Mathematics, Physics, and Computer Science
Faculty Articles and Research

Science and Technology Faculty Articles and
Research

2010

Linear Stochastic Systems: A White Noise Approach

Daniel Alpay
Chapman University, alpay@chapman.edu

David Levanony
Ben Gurion University of the Negev

Linear Stochastic Systems: A White Noise Approach

LINEAR STOCHASTIC SYSTEMS: A WHITE NOISE APPROACH

DANIEL ALPAY AND DAVID LEVANONY

ABSTRACT. Using the white noise setting, in particular the Wick product, the Hermite transform, and the Kondratiev space, we present a new approach to study linear stochastic systems, where randomness is also included in the transfer function. We prove BIBO type stability theorems for these systems, both in the discrete and continuous time cases. We also consider the case of dissipative systems for both discrete and continuous time systems. We further study ℓ_1-ℓ_2 stability in the discrete time case, and $\mathbf{L}_2$-$\mathbf{L}_\infty$ stability in the continuous time case.

CONTENTS

1. INTRODUCTION

In this paper we propose a new approach for the study of uncertainty within the theory of linear stochastic systems, and prove a number of stability theorems. To set the problems and results in perspective we

begin with a brief historical introduction. Linear system theory, operator theory and the theory of analytic functions have a long history of interactions, and two notable milestones are the work of M. Livsic on the characteristic operator function, see [45], [46], [47], and the work of R. Kalman, see [39]. The discussion of what is linear system theory would lead us too far away, and we refer the reader to [38] for more information. We also refer to [33] and [1] for surveys and to [24] for a discussion of stability results in the continuous time case.

For the purpose of this introduction, a discrete-time, time-invariant linear system will be described by an input-output relation of the form

$$(1.1) \qquad y_n = (h * u)_n = \sum_{m \in \mathbb{Z}} h_{n-m} u_m, \quad n \in \mathbb{Z}.$$

In this expression, the h_n are pre-assigned complex numbers, which stand for the impulse response of the underlying system, and the input and output are required to define a continuous map between specified spaces $\mathcal{H}_i$ and $\mathcal{H}_o$ (and of course, this entails conditions on the coefficients h_n). These various conditions are translated into properties for the $\mathcal{Z}$-transform

$$(1.2) \qquad \widehat{h}(\zeta) = \sum_{\mathbb{Z}} \zeta^n h_n$$

of the sequence (h_n). For instance, when the system (1.1) is causal, that is, when $h_n = 0$ for $n < 0$, it defines a contraction from $\ell_2(\mathbb{Z})$ into itself (the system is then called *dissipative*) if and only if the function $\widehat{h}$ is analytic and contractive in the open unit disk (such functions are called *Schur functions*), or equivalently, if and only if the operator of multiplication by $\widehat{h}$ is a contraction from the Hardy space of the open unit disk $\mathbf{H}_2(\mathbb{D})$ into itself. This allows to resort to all the tools of Schur analysis to study such systems; see for instance [20], [29], [30].

Note that $\widehat{h}$ is called in system theory the *transfer function* of the system. In certain fields (e.g. engineering) it is defined with ζ^{-1} instead of ζ.

The system (1.1) commutes with the shift operators S

$$(1.3) \qquad S(x_j) = (x_{j+1})$$

defined in the input and output spaces. In system theory terminology, it is called *time-invariant*. In fact, every time-invariant linear bounded system from $\ell_2(\mathbb{Z})$ into $\ell_2(\mathbb{Z})$ is of this form. The proof of this well

known result is recalled in the sequel; see STEP 1 in the proof of Theorem 5.1. Such a characterization does not hold when one considers $\ell_\infty(\mathbb{Z})$ instead of $\ell_2(\mathbb{Z})$, as we will explain below.

The notion of Schur function and the associated system theory interpretations, have been extended in a number of directions, well beyond the time-invariant case. We now discuss some of them. First, when considering the time-varying case (that is, when h_{n-m} in (1.1) is replaced by $h_{n,m}$), an approach originating with the work of Deprettere and Dewilde, see [17], [19], consists of replacing the complex numbers by diagonal operators. In the later works [3], [4], the Hardy space is replaced by the Hilbert space of upper-triangular operators of Hilbert-Schmidt class, and and Schur functions by upper triangular contractions. This allows, with an appropriate definition of point evaluation of an operator on a diagonal, to extend much of the function theory of the open unit disk, to the case of upper triangular operators, and hence to apply the results to time-varying systems. See [21], [9], [2], [5] for a sample of papers, and [22] for applications of this calculus on diagonals.

Among other directions of research and extensions we mention the case of multi-indexed systems and their connections to several complex variables, see for instance [10], and the non-commutative case, see for instance [11].

In all the directions outlined above, there is no randomness in the system itself, although the input (and hence the output) may be a sequence of random variables. In the present work it is a different kind of extension which we consider, allowing the h_n in (1.1) to be random variables. We use white noise space analysis, which has been introduced in 1975 by T. Hida, see [34], and the monographs [35], [36] and [41]. White noise analysis allows to translate problems from the stochastic context into problems involving analytic functions in a countable number of variables in the Fock space, or in spaces of distributions which contain the Fock space, in particular in the Kondratiev space. The Wick product is a generalization for random variables in the Kondratiev space of the pointwise product, and reduces to the pointwise product when at least one of the factors is nonrandom. It became very useful when stochastic calculus with respect to the fractional Brownian motion, and more generally with respect to processes which are not necessarily semi–martingales, began to be considered; see [36], [25], [26], [27].

Obviously, a Gaussian input into a linear system with nonrandom coefficients, will result in a Gaussian output. Here, we aim to model linear Gaussian input-output relations when the underlying linear system is random. While indeed a Gaussian input into a linear system with random coefficients cannot be expected to result in a Gaussian output, we will use the white noise space setting and replace the pointwise product by the Wick product, enabling Gaussian input-output relations when the underlying system has random coefficients. This has the advantage of preserving the Gaussian input-output relation, while allowing uncertainty in the form of randomness in the linear system under study. Thus, we replace (1.1) by

$$(1.4) \qquad y_n = \sum_{m \in \mathbb{Z}} h_{n-m} \lozenge u_m, \quad n \in \mathbb{Z},$$

where the y_n, u_n and h_n are now random variables in the Kondratiev space (or more precisely, in some Hilbert subspace of it), and where $\lozenge$ denote the Wick product. We also consider the causal case, where now

$$(1.5) \qquad y_n = \sum_{m=0}^{n} h_{n-m} \lozenge u_m, \quad n = 0, 1, 2, \ldots$$

The proposed setting can be used to model uncertainty of an otherwise deterministic linear time-invariant system, a system that maintains Gaussian input-output relation, by a *random* uncertainty in the impulse response. This is known as the Bayesian embedding approach, by which the study of a nonrandom uncertainty is carried out through an associated probabilistic analysis; see e.g. [40], [44]. We now turn to the content of the paper, and first recall three stability theorems, namely, Theorems 1.1, 1.3 and 1.4. The main aim of the paper is to develop counterpart of these (and of some other) theorems in the stochastic setting, as is explained below.

Consider a linear discrete time system of the form (1.1). Various notions of stability can be assigned to such a system; in this work we will focus in particular on BIBO stability (bounded input bounded output), ℓ_1-ℓ_2 stability and the case of dissipative systems. With BIBO stability in mind, the following result is well known; see for instance [38, p. 177].

Theorem 1.1. *There is a $M > 0$ such that the sums $\sum_{m \in \mathbb{Z}} h_{n-m} u_m$ converge absolutely for all $(u_m) \in \ell_\infty(\mathbb{Z})$, all $n \in \mathbb{Z}$, and*

$$(1.6) \qquad \sup_{n \in \mathbb{Z}} |y_n| \leq M \sup_{n \in \mathbb{Z}} |u_n|$$

if and only if

(1.7)
$$\sum_{n\in\mathbb{Z}} |h_n| \leq M.$$

Condition (1.7) means that the $\mathcal{Z}$–transform (1.2) of the impulse response is in the Wiener algebra $\mathscr{W}$; $\widehat{h}$ is in particular continuous on the unit circle, but it need not be defined in general for $|\zeta| \neq 1$. In the case of a causal system, and when h is rational, (1.7) can be given a nicer interpretation. Recall first that a rational function which has no pole on the unit circle is in the Wiener algebra; its Taylor coefficients at the origin (if the function is assumed analytic in a neighborhood of the origin) will not, in general, be equal to the Fourier coefficients of its expansion as an element in $\mathscr{W}$. They will be the same when all the $h_n = 0$ for n negative, that is, when the system is causal. Still for causal systems, condition (1.7) means that the function $\widehat{h}$ is analytic in the open unit disk $\mathbb{D}$, and continuous on the closed unit disk. Thus, when $\widehat{h}$ is assumed rational, it belongs to the subalgebra $\mathscr{W}_+$ of $\mathscr{W}$. We also note that in this case, $\widehat{h}$ has no pole in the closed unit disk. We refer to [31] for these facts and for more information on the Wiener algebra.

