + \sum_{n=0}^{\infty} p^{n+1+u} \bar{a}^{n+1} (|a| - \frac{1}{|a|}) h$$

and

$$(5.10) \quad (B_a \star g)(p) = p^v k |a| + \sum_{n=0}^{\infty} p^{n+1+v} \bar{a}^{n+1} \left(|a| - \frac{1}{|a|} \right) k.$$

If $u = v$ we have

$$[B_a \star f, B_a \star g]_2 = \bar{k} h \left(|a|^2 + \sum_{n=0}^{\infty} |a|^{2n+2} \left(|a| - \frac{1}{|a|} \right)^2 \right) = \bar{k} h = [f, g]_2.$$

To compute $[f, g]_2$ we assume that $u < v$. Then, in view of (5.10) we have

$$[p^u h |a|, B_a \star g]_2 = 0.$$

So

$$\begin{aligned} [B_a \star f, B_a \star g]_2 &= \left[\sum_{n=0}^{\infty} p^{n+1+u} \bar{a}^{n+1} \left(|a| - \frac{1}{|a|} \right) h, p^v |a| k \right]_2 + \\ &\quad + \left[\sum_{n=0}^{\infty} p^{n+1+u} \bar{a}^{n+1} \left(|a| - \frac{1}{|a|} \right) h, \sum_{m=0}^{\infty} p^{m+1+v} \bar{a}^{m+1} \left(|a| - \frac{1}{|a|} \right) k \right]_2 \\ &= |a| \bar{k} \bar{a}^{v-u} \left(|a| - \frac{1}{|a|} \right) h + \\ &\quad + \left[\sum_{m=0}^{\infty} p^{m+1+v} \bar{a}^{m+1+v-u} \left(|a| - \frac{1}{|a|} \right) h, \sum_{m=0}^{\infty} p^{m+1+v} \bar{a}^{m+1} \left(|a| - \frac{1}{|a|} \right) k \right]_2 \\ &= |a| \bar{k} \bar{a}^{v-u} \left(|a| - \frac{1}{|a|} \right) h + \bar{k} \left(|a| - \frac{1}{|a|} \right)^2 \frac{\bar{a}^{v-u} |a|^2}{1 - |a|^2} h \\ &= 0 \\ &= [f, g]_2. \end{aligned}$$

The case $v < u$ is considered by symmetry of the inner product. Hence, (5.9) holds for polynomials. By continuity, and a corollary of the Runge theorem, see [23], it holds for all $f \in \mathbf{H}_2(\mathbb{B})$. □

We mention that similar computations hold in the case of bicomplex numbers. See [5].

6. INTERPOLATION IN THE HARDY SPACE

In this section we consider the following problem:

Problem 6.1. *Given N points $a_1, \dots, a_N \in \mathbb{B}$, and M spheres $[c_1], \dots, [c_M]$ in \mathbb{B} such that the spheres $[a_1], \dots, [a_N], [c_1], \dots, [c_M]$ are pairwise non-intersecting, find all $f \in \mathbf{H}_2(\mathbb{B})$ such that*

$$(6.1) \quad f(a_i) = 0, \quad i = 1, \dots, N,$$

and

$$(6.2) \quad f([c_j]) = 0, \quad j = 1, \dots, M.$$

Theorem 6.2. *There is a Blaschke product B such that the solutions of Problem 6.1 are the functions $f = B \star g$, when g runs through $\mathbf{H}_2(\mathbb{B})$.*

We give two proofs of this theorem, the first one iterative, using formula (3.2), and the second one global.

Proof. (Iterative proof). We proceed in three steps. As a preliminary computation we consider in the first step the case $N = 1$ and $M = 0$. The problem itself will be solved by considering the interpolation at the spheres first.

STEP 1: *We solve the problem for $M = 0$ and $N = 1$.*

Let B_{a_1} be the Blaschke factor (5.2) at a_1 . By (3.2), we have $(B_{a_1} \star f)(a_1) = 0$ for all $f \in \mathbf{H}_2(\mathbb{B})$. Furthermore, by Theorem 5.17 we have that $\|B_{a_1} \star f\|_2 = \|f\|_2$. Thus for $N = 1$ the set \mathcal{M} of solutions to Problem 6.1 contains $B_{a_1} \star \mathbf{H}_2(\mathbb{B})$. We now prove that $\mathcal{M} \subseteq B_{a_1} \star \mathbf{H}_2(\mathbb{B})$. Let $f \in \mathcal{M}$. Then, by the reproducing kernel property, f is orthogonal to $(1 - p\overline{a_1})^{-*}$. The range $\text{ran } \sqrt{I - M_{a_1} M_{a_1}^*}$ is equal to the span of $(1 - p\overline{a_1})^{-*}$ (see [2]). In view of Theorem 5.17 we have $\sqrt{I - M_{a_1} M_{a_1}^*} = I - M_{a_1} M_{a_1}^*$ and thus:

$$\mathbf{H}_2(\mathbb{B}) = (I - M_{a_1} M_{a_1}^*)\mathbf{H}_2(\mathbb{B}) \oplus (M_{a_1} M_{a_1}^*)\mathbf{H}_2(\mathbb{B}).$$

Therefore $f \in (M_{a_1} M_{a_1}^*)\mathbf{H}_2(\mathbb{B})$. Hence $\mathcal{M} = B_{a_1} \star \mathbf{H}_2(\mathbb{B})$.

With this preliminary computation at hand, we solve the interpolation problem by first considering the interpolation at the spheres $[c_1], \dots, [c_M]$.

STEP 2: *Consider the sphere $[c_j]$ and let $B_{[c_j]}$ be the corresponding Blaschke factor given by (5.8), $j = 1, 2, \dots, M$. An element $f \in \mathbf{H}_2(\mathbb{B})$ vanishes on the spheres $[c_1], \dots, [c_M]$ if and only if it can be written as*

$$(6.3) \quad f = B_{[c_1]} B_{[c_2]} \cdots B_{[c_M]} g,$$

where $g \in \mathbf{H}_2(\mathbb{B})$.

