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ON DISCRETE ANALYTIC FUNCTIONS: PRODUCTS, RATIONAL FUNCTIONS, AND SOME ASSOCIATED REPRODUCING KERNEL HILBERT SPACES

DANIEL ALPAY, PALLE JORGENSEN, RON SEAGER, AND DAN VOLOK

Abstract. We introduce a family of discrete analytic functions, called expandable discrete analytic functions, which includes discrete analytic polynomials, and define two products in this family. The first one is defined in a way similar to the Cauchy-Kovalevskaya product of hyperholomorphic functions, and allows us to define rational discrete analytic functions. To define the second product we need a new space of entire functions which is contractively included in the Fock space. We study in this space some counterparts of Schur analysis.

Contents

1. Introduction

In this paper, we explore a spectral theoretic framework for representation of discrete analytic functions. While the more familiar classical case of analyticity plays

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an important role in such applications as the theory of systems and their realizations, carrying over this to the case of discrete analytic functions involves a number of operator- and spectral theoretic subtleties; for example, we show that the reproducing kernel, in the discrete case, behaves quite differently from the case of the more familiar classical kernels of Szegö and Bergman. We introduce a reproducing kernel of discrete analytic functions, which is naturally isomorphic to a Hilbert space of entire functions contractively included in the Fock space. The pointwise product of two discrete analytic functions need not be discrete analytic, and we introduce two products, each taking into account the specificities of discrete analyticity.

The first product is determined by a solution to an extension question for rational functions, extending from $\mathbb{Z}_+$ to the right half-plane in the $2D$ lattice $\mathbb{Z}^2$. Our solution to the extension problem leads to a new version of the multiplication operator $\mathcal{Z}$. We further prove that the product in $\mathcal{A}$ will be defined directly from $\mathcal{Z}$. This in turn yields a representation of the multiplier problem for the reproducing kernel Hilbert space $\mathcal{H}$. While it is possible to think of the reproducing kernel Hilbert space $\mathcal{H}$ as an extension of one of the classical Beurling-Lax theory for Hardy space, the case for discrete analytic function involves a new and different spectral analysis, departing from the classical case in several respects. For example, we show that the new multiplication operator $\mathcal{Z}$ is part of an infinite-dimensional non-Abelian Lie algebra of operators acting on the reproducing kernel Hilbert space $\mathcal{H}$. With this, we are able to find the spectral type of the operators described above.

The theory of discrete analytic functions has drawn a lot of attention recently, in part because of its connections with electrical networks and random walks. In the case of functions defined on the integer grid the notion of discrete analyticity was introduced by J. Ferrand (Lelong) in [15]:

Definition 1.1. *A function $f : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ is said to be* discrete analytic *if*

$$\forall (x,y) \in \mathbb{Z}^2, \qquad \frac{f(x+1,y+1)-f(x,y)}{1+i} = \frac{f(x+1,y)-f(x,y+1)}{1-i}. \tag{1.1}$$

The properties of discrete analytic functions were extensively investigated by R. J. Duffin in [13]. In this work it was shown that discrete analytic functions share many important properties of the classical continuous analytic functions in the complex domain, such as Cauchy integral representation and the maximum modulus principle. More recently, the notion of discrete analyticity and accompanying results were extended by C. Mercat to the case of functions defined on arbitrary graph embedded in an orientable surface; see [22].

The concept of discrete analyticity seems to cause significant difficulties in the following regard: the pointwise product of two discrete analytic functions is not necessarily discrete analytic. For example, the functions $z := x + iy$ and z^2 are discrete analytic in the sense of Definition 1.1, but z^3 is not. Thus a natural question arises, how to describe all complex polynomials in two variables x, y whose restriction to the integer grid $\mathbb{Z}^2$ is discrete analytic, and, more generally, rational discrete analytic functions. This problem was originally considered by R. Isaacs, using a definition of discrete analyticity (the so-called monodiffricity), which is algebraically simpler than Definition 1.1. In [17] R. Isaacs has posed a conjecture that *all monodiffric rational functions are polynomials.* This conjecture was disproved by C. Harman in [16], where an explicit example of a non-polynomial monodiffric function, rational in one quadrant, was constructed.

The results of R. Isaacs and C. Harman suggest that in the setting of discrete analytic functions the notion of rationality based on the pointwise product is not a suitable one. In order to introduce a class of rational discrete analytic functions, which would be sufficiently rich for applications, one needs a suitable definition of the product. This is one of the main objective of the present paper to introduce two products in the setting of discrete analytic functions.

Organization: The paper is organized as follows. Sections 2 through 4 cover our preparation of the discrete framework: analysis and tools. In Definition 2.1, the notion of discrete analyticity makes a key link between the representation in the two integral variables (x, y) in the 2-lattice $\mathbb{Z}^2$, thus making precise the interaction between the two integral variables x and y implied by analyticity. The notion of analyticity in the discrete case is a basic rule (Definition 1.1) from which one makes precise contour-summations around closed loops in $\mathbb{Z}^2$. In section 2, we introduce a basis system of polynomials (which will appear to be restriction to the positive real axis of discrete analytic polynomials ζ_n defined in section 5), see equation (2.1). We further introduce a discrete Fourier transform for functions of (x, y) in the 2-lattice $\mathbb{Z}^2$, and in the right half-plane $\mathbb{H}_+ = \mathbb{Z}_+ \times \mathbb{Z}$ in $\mathbb{Z}^2$. The Fourier representation in $\mathbb{H}_+$ is then used in sections 3 and 4; where we study extensions from $\mathbb{Z}$ to $\mathbb{Z}^2$, and from $\mathbb{Z}_+$ to $\mathbb{H}_+$. We begin our analysis in section 4 with some lemmas for the polynomial case. Theorem 4.1 offers a discrete version of the Cauchy-Riemann equations; we show that the discrete analytic functions are defined as the kernel of a del-bar operator $\overline{\mathcal{D}}$; to be studied in detail in section 7, as part of a Lie algebra representation. In Theorem 4.2 we show that every polynomial function on $\mathbb{Z}$ has a unique discrete analytic extension to $\mathbb{Z}^2$. Sections 5 and 6 deal with expandable functions (Definition 5.5), and section 7 rational discrete analytic functions. The expandable functions are defined from a basis system of discrete analytic polynomials ζ_n from section 2, and a certain Cauchy-estimate, equation (5.10). We shall need two products defined on expandable functions, the first is our Cauchy-Kovalesvskaya product (in section 6), and the second (section 10) is defined on algebra generated by the discrete analytic polynomials ζ_n. Its study makes use of realizations from linear systems theory. The definition of the Cauchy-Kovalesvskaya product relies on uniqueness of extensions for expandable functions (Corollary 5.6). Hence it is defined first for expandable functions, and then subsequently enlarged; first to the discrete analytic rational functions (section 7), and then to a new reproducing kernel Hilbert space in section 8. The latter reproducing kernel Hilbert space has its kernel defined from the discrete analytic polynomials ζ_n; see (8.1). The study of the reproducing kernel Hilbert space in turns involves such tools from analysis as representations of Lie algebras (Theorem 7.4), and of C^*-algebras (Theorem 8.4).

2. Polynomials and rational functions on the set of integers

In what follows, Ω stands for one of two sets: $\mathbb{Z}$ or $\mathbb{Z}_+$, and $x^{[n]}$ denotes the polynomial of degree n defined by

$$x^{[n]} := \prod_{j=0}^{n-1}(x-j) \tag{2.1}$$

(if $n=0$, $x^{[0]} := 1$).

They have the generating function

$$(1+t)^x = \sum_{k=0}^{x} x^{[k]}\frac{t^k}{k!}, \tag{2.2}$$

and their discrete analytic extensions $\zeta_n(x,y)$ are studied in section 5.

The purpose of this section is to prove (see Theorem 2.9 below) that any rational function $f : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ (see Definition 2.7) has a unique representation

$$f(x) = \sum_{n=1}^{\infty}\widehat{f}(n)x^{[n]}, \tag{2.3}$$

where

$$\limsup_{n\to\infty}\left(n!|\widehat{f}(n)|\right)^{\frac{1}{n}} \leq 1. \tag{2.4}$$

Definition 2.1. *The linear* difference operator δ *on the space of functions* $f : \Omega \longrightarrow \mathbb{C}$ *is defined by*

$$(\delta f)(x) = f(x+1) - f(x), \quad x \in \Omega.$$

Proposition 2.2. *Let* $f : \mathbb{Z}_+ \longrightarrow \mathbb{C}$*. Then, for every* $x \in \mathbb{Z}_+$ *the series*

$$\check{f}(x) := \sum_{n\in\mathbb{Z}_+} f(n)x^{[n]}$$

has a finite number of non zero terms.

Proof. In view of (2.1),

$$\forall x \in \mathbb{Z}_+,\ \forall n \in \mathbb{Z}_+, \qquad x < n \Longrightarrow x^{[n]} = 0.$$

Therefore, for every $x \in \mathbb{Z}_+$ the series $\check{f}(x)$ contains at most $x+1$ non-zero terms. □

Proposition 2.3. *Let* $f : \mathbb{Z}_+ \longrightarrow \mathbb{C}$*. Then there exists a unique function* $\hat{f} : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ *such that*

$$\forall x \in \mathbb{Z}_+, \qquad f(x) = \sum_{n\in\mathbb{Z}_+}\hat{f}(n)x^{[n]}. \tag{2.5}$$

The function $\hat{f}(n)$ *is given by*

$$\hat{f}(n) = \frac{(\delta^n f)(0)}{n!}. \tag{2.6}$$

Proof. To show the uniqueness of $\hat{f}(n)$, assume that (2.5) holds for some function $\hat{f}(n)$ and apply the identity

$$\delta x^{[n]} = n x^{[n-1]} \tag{2.7}$$

repeatedly to obtain (2.6).