The system (1.1) defines a linear bounded operator from $\ell_\infty(\mathbb{Z})$ into $\ell_\infty(\mathbb{Z})$ which commutes with the bilateral S defined by (1.3); this last property expresses the *time-invariance* of the system. We consider linear time-invariant systems whose input-output relation is given in the form of a convolution. We note however that not all linear time-invariant systems from $\ell_\infty(\mathbb{Z})$ into itself are given by a convolution. For instance, define $X_0 \in (\ell_\infty(\mathbb{Z}))^*$ by

$$X_0(x) = \limsup_{n\to\infty} \frac{\sum_{j=-n}^{j=n} x_j}{2n+1}.$$

The operator

(1.8)
$$X(x) = (X_0 S^{j*}(x))_{j\in\mathbb{Z}},$$

defines a bounded linear operator from $\ell_\infty(\mathbb{Z})$ into itself. which commute with S, but cannot be expressed by a convolution.

We denote

$$\mathbb{N} = \{1, 2, \ldots\} \quad \text{and} \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

To discuss ℓ_1-ℓ_2 stable and dissipative systems, it is easier to consider signals indexed by $\mathbb{N}_0$. We recall that the Hardy space $\mathbf{H}_2(\mathbb{D})$ is the

space of power series

$$f(\zeta) = \sum_{n=0}^{\infty} f_n \zeta^n, \quad f_n \in \mathbb{C},$$

with norm

$$\|f\|_{\mathbf{H}_2(\mathbb{D})} = \left(\sum_{n=0}^{\infty} |f_n|^2\right)^{1/2}.$$

The following definition and result are well known:

Definition 1.2. *The system* (1.1) *will be called ℓ_1-ℓ_2 bounded if there exists a $M < \infty$ such that*

$$\left(\sum_{n=0}^{\infty} |y_n|^2\right)^{1/2} \le M \sum_{n=0}^{\infty} |u_n|.$$

For the matrix-valued version of the theorem below, see [7, Theorem 5.1].

Theorem 1.3. *The system* (1.4) *is ℓ_1-ℓ_2 bounded if and only if its transfer function is in the Hardy space $\mathbf{H}_2(\mathbb{D})$.*

The $\ell_2(\mathbb{N}_0)$ norms of the input and output sequences are a measure of the energy of the signals, and play an important role in system theory; see [1] for a survey. The system (1.1) is called dissipative if the $\ell_2(\mathbb{N}_0)$ norm of the output is always less or equal to the $\ell_2(\mathbb{N}_0)$-norm of the input. The following result characterizes systems of the form (1.1) which are dissipative.

Theorem 1.4. *A linear system is time-invariant, causal and dissipative if and only if it is of the form* (1.1) *with a transfer function which is analytic and contractive in the open unit disk.*

In other words, the system has a transfer function which is a Schur function. Equivalently, the lower triangular Toeplitz operator

(1.9)
$$\begin{pmatrix} h_0 & 0 & \cdot & \cdots \\ h_1 & h_0 & \cdot & \cdots \\ h_2 & h_1 & h_0 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix}$$

is a contraction from $\ell_2(\mathbb{N}_0)$ into itself.

We note that, through the $\mathcal{Z}$-transform, the space $\ell_2(\mathbb{N}_0)$ is unitarily mapped onto $\mathbf{H}_2(\mathbb{D})$, that is, onto the reproducing kernel Hilbert space

with reproducing kernel

(1.10) $$K(\zeta, \nu) = \frac{1}{1 - \zeta\nu^*}, \quad \zeta, \nu \in \mathbb{D}.$$

Therefore, $\widehat{h}$ is the transfer function of a dissipative system if and only if the operator of multiplication by $\widehat{h}$ is a contraction from $\mathbf{H}_2(\mathbb{D})$ into itself, or, equivalently, if and only if the kernel

(1.11) $$\frac{1 - \widehat{h}(\zeta)\widehat{h}(\nu)^*}{1 - \zeta\nu^*}$$

is positive in the open unit disk. The associated reproducing kernel Hilbert spaces were introduced and studied by de Branges and Rovnyak, also in the operator-valued case; see [14], [15]. We will use some of their results in the sequel; see Theorem 5.2.

We wish to extend the notion of transfer function so as to include a random aspect, and present counterparts of the three theorems mentioned above in a random systems setting. We will also consider the continuous case for BIBO stability, dissipative systems, and $\mathbf{L}_2$-$\mathbf{L}_\infty$ bounded systems. Using the white noise setting and the Hermite transform this random aspect will be expressed by dependence on a countable number of independent complex variables $z_1, z_2, \ldots$. This is reviewed in the next section. First we need to recall some of the relevant notations at this stage. In the following expressions, ℓ denotes the set of finite sequences of integers indexed by $\mathbb{N}$, that is, the set of sequences $(n_1, n_2, \ldots)$ where $n_j = 0$ for all large enough j, and we set (see [36, (2.3.8) p. 29])

$$(2\mathbb{N})^\alpha = \prod_{j \in \mathbb{N}} (2j)^{\alpha_j}.$$

The product is meaningful since $\alpha_j = 0$ for all but for a finite number of indices j.

In our approach, we replace the kernel (1.10) by a kernel of the form

$$\frac{K_k(z, w)}{1 - \zeta\nu^*}$$

where $z = (z_1, z_2, \ldots)$ and $w = (w_1, w_2, \ldots)$ belong to the infinite dimensional neighborhood

$$\mathbb{K}_k = \left\{ z = (z_1, z_2, \ldots) \in \mathbb{C}^\mathbb{N} : \sum_{\alpha \in \ell} |z|^{2\alpha} (2\mathbb{N})^{k\alpha} < \infty \right\}$$

of the origin in $\mathbb{C}^{\mathbb{N}}$ (see [36, Definition 2.6.4 p. 59]), and where

$$(1.12) \qquad K_k(z, w) = \sum_{\alpha \in \ell} z^\alpha (w^*)^\alpha (2\mathbb{N})^{k\alpha},$$

with the use of the multi-index notation:

$$z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots, \quad \text{where} \quad \alpha = (\alpha_1, \alpha_2, \ldots).$$

We denote by $\mathcal{H}(K_k)$ the reproducing kernel Hilbert space with reproducing kernel $K_k(z, w)$. Elements of $\mathcal{H}(K_k)$ are analytic in $z = (z_1, z_2, \ldots)$ in the set $\mathbb{K}_k$. To take into account randomness, we replace the Hardy space $\mathbf{H}_2$ by $\mathbf{H}_2 \otimes \mathcal{H}(K_k)$. Note that

$$(1.13) \qquad \mathbf{H}_2 \otimes \mathcal{H}(K_k) = \left\{ f(\zeta) = \sum_{n=0}^{\infty} \zeta^n f_n \text{ with } f_n \in \mathcal{H}(K_k) \right\},$$

with norm

$$\|f\|_{\mathbf{H}_2 \otimes \mathcal{H}(K_k)}^2 = \sum_{n=0}^{\infty} \|f_n\|_{\mathcal{H}(K_k)}^2.$$

A transfer function will thus be a function $\mathscr{H}(\zeta, z)$ of the form

$$(1.14) \qquad \mathscr{H}(\zeta, z) = \sum_{n=0}^{\infty} \zeta^n h_n(z),$$

where now the $h_n \in \mathcal{H}(K_k)$. In (1.14), the nonrandom part of the transfer function is

$$(1.15) \qquad \mathscr{H}(\zeta, 0) = \sum_{n=0}^{\infty} \zeta^n h_n(0).$$

As we explain in the sequel, another possible generalization is to write

$$(1.16) \qquad \mathscr{H}(\zeta, z) = \widehat{h}(\zeta) + \sum_{\alpha \in \ell} z^\alpha c_\alpha(\zeta).$$

The nonrandom part in (1.16) is

$$(1.17) \qquad \mathscr{H}(\zeta, 0) = \widehat{h}(\zeta),$$

corresponding to $\alpha = 0$.

The transfer function $\mathscr{H}(\zeta, z)$ of a dissipative random system will be characterized by the positivity of the kernel

$$(1.18) \qquad (1 - \mathscr{H}(\zeta, z) \mathscr{H}(\nu, w)^*) \frac{K_k(z, w)}{(1 - \zeta \nu^*)}$$

in $\mathbb{D} \times \mathbb{K}_k$, that is, $\mathscr{H}(\zeta, z)$ is a contractive multiplicator of the reproducing kernel Hilbert space $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)$ with reproducing kernel

$$(1.19) \qquad \frac{K_k(\zeta, z)}{1 - \zeta\nu^*}.$$

A similar interpretation will hold in the continuous case, where the open unit disk is now replaced by the open upper half-plane.

The outline of the paper is as follows. The paper consists of eight sections besides the introduction. In the second section, we review the white space noise setting, and define analogs of the linear systems (1.1) and of their continuous time versions when we allow randomness both in the impulse response (h_m) and in the inputs (u_m). In Section 3 we prove the counterparts of Theorems 1.1. The cases of Theorems 1.3 and 1.4 are considered in Section 4 and 5 respectively. The next three sections are devoted to the continuous time case. In Section 6 we prove the operator-valued version of the Bochner-Chandrashekaran theorem. In Section 7 we consider the analog of BIBO continuous systems and in Section 8 the case of continuous dissipative systems. In the last section we consider the case of $\mathbf{L}_2$-$\mathbf{L}_\infty$ stability.

The theory of random linear systems is yet not well established, and it appears that the setting presented in this paper is the first which permits to handle, in an efficient way, the case where randomness is allowed in the impulse response.