Note that in (6.3) we have pointwise products since the Blaschke factors on spheres have real coefficients. By [27, Corollary 4.3.7, p. 123], f vanishes on the whole sphere $[c_1]$ if and only if $f(c_1) = f(\overline{c_1}) = 0$. By STEP 1, the first condition means that $f = B_{c_1} \star g$ for some $g \in \mathbf{H}_2(\mathbb{B})$. By (3.2), the second condition is equivalent to:

$$(6.4) \quad B_{c_1}(\overline{c_1}) g((B_{c_1}(\overline{c_1}))^{-1} \overline{c_1} B_{c_1}(\overline{c_1})) = 0.$$

Since $B_{c_1}(\overline{c_1}) \neq 0$, and taking into account (5.3), we see that (6.4) is equivalent to $g(\overline{c_1}) = 0$. Thus, once more using STEP 1, we have $g(p) = B_{\overline{c_1}} \star h$ for some $h \in \mathbf{H}_2(\mathbb{B})$. Therefore

$$f = B_{c_1} \star B_{\overline{c_1}} \star h = B_{[c_1]} h.$$

This argument can be iterated for the spheres $[c_2], \dots, [c_M]$ since $B_{[c_2]}(c_1) \neq 0$ (this last inequality in turn following from the fact that the spheres do not intersect).

We now turn to the conditions (6.1). The function f is of the form (6.3), and thus the condition $f(a_1) = 0$ becomes

$$(B_{[c_1]} B_{[c_2]} \cdots B_{[c_M]})(a_1) g(a_1) = 0,$$

and so, by STEP 1, $g = B_{a_1} \star g_1$ for some $g_1 \in \mathbf{H}_2(\mathbb{B})$. Let now $f \in \mathbf{H}_2(\mathbb{B})$ satisfying moreover $f(a_2) = 0$. By the previous argument, f is of the form

$$(B_{[c_1]} B_{[c_2]} \cdots B_{[c_M]}) B_{a_1} \star g_1$$

for some $g_1 \in \mathbf{H}_2(\mathbb{B})$. The condition $f(a_2) = 0$ and formula (3.2) gives

$$g(a'_2) = 0, \quad \text{where} \quad a'_2 = X^{-1} a_2 X,$$

with

$$X = (B_{[c_1]} B_{[c_2]} \cdots B_{[c_M]} B_{a_1}) (a_2).$$

Hence f is a solution if and only if $g_2 = B_{a'_2} \star g_2$ for some $g_2 \in \mathbf{H}_2(\mathbb{B})$. The argument can be iterated and we obtain the set of all functions $f \in \mathbf{H}_2(\mathbb{B})$ which vanish at the points a_1, \dots, a_N . \square

We now turn to the global proof of Theorem 6.2.

Proof. (Global proof). We proceed in a number of steps and first define \mathcal{M} to be the span of the functions

$$(1 - pa_1)^{-\star}, \dots, (1 - pa_N)^{-\star}, (1 - pc_1)^{-\star}, (1 - p\overline{c_1})^{-\star}, \dots, (1 - pc_M)^{-\star}, (1 - p\overline{c_M})^{-\star}.$$

Define

$$A = \text{diag} (a_1, \dots, a_N, c_1, \overline{c_1}, \dots, c_M, \overline{c_M}),$$

and

$$c = \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \end{pmatrix}}_{(N+2M) \text{ times}}.$$

Finally let \mathbf{P} denote the Gram matrix of \mathcal{M} in the $\mathbf{H}_2(\mathbb{B})$ inner product. The claim of the first step is a direct consequence of the reproducing kernel property in $\mathbf{H}_2(\mathbb{B})$.

STEP 1: $f \in \mathbf{H}_2(\mathbb{B})$ is a solution of the interpolation problem 6.1 if and only if it is orthogonal to \mathcal{M} .

STEP 2: The matrix $\mathbf{P} \in \mathbb{H}^{(N+2M) \times (N+2M)}$ is strictly positive and satisfies the matrix equation (called a Stein equation)

$$(6.5) \quad \mathbf{P} - A^* \mathbf{P} A = c^* c.$$

This step is also a consequence of the reproducing kernel property since, for $a, b \in \mathbb{B}$ it holds that:

$$[(1 - p\overline{b})^{-\star}, (1 - p\overline{a})^{-\star}]_2 = \sum_{n=0}^{\infty} a^n \overline{b}^n,$$

and so

$$[(1 - p\overline{b})^{-\star}, (1 - p\overline{a})^{-\star}]_2 - a[(1 - p\overline{b})^{-\star}, (1 - p\overline{a})^{-\star}]_2 \overline{b} = 1.$$

STEP 3: There exists a vector $b \in \mathbb{H}^{N+2M}$ and $d \in \mathbb{H}$ such that

$$(6.6) \quad \begin{pmatrix} A & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{P}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \mathbf{P}^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\mathbf{P} > 0$, it has a strictly positive square root. For this last fact, see for instance [8, Proposition 3.1.3, p. 440], and see the references [47, Corollary 6.2, p. 41] [14, 42]

for the structure of normal quaternionic matrices. Thus, the Stein equation (6.5) can be rewritten as $T^*T = I_{N+2M}$, where

$$T = \begin{pmatrix} \mathbf{P}^{1/2} \mathbf{A} \mathbf{P}^{-1/2} \\ c \mathbf{P}^{-1/2} \end{pmatrix}.$$

Thus the columns of T form an $N + 2M$ orthogonal sets of vectors in \mathbb{H}^{N+2M+1} , and we can complete it to an orthonormal basis with a vector $h \in \mathbb{H}^{N+2M+1}$ such that

$$\begin{pmatrix} T & h \end{pmatrix}^* \begin{pmatrix} T & h \end{pmatrix} = I_{N+2M+1}.$$

The claim follows from the fact that $TT^* = I_{N+2M+1}$ with

$$(6.7) \quad \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} \mathbf{P}^{-1/2} & 0 \\ 0 & 1 \end{pmatrix} h.$$

We now introduce

$$B(p) = d + pc \star (I - pA)^{-\star} b = d + \sum_{n=0}^{\infty} p^{n+1} c A^n b.$$

Step 4 below is a particular case of Proposition 9.3 below, and its proof will be omitted.