Next, let the function $\hat{f}(n)$ be defined by (2.6). Then, according to Proposition 2.2, the series

$$\tilde{f}(x) := \sum_{n\in\mathbb{Z}_+} \hat{f}(n) x^{[n]}$$

converges absolutely for every $x \in \mathbb{Z}_+$ and defines a function $\tilde{f} : \mathbb{Z}_+ \longrightarrow \mathbb{C}$. Hence (2.6) holds with $\tilde{f}$ replacing f and

$$\forall n \in \mathbb{Z}_+, \qquad (\delta^n f)(0) = n!\hat{f}(n) = (\delta^n \tilde{f})(0).$$

Now one can verify that

$$\forall x \in \mathbb{Z}_+,\ \forall n \in \mathbb{Z}_+, \qquad (\delta^n f)(x) = (\delta^n \tilde{f})(x) \tag{2.8}$$

by induction on x: If

$$\forall n \in \mathbb{Z}_+, \qquad (\delta^n f)(x) = (\delta^n \tilde{f})(x)$$

then

$$\begin{aligned}\forall n \in \mathbb{Z}_+, \qquad (\delta^n f)(x+1) &= (\delta^n f)(x) + (\delta^{n+1} f)(x) \\ &= (\delta^n \tilde{f})(x) + (\delta^{n+1} \tilde{f})(x) = (\delta^n \tilde{f})(x+1).\end{aligned}$$

It remains to set $n = 0$ in (2.8) to obtain (2.5). □

Definition 2.4. *Let $f : \mathbb{Z}_+ \longrightarrow \mathbb{C}$. Then the function $\hat{f} : \mathbb{Z}_+ \longrightarrow \mathbb{C}$, defined by* (2.6)*, is said to be the Fourier transform of f.*

Suppose that a function $f : \mathbb{Z} \longrightarrow \mathbb{C}$ is such that

$$\forall x \in \mathbb{Z}, \qquad f(x) = p(x),$$

where $p(x)$ is a polynomial with complex coefficients. Then such a polynomial $p(x)$ is unique, and we shall call the function f itself a polynomial. If $f(x) \not\equiv 0$, the degree of $f(x)$ is the same as the degree of $p(x)$.

Proposition 2.5. *Let $f : \mathbb{Z} \longrightarrow \mathbb{C}$ be a polynomial, and let $\hat{f} : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ be the Fourier transform of the restriction $f_{|\mathbb{Z}_+}$. Then the function $\hat{f}$ has a finite support and*

$$\forall x \in \mathbb{Z}, \qquad f(x) = \sum_{n\in\mathbb{Z}_+} \hat{f}(n) x^{[n]}.$$

Proof. Note that

$$\deg(\delta f) = \deg(f) - 1$$

(if $f = \text{const}$, $\delta f = 0$). Hence

$$\forall n \in \mathbb{Z}_+, \qquad n > \deg(f) \Longrightarrow \delta^n f = 0.$$

Since

$$(\delta f)_{|\mathbb{Z}_+} = \delta(f_{|\mathbb{Z}_+}),$$

(2.6) implies that

$$\forall n > \deg(f), \qquad \hat{f}(n) = 0.$$

It follows that

$$\sum_{n \in \mathbb{Z}_+} \hat{f}(n) x^{[n]}$$

is, in fact, a polynomial, which coincides with the polynomial $f(x)$ on $\mathbb{Z}_+$ and hence on $\mathbb{Z}$. □

It follows from Proposition 2.5 and identity (2.7) that if $f : \mathbb{Z} \longrightarrow \mathbb{C}$ is a polynomial then so is δf. A converse statement can be formulated as follows:

Proposition 2.6. *Let $f : \mathbb{Z} \longrightarrow \mathbb{C}$ be a polynomial. Then there exists a polynomial $g : \mathbb{Z} \longrightarrow \mathbb{C}$ such that $f(x) \equiv (\delta g)(x)$. If $f(x) \not\equiv 0$ then $\deg(g) = \deg(f) + 1$.*

Proof. By Proposition 2.5,

$$f(x) = \sum_{n \in \mathbb{Z}_+} \hat{f}(n) x^{[n]},$$

where $\hat{f} : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ is a function with finite support. Consider the polynomial

$$g(x) = \sum_{n \in \mathbb{Z}_+} \frac{\hat{f}(n)}{n+1} x^{[n+1]},$$

then, in view of (2.7),

$$\forall x \in \mathbb{Z}, \qquad (\delta g)(x) = \sum_{n \in \mathbb{Z}_+} \hat{f}(n) x^{[n]} = f(x).$$

□

Definition 2.7. *A function $f : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ is said to be* rational *if there exist polynomials $p, q : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ such that*

$$\forall x \in \mathbb{Z}_+, \qquad q(x) \neq 0,$$

and

$$\forall x \in \mathbb{Z}_+, \qquad f(x) = \frac{p(x)}{q(x)}.$$

Proposition 2.8. *Let $f : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ be given. Then $f(x)$ is a rational function if and only if there exist a polynomial $p(x)$ and matrices A, B, C such that*

$$\sigma(A) \cap \mathbb{Z}_+ = \emptyset, \tag{2.9}$$

and

$$f(x) = p(x) + C(xI - A)^{-1}B. \tag{2.10}$$

Proof. We first recall that a matrix-valued rational function of a complex variable can always be written in the form

$$r(z) = p(z) + C(zI - A)^{-1}B, \tag{2.11}$$

where the matrix-valued polynomial p takes care of the pole at infinity, and A, B and C are matrices of appropriate sizes. Furthermore, when the dimension of A is minimal, the (finite) poles of r coincide with the spectrum of A; see [8, 19]. Here, we consider complex-valued functions, and thus C and B are respectively a row and column vector.

Now, by Definition 2.7, f is a rational function if and only if it is the restriction on $\mathbb{Z}_+$ of a rational function of a complex variable, with no poles on $\mathbb{Z}_+$, that is if and only if it can be written as (2.10) with the matrix A satisfying furthermore (2.9). □

Theorem 2.9. *Let $f : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ be rational, and let $\hat{f} : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ be the Fourier transform of f. Then*

$$\limsup_{n\to\infty}(|\hat{f}(n)|n!)^{1/n} \leq 1.$$

Proof. According to Proposition 2.8, there exist a polynomial $p(x)$ and matrices A, B, C such that $\sigma(A) \cap \mathbb{Z}_+ = \emptyset$ and

$$f(x) = p(x) + C(xI - A)^{-1}B.$$

Hence, for $n > \deg(p)$,

$$|\hat{f}(n)n!| = |(\delta^n f)(0)| \leq \|C\| \cdot \|B\| \cdot \|A^{-1}\| \cdot \prod_{j=1}^{n} \| \left(I - \frac{1}{n}A\right)^{-1} \|.$$

Since

$$\lim_{n\to\infty} \| \left(I - \frac{1}{n}A\right)^{-1} \| = 1,$$

$$\forall \epsilon > 0,\ \exists M, N \in \mathbb{Z}_+, \forall n \in \mathbb{Z}_+ \quad : \quad n \geq N \Longrightarrow |\hat{f}(n)n!| \leq M(1+\epsilon)^n,$$

and the conclusion follows. □

3. Discrete polynomials of two variables

Definition 3.1. *The linear* difference operators δ_x, δ_y *on the space of functions $f : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{C}$ are defined by*

$$(\delta_x f)(x, y) := f(x+1, y) - f(x, y), \quad (\delta_y f)(x, y) := f(x, y+1) - f(x, y).$$

Note that the difference operators δ_x and δ_y commute:

(3.1)

$$(\delta_x\delta_y f)(x, y) = (\delta_y\delta_x f)(x, y) = f(x+1, y+1) - f(x, y+1) - f(x+1, y) + f(x, y).$$

Proposition 3.2. *Let $f : \mathbb{Z}_+^2 \longrightarrow \mathbb{C}$. Then for every $(x, y) \in \mathbb{Z}_+^2$, the series*

$$\check{f}(x, y) := \sum_{(m,n)\in\mathbb{Z}_+^2} f(m, n)x^{[m]}y^{[n]} \tag{3.2}$$

contains finitely many non-zero terms.

Proof. In view of (2.1),

$$\forall x \in \mathbb{Z}_+, \forall n \in \mathbb{Z}_+, \quad x < n \Longrightarrow x^{[n]} = 0.$$

Therefore, for every $(x, y) \in \mathbb{Z}_+^2$ the series $\check{f}(x)$ contains at most $(x+1)(y+1)$ non-zero terms. □

Formula (3.2) can be viewed as a transform of a discrete function. The inverse transform is calculated in the next proposition.