We denote by

$$Uf(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{ixt}dx$$

the Fourier transform, where $f \in \mathbf{L}_2(\mathbb{R})$, or, more generally, belongs to $\mathbf{L}_2(\mathbb{R}) \otimes \mathcal{H}$ for some Hilbert space $\mathcal{H}$. For a Hilbert space $\mathcal{H}$, the symbol $\mathbf{L}(\mathcal{H})$ denotes the set of bounded operators from $\mathcal{H}$ into itself.

2. The white noise space setting

We now briefly recall the definitions of the white noise space and of the Kondratiev space. We refer to [35], [36],[41], [12] and [28] for more information.

The function

$$K(s_1 - s_2) = \exp(-\|s_1 - s_2\|^2_{\mathbf{L}_2(\mathbb{R})}/2)$$

is positive on the Schwartz space $\mathcal{S}$ of rapidly decreasing, infinitely differentiable functions. Since $\mathcal{S}$ is a nuclear space, the Bochner-Minlos theorem (see [32, Théorm̀e 2 p. 342]) ensures the existence of a probability measure P on the Borel sets $\mathcal{F}$ of $\Omega = \mathcal{S}'$ such that

$$\int_\Omega e^{i\langle s,\omega\rangle_{\mathbf{L}_2(\mathbb{R})}} dP(\omega) = \exp(-\|s\|_{\mathbf{L}_2(\mathbb{R})}^2/2).$$

The white noise space is $\mathcal{W} = \mathbf{L}_2(\Omega, \mathcal{F}, P)$. An orthogonal basis of the white noise space is given by the Hermite functions $(H_\alpha)_{\alpha \in \ell}$. These functions are computed in terms of the Hermite polynomials, and we refer to [36, Definition 2.2.1 p. 19] for their definition. Every element in $\mathcal{W}$ can be written as

(2.1)
$$F(\omega) = \sum_{\alpha \in \ell} c_\alpha H_\alpha(\omega), \quad c_\alpha \in \mathbb{C}, \quad \text{with} \quad \|F\|_{\mathcal{W}}^2 = \sum_{\alpha \in \ell} |c_\alpha|^2 \alpha! < \infty$$

In general, one takes real c_α. Here we take complex coefficients, that is, we consider the complexified real white noise space. A similar remark holds for the spaces $\mathcal{H}(K_k)$ defined below.

Let $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$. The linear map which to H_α associates the polynomial $\mathbf{I}(H_\alpha) = z^\alpha$ extends to a unitary map between the white noise space and the real reproducing kernel Hilbert space with reproducing kernel $\exp \langle z, w \rangle_{\ell_2}$. This space is called the Fock space. The map $\mathbf{I}$ is called the *Hermite transform*. The Wick product is defined through the Hermite functions by

$$H_\alpha \lozenge H_\beta = H_{\alpha+\beta}, \quad \alpha, \beta \in \ell.$$

It is in fact independent of the chosen basis in the white noise space; see [36, Appendix D, pp. 209-215]. In general, the Wick product of two elements in the white noise space need not be in the white noise space. The most convenient space which is stable with respect to the Wick product is the Kondratiev space S_{-1}. To define S_{-1} we first introduce for $k \in \mathbb{N} = \{1, 2, \ldots\}$ the Hilbert space $\mathcal{H}_k$ which consist of series of the form (2.1) such that

(2.2)
$$\|f\|_k \stackrel{\text{def.}}{=} \left(\sum_{\alpha \in \ell} |c_\alpha|^2 (2\mathbb{N})^{-k\alpha} \right)^{1/2} < \infty.$$

Note that the the Hermite transform is a unitary mapping from $\mathcal{H}_k$ onto the reproducing kernel Hilbert space with reproducing kernel $K_k(z, w)$, where K_k is defined in (1.12). The Kondratiev space S_{-1} is the inductive limit of the spaces $\mathcal{H}_k$. We note that when either one of the factors

f or g in S_{-1} is nonrandom, the Wick product $f \lozenge g$ reduces to the point-wise product fg.

We will consider stochastic processes in the series form

$$(2.3) \qquad X(\tau, \omega) = \sum_{\alpha \in \ell} c_\alpha(\tau) H_\alpha(\omega),$$

where the $c_\alpha(\tau)$ are nonrandom functions depending on a parameter τ. We require that the series (2.3) belongs to the Kondratiev space S_{-1} for every value of τ. Here τ belongs to the integers or to the real numbers. We note that other choices of indices are possible (for example, $\mathbb{Z}^2$, or the case of the vertices of a binary tree). We also note that a case of special interest arises when one applies the Hermite transform and the Laplace transform (for the continuous time case) or the $\mathcal{Z}$-transform (for the discrete time case), to obtain a function which depends on a finite number of variables z_i, which, in addition, is a rational function in these variables, that is, is a function of the form

$$(2.4) \qquad D(\zeta) + C(\zeta)(I_N - \sum_{k=1}^{M} z_k A_k(\zeta))^{-1}(\sum_{k=1}^{M} z_k B_k(\zeta)).$$

Here, ζ denotes the variable corresponding either to the Laplace transform, or the $\mathcal{Z}$-transform. The functions of ζ in that expression are also assumed rational. See [8]. Other type of realizations are possible; see Theorem 5.2 below.

3. BIBO STABLE LINEAR DISCRETE TIME STOCHASTIC SYSTEMS

Fix some integer $l > 0$, and let $k > l+1$. Consider $h \in \mathcal{H}_l$ and $u \in \mathcal{H}_k$. Then, Våge's inequality (see [36, Proposition 3.3.2 p. 118]) is in the form

$$(3.1) \qquad \|h \lozenge u\|_k \leq A(k - l)\|h\|_l \|u\|_k,$$

where

$$(3.2) \qquad A(k - l) = \sum_{\alpha \in \ell} (2\mathbb{N})^{(l-k)\alpha}.$$

It is not a trivial fact that $A(k-l)$ is finite; see [52] and [36, Proposition 2.3.3 p. 31] for a proof.

Inequality (3.1) expresses the fact that the multiplication operator

$$T_h : u \mapsto h \lozenge u$$

is a bounded map from the Hilbert space $\mathcal{H}_k$ into itself, and that its operator norm $\|T_h\|_{\mathrm{op},l,k}$ satisfies the inequality

$$(3.3) \qquad\qquad \|T_h\|_{\mathrm{op},l,k} \leq A(k-l)\|h\|_l.$$

The norm of T_h depends on k and l and will be in general different from $\|h\|_l$. It implies in particular that $\mathcal{H}_l$ endowed with the norm

$$\|h\| \overset{\mathrm{def.}}{=} \|T_h\|_{\mathrm{op},l,k}$$

is a normed algebra.

To simplify the notation, we set

$$\|T_h\|_{\mathrm{op},l,k} = \|T_h\|.$$

Definition 3.1. *A random discrete time signal will be a sequence (u_n) indexed by $\mathbb{Z}$, of elements in the Kondratiev space, such that there exists a $k \in \mathbb{N}$ (depending on the signal) such that*

$$u_n \in \mathcal{H}_k, \quad \forall n \in \mathbb{Z}.$$

We note that k is imposed to be independent of n.

Theorem 3.2. *Let $k > l+1$ and let (h_n) be a sequence of elements in $\mathcal{H}_l$ indexed by $\mathbb{Z}$. Then*
(a) The sums (1.4)

$$y_n = \sum_{m \in \mathbb{Z}} h_{n-m} \Diamond u_m, \quad n \in \mathbb{Z},$$

converge absolutely in $\mathcal{H}_k$ for all inputs $(u_m)_{m \in \mathbb{Z}}$ such that $\sup_{m \in \mathbb{Z}} \|u_m\|_k < \infty$, and
(b) There exists an $M > 0$ such that, for all such inputs $(u_n)_{n \in \mathbb{Z}}$, it holds that

$$(3.4) \qquad\qquad \sup_{n \in \mathbb{Z}} \|y_n\|_k \leq M \sup_{n \in \mathbb{Z}} \|u_n\|_k$$

if and only if for all $v \in \mathcal{H}_k$ with $\|v\|_k = 1$ it holds that

$$(3.5) \qquad\qquad \sum_{n \in \mathbb{Z}} \|T_{h_n}^*(v)\|_k \leq M.$$

We note that, in the nonrandom case, where the h_n are (nonrandom) complex numbers, the Wick product reduces to a pointwise product, and we have systems of the form (1.1). Furthermore, in this case,

$$\|T_{h_n}^* v\|_k = |h_n| \cdot \|v\|_k,$$

and we retrieve the well known BIBO stability condition (1.7), with Theorem 3.2 reduced to Theorem 1.1.

Furthermore, we notice that condition

$$\sum_{n\in\mathbb{Z}} \|T_{h_n}\| \leq M$$

on the norms of the operators T_{h_n} implies condition (3.5). Expressions of the form

$$\sum_{n\in\mathbb{Z}} \zeta^n T_{h_n}$$

with

$$\sum_{n\in\mathbb{Z}} \|T_{h_n}\| < \infty$$

form an algebra, which appears to be the counterpart of the classical Wiener algebra in the present setting.

Proof of Theorem 3.2: First note that, in view of the restriction $k > l + 1$, each of the terms in (1.4) belongs to $\mathcal{H}_k$.