STEP 4: *The function B satisfies*

$$(6.8) \quad C(I_{N+2M} - pA)^{-\star} \mathbf{P}^{-1} (C(I_{N+2M} - qA)^{-\star})^* = (1 - B(p) \overline{B(q)}) \star (1 - p\overline{q})^{-\star}.$$

Since $\mathbf{P}^{-1} > 0$ it follows from (6.8) that B is a Schur multiplier, and in particular, $\mathcal{M} = \text{ran } \sqrt{I - M_B M_B^*}$, where M_B denotes the operator of slice multiplication by B on the left (the square root exists because the operator has finite rank; more generally, any positive operator in a quaternionic Hilbert space has a positive square root. We will not need this general fact here). Since \mathcal{M} is finite dimensional, we have more precisely

$$(6.9) \quad \mathcal{M} = \text{ran } \sqrt{I - M_B M_B^*} = \text{ran } (I - M_B M_B^*).$$

STEP 5: *The function $B \star g$ satisfies the interpolation conditions (6.1)-(6.2) for every $g \in \mathbf{H}_2(\mathbb{B})$.*

We first prove that $(B \star g)(a_1) = 0$. The proof that $B \star g$ vanishes at the points a_2, \dots, a_N and $c_1, \overline{c_1}, \dots, c_M, \overline{c_M}$ is the same.

$$\begin{aligned} B(a_1) &= d + \sum_{n=0}^{\infty} a_1^{n+1} \begin{pmatrix} \overline{a_1}^n & \overline{a_2}^n & \cdots & \overline{a_N}^n \end{pmatrix} b \\ &= d + a_1 \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \mathbf{P} b \\ &= \begin{pmatrix} a_1 \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \mathbf{P} & 1 \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} \\ &= \begin{pmatrix} a_1 \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \mathbf{P}^{1/2} & 1 \end{pmatrix} h \quad (\text{where we have used (6.7)}) \\ &= 0 \end{aligned}$$

by definition of h since

$$\begin{pmatrix} \mathbf{P}^{1/2} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ 1 \end{pmatrix} \overline{a_1} = T \begin{pmatrix} \mathbf{P}^{1/2} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{pmatrix} \in \text{ran } T.$$

Let now $g(p) = p^m k$ with $m \in \mathbb{N}$ and $k \in \mathbb{H}$. Then,

$$(B \star g)(p) = p^m B(p)k$$

and so the interpolation conditions (6.1)-(6.2) are met. The result is thus true for all slice hyperholomorphic polynomials in p , and hence, in view of the preceding step, for every element g in $\mathbf{H}_2(\mathbb{B})$ since convergence in norm implies pointwise convergence in a reproducing kernel Hilbert space.

STEP 6: *The set of solutions is given by $f = B \star g$, when g runs through $\mathbf{H}_2(\mathbb{B})$.*

By definition of \mathcal{M} and using (6.9) and the reproducing kernel property we see that $\text{ran } (I - M_B M_B^*)$ and $\text{ran } M_B M_B^*$ are orthogonal in $\mathbf{H}_2(\mathbb{B})$. Since the sum of these two ranges is the whole of $\mathbf{H}_2(\mathbb{B})$, we deduce that $\mathcal{M}^\perp = \text{ran } M_B M_B^*$ and this ends the proof since $\text{ran } (M_B M_B^*) = \text{ran } M_B$. \square

7. QUATERNIONIC PONTRYAGIN SPACES

Quaternionic Pontryagin spaces have been studied in [8]. In this section we review the main definitions, and prove in the setting of quaternionic spaces, an important result due to Shmulyan in the complex setting; see [46] and [3, Theorem 1.4.2, p. 29]. Consider a right vector space \mathcal{P} on the quaternions, endowed with a \mathbb{H} -valued Hermitian form $[\cdot, \cdot]$, meaning that

$$[va, wb] = \bar{b}[v, w]a, \quad \forall a, b \in \mathbb{H} \quad \text{and} \quad \forall v, w \in \mathcal{P}.$$

\mathcal{P} is called a (right, quaternionic) Pontryagin space if it admits a decomposition

$$(7.1) \quad \mathcal{P} = \mathcal{P}_+ + \mathcal{P}_-,$$

into a sum of two vector subspaces \mathcal{P}_+ and \mathcal{P}_- with the following properties:

- (1) $(\mathcal{P}_+, [\cdot, \cdot])$ is a (right, quaternionic) Hilbert space.
- (2) $(\mathcal{P}_-, -[\cdot, \cdot])$ is a finite dimensional (right, quaternionic) Hilbert space.
- (3) The sum (7.1) is direct and orthogonal: $\mathcal{P}_+ \cap \mathcal{P}_- = \{0\}$ and

$$[v_+, v_-] = 0, \quad \forall v_+ \in \mathcal{P}_+ \quad \text{and} \quad \forall v_- \in \mathcal{P}_-.$$

The space \mathcal{P} endowed with the form

$$\langle v, w \rangle = [v_+, v_-] - [w_+, w_-], \quad v = v_+ + v_-, \quad w = w_+ + w_-,$$

is a (right quaternionic) Hilbert space. The decomposition (7.1) is called a fundamental decomposition. It is not unique (except for the case where one of the components reduces to $\{0\}$), but all the corresponding Hilbert space topologies are equivalent; see [8, Theorem 12.3, p. 467]. The number $\kappa = \dim \mathcal{P}_-$ is called the index of the Pontryagin space \mathcal{P} . It is the same for all the decompositions; see [8, Proposition 12.6, p. 469]. The reader should be aware that in some sources on the complex valued case, and in

particular in [13, 35], the convention is the opposite, and it is the space \mathcal{P}_+ which is required to be finite dimensional.

An important example of finite dimensional Pontryagin space is:

Example 7.1. *Let $J \in \mathbb{H}^{N \times N}$ be a signature matrix. The space \mathbb{H}^N endowed with the Hermitian form*

$$[v, w]_J = w^* J v.$$

is a right quaternionic Pontryagin space, which we will denote by \mathbb{H}_J^N .