Proposition 3.3. *Let $f : \mathbb{Z}_+^2 \longrightarrow \mathbb{C}$. Then there exists a unique function $\hat{f} : \mathbb{Z}_+^2 \longrightarrow \mathbb{C}$ such that*

$$\forall (x,y) \in \mathbb{Z}_+^2, \qquad f(x,y) = \sum_{(m,n)\in\mathbb{Z}_+^2} \hat{f}(m,n) x^{[m]} y^{[n]}. \tag{3.3}$$

The function $\hat{f}(m,n)$ is given by

$$\hat{f}(m,n) = \frac{(\delta_x^m \delta_y^n f)(0,0)}{m!n!}, \quad (m,n) \in \mathbb{Z}_+^2. \tag{3.4}$$

Proof. First, fix $x \in \mathbb{Z}_+$ and consider the function $f_x : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ given by

$$f_x(y) = f(x,y), \quad y \in \mathbb{Z}_+.$$

Then, according to Proposition 2.3, there is a unique function $\hat{f}_x : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ such that

$$\forall y \in \mathbb{Z}_+, \qquad f_x(y) = \sum_{n\in\mathbb{Z}_+} \hat{f}_x(n) y^{[n]};$$

the function $\hat{f}_x(n)$ is given by

$$\hat{f}_x(n) = \frac{(\delta_y^n f)(x,0)}{n!}, \quad n \in \mathbb{Z}_+.$$

Next, fix $n \in \mathbb{Z}_+$ and consider the function $g_n : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ given by

$$g_n(x) = \hat{f}_x(n), \quad x \in \mathbb{Z}_+.$$

By the same Proposition 2.3, there is a unique function $\hat{g_n} : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ such that

$$\forall x \in \mathbb{Z}_+, \qquad g_n(x) = \sum_{m\in\mathbb{Z}_+} \hat{g_n}(m) x^{[m]};$$

the function $\hat{g_n}(m)$ is given by

$$\hat{g_n}(m) = \frac{(\delta_x^m \delta_y^n f)(0,0)}{m!n!}, \quad m \in \mathbb{Z}_+.$$

Thus

$$\forall (x,y) \in \mathbb{Z}_+^2, \qquad f(x,y) = \sum_{(m,n)\in\mathbb{Z}_+^2} \hat{g_n}(m) x^{[m]} y^{[n]}.$$

It remains to set $\hat{f}(m,n) = \hat{g_n}(m)$. ☐

Definition 3.4. *Let $f : \mathbb{Z}_+^2 \longrightarrow \mathbb{C}$. Then the function $\hat{f} : \mathbb{Z}_+^2 \longrightarrow \mathbb{C}$, defined by* (3.4)*, is said to be the Fourier transform of f.*

Theorem 3.5. *Let $p(z,w)$ be a complex polynomial in two variables. Then, $p\big|_{\mathbb{Z}^2} = 0$ if and only if $p \equiv 0$.*

Proof. Write

$$p(z,w) = \sum_{n=0}^{N} p_n(z) w^n, \tag{3.5}$$

where the p_n are polynomials in z. The equations $p(z,w) \equiv 0$ for $w = 0, 1, \ldots N$ lead to (using a Vandermonde determinant) that $p_0(z), p_1(z), \ldots, p_N(z)$ vanish on $\mathbb{Z}$ and hence identically. ☐

Suppose that a function $f : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ is such that

$$\forall (x, y) \in \mathbb{Z}^2, \qquad f(x, y) = p(x, y),$$

where $p(x, y)$ is a polynomial with complex coefficients. Then in view of Theorem 3.5, such a polynomial $p(x, y)$ is unique, and we shall call the function f itself a polynomial. If $f(x, y) \not\equiv 0$, the degree of $f(x, y)$ is the same as the degree of $p(x, y)$.

Proposition 3.6. *Let $f : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ be a polynomial, and let $\hat{f} : \mathbb{Z}_+^2 \longrightarrow \mathbb{C}$ be the Fourier transform of the restriction $f_{|_{\mathbb{Z}_+^2}}$. Then the function $\hat{f}$ has a finite support and*

$$\forall (x, y) \in \mathbb{Z}^2, \quad f(x, y) = \sum_{(m,n) \in \mathbb{Z}_+^2} \hat{f}(m, n) x^{[m]} y^{[n]}.$$

Proof. First, in view of (3.4) and of the fact that

$$\forall (m, n) \in \mathbb{Z}_+^2, \ \forall (x, y) \in \mathbb{Z}^2, \quad m + n > \deg(f) \Longrightarrow (\delta_x^m \delta_y^n f)(x, y) = 0,$$

the function $\hat{f}$ has a finite support. It follows that

$$\sum_{(m,n) \in \mathbb{Z}_+^2} \hat{f}(m, n) x^{[m]} y^{[n]}$$

is, in fact, a polynomial, which coincides with the polynomial $f(x, y)$ on $\mathbb{Z}_+^2$ and hence on $\mathbb{Z}^2$. □

It follows from Proposition 3.6 and identity (2.7) that if $f : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ is a polynomial then so are $\delta_x f$ and $\delta_y f$. A converse statement can be formulated as follows. It will be used in the proof of Theorem 4.2,

Proposition 3.7. *Let $f, g : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ be two polynomials, such that*

$$(\delta_y f)(x, y) \equiv (\delta_x g)(x, y). \tag{3.6}$$

Then there exists a polynomial $h : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ such that

$$(\delta_x h)(x, y) \equiv f(x, y), \quad (\delta_y h)(x, y) \equiv g(x, y). \tag{3.7}$$

Proof. By Proposition 3.6, there exist functions $\hat{f}, \hat{g} : \mathbb{Z}_+^2 \longrightarrow \mathbb{C}$ with finite support, such that

$$f(x, y) = \sum_{(m,n) \in \mathbb{Z}_+^2} \hat{f}(m, n) x^{[m]} y^{[n]}, \quad g(x, y) = \sum_{(m,n) \in \mathbb{Z}_+^2} \hat{g}(m, n) x^{[m]} y^{[n]}.$$

Then identity (3.6) implies that

$$\forall (m, n) \in \mathbb{Z}_+^2, \qquad (n + 1) \hat{f}(m, n + 1) = (m + 1) \hat{g}(m + 1, n). \tag{3.8}$$

Consider the polynomial

$$h(x, y) = \sum_{(m,n) \in \mathbb{Z}_+^2} \frac{\hat{f}(m, n)}{m + 1} x^{[m+1]} y^{[n]} + \sum_{n \in \mathbb{Z}_+} \frac{\hat{g}(0, n)}{n + 1} y^{[n+1]},$$

then

$$\forall (x, y) \in \mathbb{Z}^2, \qquad (\delta_x h)(x, y) = \sum_{(m,n) \in \mathbb{Z}_+^2} \hat{f}(m, n) x^{[m]} y^{[n]} = f(x, y).$$

On the other hand, in view of (3.8),

$$h(x,y) = \sum_{(m,n)\in\mathbb{Z}_+^2} \frac{\hat{g}(m,n)}{n+1} x^{[m]} y^{[n+1]} + \sum_{n\in\mathbb{Z}_+} \frac{\hat{f}(m,0)}{m+1} x^{[m+1]},$$

hence

$$\forall (x,y)\in\mathbb{Z}^2, \qquad (\delta_y h)(x,y) = \sum_{(m,n)\in\mathbb{Z}_+^2} \hat{g}(m,n) x^{[m]} y^{[n]} = g(x,y).$$

□

4. Discrete analytic polynomials

Difference operators: It is convenient to recast Definition 1.1 in terms of the difference operators. In what follows, each of the sets Ω_1, Ω_2 is either $\mathbb{Z}$ or $\mathbb{Z}_+$.

Theorem 4.1. *A function $f : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{C}$ is discrete analytic if and only if*

$$\forall (x,y)\in \Omega_1\times\Omega_2, \qquad (\bar{\mathcal{D}} f)(x,y) = 0,$$

where

$$\bar{\mathcal{D}} := (1-i)\delta_x + (1+i)\delta_y + \delta_x\delta_y. \tag{4.1}$$

Proof. In view of (3.1),

$$\begin{aligned}
\forall (x,y)\in\Omega_1\times\Omega_2, \quad & \frac{f(x+1,y+1)-f(x,y)}{1+i} - \frac{f(x+1,y)-f(x,y+1)}{1-i} \\
&= \frac{1-i}{2}(f(x+1,y+1) - f(x,y) - if(x+1,y) + if(x,y+1)) \\
&= \frac{1-i}{2}((\delta_x\delta_y f)(x,y) - 2f(x,y) + (1-i)f(x+1,y) + (1+i)f(x,y+1)) \\
&= \frac{1-i}{2}((\delta_x\delta_y f)(x,y) + (1-i)(\delta_x f)(x,y) + (1+i)(\delta_y f)(x,y)).
\end{aligned}$$

□

Extension: In view of Definition 1.1, given $f_0 : \mathbb{Z} \longrightarrow \mathbb{C}$ there are infinitely discrete analytic functions f on $\mathbb{Z}^2$ such that $f(x,0) = f_0(x)$. However, the following theorem show that in the case when f_0 is a polynomial, only one of these discrete analytic extensions will be a polynomial in x, y. The result itself originates with the work of Duffin [13], and we give a new proof.

Theorem 4.2. *Let $p : \mathbb{Z} \longrightarrow \mathbb{C}$ be a polynomial. Then there exists a unique discrete analytic polynomial $q : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ such that*

$$\forall x\in\mathbb{Z}, \qquad q(x,0) = p(x).$$

In particular, $q(x,y) \equiv 0$ if and only if $p(x) \equiv 0$. If this is not the case,

$$\deg(q) = \deg(p).$$

Lemma 4.3. *Let $q : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ be a discrete analytic polynomial, such that $q(x,0) \equiv 0$. Then $q(x,y) \equiv 0$.*

Proof. Assume the opposite, then $\deg(q) > 0$, and q can be chosen so that $\deg q$ is the smallest possible. Observe that $(\delta_x q)(x, y)$ is also a discrete analytic polynomial, that $(\delta_x q)(x, 0) \equiv 0$, and that $\deg(\delta_x p) < \deg(p)$. Hence $(\delta_x q)(x, y) \equiv 0$. Since $q(x, y)$ is discrete analytic, Definition 4.1 implies that

$$\forall (x, y) \in \mathbb{Z}^2, \qquad (\delta_y q)(x, y) = \frac{i-1}{2}((1 - i + \delta_y)\delta_x q)(x, y) = 0.$$