Assume that (3.4) is in force. Then, for every $n \in \mathbb{Z}$, taking sequences (u_m) which have only a finite number of non zero entries, one has for any preassigned $n \in \mathbb{Z}$,

$$
\begin{aligned}
M \sup_{m\in\mathbb{Z}} \|u_m\|_k &\geq \| \sum_{m\in\mathbb{Z}} T_{h_{n-m}} \lozenge u_m \|_k \\
&= \sup_{\substack{v\in\mathcal{H}_k \\ \|v\|_k=1}} \langle \sum_{m\in\mathbb{Z}} T_{h_{n-m}} \lozenge u_m, v \rangle_k \\
&= \sup_{\substack{v\in\mathcal{H}_k \\ \|v\|_k=1}} \langle \sum_{\substack{m\in\mathbb{Z} \\ u_m\neq 0}} u_m, T^*_{h_{n-m}}(v) \rangle_k.
\end{aligned}
$$

The special choice

$$
u_m = \begin{cases} \dfrac{T^*_{h_{n-m}}(v)}{\|T^*_{h_{n-m}}(v)\|_k}, & \text{if} \quad \|T^*_{h_{n-m}}(v)\|_k \neq 0 \\ 0, & \text{if} \quad \|T^*_{h_{n-m}}(v)\|_k = 0 \end{cases}
$$

leads to

$$\sum_{\substack{m\in\mathbb{Z} \\ u_m\neq 0}} \|T^*_{h_{n-m}}(v)\|_k \leq M.$$

This expression stays the same also for the indices m such that $u_m \neq 0$ since $u_m \neq 0$ if and only if $\|T^*_{h_{n-m}}(v)\|_k \neq 0$. Finally, since the right handside of the above inequality is independent of the support of (u_m) we get the result.

Conversely, assume that (3.5) is in force. Then, still for sequences (u_m) with only a finite number of non zero entries, we have

$$
\begin{aligned}
\| \sum_{\substack{m \in \mathbb{Z} \\ u_m \neq 0}} T_{h_{n-m}} u_m \|_k &= \sup_{\substack{v \in \mathcal{H}_k \\ \|v\|_k = 1}} \langle \sum_{\substack{m \in \mathbb{Z} \\ u_m \neq 0}} u_m, T^*_{h_{n-m}}(v) \rangle_k \\
&\leq \sup_{\substack{v \in \mathcal{H}_k \\ \|v\|_k = 1}} \sum_{\substack{m \in \mathbb{Z} \\ u_m \neq 0}} \|u_m\|_k \|T^*_{h_{n-m}} v\|_k \\
&\leq \sup_{\substack{m \in \mathbb{Z} \\ u_m \neq 0}} \|u_m\|_k \cdot \sup_{\substack{v \in \mathcal{H}_k \\ \|v\|_k = 1}} \left(\sum_{m \in \mathbb{Z}} \|T^*_{h_m}(v)\|_k \right) \\
&\leq M \sup_{\substack{m \in \mathbb{Z} \\ u_m \neq 0}} \|u_m\|_k.
\end{aligned}
$$

Assume now that the sum

$$
\sum_{m \in \mathbb{Z}} T_{h_{n-m}} u_m
$$

converges absolutely in $\mathcal{H}_k$. The result is then obtained by continuity by considering partial finite sums. $\qquad\square$

Applying the Hermite transform to equations (1.4), that is to

$$
y_n = \sum_{m \in \mathbb{Z}} h_{n-m} \Diamond u_m, \quad n \in \mathbb{Z},
$$

leads to the following: let

$$
y_n(\omega) = \sum_{\alpha \in \ell} y_\alpha(n) H_\alpha(\omega) \quad \text{and} \quad h_n(\omega) = \sum_{\alpha \in \ell} h_\alpha(n) H_\alpha(\omega),
$$

where the coefficients $y_\alpha(n)$ and $h_\alpha(n)$ are nonrandom complex numbers. Then,

$$
y_n = \sum_{\alpha \in \ell} H_\alpha(\omega) \left(\sum_{m \in \mathbb{Z}} \sum_{\beta \leq \alpha} h_{\alpha-\beta}(n-m) u_\beta(m) \right), \quad n \in \mathbb{Z},
$$

that is, after taking the Hermite transform

$$\mathbf{I}(y_n) = \sum_{\alpha \in \ell} z^\alpha \left(\sum_{m \in \mathbb{Z}} \sum_{\beta \leq \alpha} h_{\alpha-\beta}(n-m) u_\beta(m) \right), \quad n \in \mathbb{Z},$$

and hence,

$$(3.6) \qquad y_\alpha(n) = \sum_{m \in \mathbb{Z}} \sum_{\beta \leq \alpha} h_{\alpha-\beta}(n-m) u_\beta(m), \quad n \in \mathbb{Z}.$$

Equation (3.6) exhibits two convolutions: the first is with respect to the index in ℓ, which is related to the stochastic aspect of the system; the second is with respect to the time variable.

The $\mathcal{Z}$ transform (denoted by $\widehat{y}$, with variable ζ) then leads to

$$\begin{aligned}
\widehat{y}(\zeta, z) &\overset{\text{def.}}{=} \sum_{n \in \mathbb{Z}} \mathbf{I}(y_n) \zeta^n \\
&= \sum_{\alpha \in \ell} z^\alpha \sum_{\beta \leq \alpha} \widehat{h}_{\alpha-\beta}(\zeta) \widehat{u}_\beta(\zeta) \\
&= \left(\sum_{\alpha \in \ell} z^\alpha \widehat{h}_\alpha(\zeta) \right) \left(\sum_{\alpha \in \ell} z^\alpha \widehat{u}_\alpha(\zeta) \right).
\end{aligned}$$

Definition 3.3. *The function*

$$\mathscr{H}(\zeta, z) = \sum_{\alpha \in \ell} z^\alpha \widehat{h}_\alpha(\zeta) = \sum_{n \in \mathbb{Z}} \zeta^n (\mathbf{I}(h_n))(z)$$

is the called generalized transfer function of the system.

When all $\widehat{h}_\alpha(\zeta) = 0$ for $\alpha \neq 0$, we retrieve the classical notion of the transfer function. We can thus define a hierarchy of systems, depending on the properties of the function $\mathscr{H}(\zeta, z)$. The rational case will be when the function $\mathscr{H}(\zeta, z)$ is of the form (2.4). Another case of interest would be the isospectral case, when the function $A(\zeta)$ in (2.4) does not depend on the variable ζ.

4. ℓ_1-ℓ_2 STABLE RANDOM SYSTEMS

The analog of Theorem 1.3 is the following:

Theorem 4.1. *Let $l > k + 1$ and assume that in the system* (1.5) $h_n \in \mathcal{H}_l$. *Then there exists $M > 0$ such that*

$$\left(\sum_{n=0}^{\infty} \|y_n\|_k^2 \right)^{1/2} \leq M \sum_{n=0}^{\infty} \|u_n\|_k$$

for all inputs (u_n) such that the right handside of the above equation is finite, if and only if its transfer function belongs to $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)$, i.e. if and only if there exists a number $c > 0$ such that the kernel

$$(4.1) \qquad \frac{K_l(z, w)}{1 - \zeta \nu^*} - c\mathscr{H}(\zeta, z)\mathscr{H}(\nu, w)^*$$

is positive in $\mathbb{D} \times \mathbb{K}_l$.

The system (1.4) is then called ℓ_1-ℓ_2 bounded. In the proof of the theorem, we use the already mentioned fact that the spaces $\mathcal{H}_k$ and $\mathcal{H}(K_k)$ are unitarily equivalent via the Hermite transform. This allows us to make use of Våge's inequality in the spaces $\mathcal{H}(K_k)$.

Proof of Theorem (4.1)**:** We first remark that the equivalence of (4.1) with the condition $\mathscr{H} \in \mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)$ directly follows the characterization of the elements of a reproducing kernel space. We proceed in three steps. Recall that $A(k - l)$ has been defined by (3.2).