Before turning to Shmulyan's theorem we recall the following definition: Given two right quaternionic Pontryagin spaces $(\mathcal{P}_1, [\cdot, \cdot]_1)$ and $(\mathcal{P}_2, [\cdot, \cdot]_2)$ a linear relation between \mathcal{P}_1 and \mathcal{P}_2 is a right linear subspace, say R , of the product $\mathcal{P}_1 \times \mathcal{P}_2$. The domain of the relation is the set of elements $v_1 \in \mathcal{P}_1$ such that there exists a (not necessarily unique) $v_2 \in \mathcal{P}_2$ such that $(v_1, v_2) \in R$. The relation is called contractive if

$$[v_1, v_1]_1 \leq [v_2, v_2]_2, \quad \forall (v_1, v_2) \in R.$$

The graph of an operator is a relation. A relation will be the graph of an operator if and only if it has no elements of the form $(0, v_2)$ with $v_2 \neq 0$.

Theorem 7.2. *A densely defined contractive relation between quaternionic Pontryagin spaces of same index extends to the graph of a contraction from \mathcal{P}_1 into \mathcal{P}_2 .*

Proof. We follow the strategy of [3, p. 29-30] we divide the proof into a number of steps. We recall that a strictly negative subspace is a linear subspace V such that $[v, v] < 0$ for every non zero element of V .

STEP 1: *The domain of the relation contains a maximum negative subspace.*

Indeed, every dense linear subspace of a right quaternionic Pontryagin space of index $\kappa > 0$ contains a κ dimensional strictly negative subspace. See [8, Theorem 12.8 p. 470]. We denote by \mathcal{V}_- such a subspace of the domain of R .

STEP 2: *The relation R restricted to \mathcal{V}_- has a zero kernel, and the image of \mathcal{V}_- is a strictly negative subspace of \mathcal{P}_2 of dimension κ .*

Let $(v_1, v_2) \in R$ with $v_1 \in \mathcal{V}_-$. Since R is contractive we have

$$[v_2, v_2]_2 \leq [v_1, v_1]_1 \leq 0,$$

and the second inequality is strict when $v_1 \neq 0$. Thus, the image of \mathcal{V}_- is a strictly negative subspace of \mathcal{P}_2 . Next, let $(v, w) \in R$ and (\tilde{v}, w) with $v, \tilde{v} \in \mathcal{V}_-$ and $w \in \mathcal{P}_2$. Then, $(v - \tilde{v}, 0) \in R$. Since R is contractive we have

$$[0, 0]_2 \leq [v - \tilde{v}, v - \tilde{v}]_1$$

This forces $v = \tilde{v}$ since \mathcal{V}_- is strictly negative, and proves the second step.

STEP 3: *R is the graph of a densely defined contraction.*

We choose \mathcal{V}_- as in the first two steps, and take v_1, \dots, v_κ a basis of \mathcal{V}_- . Then, there exists uniquely defined vectors $w_1, \dots, w_\kappa \in \mathcal{P}_2$ such that $(v_i, w_i) \in R$ for $i = 1, \dots, \kappa$. Set \mathcal{W}_- to be the linear span of w_1, \dots, w_κ . By Step 2 and since the spaces \mathcal{P}_1 and \mathcal{P}_2 have the same negative index

$$\dim \mathcal{V}_- = \dim \mathcal{W}_- = \text{ind}_- \mathcal{P}_1 = \text{ind}_- \mathcal{P}_2,$$

and there exists fundamental decompositions

$$\mathcal{P}_1 = \mathcal{V}_- + \mathcal{V}_+ \quad \text{and} \quad \mathcal{P}_2 = \mathcal{W}_- + \mathcal{W}_+,$$

where $(\mathcal{V}_+, [\cdot, \cdot]_1)$ and $(\mathcal{W}_+, [\cdot, \cdot]_2)$ are right quaternionic Hilbert spaces. Let now $(0, w) \in R$. We need to show that $w = 0$. Still following [3, p. 30] we write $w = w_- + w_+$ where $w_- \in \mathcal{W}_-$ and $w_+ \in \mathcal{W}_+$. Let $w_- = \sum_{n=1}^\kappa w_j q_j$ where the $q_j \in \mathbb{H}$, and set $v_- = \sum_{n=1}^\kappa v_j q_j$. Then, $(v_-, w_-) \in R$ and

$$(0, w) = (v_-, w_-) + (-v_-, w_+).$$

It follows that $(-v_-, w_+) \in R$. Since R is contractive, we have

$$[w_+, w_+]_2 \leq [v_-, v_-]_1,$$

and so $[w_+, w_+]_2 \leq 0$. Thus $w_+ = 0$. It follows that $(0, w_-) \in R$ and so $w_- = 0$, because R is one-to-one on \mathcal{V}_- , as follows from STEP 2.

STEP 4: *R extends to the graph of an everywhere defined contraction.*

In the complex case, this is [3, Theorem 1.4.1 p. 27]. We follow the arguments there. We consider the orthogonal projection from \mathcal{P}_2 onto \mathcal{W}_- . Let T be the densely defined contraction with graph the relation R . There exist \mathbb{H} -valued right linear functionals c_1, \dots, c_κ , defined on the domain of R , and such that

$$Tv = \sum_{n=1}^\kappa f_n c_n(v) + w_+,$$

where $w_+ \in \mathcal{W}_+$ is such that $[f_n, w_+]_2 = 0$ for $n = 1, 2, \dots, \kappa$. Assume that c_1 is not bounded on its domain, let v_+ be such that $c_1(v_+) = 1$, and let v_n be vectors in \mathcal{V}_+ such that $c_1(v_n) = 1$ and $\lim_{n \rightarrow \infty} [v_+ - v_n, v_+ - v_n]_1 = 0$. Then v_+ belongs to the closure of $\ker c_1$ and so, we have that the closure of $\ker c_1 = \mathcal{V}_+$. Thus $\ker c_1$ contains a strictly negative subspace of dimension κ , say \mathcal{K}_- . For $v \in \mathcal{K}_-$, we have

$$Tv = \sum_{n=2}^\kappa f_n c_n(v).$$

This contradicts STEP 2 and thus completes the proof of the theorem. \square

8. NEGATIVE SQUARES

The notion of kernels with a finite number of negative squares extend the notion of positive definite kernels. For this notion in the quaternionic case, we send the reader to [8, §11]. We recall that a $\mathbb{H}^{N \times N}$ Hermitian matrix A has only real (right) eigenvalues. We denote by $\text{sq}_-(A)$ the number of its strictly negative eigenvalues (if any).