Thus

$$(\delta_x q)(x, y) \equiv (\delta_y q)(x, y) \equiv 0$$

and hence $q =$ const - a contradiction. □

Proof of Theorem 4.2. The uniqueness of the polynomial $q(x, y)$ follows from Lemma 4.3. The existence in the case $p =$ const is clear: it suffices to set

$$q(x, y) = p(0).$$

If $p \neq$ const we proceed by induction on $d = \deg(p)$. According to Proposition 2.6, $(\delta p)(x)$ is a polynomial and $\deg(\delta p) = d - 1$. Therefore, by the induction assumption, there is a discrete analytic polynomial $f(x, y)$, such that

$$\forall x \in \mathbb{Z}, \qquad f(x, 0) = (\delta p)(x)$$

and $\deg(f) = d - 1$. Let $g : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ be defined by

$$g(x, y) = if(x, y) - \frac{1-i}{2}(\delta_y f)(x, y),$$

then $g(x, y)$ is also a discrete analytic polynomial, $\deg(g) = d - 1$. Furthermore, since $(\bar{D}f)(x, y) \equiv 0$,

$$\forall (x, y) \in \mathbb{Z}^2, \quad (\delta_x g)(x, y) = i(\delta_x f)(x, y) - \frac{1-i}{2}(\delta_x \delta_y f)(x, y)$$
$$= i(\delta_x f)(x, y) + \frac{1-i}{2}((1 - i)(\delta_x f)(x, y) + (1 + i)(\delta_y f)(x, y)) = (\delta_y f)(x, y).$$

Hence, according to Proposition 3.7, there exists a polynomial $h : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ such that

$$(\delta_x h)(x, y) \equiv f(x, y), \quad (\delta_y h)(x, y) \equiv g(x, y).$$

Since

$$\forall (x, y) \in \mathbb{Z}^2, \qquad (\bar{D}h)(x, y) = (1 - i)f(x, y) + (1 + i)g(x, y) + (\delta_y f)(x, y) = 0,$$

the polynomial $h(x, y)$ is discrete analytic. Finally, since

$$\forall x \in \mathbb{Z}, \qquad (\delta_x h)(x, 0) = f(x, 0) = (\delta p)(x),$$

$h(x, 0) - p(x)$ is a constant function. Thus it suffices to set

$$q(x, y) = h(x, y) - h(0, 0) + p(0)$$

to complete the proof. □

5. Expandable discrete analytic functions

In view of Theorem 4.2, there exists a unique discrete analytic polynomial $\zeta_n(x,y)$ determined by

$$\zeta_n(x,0) \equiv x^{[n]}.$$

Then (as follows from Proposition 3.6 and identities (2.7), (3.1)) $(\delta_x\zeta_n)(x,y)$ is also a discrete analytic polynomial such that

$$(\delta_x\zeta_n)(x,0) \equiv \delta x^{[n]} \equiv n x^{[n-1]}.$$

Hence, by the uniqueness part of Theorem 4.2,

$$(\delta_x\zeta_n)(x,y) \equiv n\zeta_{n-1}(x,y) \tag{5.1}$$

(if $n=0$, $\zeta_0(x,y)\equiv 1$ and $(\delta_x\zeta_0)(x,y)\equiv 0$).

Proposition 5.1. *For each $(x,y)\in\mathbb{Z}^2$ the function*

$$e_{x,y}(z) = (1+z)^x\left(\frac{1+i+iz}{1+i+z}\right)^y, \tag{5.2}$$

is analytic (in the usual sense) in the variable z in the open unit disk $\mathbb{D}$, and admits the Taylor expansion

$$e_{x,y}(z) = \sum_{n\in\mathbb{Z}_+}\frac{z^n\zeta_n(x,y)}{n!}, \quad \forall z\in\mathbb{D}. \tag{5.3}$$

Proof. The analyticity is clear because $x,y\in\mathbb{Z}$ and we have

$$e_{x,y}(z) = \sum_{n\in\mathbb{Z}_+}\frac{z^n c_n(x,y)}{n!}, \quad \forall z\in\mathbb{D}.$$

where

$$c_n(x,y) = \frac{d^n}{dz^n}e_{x,y}\Big|_{z=0}.$$

Since

$$e_{x+1,y}(z) - e_{x,y}(z) = z e_{x,y}(z) = \sum_{n\in\mathbb{Z}_+}\frac{z^{n+1}c_n(x,y)}{n!},$$

$$\forall (x,y)\in\mathbb{Z}^2,\ \forall n\in\mathbb{Z}_+, \qquad (\delta_x c_n)(x,y) = n c_{n-1}(x,y)$$

(if $n=0$, $c_0(x,y)\equiv 1$ and $(\delta_x c_0)(x,y)\equiv 0$). Similarly, since

$$\begin{aligned}\frac{e_{x+1,y+1}(z)-e_{x,y}(z)}{1+i} - \frac{e_{x+1,y}(z)-e_{x,y+1}(z)}{1-i} &= \\ = \left((1+z)\frac{1+i+iz}{1+i+z}-1\right)\frac{e_{x,y}(z)}{1+i} &- \left(1+z-\frac{1+i+iz}{1+i+z}\right)\frac{e_{x,y}(z)}{1-i} \\ = 0,&\end{aligned} \tag{5.4}$$

we have

$$\forall (x,y)\in\mathbb{Z}^2, \text{ and } \forall n\in\mathbb{Z}_+, \qquad (\bar{D}c_n)(x,y) = 0.$$

Thus, for every $n\in\mathbb{Z}_+$, the function $c_n:\mathbb{Z}^2\longrightarrow\mathbb{C}$ is discrete analytic. Next, we show by induction that

$$\forall n\in\mathbb{Z}_+, \quad c_n(x,y)\equiv\zeta_n(x,y). \tag{5.5}$$

Indeed, for $n=0$,

$$c_0(x,y)\equiv 1\equiv\zeta_0(x,y).$$

Assume that, for some $n \in \mathbb{Z}_+$,

$$c_n(x,y) \equiv \zeta_n(x,y),$$

then, in view of (5.1),

$$(\delta_x c_{n+1}(x,y) \equiv (n+1)c_n(x,y) \equiv (n+1)\zeta_n(x,y) \equiv (\delta_x \zeta_{n+1}(x,y),$$

hence

$$(\delta_x(\zeta_{n+1} - c_{n+1}))(x,y) \equiv 0.$$

But the functions c_{n+1} and ζ_{n+1} are discrete analytic, hence

$$(\bar{D}(\zeta_{n+1} - c_{n+1}))(x,y) \equiv 0$$

and

$$(\delta_y(\zeta_{n+1} - c_{n+1}))(x,y) \equiv 0.$$

It follows that

$$\zeta_{n+1} - c_{n+1} = \text{const};$$

since

$$\zeta_{n+1}(0,0) = 0 = c_{n+1}(0,0),$$

one concludes that

$$c_{n+1}(x,y) \equiv \zeta_{n+1}(x,y),$$

and (5.5) follows. □

Corollary 5.2. *Let $x, n \in \mathbb{Z}_+$. Then,*

$$\zeta_n(x,0) = \begin{cases} x(x-1)\cdots(x-n+1) = x^{[n]}, & if \quad n \le x \\ 0, & if \quad n > x. \end{cases} \tag{5.6}$$

Proof. Set $y = 0$ in $e_{x,y}(z)$ in (5.2). By (5.5) we get

$$(1+z)^x = \sum_{n \in \mathbb{Z}_+} \frac{z^n}{n!} \zeta_n(x,0). \tag{5.7}$$

(5.6) follows by comparing the coefficients of z^n in (5.7). □

Theorem 5.3. *It holds that*

$$\forall (x,y) \in \mathbb{Z}_+ \times (\mathbb{Z} \setminus \{0\}), \qquad \limsup_{n \to \infty} \left(\frac{|\zeta_n(x,y)|}{n!} \right)^{1/n} = \frac{1}{\sqrt{2}}.$$

Proof. When $x \ge 0$ the function (5.2) is analytic in the variable z in the disk centered at the origin and of radius $\sqrt{2}$, and has a pole on the boundary of this disk. Hence the radius of convergence of the McLaurin series is precisely $\sqrt{2}$, and

$$\limsup_{n \to \infty} \left(\frac{|\zeta_n(x,y)|}{n!} \right)^{1/n} = \frac{1}{\sqrt{2}}.$$

□

Theorem 5.4. *Let $g : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ be such that for every $(x,y) \in \mathbb{Z}_+ \times \mathbb{Z}$ the series*

$$f(x,y) = \sum_{n \in \mathbb{Z}_+} g(n)\zeta_n(x,y) \tag{5.8}$$

converges absolutely. Then the function $f : \mathbb{Z}_+ \times \mathbb{Z} \longrightarrow \mathbb{C}$, defined by (5.8), is discrete analytic, and it holds that

$$g(n) \equiv \hat{f}_0(n),$$

where the function $f_0 : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ *is given by*

$$f_0(x) = f(x,0), \quad x \in \mathbb{Z}_+. \tag{5.9}$$

Proof. The discrete analyticity of f follows directly from the discrete analyticity of the polynomials ζ_n. Furthermore, when $y = 0$ the formula (5.8) becomes

$$f_0(x) = \sum_{n \in \mathbb{Z}_+} g(n) x^{[n]},$$

hence, according to Proposition 2.3, $g = \hat{f}_0$. □

Now we can introduce the main class of functions to be considered in this paper.

Definition 5.5. *A function* $f : \mathbb{Z}_+ \times \mathbb{Z} \longrightarrow \mathbb{C}$ *is said to be* expandable *if:*

(1) *the Fourier transform* $\hat{f}_0$ *of the function* $f_0 : \mathbb{Z}_+ \longrightarrow \mathbb{C}$*, given by* (5.9)*, satisfies the estimate*

$$\limsup_{n \to \infty} (|\hat{f}_0(n)| n!)^{1/n} < \sqrt{2}; \tag{5.10}$$

(2) *the function* f *admits the representation*

$$f(x,y) = \sum_{n \in \mathbb{Z}_+} \hat{f}_0(n) \zeta_n(x,y), \quad (x,y) \in \mathbb{Z}_+ \times \mathbb{Z}.$$

The class of expandable functions contains all discrete analytic polynomials, and elements of this class are determined by their values on the positive horizontal axis.