STEP 1: *Assume that* (4.1) *holds, and let $u \in \mathcal{H}(K_k)$. Then,*

$$(4.2) \qquad \|\mathscr{H}x\|_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)} \leq A(k - \ell)\|\mathscr{H}\|_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)} \cdot \|u\|_{\mathcal{H}(K_k)}.$$

Indeed, let $\mathscr{H}(\zeta, z) = \sum_{n=0}^{\infty} \zeta^n h_n(z)$, with $h_n \in \mathcal{H}(K_\ell)$. Then,

$$\|\mathscr{H}u\|_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)}^2 = \sum_{n=0}^{\infty} \|h_n u\|_k^2$$

$$\leq A(k - l)^2 \sum_{n=0}^{\infty} \|h_n\|_{\mathcal{H}(K_l)}^2 \|u\|_{\mathcal{H}(K_k)}^2$$

$$= A(k - l)^2 \|\mathscr{H}\|_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)} \cdot \|u\|_{\mathcal{H}(K_k)}^2.$$

STEP 2: *Assume that* (4.1) *holds. Then, the system is ℓ_1-ℓ_2 bounded.*

We first note that, from the definition of the norm in $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)$, the operator M_ζ of multiplication by ζ is an isometry from $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)$ into itself:

$$(4.3) \quad \|M_\zeta F\|_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)} = \|F\|_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)}, \quad \forall F \in \mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l).$$

Let $\widehat{u}(\zeta, z) = \sum_{n=0}^{\infty} \zeta^n u_n(z)$, where the $u_n \in \mathcal{H}(K_k)$. Then,

$$\widehat{y}(\zeta, z) = \sum_{n=0}^{\infty} \mathcal{H}(\zeta, z)\zeta^n u_n(z),$$

where the convergence is in $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)$, as will be justified by equation (4.4) below. So, using (4.2) and (4.3), we have, with $\widehat{y}(\zeta, z) = \sum_{n=0}^{\infty} \zeta^n y_n(z)$,

$$\left(\sum_{n=0}^{\infty} \|y_n\|_{\mathcal{H}(K_k)}^2\right)^{1/2} = \|\widehat{y}\|_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)}$$

$$= \sum_{n=0}^{\infty} \|\mathcal{H} M_\zeta^n u_n\|_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)}$$

(4.4)
$$\leq \sum_{n=0}^{\infty} \|\mathcal{H} u_n\|_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)}$$

$$\leq A(k - l) \sum_{n=0}^{\infty} \|\mathcal{H}\|_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)} \|u_n\|_{\mathcal{H}(K_k)}$$

$$= M\left(\sum_{n=0}^{\infty} \|u_n\|_{\mathcal{H}(K_k)}\right),$$

with $M = A(k - l)\|\mathcal{H}\|_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_l)}$.

STEP 3: *We complete the proof.*

Assume the system ℓ_1-ℓ_2 bounded. Since $1 \in \mathcal{H}(K_k)$ we have that $\mathcal{H}(\zeta, z) \in \mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)$, and we can use the previous step to conclude the proof. $\square$

When the system is not random, that is, when $\mathcal{H}$ is only a function of ζ, the positivity of the kernel (4.1) is equivalent to the fact that $\mathcal{H} \in \mathbf{H}_2(\mathbb{D})$. Indeed, the tensor product space $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)$ can be seen not only as the representation (1.13), but also as the space of functions of the form

$$F(\zeta, z) = \sum_{\alpha \in \ell} z^\alpha f_\alpha, \quad f_\alpha \in \mathbf{H}_2(\mathbb{D}),$$

with the norm

$$\|F\|_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)}^2 = \sum_{\alpha \in \ell} \frac{\|f_\alpha\|_{\mathbf{H}_2(\mathbb{D})}^2}{(2\mathbb{N})^{k\alpha}}.$$

5. Dissipative discrete time random systems

We denote by M_ζ the operator of multiplication by ζ and by M_{z_j} the operator of multiplication by z_j (in both cases, the domain and range of these operators is given below). In the next theorem, the first condition expresses the time-invariance of the system. The second condition is needed to ensure that we get a multiplication operator. It can be interpreted as the property of invariance with respect to randomness. In the following statement, recall that the kernel K_k has been defined by (1.12).

Theorem 5.1. *Let $k \in \mathbb{N}$. The operators M_{z_j} are bounded from $\mathcal{H}(K_k)$ into itself. A linear operator from $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)$ into itself is contractive and such that*

(5.1)
$$T(M_\zeta f) = M_\zeta T f$$
$$T(M_{z_j} f) = M_{z_j} T f$$

if and only if it is of the form

$$(Tf)(\zeta, z) = \mathscr{S}(\zeta, z) f(\zeta, z)$$

where $\mathscr{S}$ is such that the kernel (1.18):

$$(1 - \mathscr{S}(\zeta, z) \mathscr{S}(\nu, w)^*) \frac{K_k(z, w)}{(1 - \zeta \nu^*)}$$

is positive in $\mathbb{D} \times \mathbb{K}_k$.

Proof: We divide the proofs into several steps.

STEP 1: *Let $\mathcal{H}$ be a Hilbert space, and let T be a bounded operator from $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}$ into itself which commutes with multiplication by ζ. Then, T is of the form*

$$Tf(\zeta) = S(\zeta) f(\zeta),$$

where S is an $\mathbf{L}(\mathcal{H})$-valued function analytic and contractive in the open unit disk.

This is a well known fact (see for instance [43, Lemma 1 p. 301]), being the discrete analog of the Bochner-Chandrasekharan theorem (see [13, Theorem 72, p. 144] and Section 6 below for the latter). We briefly review its proof for completeness. Let $c \in \mathcal{H}$. We have

$$(Tc)(\zeta) = \sum_{n=0}^{\infty} \zeta^n T_n(c),$$

where the operators T_n are readily seen to be linear bounded operators from $\mathcal{H}$ into itself. Furthermore,

$$\|T_n\| \le \|T\|, \quad \forall n \in \mathbb{N}_0,$$

since

$$\|Tc\|^2_{\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}} = \sum_{m=0}^{\infty} \|T_m c\|^2_{\mathcal{H}} \ge \|T_n c\|^2_{\mathcal{H}}, \quad \forall c \in \mathcal{H} \quad \text{and} \quad \forall n \in \mathbb{N}_0.$$

The $\mathbf{L}(\mathcal{H})$–valued function

$$(5.2) \qquad\qquad S(\zeta) = \sum_{n=0}^{\infty} \zeta^n T_n$$

is analytic in the open unit disk (we refer to [49, pp. 189–190] for a review of operator-valued analytic functions, and the equivalence between strong and weak analyticity). We now show that it takes contractive values and that $T = M_S$. Since $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}$ is the reproducing kernel Hilbert space with reproducing kernel

$$\frac{I_{\mathcal{H}}}{1 - \zeta \nu^*},$$

convergence in norm implies weak pointwise convergence in the coefficient space $\mathcal{H}$. Thus, for every $c \in \mathcal{H}$, and every $f \in \mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}$ in the form

$$f(\zeta) = \sum_{n=0}^{\infty} \zeta^n f_n,$$

using continuity of T, we have

$$\langle (Tf)(\zeta), c \rangle_{\mathcal{H}} = \langle (T(\sum_{n=0}^{\infty} M_\zeta^n f_n))(\zeta), c \rangle_{\mathcal{H}}$$

$$= \sum_{n=0}^{\infty} \langle (M_\zeta^n T(f_n))(\zeta), c \rangle_{\mathcal{H}}$$

$$= \sum_{n=0}^{\infty} \langle \zeta^n (Tf_n)(\zeta), c \rangle_{\mathcal{H}}$$

$$= \sum_{n=0}^{\infty} \langle \zeta^n S(\zeta) f_n, c \rangle_{\mathcal{H}}$$

$$= \langle S(\zeta) f(\zeta) c \rangle_{\mathcal{H}}.$$

Thus $T = M_S$. Furthermore, the formula

$$(M_S^* \frac{c}{1 - \cdot \nu^*})(\zeta) = \frac{S(\nu)^* c}{1 - \zeta \nu^*}$$

implies that the kernel

$$(5.3) \qquad \frac{I_{\mathcal{H}} - S(\zeta)S(\nu)^*}{1 - \zeta\nu^*}$$

is positive in $\mathbb{D}$ and in particular S takes contractive values.

STEP 2: *The operators M_{z_j} are bounded in $\mathcal{H}(K_k)$.*

Indeed, let $g(z) = \sum_{\alpha \in \ell} c_\alpha z^\alpha$ with

$$\|g\|^2_{\mathcal{H}(K_k)} = \sum_{\alpha \in \ell} |c_\alpha|^2 (2\mathbb{N})^{-k\alpha} < \infty.$$

Then, with $e_j \in \ell$ denoting the sequence with all elements equal to 0, at the exception of the j-th, equal to 1, we have

$$z_j g(z) = \sum_{\alpha \in \ell} c_\alpha z^{\alpha + e_j},$$

and

$$\sum_{\alpha \in \ell} |c_\alpha|^2 (2\mathbb{N})^{-k(\alpha + e_j)} = \sum_{\alpha \in \ell} |c_\alpha|^2 (2\mathbb{N})^{-k\alpha} \left((2\mathbb{N})^{k\alpha} (2\mathbb{N})^{-k(\alpha + e_j)} \right)$$

$$\leq \sum_{\alpha \in \ell} |c_\alpha|^2 (2\mathbb{N})^{-k\alpha},$$

and so M_{z_j} is a contractive operator in $\mathcal{H}(K_k)$.

STEP 3: *We now take $\mathcal{H} = \mathcal{H}(K_k)$. The fact that T commutes with the operators of multiplication by z_j implies that*

$$(T_n f)(z) = s_n(z) f(z),$$

for some $\mathbf{L}(\mathcal{H}(K_k))$-valued function $s_n(z)$.

Indeed, we have in particular

$$T_n(z^\alpha) = z^\alpha T_n(1).$$

Thus, for $f(z) = \sum_{\alpha \in \ell} c_\alpha z^\alpha \in \mathcal{H}(K_k)$ we have

$$(T_n f)(z) = \sum_{\alpha \in \ell} c_\alpha T_n(1) = T_n(1) f(z),$$

first for finite sums, and then using continuity of the operator T_n for all $f \in \mathcal{H}(K_k)$ in the norm topology, and finally pointwise, since $\mathcal{H}(K_k)$ is a reproducing kernel Hilbert space.