Definition 8.1. Let $\kappa \in \mathbb{N}_0$. A $\mathbb{H}^{N \times N}$ -valued function $K(z, w)$ defined in a set Ω is said to have κ negative squares if it is Hermitian:

$$(8.1) \quad K(z, w) = K(w, z)^*, \quad \forall z, w \in \Omega,$$

and if, for every $N \in \mathbb{N}$ and every choice of $z_1, \dots, z_N \in \Omega$ and $c_1, \dots, c_N \in \mathbb{H}^N$, the $N \times N$ Hermitian matrix with ℓ, j entry equal to $c_\ell^* K(z_\ell, z_j) c_j$, has at most κ strictly negative eigenvalues, and has exactly κ strictly negative eigenvalues for some choice of $N, z_1, \dots, z_N, c_1, \dots, c_N$.

We will usually use the term *kernel* rather than *function* to denote such $K(z, w)$'s. When $\kappa = 0$, the kernel $K(z, w)$ is positive definite. In the following theorem we recall the following quaternionic counterparts of results well known in the complex case. First a definition. A positive definite $\mathbb{H}^{N \times N}$ -valued function $Q(z, w)$ is said to be of *finite rank* if it can be factored as

$$Q(z, w) = N(z, w)^* N(z, w),$$

where $N(z, w)$ is $\mathbb{H}^{N \times M}$ -valued for some $M \in \mathbb{N}$. The smallest such M is called the rank of Q .

Theorem 8.2.

(a) Let $K(z, w)$ be an Hermitian $\mathbb{H}^{N \times N}$ -valued function (see (8.1)) for z, w in some set Ω . Then, K has κ negative squares if and only if it can be written as a difference

$$K(z, w) = K_+(z, w) - K_-(z, w),$$

where both K_+ and K_- are positive definite in Ω , with moreover K_- of finite rank.

(b) There is a one-to-one correspondence between right quaternionic reproducing kernel Pontryagin spaces of index κ , of \mathbb{H}^N -valued functions on a set Ω , and $\mathbb{H}^{N \times N}$ -valued functions with κ negative squares in Ω .

For a proof of these facts, see [8, Theorems 11.5, p. 466 and 13.1, p. 472].

Theorem 8.3. Let $K(p, q)$ be a $\mathbb{H}^{N \times N}$ -valued function with κ negative squares in an open nonempty subset Ω of \mathbb{H} . Then there exists a unique right quaternionic reproducing kernel Pontryagin space \mathcal{P} consisting of \mathbb{H}^N -valued function slice hyperholomorphic in Ω and with reproducing kernel $K(p, q)$.

Proof. The fact that there exists a unique Pontryagin space \mathcal{P} associated to K follows as in Theorem 13.1 in [8]. We have to show that the elements in \mathcal{P} are slice hyperholomorphic. Let $\mathring{\mathcal{P}}(K)$ be the linear span of the functions of the form $p \mapsto K(p, q)a$ where $q \in \Omega$ and $a \in \mathbb{H}^N$. Since K has κ negative squares, $\mathring{\mathcal{P}}(K)$ has a maximal strictly negative subspace \mathcal{N}_- of dimension κ . By Proposition 10.3 in [8] it is possible to write

$$\mathring{\mathcal{P}}(K) = \mathcal{N}_- + \mathcal{N}_-^{[\perp]},$$

where $\mathcal{N}_-^{[\perp]}$ is a quaternionic pre-Hilbert space. The space $\mathcal{N}_-^{[\perp]}$ has a unique completion, denoted by \mathcal{N}_+ . Let us define

$$\mathcal{P} := \mathcal{N}_+ + \mathcal{N}_-,$$

with the inner product

$$[f, f] := [f_+, f_+]_{\mathcal{N}_+} + [f_-, f_-]_{\mathcal{N}_-}, \quad \text{where } f = f_+ + f_-, \quad f_{\pm} \in \mathcal{N}_{\pm}.$$

If f_1, \dots, f_κ is an orthonormal basis of \mathcal{N}_- , then

$$(8.2) \quad K(p, q) - \sum_{j=1}^{\kappa} f_j(p) f_j(q)^*$$

is a reproducing kernel for \mathcal{N}_+ . The functions $f_j(p)$ are clearly slice hyperholomorphic in p since they belong to $\mathcal{P}(K)$ and so are the products $f_j(p) f_j(q)^*$ as well as the kernel (8.2). Therefore the elements in \mathcal{N}_+ are slice hyperholomorphic and so are the elements in \mathcal{P} . This concludes the proof. \square

9. GENERALIZED SCHUR FUNCTIONS

Definition 9.1. Let J_1 and J_2 be two signature matrices, respectively in $\mathbb{H}^{N \times N}$ and $\mathbb{H}^{M \times M}$, and assume that $\text{sq}_- J_1 = \text{sq}_- J_2$. A $\mathbb{H}^{N \times M}$ -valued function Θ , slice hyperholomorphic in a neighborhood \mathcal{V} of the origin, is called a *generalized Schur function* if the kernel

$$K_\Theta(p, q) = \sum_{\ell=0}^{\infty} p^\ell (J_2 - \Theta(p) J_1 \Theta(q)^*) \bar{q}^\ell$$

has a finite number, say κ , of negative squares in \mathcal{V} .

We will use the notation $\mathcal{S}_\kappa(J_2, J_1)$ for the class of such functions. When $N = M = 1$, $\kappa = 0$ and $J_1 = J_2 = 1$, this class was introduced in [2]. In the statement, a pair of operators (C, A) between appropriate spaces is called *observable* if

$$(9.1) \quad \bigcap_{n=0}^{\infty} \ker C A^n = \{0\}.$$

Theorem 9.2. Let Θ be slice hyperholomorphic in a neighborhood of the origin. Then, it is in $\mathcal{S}_\kappa(J_2, J_1)$ if and only if it can be written in the form

$$\Theta(p) = D + pC \star (I_{\mathcal{P}} - pA)^{-*} B,$$

where \mathcal{P} is a right quaternionic Pontryagin space of index κ , where the pair (C, A) is observable, and the operator matrix satisfies

$$(9.2) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_{\mathcal{P}} & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} I_{\mathcal{P}} & 0 \\ 0 & J_1 \end{pmatrix}.$$