Corollary 5.6. *Suppose that* $f_0 : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ *is rational. Then there exists a unique expandable function* $f : \mathbb{Z}_+ \times \mathbb{Z} \longrightarrow \mathbb{C}$ *such that*

$$f(x,0) \equiv f_0(x).$$

Proof. This is a consequence of Theorem 2.9. □

6. The Cauchy-Kovalevskaya product

Theorem 5.4 and Corollary 5.6 allows us to define a (partially) defined product on expandable functions, which is everywhere defined on rational functions. This product will be denoted by $\odot$ and called, for reasons to be explain later in the section, the Cauchy-Kovalesvskaya product. Consider f_1 and f_2 two expandable functions, and assume that the Fourier transform of the pointwise product $f_1(x,0)f_2(x,0)$ satisfy (5.10). Then there exists a unique discrete analytic expandable function g such that

$$g(x,0) = f_1(x,0) f_2(x,0) \tag{6.1}$$

g is called the Cauchy-Kovalesvskaya product of f_1 and f_2 and is denoted by $f_1 \odot f_2$. Note that the Cauchy-Kovalesvskaya product of two rational functions always exist. We now give a more formal definition of the product:

Definition 6.1. *Let* $f : \mathbb{Z}^2 \longrightarrow \mathbb{C}$ *be a polynomial, such that*

$$f(x,0) \equiv c_0 + c_1 x + \cdots + c_n x^n,$$

and let $g : \Omega \times \mathbb{Z} \longrightarrow \mathbb{C}$ *be given. The Cauchy-Kovalevskaya (C-K) product of* f *and* g *is defined by*

$$(g \odot f)(x,y) = (f \odot g)(x,y) = c_0 g(x,y) + c_1 (\mathcal{Z} g)(x,y) + \cdots + c_n (\mathcal{Z}^n g)(x,y). \tag{6.2}$$

We shall abbreviate this as

$$f \odot g = f(\mathcal{Z})g. \tag{6.3}$$

The commutativity asserted in the definition is proved in the following theorem.

Theorem 6.2.
(1) *The restriction of the C-K product to* $\mathbb{Z}_+$ *is the product of the restriction.*
(2) *The C-K product is the unique discrete analytic extension corresponding to the product of the restrictions.*
(3) *The C-K product is commutative, i.e.*

$$g \odot f = f \odot g \tag{6.4}$$

for all choices of discrete analytic polynomials

Proof.
(1) Setting $y = 0$ in (6.2) and taking into account the definition of $\mathcal{Z}$ we have

$$\begin{aligned}(g \odot f)(x,0) &= c_0 g(x,0) + c_1(\mathcal{Z}g)(x,0) + \cdots + c_n(\mathcal{Z}^n g)(x,0)\\ &= c_0 g(x,0) + c_1 x g(x,0) + \cdots + c_n x^n g(x,0)\\ &= f(x,0)g(x,0).\end{aligned} \tag{6.5}$$

(2) For an expandable function the discrete Cauchy-Riemann equation $\overline{D}f = 0$ with prescribed initial values on the horizontal positive axis has a unique solution (see Theorem 5.4).

(3) is then clear from (1) and (2). □

We now explain the name given to this product. Recall that the classical Cauchy-Kovalevskaya theorem concerns uniqueness of solutions of certain partial differential equations with given initial conditions. See for instance [20]. In Clifford analysis, where the pointwise product of hyperholomorphic functions need not be hyperholomorphic, this theorem was used by F. Sommen in [26] (see also [9]) to define the product of hyperholomorphic quaternionic-valued functions in $\mathbb{R}^4$ by extending the pointwise product from an hyperplane. In the present setting of expandable functions the discrete Cauchy-Riemann equation $\overline{D}f = 0$ with prescribed initial values on the horizontal positive axis also has a unique solution. This is why the pointwise product on the horizontal positive axis can be extended to a unique expandable function, which we call the C-K product.

If p is a discrete analytic polynomial and f is an expandable function, $p \odot f$ is the expandable function determined by

$$(p \odot f)(x,0) = p(x,0)f(x,0). \tag{6.6}$$

However it is not true in general that the pointwise product of the restrictions of two expandable functions, say f and g, is itself the restriction of an expandable function, as is illustrated by the example

$$f(x,y) = g(x,y) = e_{x,y}(t), \tag{6.7}$$

where $e_{x,y}(t)$ is defined by (5.2) and $|t| >$. Indeed,

$$e_{x,0}(t) = (1+t)^x \tag{6.8}$$

is the restriction of an expandable function whenever $|t| < \sqrt{2}$. On the other hand, $(e_{x,0}(t))^2 = e_{x,0}(2t + t^2)$ will not be the restriction of an expandable function for $|t| > \sqrt{1+\sqrt{2}} - 1$ which is strictly smaller than $\sqrt{2}$.

We note that the C-K product for hyperholomorphic functions was used in [3, 4, 5] to define and study rational hyperholomorphic functions, and some related reproducing Hilbert spaces.

Proposition 6.3. *For $m, n \in \mathbb{Z}_+$ and $j \in \{0, \ldots, m+n\}$, set*

$$c_j^{m,n} = \frac{\delta^j \left(x^{[m]} x^{[n]}\right)}{j!}|_{x=0}. \tag{6.9}$$

Then,

$$\zeta_m \odot \zeta_n = \sum_{j=0}^{m+n} c_j^{m,n} \zeta_j. \tag{6.10}$$

Proof. It suffices to note that (6.10) holds for $y = 0$, thanks to Proposition 2.3. □

Systems like (6.10) occur in the theory of discrete hypergroups. See [21].

7. Rational discrete analytic functions

As we already mentioned, the pointwise product of two discrete analytic functions need not be discrete analytic. In the sequel of the section we define a product on discrete analytic functions when one of the terms is a polynomial, and show that a rational function is a quotient of discrete analytic polynomials with respect to this product. We first need the counterpart of multiplication by the complex variable.

Definition 7.1. *The* multiplication operator $\mathcal{Z}$ *on the class of functions $f : \Omega \times \mathbb{Z} \longrightarrow \mathbb{C}$ is given by*

$$(\mathcal{Z}f)(x, y) = xf(x, y) + iy\frac{f(x, y+1) + f(x, y-1)}{2}.$$

Proposition 7.2. *Let f be a function from $\Omega \times \mathbb{Z}$ into $\mathbb{C}$. Then*
(1)

$$(\mathcal{Z}f)(x, 0) \equiv xf(x, 0). \tag{7.1}$$

Furthermore:
(2) *If f is a polynomial, then so is $\mathcal{Z}f$.*
(3) *If f is discrete analytic, then so is $\mathcal{Z}f$.*

Proof. The proofs of (1) and (2) are clear from the definition. The proof of (3) follows from the identity (7.6) in Theorem 7.4. □

Proposition 7.3. *Let $f : \mathbb{Z}_+ \times \mathbb{Z} \longrightarrow \mathbb{C}$ be expandable. Then so is $\mathcal{Z}f$. In particular,*

$$\forall n \in \mathbb{Z}_+, \qquad (\mathcal{Z}\zeta_n)(x, y) \equiv \zeta_{n+1}(x, y) + n\zeta_n(x, y). \tag{7.2}$$

Proof. By Proposition 7.2, for every $n \in \mathbb{Z}_+$, $(\mathcal{Z}\zeta_n)(x, y)$ is a discrete analytic polynomial. In view of (7.1),

$$(\mathcal{Z}\zeta_n)(x, 0) = x \cdot x^{[n]} = x^{[n+1]} + nx^{[n]},$$

hence formula (7.2) follows from Theorem 4.2.

Let $f_0 : \mathbb{Z}_+ \longrightarrow \mathbb{C}$ be defined by (5.9), then

$$(\mathcal{Z}f)(x,0) = xf_0(x) = \sum_{n\in\mathbb{Z}_+} (n\hat{f}_0(n) + \hat{f}_0(n-1))x^{[n]},$$

where $\hat{f}_0(-1) := 0$. Since

$$\limsup_{n\to\infty}(|\hat{f}_0(n)|n!)^{1/n} < \sqrt{2},$$

$$\limsup_{n\to\infty}(|n\hat{f}_0(n) + \hat{f}_0(n-1)|n!)^{1/n} < \sqrt{2}.$$

Finally, since

$$f(x,y) = \sum_{n\in\mathbb{Z}_+} \hat{f}_0(n)\zeta_n(x,y),$$

where the convergence is absolute,

$$(\mathcal{Z}f)(x,y) = \sum_{n\in\mathbb{Z}_+} \hat{f}_0(n)(\mathcal{Z}\zeta_n)(x,y) = \sum_{n\in\mathbb{Z}_+} (n\hat{f}_0(n) + \hat{f}_0(n-1))\zeta_n(x,y).$$

□

From the preceeding proof we note that

$$\zeta_1 \odot \zeta_n = \mathcal{Z}\zeta_n = n\zeta_n + \zeta_{n+1}. \tag{7.3}$$

Theorem 7.4. *The operators $\delta_x, \delta_y, \mathcal{Z}$ and $\overline{D}$ generate a Lie algebra of linear operators on the space of all functions from $\mathbb{Z}^2$ into $\mathbb{C}$. The Lie bracket is $[A,B] = AB - BA$ and the relations on the generators are*

$$[\delta_x, \mathcal{Z}] = 1 + \delta_x, \tag{7.4}$$

$$[\delta_y, \mathcal{Z}] = i(1 + \delta_y + \delta_y^2), \tag{7.5}$$

$$[\overline{D}, \mathcal{Z}] = \left(\frac{1+i}{2} + \frac{i}{2}\delta_y\right)\overline{D}, \tag{7.6}$$

$$[\overline{D}, \delta_x] = [\overline{D}, \delta_y] = [\delta_x, \delta_y] = 0. \tag{7.7}$$