We set $(T_n(1))(z) = s_n(z)$. The result follows with

$$\mathscr{S}(\zeta, z) = \sum_{n=0}^{\infty} \zeta^n s_n(z).$$

Therefore T is a multiplication operator in the reproducing kernel Hilbert space $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)$. Since it is a contraction, (1.18) is in force.

The converse is clear: if a function $\mathscr{S}$ is such that the kernel (1.18) is positive on $\mathbb{D} \times \mathbb{K}_k$, then the operator of multiplication by $\mathscr{S}$ is a contraction from $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)$ into itself, and satisfies the commutativity hypothesis (5.1). $\qquad \square$

When the system is not random, $\mathscr{S}(\zeta, z)$ depends only on ζ and is a Schur function.

We further remark that the function $S(\zeta)$ defined in (5.2) is a $\mathbf{L}(\mathcal{H}(K_k))$-valued function, and the theory of realization for such functions is thus applicable to it; see [6]. Using the results of [6] we have:

Theorem 5.2. *Let $\mathscr{S}(\zeta, z)$ be a Schur multiplier of the space $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)$, and let $\mathcal{H}(\mathscr{S})$ the associated reproducing kernel Hilbert space. Then*

$$\mathscr{S}(\zeta, z) = D + \zeta C(I_{\mathcal{H}(\mathscr{S})} - \zeta A)^{-1} B,$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H}(\mathscr{S}) \\ \mathcal{H}(K_k) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}(\mathscr{S}) \\ \mathcal{H}(K_k) \end{pmatrix}$$

with

$$(Af)(\zeta, z) = \frac{f(\zeta, z) - f(0, z)}{\zeta},$$

$$(Bx)(\zeta, z) = \frac{\mathscr{S}(\zeta, z) - \mathscr{S}(0, z)}{\zeta} x(z),$$

$$(Cf)(\zeta, z) = f(0, z),$$

$$(Dx)(\zeta, z) = \mathscr{S}(0, z) x(z),$$

defining a coisometric realization of $\mathscr{S}$.

Proof: Indeed, let $\mathcal{H}$ denote a Hilbert space, and let S be an $\mathbf{L}(\mathcal{H})$-valued function analytic and contgractive in $\mathbb{D}$. Then the kernel (5.3) is positive in the open unit disk. Let $\mathcal{H}(S)$ be the associated reproducing kernel Hilbert space. The formulas

$$(Af)(\zeta) = \frac{f(\zeta) - f(0)}{\zeta},$$

$$(B\xi)(\zeta) = \frac{S(\zeta) - S(0)}{\zeta}\xi,$$

$$Cf = f(0),$$

$$D\xi = \mathscr{S}(0)\xi,$$

define a coisometric realization with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H}(S) \\ \mathcal{H} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}(S) \\ \mathcal{H} \end{pmatrix}$$

in the form:

$$S(\zeta) = D + \zeta C(I_{\mathcal{H}(S)} - \zeta A)^{-1}B.$$

Now, there is here an extra structure, which does not appear in the general case: there is a function $\mathscr{S}(\zeta, z)$ such that

$$(S(\lambda)x)(z) = \mathscr{S}(\zeta, z)x(z), \quad x \in \mathcal{H}(K_k).$$

The formulas for the realization of $\mathscr{S}$ then easily follow. $\qquad\square$

Since constants belong to $\mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)$, it follows in particular that $\mathscr{S}(\zeta, z) \in \mathbf{H}_2(\mathbb{D}) \otimes \mathcal{H}(K_k)$, and thus can be written as

$$\mathscr{S}(\zeta, z) = \sum_{n=0}^{\infty} \zeta^n h_n(z).$$

The fact that the operator is a contraction is then equivalent to the fact that the block Toeplitz operator

$$(5.4) \qquad \begin{pmatrix} T_{h_0} & 0 & \cdot & \cdot & \\ T_{h_1} & T_{h_0} & \cdot & \cdot & \\ T_{h_2} & T_{h_1} & T_{h_0} & \cdot & \end{pmatrix}$$

is a contraction from the Hilbert space $\ell_2(\mathbb{N}_0) \otimes \mathcal{H}(K_k)$ into itself.

As a consequence, we have:

Theorem 5.3. *Let $k > l + 1$ and let (h_n) be a sequence of elements in $\mathcal{H}_l$ indexed by $\mathbb{N}_0$. Let*

$$y_n = \sum_{m=0}^{n} h_{n-m} \Diamond u_m, \quad n = 0, 1, \ldots$$

Then there exists an $M \in [0, 1]$ such that, for all signals

$$(5.5) \qquad \sum_{n=0}^{\infty} \|y_n\|_k^2 \leq M \sum_{n=0}^{\infty} \|u_n\|_k^2$$

if and only if (5.4) *is a contraction.*

6. The Bochner-Chandrasekharan theorem

Let $\mathcal{H}$ be a Hilbert space. We denote for $h \in \mathbb{R}$ by T_h the translation operator:

$$(T_h f)(t) = f(t + h),$$

where f is in $\mathbf{L}_2(\mathbb{R}) \otimes \mathcal{H}$. The fact that a bounded operator from $\mathbf{L}_2(\mathbb{R}) \otimes \mathcal{H}$ into itself commutes with all the T_h, expresses the time-invariance of the underlying linear system. When $\mathcal{H} = \mathbb{C}$, the Bochner-Chandrasekharan theorem provides a characterization of all bounded operators from $\mathbf{L}_2(\mathbb{R})$ into itself which commute with the T_h; see [13, Theorem 72, p. 144]. In the present section we show how the strategy of [13] allows to prove a version of this theorem in the case of Hilbert space valued functions. Our approach consists of reducing the operator-valued case to the scalar case, but we need an extra assumption (the invariance property (6.1)), which means that we consider causal systems, a property which holds in the cases considered in the paper. We first prove the following result. Its scalar version is [13, Theorem 70, p. 140]. In the statement of Theorem 6.1, for a given Hilbert space $\mathcal{H}$, the Banach space of bounded operator from $\mathcal{H}$ into itself is denoted by $\mathbf{L}(\mathcal{H})$. Moreover, we denote by $\mathbf{H}_2(\mathbb{C}_+)$ the Hardy space of the open upper half plane $\mathbb{C}_+$.

In connection with this theorem, it is of interest to mention the following result proved in [37, Theorem 1 p. 397]: First recall that $\mathcal{E}'$ denotes the space of distributions with compact support, and that $\mathcal{D}'$ denotes the dual of the space $\mathcal{D}$ of infinitely differentiable functions with compact support. Then, any linear continuous map from $\mathcal{E}'$ into $\mathcal{D}'$ which commutes with the translation operators is a convolution by a tempered distribution. See also the discussion in [42, p. 35].

Theorem 6.1. *Let $\mathcal{H}$ be a separable Hilbert space, and let X be a bounded linear operator from $\mathbf{L}_2(\mathbb{R}) \otimes \mathcal{H}$ into itself. Assume that if f vanishes outside an interval, then Xf vanishes outside the same interval. Assume moreover that*

$$(6.1) \qquad X(\mathbf{H}_2(\mathbb{C}_+) \otimes \mathcal{H}) \subset \mathbf{H}_2(\mathbb{C}_+) \otimes \mathcal{H}.$$

Then, there exists a $\mathbf{L}(\mathcal{H})$-valued function S bounded in norm and an-alytic in $\mathbb{C}_+$ such that X is the operator of multiplication by S:

$$Xf(\lambda) = S(\lambda)f(\lambda), \quad f \in \mathbf{H}_2(\mathbb{C}_+) \otimes \mathcal{H}, \quad \lambda \in \mathbb{C}_+.$$

Remark 6.2. *Bochner and Chandrasekharan call an operator which has the vanishing property of Theorem 6.1 a transformation of simple type.*

Proof of Theorem 6.1: Let $e_0, e_1, \ldots$ be an orthonormal basis of $\mathcal{H}$. For $n, m \in \mathbb{N}_0$, the expressions

$$(6.2) \qquad (X_{n,m}(u))(t) = \langle (X(ue_n))(t), e_m \rangle_{\mathcal{H}}, \quad u \in \mathbf{L}_2(\mathbb{R}).$$

define bounded linear operators from $\mathbf{L}_2(\mathbb{R})$ into itself since, by the Cauchy-Schwarz inequality

$$|(X_{n,m}(u))(t)| \le \|(X(ue_n))(t)\|_{\mathcal{H}},$$

and so

$$\begin{aligned}
\int_{\mathbb{R}} |(X_{n,m}(u))(t)|^2 dt &\le \int_{\mathbb{R}} \|X(ue_n)\|_{\mathcal{H}}^2(t) dt \\
&= \|X(ue_n)\|_{\mathbf{L}_2(\mathbb{R}) \otimes \mathcal{H}}^2 \\
&\le \|X\|^2 \|ue_n\|_{\mathbf{L}_2(\mathbb{R}) \otimes \mathcal{H}}^2 \\
&= \|X\|^2 \cdot \|u\|_{\mathbf{L}_2(\mathbb{R})}^2.
\end{aligned}$$

Moreover, the $X_{n,m}$ are transformations of simple type since X is a transformation of simple type; therefore, by [13, Theorem 70, p. 140], there exist bounded measurable functions $s_{m,n}$ such that

$$(6.3) \qquad \langle (X(ue_n))(t), e_m \rangle_{\mathcal{H}} = s_{m,n}(t) \cdot u(t) \quad u \in \mathbf{L}_2(\mathbb{R}).$$

From the proof in [13] we moreover have that

$$\sup_{t \in \mathbb{R}} |s_{m,n}(t)| \le \|T\|.$$

Setting $u = \langle f, e_n \rangle_{\mathcal{H}}$ with $f \in \mathbf{L}_2(\mathbb{R}) \otimes \mathcal{H}$ in (6.3), we obtain

$$(6.4) \qquad \langle (X(\langle f, e_n \rangle_{\mathcal{H}} e_n))(t), e_m \rangle_{\mathcal{H}} = s_{n,m}(t) \cdot \langle f(t), e_n \rangle_{\mathcal{H}}$$

for every $f \in \mathbf{L}_2(\mathbb{R}) \otimes \mathcal{H}$.