Proof. We denote by $\mathcal{P}(\Theta)$ the right quaternionic reproducing kernel Pontryagin space with reproducing $K_\Theta(p, q)$. We follow the proof of [3, Theorem 2.2.1, p. 49], and we use the same densely defined linear relation as [2], but this time in $(\mathcal{P}(\Theta) \oplus \mathbb{H}_{J_2}^M) \times (\mathcal{P}(\Theta) \oplus \mathbb{H}_{J_1}^N)$. More precisely, R is now

$$\left\{ \begin{pmatrix} K_\Theta(p, q) \bar{q} u \\ \bar{q} v \end{pmatrix}, \begin{pmatrix} (K_\Theta(p, q) - K_\Theta(p, 0))u + K_\Theta(p, 0) \bar{q} v \\ (\Theta(q)^* - \Theta(0)^*)u + \Theta(0)^* \bar{q} v \end{pmatrix} \right\}.$$

Since $\text{sq}_-(J_1) = \text{sq}_-(J_2)$, these Pontryagin spaces have same negative index, and we then use Shmulyan's result to conclude. The arguments are similar to those in [2] and will be omitted. \square

We now characterize finite dimensional $\mathcal{P}(s)$ spaces. We begin with a preliminary proposition.

Proposition 9.3. *Let*

$$(9.3) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & J_1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} H & 0 \\ 0 & J_2 \end{pmatrix}$$

and

$$(9.4) \quad s(p) = D + pC \star (I - pA)^{-\star} B.$$

Then it holds that

$$J_2 - s(p)J_1s(q)^* = C \star (I - pA)^{-\star} \star (H - pH\bar{q}) \star_r (I - qA)^{-\star} \star_r C^*.$$

Proof. We rewrite the matrix identity (9.3) as:

$$J_2 - DJ_1D^* = CHC^*$$

$$BJ_1B^* = H - AHA^*$$

$$AHC^* = -BJ_1D^*.$$

In the sequel, for the sake of simplicity, we will write \star instead of \star_l . Let $s(p)$ be given by (9.4), and consider the function $J_2 - s(p)J_1s(q)^*$ which is slice hyperholomorphic in p and \bar{q} on the left and on the right, respectively. Let us compute

$$\begin{aligned} J_2 - s(p)J_1s(q)^* &= \\ &= J_2 - (D + pC \star (I - pA)^{-\star} B)J_1(D + qC \star (I - qA)^{-\star} B)^*. \end{aligned}$$

In order to preserve the hyperholomorphicity in p, \bar{q} we take, accordingly, the \star -product in p and \star_r -product in \bar{q} and we obtain:

$$\begin{aligned} J_2 - s(p)J_1s(q)^* &= J_2 - (D + pC \star (I - pA)^{-\star} B)J_1(D^* + B^* \star_r ((I - qA)^{-\star})^* \star_r C^* \bar{q}) \\ &= J_2 - DJ_1D^* - pC \star (I - pA)^{-\star} BJD^* - DJ_1B^* \star_r ((I - qA)^{-\star})^* \star_r C^* \bar{q} \\ &\quad - pC \star (I - pA)^{-\star} BJ_1B^* \star_r ((I - qA)^{-\star})^* \star_r C^* \bar{q}. \end{aligned}$$

Using the relations implied by (9.3) and the identities (9.6), we obtain

$$\begin{aligned} J_2 - s(p)J_1s(q)^* &= CHC^* + pC \star (I - pA)^{-\star} AHC^* + CHA^* \star_r ((I - qA)^{-\star})^* \star_r C^* \bar{q} \\ &\quad - pC \star (I - pA)^{-\star} (H - AHA^*) \star_r ((I - qA)^{-\star})^* \star_r C^* \bar{q} \\ &= C \star (I - pA)^{-\star} \star \left[(I - pA)HC^* + pAHC^* + (I - pA)HA^* \star_r ((I - qA)^{-\star})^* \star_r C^* \bar{q} \right. \\ &\quad \left. - p(H - AHA^*) \star_r ((I - qA)^{-\star})^* \star_r C^* \bar{q} \right] \\ &= C \star (I - pA)^{-\star} \star \left[(I - pA)H(I - qA)^* + pAH(I - qA)^* + (I - pA)HA^* \bar{q} \right. \\ &\quad \left. - p(H - AHA^*) \bar{q} \right] \star_r ((I - qA)^{-\star})^* \star_r C^* \\ &= C \star (I - pA)^{-\star} \star \left[H - HA^* \bar{q} - pAH + pAHA^* \bar{q} + pAH \right. \\ &\quad \left. - pAHA^* \bar{q} + HA^* \bar{q} - pAHA^* \bar{q} - pH\bar{q} + pAHA^* \bar{q} \right] \star_r ((I - qA)^{-\star})^* \star_r C^* \\ &= C \star (I - pA)^{-\star} \star (H - pH\bar{q}) \star_r ((I - qA)^{-\star})^* \star_r C^*. \end{aligned}$$

We can also write, in an equivalent way:

$$\begin{aligned} J_2 - s(p)J_1s(q)^* &= C \star (I - pA)^{-\star} H \star (1 - p\bar{q}) \star_r (I - qA)^{-\star} \star_r C^* \\ &= C \star (I - pA)^{-\star} \star (1 - p\bar{q}) \star_r H((I - qA)^{-\star})^* \star_r C^*, \end{aligned}$$

or

$$\begin{aligned} J_2 - s(p)J_1s(q)^* &= (C \star (I - pA)^{-\star}) \star (H - pH\bar{q}) \star_r (C \star (I - qA)^{-\star})^*. \end{aligned}$$

□

Specializing Theorem 9.2 to the finite dimensional case we obtain:

Theorem 9.4. *Let s be a generalized Schur function. The associated space right reproducing kernel Pontryagin space $\mathcal{P}(s)$ is finite dimensional if and only there exists a finite dimensional right Pontryagin space \mathcal{P} such that:*

$$s(p) = D + pC \star (I - pA)^{-\star} B,$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{P} \oplus \mathbb{H}_{J_2}^M \longrightarrow \mathcal{P} \oplus \mathbb{H}_{J_1}^N$$

is coisometric, that is:

$$(9.5) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_{\mathcal{P}} & 0 \\ 0 & J_2 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} I_{\mathcal{P}} & 0 \\ 0 & J_1 \end{pmatrix}.$$

Proof. One half of the theorem follows from the preceding proposition, while the other half is a special case of Theorem 9.2. □

Here we focus on the case $M = N$ and $\mathcal{P}(s)$ finite dimensional.