Proof. The identities (7.4)-(7.7) can be verified by the calculations in the proofs of the two preceding propositions. □

Definition 7.5. *A function $f : \mathbb{Z}_+ \times \mathbb{Z} \longrightarrow \mathbb{C}$ is said to be a* rational discrete analytic *function if $f(x,y)$ is expandable and $f(x,0)$ is rational.*

Theorem 7.6. *An expandable function $f : \mathbb{Z}_+ \times \mathbb{Z} \longrightarrow \mathbb{C}$ is rational if and only if it is a C-K quotient of discrete analytic polynomials function that is, if and only if there exist discrete analytic polynomials $p(x,y)$ and $q(x,y)$ such that*

$$\forall x \in \mathbb{Z}_+, \qquad q(x,0) \neq 0$$

and

$$(q \odot f)(x,y) \equiv p(x,y).$$

Proof. Suppose first that f is rational, and let $f_0(x)$ denote the restriction of f to the horizontal positive axis. By definition there exists two polynomials $p_0(x)$ and $q_0(x)$ such that $q_0(x) \neq 0$ (on $\mathbb{Z}_+$) and

$$q_0(x)f_0(x) = p_0(x), \quad x \in \mathbb{Z}_+. \tag{7.8}$$

Let $p(x,y)$ and $q(x,y)$ denote the discrete analytic polynomials extending $p_0(x)$ and $q_0(x)$ respectively. Then both p and $q \odot f$ are expandable functions, which coincide on $\mathbb{Z}_+$, and therefore everywhere.

Conversely, let $p(x,y)$ and $q(x,y)$ be the discrete analytic polynomials such that

$$\forall x \in \mathbb{Z}_+, \qquad q(x,0) \neq 0$$

and

$$(q \odot f)(x,y) \equiv p(x,y).$$

Setting $y = 0$ leads to

$$\forall x \in \mathbb{Z}_+, \quad q(x,0)f(x,0) = p(x,0),$$

which ends the proof. □

Theorem 7.7. *Let $p(x,y)$ and $q(x,y)$ be the discrete analytic polynomials such that*

$$\forall x \in \mathbb{Z}_+, \qquad q(x,0) \neq 0$$

Then there is a unique expandable rational function f such that

$$(q \odot f)(x,y) \equiv p(x,y).$$

Proof. Denote

$$g(x,y) = (q \odot f)(x,y).$$

According to Proposition 7.3, the function $g : \mathbb{Z}_+ \times \mathbb{Z} \longrightarrow \mathbb{C}$ is expandable, and therefore can be written as

$$g(x,y) = \sum_{n \in \mathbb{Z}_+} \hat{g_0}(n) \zeta_n(x,y),$$

where $\hat{g_0}$ is the Fourier transform of its restriction

$$g_0(x) = g(x,0).$$

In particular, by Theorem 5.4, g is discrete analytic. In view of (7.1),

$$g_0(x) = q(x,0)f(x,0) = p(x,0),$$

hence, by Proposition 2.5, $\hat{g_0}$ has finite support and g is a discrete analytic polynomial. In view of Theorem 4.2,

$$g(x,y) \equiv p(x,y).$$

□

8. The C^*-algebra associated to expandable discrete analytic functions

. We denote by $\mathcal{H}_{DA}$ the reproducing kernel Hilbert space with reproducing kernel

$$K((x_1,y_1),(x_2,y_2)) = \sum_{n=0}^{\infty} \frac{\zeta_n(x_1,y_1)\zeta_n(x_2,y_2)^*}{(n!)^2}, \tag{8.1}$$

and let $e_n := \frac{1}{n!}\zeta_n$ be the corresponding ONB in $\mathcal{H}_{DA}$. Then,

Theorem 8.1.

$$\delta_x e_1 = 0 \quad \textit{and} \quad \delta_x e_n = e_{n-1}, \quad n > 1, \tag{8.2}$$

i.e., δ_x is a copy of the backwards shift.

Proof. Using Proposition 5.1, we get

$$\delta_x e_{x,y}(z) = e_{x+1,y}(z) - e_{x,y}(z) = z e_{x,y}(z). \tag{8.3}$$

Substituting the expression $e_{x,y}(z) = \sum_{n\in\mathbb{Z}_+} \frac{z^n}{n!}\zeta_n(x,y)$ into (8.3), we get $\delta_x\zeta_1 = 0$ and $\delta_x\zeta_n = n\zeta_{n-1}$ if $n > 1$. The result (8.2) follows. □

Proposition 8.2. *In $\mathcal{H}_{DA}$ we have*

$$\begin{aligned}\delta_y &= \delta_x\left(I - \frac{i-1}{2}\delta_x\right)^{-1}\\ &= \sum_{n=0}^{\infty}\left(\frac{i-1}{2}\right)^n \delta_x^{n+1},\end{aligned} \tag{8.4}$$

where the convergence of the above series is in the operator norm

Proof. Recall that the operator $\overline{\mathcal{D}}$ was defined in (4.1). Since the elements of $\mathcal{H}_{DA}$ are discrete analytic we have $\overline{\mathcal{D}} = 0$ in $\mathcal{H}_{DA}$, that is,

$$(1-i)\delta_x + (1+i)\delta_y + \delta_x\delta_y = 0, \tag{8.5}$$

and thus

$$\delta_y\left((1+i)I + \delta_x\right) = (1-i)\delta_x. \tag{8.6}$$

Since δ_x is an isometry, and has in particular norm 1, we can solve equation (8.5) and obtain (8.4). The power expansion converges in the operator norm since

$$\|\left(\frac{i-1}{2}\right)\delta_x\| = \frac{1}{\sqrt{2}} < 1. \tag{8.7}$$

□

Theorem 8.3. *The C^*-algebra generated by δ_x, or equivalently by δ_x and δ_y is the Toeplitz C^*-algebra.*

Proof. This follows from the preceding proposition and from [11]. Indeed it is known [11] that the Toeplitz C^*-algebra $\mathcal{T}$ is the unique C^*-algebra generated by the shift. Since δ_x^* is a copy of the shift, and $\delta_y \in C^*(\delta_x^*)$, the result follows. □

We now consider

$$A = \text{Re } \mathcal{Z} = \frac{1}{2}\left(\mathcal{Z} + \mathcal{Z}^*\right), \tag{8.8}$$

where $\mathcal{Z}$ is defined from Definition 7.1.

Theorem 8.4.
(*i*) *The operator A is essentially self-adjoint on the linear span $\mathcal{D}$ of the functions ζ_n, $n \in \mathbb{Z}_+$.*
(*ii*) *On $\mathcal{D}$ it holds that*

$$[\delta_x, A] = \frac{1}{2}\left(I + \delta_x + \delta_x^2\right). \tag{8.9}$$

(*iii*) *There exists a strongly continuous one parameter semi-group $\alpha_t : \mathcal{T} \longrightarrow \mathcal{T}$ such that*

$$(e^{itA})b(e^{-itA}) = \alpha_t(b), \quad \forall t \in \mathbb{R},\ \forall b \in \mathcal{T}, \tag{8.10}$$

where $\mathcal{T}$ denotes the Toeplitz C^*-algebra.

Proof. In (iii), we denote by e^{itA} the unitary one-parameter group generated by the self-adjoint operator A from (i). The matrix representation of A with respect to the ONB $e_n := \frac{1}{n!}\zeta_n$ is:

$$\begin{pmatrix} & 0 & & & \\ n-2 & n-1 & 0 & & \\ n-1 & n-1 & n & 0 & \\ & n & \boxed{n} & n+1 & \\ & 0 & n+1 & n+1 & n+2 \\ & 0 & 0 & n+2 & n+2 \\ & & & 0 & \end{pmatrix} \tag{8.11}$$

It is therefore a banded infinite matrix with terms going to infinity linearly with n. It follows from [18] that A is essentially self-adjoint.

(ii) By definition of $\mathcal{Z}$ and δ_x

$$\begin{aligned} [A, \delta_x^*](e_n) &= (A\delta_x^* - \delta_x^* A)(e_n) \\ &= \frac{1}{2}(e_n + e_{n+1} + e_{n+2}) \\ &= \frac{1}{2}(I + \delta_x^* + \delta_x^{*2})e_n, \end{aligned} \tag{8.12}$$

and hence the result.

(iii) We have

$$e^{itA}be^{-itA} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} (\text{ad } A)^n (b), \tag{8.13}$$

and

$$(\text{ad } A)^{n+1} (b) = [A, (\text{ad } A)^n b], \tag{8.14}$$

We verify (8.13) on monomials of δ_x and δ_x^* using (8.14) and induction. See [10, 24] for more details regarding limits. □

The next corollary deals with a flow. For more information on this topic, see [23].

Corollary 8.5. *The one-parameter group* $\{\alpha_t\} \subset \text{Aut } (\mathcal{T})$ *passes to a flow on the circle group* $\mathbb{T} = \{z \in \mathbb{C} \ ; \ |z| = 1\}$.

Proof. By [11], the Toeplitz algebra $\mathcal{T}$ has a represnetation as a short exact sequence

$$0 \longrightarrow \mathcal{T}/\mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(\mathbb{T}) \longrightarrow 0, \tag{8.15}$$

where $\mathcal{K}$ is a copy of the C^*-algebra of all compact operators on $\mathcal{H}_{DA}$. Hence,

$$\mathcal{T}/\mathcal{K} \simeq C(\mathbb{T}). \tag{8.16}$$

Since the left hand-side of (8.10) leaves $\mathcal{K}$ invariant, it follows that

$$\alpha_t : \mathcal{T} \longrightarrow \mathcal{T} \tag{8.17}$$

passes to the quotient $\mathcal{T}/\mathcal{K} \simeq C(\mathbb{T})$. □

9. A reproducing kernel Hilbert space of entire functions

As we stated in the previous section, the C-K product has the disadvantage of not being defined for all pairs of expandable functions. In Section 10 we introduce a different product which turns the space of expandable functions into a ring, and consider a related reproducing kernel Hilbert space. In preparation we introduce in the present section a reproducing kernel Hilbert space of entire functions of a complex variable within which the results of Section 10 can be set in a natural way.