Summing (6.4) with respect to n we expect to obtain

$$(6.5) \quad \langle (Xf)(t), e_m \rangle_{\mathcal{H}} = \sum_{n=0}^{\infty} s_{m,n}(t) \cdot \langle f(t), e_n \rangle_{\mathcal{H}}, \quad m = 0, 1, 2, \ldots,$$

to show that

$$(Xf)(t) = S(t)f(t),$$

where $S(t)$ is the operator with matrix $(s_{m,n}(t))$. We cannot justify these computations in the general case (see also the remark following the proof of the theorem), and at this stage of the proof, hypothesis (6.1) is applied.

In view of (6.1), the functions $s_{m,n}$ are analytic in the open upper half-plane. The space $\mathbf{H}_2(\mathbb{C}_+)$ is the reproducing kernel Hilbert space with reproducing kernel

$$\frac{I_{\mathcal{H}}}{-i(\lambda - \mu^*)}, \quad \lambda, \mu \in \mathbb{C}_+.$$

and therefore convergence in norm in this space implies pointwise convergence. Thus it follows from

$$(6.6) \qquad\qquad Xf = \sum_{0}^{\infty} \langle Xf, e_n \rangle_{\mathcal{H}} e_n,$$

where the convergence is in $\mathbf{H}_2(\mathbb{C}_+) \otimes \mathcal{H}$, that for every $m \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}_+$,

$$\langle Xf, \frac{e_n}{-i(\cdot - \lambda^*)} \rangle_{\mathbf{L}_2 \otimes \mathcal{H}} = \langle (Xf)(\lambda), e_m \rangle_{\mathcal{H}}$$

$$= \sum_{m=0}^{\infty} \langle \langle (Xf)(\lambda), e_n \rangle_{\mathcal{H}} e_n, e_m \rangle_{\mathcal{H}}$$

$$= \sum_{m=0}^{\infty} s_{m,n}(\lambda) \langle f(\lambda), e_n \rangle_{\mathcal{H}}$$

$$= \langle S(\lambda) f(\lambda), e_n \rangle_{\mathcal{H}}.$$

$\square$

Remark 6.3. Given a sequence of $\mathbf{L}_2(\mathbb{R})$ functions which converges in $\mathbf{L}_2(\mathbb{R})$, there exists a subsequence of this sequence which converges almost everywhere; see for instance [18, Théorème 2.3 p. 95]. Therefore, without assumption (6.1), convergence in norm of the series (6.6) and a diagonal argument allows to assert that, for a given $f \in \mathbf{L}_2(\mathbb{R}) \otimes \mathcal{H}$, (6.5) holds almost everywhere on the real line when the limit is taken via a subsequence.

Recall that we denote by U the Fourier transform. We can now prove:

Theorem 6.4. *Let X be a bounded operator from $\mathbf{L}_2(\mathbb{R}) \otimes \mathcal{H}$ into itself which commute with translation operators, and assume that UXU^{-1}*

satisfies (6.1). *Then there exists a* $\mathbf{L}(\mathcal{H})$*-valued function* S *such that*

$$UXU^{-1}f = S(t)f(t), \quad f \in \mathbf{H}_2(\mathbb{C}_+) \otimes \mathcal{H}.$$

Proof: Let $X_{m,n}$ be defined by (6.2) and $h \in \mathbb{R}$. Since X commutes with the T_h, we have for $u \in \mathbf{L}_2(\mathbb{R}$

$$
\begin{aligned}
(T_h(X_{m,n}u))(t) &= \langle (X(ue_n))(t+h), e_m \rangle_{\mathcal{H}} \\
&= \langle (X(T_hu))(t)e_m \rangle_{\mathcal{H}} \\
&= (X_{m,n}(T_hu))(t),
\end{aligned}
$$

and thus $X_{m,n}$ also commutes with the translation operators. We now apply [13, Theorem 72, p. 144-147] to show that the image under the Fourier transform of $X_{m,n}$ is of simple type, which allows to conclude the thanks to the preceding Theorem 6.1. $\qquad\square$

7. BIBO STABILITY: THE CONTINUOUS TIME CASE

In the classical case, one considers the Hardy space $\mathbf{H}_2(\mathbb{C}_+)$ of the open upper half-plane, and, in the stochastic case, we will consider the space $\mathbf{H}_2(\mathbb{C}_+) \otimes \mathcal{H}_k$.

In this section we restrict ourselves to the case of continuous functions $t \mapsto f(t)$ from the real line to $\mathcal{H}_k$ for some $k \geq 1$. Then, there is no difficulty to define the integral $\int_a^b f(t)dt$ on a compact interval, and, with appropriate hypothesis, also on the real line. Obviously, more general situations can be considered. These will not be pursued in the present paper.

A continuous signal will be an $\mathcal{H}_k$-valued continuous function $t \mapsto u(t)$ defined for $t \in \mathbb{R}$, and such that

$$\sup_{t \in \mathbb{R}} \|u(t)\|_k < \infty.$$

Lemma 7.1. *Let* $k > l + 1$. *Assume that* $t \mapsto f(t)$ *is a continuous* $\mathcal{H}_l$*-valued function and* $t \mapsto g(t)$ *is a continuous* $\mathcal{H}_k$*-valued function. The* $\mathcal{H}_k$*-valued function* $t \mapsto f(t)\lozenge g(t)$ *is continuous with respect to the norm of* $\mathcal{H}_k$.

Proof: to prove the claim, it suffices to write

$$f(t_1)\lozenge g(t_1) - f(t_2)\lozenge g(t_2) = (f(t_1) - f(t_2))\lozenge g(t_1) + f(t_2)\lozenge(g(t_1) - g(t_2))$$

for $t_1, t_2 \in \mathbb{R}$ and use inequality (3.1). $\qquad\square$

Definition 7.2. *A continuous linear time-invariant stochastic system will be defined by its S_{-1}-valued impulse response*

$$(7.1) \qquad\qquad h(\tau, \omega) = \sum_{\alpha \in \ell} h_\alpha(\tau) H_\alpha(\omega).$$

The associated input-output relation is in the Wick convolution form

$$y(t) = \int_{\mathbb{R}} h(t - \tau) \Diamond u(\tau) d\tau$$

Theorem 7.3. *Let $k > l + 1$. Let $t \mapsto h(t)$ be a continuous $\mathcal{H}_l$-valued function $(t \in \mathbb{R})$. Then:*
There exists $M > 0$ such that
(a) The integrals

$$(7.2) \qquad\qquad y(t) = \int_{\mathbb{R}} h(s) \Diamond u(t - s) ds, \quad \forall t \in \mathbb{R},$$

exist for all continuous $\mathcal{H}_k$-valued functions u such that

$$\sup_{\mathbb{R}} \|u(t)\|_k < \infty$$

and
(b) it holds that

$$(7.3) \qquad \sup_{t \in \mathbb{R}} \| \int_{\mathbb{R}} h(s) \Diamond u(t - s) ds \|_k \le M \sup_{t \in \mathbb{R}} \|u(t)\|_k$$

if and only if for every continuous $g \in \mathcal{H}_k$ of norm less or equal to 1, it holds that

$$(7.4) \qquad\qquad \int_{\mathbb{R}} \|T^*_{h(t)} g\|_k dt \le M$$

Proof: We first remark that the continuity of the function $t \mapsto h(t)$ implies the continuity of the function $t \mapsto \|T^*_{h(t)} g\|_k$. Indeed, we have for $t_1, t_2 \in \mathbb{R}$

$$\begin{aligned}
\left| \|T^*_{h(t_1)} g\|_k - \|T^*_{h(t_2)} g\|_k \right| &\le \|(T^*_{h(t_1)} - T^*_{h(t_2)}) g\|_k \\
&\le \|g\|_k \cdot \|T^*_{h(t_1)} - T^*_{h(t_2)}\| \\
&= \|g\|_k \cdot \|T^*_{h(t_1) - h(t_2)}\| \\
&= \|g\|_k \cdot \|T_{h(t_1) - h(t_2)}\| \\
&\le A(k - l) \|g\|_k \cdot \|h(t_1) - h(t_2)\|_l
\end{aligned}$$

where we have used (3.3) to obtain to the last inequality. Thus, the integral on the left handside of (7.4) makes sense as a possibly divergent integral. We now show that it converges.