Definition 9.5. *Let $J \in \mathbb{H}^{N \times N}$ be a signature function. The $\mathbb{H}^{N \times N}$ -valued generalized function s belongs to $s \in \mathcal{U}_{\kappa}(J)$ if the space $\mathcal{P}(s)$ is finite dimensional and if $\text{sq}_{-}(s) = \kappa$.*

Theorem 9.6. *$s \in \mathcal{U}_{\kappa}(J)$ and it is slice hyperholomorphic in a neighborhood of the origin and only if it admits a realization*

$$s(p) = D + pC \star (I - pA)^{-\star} B$$

where A, B, C and D are matrices such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix}$$

for some Hermitian matrix $H \in \mathbb{H}^{N \times N}$.

Proof. First of all we observe that, if $f(p)$ is a (left) slice hyperholomorphic function, and C is a matrix, we have the following identities which immediately follow from the definition of (left) slice hyperholomorphic product:

$$(9.6) \quad pC \star_l f(p) = C \star_l pf(p) = C \star_l f(p) \star_l p.$$

An analogous property holds for the right slice hyperholomorphic product. It is also useful to recall that (compare with Section 3), if f, g are left slice hyperholomorphic functions, then $(f \star_l g)^* = g^* \star_r f^*$ and that $f \star_l C = fC$, and analogously, if h is right slice hyperholomorphic then $C \star_r h = Ch$. \square

For the complex-valued counterparts of the results in this section we refer to [6, 4]. These last papers also suggest factorization results, which will be considered elsewhere.

10. GENERALIZED CARATHÉODORY FUNCTIONS

To conclude this paper we briefly study the counterparts of the kernels (1.2).

Definition 10.1. *Let $J \in \mathbb{H}^{N \times N}$ be a signature matrix. A $\mathbb{C}^{N \times N}$ -valued function φ slice hyperholomorphic in a neighborhood \mathcal{V} of the origin is called a generalized Carathéodory function if the kernel*

$$k_\varphi(p, q) = \sum_{\ell=0}^{\infty} p^\ell (\varphi(p)J + J\varphi(q)^*) \bar{q}^\ell$$

has a finite number, say κ , of negative squares in \mathcal{V} .

We will use the notation $\mathcal{C}_\kappa(J)$ for the class of such functions. In the case of analytic functions, and for $N = 1$ and $\kappa = 0$, these functions appear in particular in the work of Herglotz, see [34], [30]. Still for analytic functions, these classes were introduced and studied by Krein and Langer, also in the operator-valued case. See [36]. We now give a realization theorem for such functions, which is the counterpart in the present setting of a result of Krein and Langer, see [36]. As for the realization of generalized Schur functions, we build a densely defined relation, and apply Shmulyan's theorem (Theorem 7.2 above). We follow in the present setting the arguments in [12, Theorem 5.2, p. 708]. For the notion of observability in the statement of the theorem, see (9.1). It is equivalent to:

$$(10.1) \quad C \star (I - pV)^{-*} f \equiv 0 \implies f = 0.$$

Theorem 10.2. *A $\mathbb{C}^{N \times N}$ -valued function φ slice hyperholomorphic in a neighborhood \mathcal{V} of the origin belongs to $\mathcal{C}_\kappa(J)$ if and only if it can be written as*

$$(10.2) \quad \varphi(p) = \frac{1}{2} C \star (I_{\mathcal{P}} + pV) \star (I_{\mathcal{P}} - pV)^{-*} C^* J + \frac{\varphi(0) - J\varphi(0)^* J}{2},$$

where \mathcal{P} is a right quaternionic Pontryagin space of index κ , V is a co-isometry in \mathcal{P} , and C is a bounded operator from \mathcal{P} into \mathbb{H}^N , and the pair (C, A) is observable.

Proof. Let $\mathcal{L}(\varphi)$ denote the reproducing kernel right quaternionic Pontryagin space of functions slice hyperholomorphic in \mathcal{V} , with reproducing kernel $k_\varphi(p, q)$, and proceed in a number of steps. Note that in the sequel, for the sake of simplicity, we will write I to denote the identity without specifying the space on which it is defined.

STEP 1: *The linear relation consisting of the pairs $(F, G) \in \mathcal{L}(\varphi) \times \mathcal{L}(\varphi)$ with*

$$F(p) = \sum_{j=1}^n k_\varphi(p, p_j) \bar{p}_j b_j, \quad \text{and} \quad G(p) = \sum_{j=1}^n k_\varphi(p, p_j) b_j - k_\varphi(p, 0) \left(\sum_{\ell=1}^n b_\ell \right),$$

where n varies in \mathbb{N} , $p_1, \dots, p_n \in \mathcal{V} \subset \mathbb{H}$ and $b_1, \dots, b_n \in \mathbb{H}^N$ is isometric, and where by $\overline{p_j}b_j$ we mean multiplication on the right by p_j on all the components of b_j .

We need to check that

$$(10.3) \quad [F, F]_{\mathcal{L}(\varphi)} = [G, G]_{\mathcal{L}(\varphi)}.$$

We have

$$\begin{aligned} [F, F]_{\mathcal{L}(\varphi)} &= \left[\sum_{j=1}^n k_{\varphi}(p, p_j) \overline{p_j} b_j, \sum_{k=1}^n k_{\varphi}(p, p_k) \overline{p_k} b_k \right]_{\mathcal{L}(\varphi)} \\ &= \sum_{j,k=1}^n b_k^* p_k k_{\varphi}(p_k, p_j) \overline{p_j} b_j \\ &= \sum_{\ell=1}^{\infty} \sum_{j,k=1}^n b_k^* p_k^{\ell+1} (\varphi(p_k) J + J \varphi(p_j)^*) \overline{p_j}^{\ell+1} b_j, \end{aligned}$$

while the inner product $[G, G]_{\mathcal{L}(\varphi)}$ is a sum of four terms: The first is

$$\sum_{j,k=1}^n b_k^* k_{\varphi}(p_k, p_j) b_j = \sum_{\ell=1}^{\infty} \sum_{j,k=1}^n b_k^* p_k^{\ell} (\varphi(p_k) J + J \varphi(p_j)^*) \overline{p_j}^{\ell} b_j.$$