To set these results in a wider setting, let us recall a few facts on Schur analysis, that is, on the study of functions analytic and contractive in the open unit disk. If s_0 is such a function (in the sequel, we write $s_0 \in \mathscr{S}$), the operator of multiplication by s_0 is a contraction from the Hardy space of the open unit disk $\mathbf{H}_2$ into itself. The kernel

$$\frac{1 - s_0(z)s_0(w)^*}{1 - zw^*} \tag{9.1}$$

is then positive definite in the open unit $\mathbb{D}$, and its associated reproducing kernel Hilbert space $H(s_0)$ was first studied by de Branges and Rovnyak. Spaces $H(s_0)$ and their various generalizations play an important role in linear system theory and in operator theory. See for instance [1, 2, 7, 14] for more information. Here we replace $\mathbf{H}_2$ by two spaces, a space of entire functions in the present section and a space of discrete analytic functions in the next section.

Thus, let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{O}$ denote the space of $\mathbf{L}(\mathcal{H})$-valued functions analytic at the origin, and consider the linear operator T on $\mathcal{O}$ defined by

$$T(z^n A_n) = \frac{z^n}{n!} A_n, \quad A_n \in \mathbf{L}(\mathcal{H}). \tag{9.2}$$

Then $T\mathcal{O}$ is a space of $\mathbf{L}(\mathcal{H})$-valued entire functions. The operator T induces a product $\lozenge$ of elements in $T\mathcal{O}$ via

$$(Tf)\lozenge(Tg) = T(fg).$$

Theorem 9.1. *Let $\mathcal{H}$ be a Hilbert space and let A is a bounded operator from $\mathcal{H}$ into itself. Then the $\mathbf{L}(\mathcal{H})$-valued entire function*

$$(I_{\mathcal{H}} - zA)^{-\lozenge} = (\sum_{n=0}^{\infty} \frac{z^n A^n}{n!}) = e^{zA}$$

satisfies

$$(I_{\mathcal{H}} - zA)\lozenge(I_{\mathcal{H}} - zA)^{-\lozenge} = I_{\mathcal{H}}, \tag{9.3}$$

and it is the only function in $T\mathcal{O}$ with this property.

Proof. This comes from the power expansion and norm estimates. □

Take now $\mathcal{H} = \mathbb{C}$ and let $\mathbf{H}_2$ denote the Hardy space of the unit disk. Then T is a positive contractive injection from $\mathbf{H}_2$ into itself. Denote by $\mathbf{H}$ the space $T\mathbf{H}_2$ equipped with the range norm:

$$\forall f \in \mathbf{H}_2, \qquad \|Tf\|_{\mathbf{H}} = \|f\|_2.$$

Then $T : \mathbf{H}_2 \longrightarrow \mathbf{H}$ is unitary, and $\mathbf{H}$ is a reproducing kernel Hilbert space of entire functions with the reproducing kernel

$$K_{\mathbf{H}}(z,w) = \sum_{n=0}^{\infty} \frac{(zw^*)^n}{(n!)^2}.$$

Proposition 9.2. $\mathbf{H}$ *is the Hilbert space of entire functions such that*

$$\int_{\mathbb{C}} |f(z)|^2 K_0(2|z|) dA(z) < \infty,$$

where

$$K_0(r) = \frac{1}{\pi}\int_{\mathbb{R}} \exp(-r\cosh t)dt$$

is the modified Bessel function of the second kind of order 0.

Proof. This follows from the fact that the Mellin transform of the square of the function Γ is the function $K_0(2\sqrt{x})$. See for instance [12, p. 50] for the latter. □

We note that $\mathbf{H}$ is contractively included in the Fock space since the reproducing kernel of the latter is

$$K_F(z,w) = \sum_{n=0}^{\infty} \frac{z^n w^{*n}}{n!}, \tag{9.4}$$

and

$$K_F(z,w) - K_{\mathbf{H}}(z,w) = \sum_{n=0}^{\infty} (zw^*)^n \left(\frac{1}{n!} - \frac{1}{(n!)^2}\right) \tag{9.5}$$

is positive definite in $\mathbb{C}$. See for instance [6, Theorem I, p. 354], [25] for differences of positive definite functions.

In view of Liouville's theorem, the only multipliers on $\mathbf{H}$ in the sense of the usual pointwise product are constants. The class of multipliers in the sense of the $\lozenge$ product is more interesting.

Theorem 9.3. *A function* $s \in \mathcal{O}$ *is a contractive* $\lozenge$*-multiplier on* $\mathbf{H}$ *if and only if it is of the form*

$$s = Ts_0, \quad s_0 \in \mathscr{S},$$

where $\mathscr{S}$ *denotes the Schur class of functions analytic and contractive in the open unit disk.*

Proof. Assume first that $s \in \mathcal{O}$ is a contractive $\lozenge$-multiplier on $\mathbf{H}$. Then $s = s\lozenge 1 \in \mathbf{H}$ and hence $s = Ts_0$ for some $s_0 \in \mathbf{H}_2$. Furthermore, let $f \in \mathbf{H}_2$. Since $s\lozenge(Tf) = T(s_0 f) \in \mathbf{H}$, $s_0 f \in \mathbf{H}_2$. Since

$$\|f\|_{\mathbf{H}_2} = \|Tf\|_{\mathbf{H}} \geq \|s\lozenge(Tf)\|_{\mathbf{H}} = \|T(s_0 f)\|_{\mathbf{H}} = \|s_0 f\|_{\mathbf{H}_2},$$

$s_0 \in \mathscr{S}$.
Conversely, if $s = Ts_0$ where $s_0 \in \mathscr{S}$, and $f \in \mathbf{H}_2$ then $s\lozenge(Tf) = T(s_0 f) \in \mathbf{H}$ and

$$\|s\lozenge(Tf)\|_{\mathbf{H}} = \|T(s_0 f)\|_{\mathbf{H}} = \|s_0 f\|_{\mathbf{H}_2} \leq \|f\|_{\mathbf{H}_2} = \|Tf\|_{\mathbf{H}}.$$

Thus s is a contractive $\lozenge$-multiplier on $\mathbf{H}$. □

Let $s_0 \in \mathscr{S}$. The operator M_{s_0} of pointwise multiplication is a contraction from $\mathbf{H}_2$ into itself. The operator range $\sqrt{I - M_{s_0}M_{s_0}^*}$ endowed with the range norm is called the associated de Branges-Rovnyak space. We denote it by $H(s_0)$. Similarly one can associate with $s \in T\mathscr{S}$ a reproducing kernel

$$K_s(z,w) = \left((I - M_sM_s^*)K_{\mathbf{H}}(\cdot,w)\right)(z),$$

where M_s denotes the operator of $\Diamond$-multiplication by s on $\mathbf{H}$. The corresponding reproducing kernel Hilbert space is $\mathrm{ran}(\sqrt{I - M_sM_s^*})$ with the range norm; it will be denoted by $H(s)$.

Theorem 9.4. *The mapping $f \mapsto Tf$ is unitary from de Branges - Rovnyak space $H(s_0)$ onto $H(s)$.*

Proof. Since

$$M_sT = TM_{s_0},$$
$$\sqrt{I - M_sM_s^*}T = T\sqrt{I - M_{s_0}M_{s_0}^*}.$$

□

The $H(s)$ spaces can be characterized in terms of ∂-invariance, where ∂ is the differentiation operator:

$$\partial f = f'.$$

Lemma 9.5. *The operator ∂ is bounded on $\mathbf{H}$; moreover,*

$$\partial T = TR_0, \tag{9.6}$$

where R_0 is the backward shift operator, and

$$\partial^*\partial = I_{\mathbf{H}} - C^*C, \quad \partial\partial^* = I_{\mathbf{H}},$$

where $Cf := f(0)$. Furthermore, the reproducing kernel of $\mathbf{H}$ is given by

$$K_{\mathbf{H}}(z,w) = Ce^{z\partial}e^{w^*\partial^*}C^*.$$

Proof. The claims follow from the definition of the operator T in (9.2). We prove only (9.6). Let $f \in \mathbf{H}_2$ with power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_nz^n. \tag{9.7}$$

Then,

$$(R_0f)(z) = \sum_{n=1}^{\infty} a_nz^{n-1}, \tag{9.8}$$

and therefore

$$\begin{aligned}(TR_0f)(z) &= \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!}z^{n-1}\\ &= \frac{d}{dz}\left(\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n\right)\\ &= (\partial Tf)(z).\end{aligned} \tag{9.9}$$

□

Theorem 9.6. *A closed subspace H of $\mathbf{H}$ is ∂-invariant if and only if*

$$H = \mathbf{H} \ominus M_{Ts_0}\mathbf{H},$$

where $s_0(z)$ is an inner function.