Let

$$b = \sum_{\ell=1}^n b_{\ell}.$$

The second and third terms are

$$\begin{aligned} - \left(\sum_{k=1}^n b_k^* k_{\varphi}(p_k, 0) \right) b &= - \sum_{k=1}^n b_k^* (\varphi(p_k) J + J \varphi(0)^*) b \\ &= - \left(\sum_{k=1}^n b_k^* \varphi(p_k) J \right) b - b^* J \varphi(0)^* b, \end{aligned}$$

and

$$\begin{aligned} -b^* \left(\sum_{k=1}^n k_{\varphi}(0, p_j) b_j \right) &= \sum_{k=1}^n b^* (\varphi(0) J + J \varphi(p_j)^*) b_k \\ &= -b^* \varphi(0) J b - b^* \left(\sum_{j=1}^n J \varphi(p_j)^* b_j \right), \end{aligned}$$

respectively, and the fourth term is

$$b^* k_{\varphi}(0, 0) b = b^* (\varphi(0) J + J \varphi(0)^*) b.$$

Equation (10.3) follows since

$$\begin{aligned}
[F, F]_{\mathcal{L}(\varphi)} - \sum_{j,k=1}^n b_k^* k_\varphi(p_k, p_j) b_j &= \sum_{\ell=1}^{\infty} \sum_{j,k=1}^n b_k^* p_k^{\ell+1} (\varphi(p_k) J + J \varphi(p_j)^*) \overline{p_j}^{\ell+1} b_j - \\
&\quad - \sum_{\ell=1}^{\infty} \sum_{j,k=1}^n b_k^* p_k^{\ell} (\varphi(p_k) J + J \varphi(p_j)^*) \overline{p_j}^{\ell} b_j \\
&= \sum_{j,k=1}^n b_k^* (\varphi(p_k) J + J \varphi(p_j)^*) b_j.
\end{aligned}$$

The domain of R is dense. Thus by Shmulyan's theorem (Theorem 7.2 above), R is the graph of a densely defined isometry, which extends to an isometry to all of $\mathcal{L}(\varphi)$. We denote by T this extension.

STEP 2: *We compute the adjoint of the operator T .*

Let $f \in \mathcal{L}(\varphi)$, $h \in \mathbb{H}^N$ and $p \in \mathcal{V}$. We have:

$$\begin{aligned}
h^* p ((T^* f)(p)) &= [T^* f, k_\varphi(\cdot, p) \overline{p} h]_{\mathcal{L}(\varphi)} \\
&= [f, T(k_\varphi(\cdot, p) h)]_{\mathcal{L}(\varphi)} \\
&= [f, k_\varphi(\cdot, p) h - k_\varphi(\cdot, 0) h]_{\mathcal{L}(\varphi)} \\
&= h^* (f(p) - f(0)),
\end{aligned}$$

and hence (with $f(p) = \sum_{\ell=0}^{\infty} p^\ell f_\ell$)

$$(T^* f)(p) = \begin{cases} p^{-1}(f(p) - f(0)), & p \neq 0, \\ f_1, & p = 0. \end{cases}$$

STEP 3: *Formula (10.2) is in force.*

We first note that $f_\ell = C R_0^\ell f$, and so

$$f(p) = \sum_{\ell=0}^{\infty} p^\ell C R_0^\ell f = C \star (I - p R_0)^{-*} f.$$

Applying this formula to the function $C^* 1 = k_\varphi(\cdot, 0)$ we obtain:

$$\varphi(p) J + J \varphi(0)^* = C \star (I - p R_0)^{-*} C^* 1 \quad \text{and} \quad \varphi(0) J + J \varphi(0)^* = C C^* 1.$$

Multiplying the second equality by $1/2$ and making the difference with the first equality we get

$$\varphi(p) J + \frac{1}{2} (J \varphi(0)^* - \varphi(0) J) = \frac{1}{2} C \star (I - p R_0)^{-*} \star (I + p R_0) C^*.$$

STEP 4: *We show that conversely, every function of the form (10.2) is in $\mathcal{C}_\kappa(J)$.*

From (10.2) we obtain

$$(10.4) \quad \varphi(p) J + J \varphi(q)^* J = C \star (I - p V)^{-*} \star (1 - p \overline{q}) \star_r ((I - q V)^{-*})^* \star_r C^*.$$

So the reproducing kernel of $\mathcal{L}(\varphi)$ can be written as

$$k_\varphi(p, q) = C \star (I - p V)^{-*} ((I - q V)^{-*})^* \star_r C^*,$$

since, in view of (10.4), the right side of the above equation satisfies

$$k_\varphi(p, q) - pk_\varphi(p, q)\bar{q} = \varphi(p)J + J\varphi(q)^*J.$$

In view of (10.1), $\mathcal{L}(\varphi)$ consists of the functions of the form

$$f(p) = C \star (I - pV)^{-*}\xi, \quad \xi \in \mathcal{P},$$

with the inner product

$$[f, g]_{\mathcal{L}(\varphi)} = [\xi, \eta]_{\mathcal{P}} \quad (\text{with } g(p) = C \star (I - pV)^{-*}\eta),$$

and so the kernel k_φ has exactly κ negative squares. \square

Corollary 10.3. *When $J = I_N$ and $\kappa = 0$, the function φ has a slice holomorphic extension to all of the unit ball of \mathbb{H} .*

Proof. This follows from (10.2) since V is then contractive. \square

When $J = I_N$, and in the complex variable setting generalized Carathéodory functions admit another representation, namely

$$(10.5) \quad \varphi(z) = g(z)\varphi_0(z)g(1/\bar{z})^*,$$

where φ_0 is a Carathéodory function (that is, the corresponding kernel is positive definite) and g is analytic and invertible in the open unit disk. See [31, 29, 28]. We note that in the rational case, generalized Carathéodory functions are called generalized positive functions, and play an important role in linear system theory. We refer to [7] for a survey of the literature and a constructive proof of the factorization (10.5) (in the half-line case) in the scalar rational case.

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