Proof. Let H be a closed subspace of $\mathbf{H}$ then $H = TH_0$ where H_0 is a closed subspace of $\mathbf{H}_2$. H is ∂-invariant if and only if H_0 is R_0-invariant, which is equivalent to $\mathbf{H}_2 \ominus H_0$ being invariant under multiplication by z. By the Beurling-Lax theorem, the last condition holds if and only if $\mathbf{H}_2 \ominus H_0 = M_{s_0}\mathbf{H}_2$, where s_0 is an inner function. □

Theorem 9.7. *Let $s \in T\mathscr{S}$. Then s admits the representation*

$$s(z) = D + \int_0^z Ce^{t\partial}Bdt,$$

where

$$\begin{pmatrix} \partial & B \\ C & D \end{pmatrix} : \begin{pmatrix} H(s) \\ \mathbb{C} \end{pmatrix} \longrightarrow \begin{pmatrix} H(s) \\ \mathbb{C} \end{pmatrix}$$

is a coisometry given by

$$\begin{aligned} \partial f &= f', \\ B1 &= s', \\ Cf &= f(0), \\ D1 &= s(0). \end{aligned}$$

Proof. Write $s = Ts_0$, where $s_0 \in \mathscr{S}$. Then

$$s_0(z) = D_0 + zC_0(I - zR_0)^{-1}B_0,$$

where

$$\begin{pmatrix} R_0 & B_0 \\ C_0 & D_0 \end{pmatrix} : \begin{pmatrix} H(s_0) \\ \mathbb{C} \end{pmatrix} \longrightarrow \begin{pmatrix} H(s_0) \\ \mathbb{C} \end{pmatrix}$$

is a coisometry given by

$$\begin{aligned} R_0 f &= (f - f(0))/z, \\ B_0 1 &= R_0 s_0, \\ C_0 f &= f(0), \\ D_0 1 &= s_0(0) = s(0). \end{aligned}$$

Hence

$$\begin{aligned} s(z) &= D_0 + \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)!} C_0 R_0^n B_0 \\ &= D_0 + \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)!} C_0 T^{-1}(TR_0T^{-1})^n TB_0 \\ &= D + \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)!} C\partial^n B \\ &= D + \int_0^z Ce^{t\partial}Bdt. \end{aligned}$$

□

Theorem 9.8. *Let H be a Hilbert space and let*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} H \\ \mathbb{C} \end{pmatrix} \longrightarrow \begin{pmatrix} H \\ \mathbb{C} \end{pmatrix}$$

be a coisometry. Then the function

$$s(z) = D + \int_0^z Ce^{tA}Bdt$$

is a contractive $\lozenge$-multiplier on $\mathbf{H}$, and the corresponding reproducing kernel is given by

$$K_s(z,w) = Ce^{zA}e^{w^*A^*}C^*.$$

Proof. Set

$$s_0(z) = D + zC(I - zA)^{-1}B,$$

then $s_0 \in \mathscr{S}$ and $s = Ts_0$. Since

$$K_{s_0}(z,w) = C(I - zA)^{-1}(I - wA)^{-*}C^*,$$

the formula for $K_s(z,w)$ follows. □

Theorem 9.9. *A reproducing kernel Hilbert space H of functions in $\mathcal{O}$ is of the form $H = H(s)$ for some $s \in T\mathscr{S}$ if and only if*

(1) *H is ∂-invariant;*
(2) *for every $f \in H$*

$$\|\partial f\|_H^2 \le \|f\|_H^2 - |f(0)|^2.$$

Proof. One direction follows immediately from Theorem 9.4. The proof of the other direction is modelled after the proof of [2, Theorem 3.1.2, p. 85] and is done as follows: let H be ∂-invariant; then for every $f \in H$

$$Ce^{z\partial}f = f(z),$$

where $Cf = f(0)$. Hence the reproducing kernel of H is given by

$$L_((z,w) = Ce^{z\partial}e^{w^*\partial^*}C^*.$$

Since

$$\partial^*\partial + C^*C \le I,$$

there exists a coisometry

$$\begin{pmatrix} \partial & B \\ C & D \end{pmatrix} : \begin{pmatrix} H \\ \mathbb{C} \end{pmatrix} \longrightarrow \begin{pmatrix} H \\ \mathbb{C} \end{pmatrix}.$$

But the the function

$$s(z) = D + \int_0^z Ce^{t\partial}Bdt$$

is a contractive $\lozenge$-multiplier and the associated kernel $K_s(z,w)$ coincides with $L(z,w)$. Hence $H = H(s)$. □

It is also of interest to consider $\lozenge$-rational matrix valued functions.

Theorem 9.10. *Tne following are equivalent:*

(1) *A function $f \in T\mathcal{O}$ is $\lozenge$-rational in the sense that for some polynomial $p(z)$, not vanishing at the origin, $p\lozenge f$ is also a polynomial.*

(2) *$f(z)$ is of the form*

$$f(z) = D + \int_0^z Ce^{tA}Bdt$$

with A, B, C, D - matrices of suitable dimensions;

(3) *the columns of ∂f belong to a finite-dimensional ∂-invariant space.*

Proof. It suffices to observe that a function $f \in T\mathcal{O}$ is $\diamondsuit$- rational if and only if it is of the form $f = Tf_0$ where$f_0 \in \mathcal{O}$ is rational in the usual sense. $\square$

10. A reproducing kernel Hilbert space of expandable discrete analytic function

In parallel with the previous section, we introduce the product $\boxdot$ of expandable discrete analytic functions by

$$\zeta_n \boxdot \zeta_m = \frac{m!n!\zeta_{m+n}}{(m+n)!}. \tag{10.1}$$

The advantage of this product versus the C-K one is that the space of expandable discrete analytic functions forms a ring.

Consider the linear mapping $V : z^n \mapsto \zeta_n$. Then VT maps, in particular, the space of functions analytic in a neighborhood of the closed disk $\{z : |z| \leq 1/\sqrt{2}\}$ onto the space of expandable functions. Then $V\mathbf{H}$ with the range norm is the reproducing kernel Hilbert space $\mathcal{H}_{DA}$ with the reproducing kernel (8.1)

$$K((x_1, y_1), (x_2, y_2)) = \sum_{n=0}^{\infty} \frac{\zeta_n(x_1, y_1)\zeta_n(x_2, y_2)^*}{(n!)^2}.$$

Note that

$$V\partial = \delta_x V, \quad V(e^z A) = e_{x,y}(A).$$

Since $V : \mathbf{H} \longrightarrow \mathcal{H}_{DA}$ is unitary, the following theorems are direct consequences of Theorems 9.3-9.10 in the previous section. We state them here in order to emphasize the new product.

Theorem 10.1. *A closed subspace H of $\mathcal{H}_{DA}$ is δ_x-invariant if and only if*

$$H = \mathcal{H}_{DA} \ominus M_{VTs_0}\mathcal{H}_{DA},$$

where $s_0(z)$ is an inner function.

Theorem 10.2. *Let $s \in VT\mathscr{S}$. Then s admits the representation*

$$s(x, y) = D + Ce_{x,y}(\delta_x) \boxdot (\zeta_1(x, y)B),$$

where

$$\begin{pmatrix} \delta_x & B \\ C & D \end{pmatrix} : \begin{pmatrix} H(s) \\ \mathbb{C} \end{pmatrix} \longrightarrow \begin{pmatrix} H(s) \\ \mathbb{C} \end{pmatrix}$$

is a coisometry given by

$$B1 = \delta_x s,$$
$$Cf = f(0),$$
$$D1 = s(0).$$

Theorem 10.3. *Let H be a Hilbert space and let*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} H \\ \mathbb{C} \end{pmatrix} \longrightarrow \begin{pmatrix} H \\ \mathbb{C} \end{pmatrix}$$

be a coisometry. Then the function

$$s(z) = D + Ce_{x,y}(A) \boxdot (\zeta_1(x,y)B),$$

is a contractive $\boxdot$-multiplier on $\mathcal{H}_{DA}$, and the corresponding reproducing kernel is given by

$$K_s((x_1,y_1),(x_2,y_2)) = Ce_{x_1,y_1}(A)(e_{x_2,y_2}(A))^*C^*.$$

Theorem 10.4. *A reproducing kernel Hilbert space H of expandable functions is of the form $H = H(s)$ for some $s \in VT\mathscr{S}$ if and only if*

1. *H is δ_x-invariant;*
2. *for every $f \in H$*

$$\|\delta_x f\|_H^2 \le \|f\|_H^2 - |f(0,0)|^2.$$

Theorem 10.5. *The following are equivalent:*

1. *An expandable function f is $\boxdot$-rational in the sense that for some discrete analytic polynomial $p(x,y)$, not vanishing at the origin, $p \boxdot f$ is also a discrete analytic polynomial.*
2. *$f(x,y)$ is of the form*

$$f(x,y) = D + Ce_{x,y}(A) \boxdot (\zeta_1(x,y)B),$$

with A, B, C, D - matrices of suitable dimensions, and $\|A\| < \sqrt{2}$, and $e_{x,y}(A)$ is as in (5.2).

3. *the columns of $\delta_x f$ belong to a finite-dimensional δ_x-invariant space of expandable functions.*

Theorem 10.6. *Let $s \in T\mathscr{S}$. Then s admits the representation*

$$s(x,y) = D + Ce_{x,y}(\delta_x) \boxdot (\zeta_1(x,y)B),$$

where

$$\begin{pmatrix} \delta_x & B \\ C & D \end{pmatrix} : \begin{pmatrix} H(s) \\ \mathbb{C} \end{pmatrix} \longrightarrow \begin{pmatrix} H(s) \\ \mathbb{C} \end{pmatrix}$$

is a coisometry given by

$$\begin{aligned} B1 &= s', \\ Cf &= f(0), \\ D1 &= s(0). \end{aligned}$$

Theorem 10.7. *The following are equivalent:*

1. *An expandable function f is $\boxdot$-rational in the sense that for some discrete analytic polynomial $p(x,y)$, not vanishing at the origin, $p \boxdot f$ is also a discrete analytic polynomial.*
2. *$f(x,y)$ is of the form*

$$f(x,y) = D + Ce_{x,y}(A) \boxdot (\zeta_1(x,y)B),$$

with A, B, C, D - matrices of suitable dimensions, and $\|A\| < \sqrt{2}$, and $e_{x,y}(A)$ is as in (5.2).

3. *the columns of $\delta_x f$ belong to a finite-dimensional δ_x-invariant space.*

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