Assume that (7.3) is in force. Then, for every real t,

$$\int_{\mathbb{R}} \|h(\tau)\Diamond u(t-\tau)d\tau\|_k \leq M \sup_{t\in\mathbb{R}} \|u(t)\|_k.$$

Hence, we have that

$$\int_{\mathbb{R}} |\langle h(\tau)\Diamond u(t-\tau)d\tau, g\rangle_k|du \leq M \sup_{t\in\mathbb{R}} \|u(t)\|_k,$$

and thus

$$|\int_{\mathbb{R}} \langle h(s)\Diamond u(t-s), g\rangle_k ds| \leq M \sup_{\mathbb{R}} \|u(t)\|_k,$$

for every $g \in \mathcal{H}_k$ of norm less or equal to 1. Hence, for such g,

$$(7.5) \qquad |\int_{\mathbb{R}} \langle u(t-s), T^*_{h(s)}(g)\rangle_k ds| \leq M \sup_{\mathbb{R}} \|u(t)\|_k.$$

Let $\epsilon > 0$ and t fixed in $\mathbb{R}$. The function

$$u(s) = \frac{T^*_{h(t-s)}(g)}{\|T^*_{h(t-s)}(g)\|_k + \epsilon}$$

is continuous in s with respect to the norm of $\mathcal{H}_k$. Its substitution in (7.5) leads to

$$\int_{\mathbb{R}} \frac{\|T^*_{h(s)}(g)\|^2_k}{\|T^*_{h(s)}(g)\|_k + \epsilon} ds \leq M.$$

Taking $\epsilon = 1/n$, $n = 1, 2, \ldots$ and using the Monotone Convergence Theorem then leads to (7.4).

Conversely, assume that (7.4) holds, and let u be a $\mathcal{H}_k$-valued continuous and bounded function. From

$$|\langle h(s)\Diamond u(t-s), g\rangle_k| = |\langle u(t-s), T^*_{h(s)}(g)\rangle_k|$$
$$\leq \sup_{s\in\mathbb{R}} \|u(s)\|_k \|T^*_{h(s)}(g)\|_k,$$

we obtain that

$$\int_{\mathbb{R}} |\langle h(s)\Diamond u(t-s), g\rangle_k| ds \leq M \sup_{s\in\mathbb{R}} \|u(s)\|_k,$$

and thus

$$\int_{\mathbb{R}} |\langle h(s)\Diamond u(t-s), g\rangle_k| ds \leq M \sup_{s\in\mathbb{R}} \|u(s)\|_k.$$

Since this inequality holds for all $g \in \mathcal{H}_k$ of norm less or equal to 1, we obtain (7.3). In the above chain of equalities we have used the

following (see [23, (8.7.6) p. 169]): given a Hilbert space $\mathcal{H}$ and a $\mathcal{H}$-valued continuous function h such that $\int_\mathbb{R} h(s)ds$ exists, then, for any $g \in \mathcal{H}$, it holds that

$$\langle \int_\mathbb{R} h(s)ds, g \rangle_\mathcal{H} = \int_\mathbb{R} \langle h(s), g \rangle_\mathcal{H} ds,$$

which concludes the proof of the theorem. $\qquad\qquad\square$

We note that a weaker condition than (7.4) is given by

(7.6)
$$\int_\mathbb{R} \|T_{h(t)}\| dt \leq M,$$

where, in view of the continuity of h, the integral makes sense since, by (3.3),

$$\|T_{h(t_1)-h(t_2)}\| \leq A(k-l)\|h(t_1)-h(t_2)\|_l.$$

We note that, when h is not random, the Wick product reduces to a pointwise product, and so (7.4) reduces to the well known condition

$$\int_\mathbb{R} \|h\|(t)dt < \infty.$$

See for instance [38, §2.6.1. p. 175] or [48, Corollary 3, p. 585].

Finally, write $u(t) = \sum_\alpha u_\alpha(t)H_\alpha$, and similarly for the output. Taking the Laplace transform (denoted by $\widehat{u}$), followed by the Hermite transform, we obtain an expression of the form

$$\sum_{\alpha \in \ell} \widehat{u_\alpha}(\xi)z^\alpha,$$

where we have denoted by ξ the variable corresponding to the Laplace transform.

Proposition 7.4. *It holds that*

(7.7)
$$\widehat{y}(z, \zeta) = \sum_{\alpha \in \ell} z^\alpha \left\{ \sum_{\beta \leq \alpha} \widehat{h}_\beta(\zeta)\widehat{u}_{\alpha-\beta}(\zeta) \right\}.$$

Proof: For given real numbers τ and t we have

$$h(t-\tau)\Diamond u(\tau) = \sum_{\alpha \in \ell} \left\{ \sum_{\beta \leq \alpha} h_\beta(t-\tau)u_{\alpha-\beta}(\tau) \right\} H_\alpha(\omega),$$

and so, y_α is given by

$$(7.8) \qquad y_\alpha(\tau) = \int_{\mathbb{R}} \sum_{\beta \leq \alpha} h_\beta(t - \tau) u_{\alpha-\beta}(\tau) d\tau$$

Taking Hermite and Laplace transforms leads to (7.7).   □

We see that as in the discrete time case, in continuous time, we also have two convolutions, one with respect to time and one reflecting the stochastic aspect of the system.

8. DISSIPATIVE CONTINUOUS TIME RANDOM SYSTEMS

For $t \geq 0$ let χ_t denote the function

$$\chi_t(\lambda) = e^{it\lambda}.$$

Note that, for $t \geq 0$, the operator M_{χ_t} of multiplication by χ_t is an isometry from $\mathbf{H}_2(\mathbb{C}_+)$ into itself, and that, by the Bochner-Chandrasekharan theorem and the properties of the Fourier transform, a bounded linear operator from $\mathbf{H}_2(\mathbb{C}_+)$ into itself commutes with the operators M_{χ_t} if and only if it is a multiplication operator, or, in systems terminology, if and only if it is time-invariant.

Theorem 8.1. *Let T be a linear contractive operator from $\mathbf{H}_2(\mathbb{C}_+) \otimes \mathcal{H}(K_k)$ into itself and such that*

$$(8.1) \qquad \begin{aligned} T(\chi_t f) &= \chi_t f \\ T(z_j f) &= z_j T f. \end{aligned}$$

Then, T is a multiplication operator by a function $\mathscr{H}(\lambda, z)$ which is such that the kernel

$$(8.2) \qquad (1 - \mathscr{H}(\lambda, z)\mathscr{H}(\nu, w)^*)\frac{K_k(z, w)}{-i(\lambda - \nu^*)}$$

is positive in $\mathbb{C}_+ \times \mathbb{K}_k$.

Proof: The proof follows the lines of the proof of Theorem 5.1. We use the Hilbert space version of the Bochner-Chandrasekharan theorem to assert the existence of a $\mathbf{L}(\mathcal{H}(K_k))$-valued function S such that

$$T(\lambda) = S(\lambda)f(\lambda), \quad f \in \mathbf{H}_2(\mathbb{C}_+) \otimes \mathcal{H}(K_k),$$

for which $\|S(\lambda)\|_{\mathbf{L}(\mathcal{H}(K_k))} \leq 1, \quad \forall \lambda \in \mathbb{C}_+$. This last norm condition on S is equivalent to the positivity of the kernel

$$\frac{I_{\mathcal{H}(K_k)} - S(\lambda)S(\nu)^*}{-i(\lambda - \nu^*)}$$

in the open upper half-plane $\mathbb{C}_+$. Let $\lambda \in \mathbb{C}_+$. Since the operator

$$S(\lambda) \; : \; \mathcal{H}(K_k) \longrightarrow \mathcal{H}(K_k)$$

commutes with the operators of multiplication by the variables z_j we have that, for every $\alpha \in \ell$,

$$(8.3) \qquad ((S(\lambda))(M_{z^\alpha}))(z) = z^\alpha ((S(\lambda)(1)))(z).$$

We set:

$$\mathscr{S}(\lambda, z) = S(\lambda)(1).$$

Let $f \in \mathcal{H}(K_k)$. It follows from (8.3) that

$$(S(\lambda)(f))(\lambda, z) = \mathscr{S}(\lambda, z) \cdot f(\lambda, z).$$

The operator T is thus a contractive multiplication operator in the reproducing kernel Hilbert space $\mathbf{H}_2(\mathbb{C}_+) \otimes \mathcal{H}(K_k)$, and the kernel (8.2) is positive in $\mathbb{C}_+ \times \mathbb{K}_k$. $\qquad \square$

As in the disk case, one can write a realization for $\mathscr{S}$ in terms of an associated reproducing kernel Hilbert space. This will not be pursued here.

When the system is not random, setting $z = w$ and $\zeta = \nu$ in (8.2), we get that the function $\mathscr{S}(\zeta)$ is contractive in $\mathbb{C}_+$.

9. $\mathbf{L}_2$-$\mathbf{L}_\infty$ STABILITY

In this section we prove the analog of Theorem 4.1 in the continuous time case.

Definition 9.1. *Let $k > l+1$. The continuous time system (7.2) (with a continuous h) is said to be $\mathbf{L}_2$-$\mathbf{L}_\infty$ stable if there exists an $M < \infty$ such that*

$$(9.4) \qquad \sup_{t \in \mathbb{R}} \| \int_{\mathbb{R}} h(s) \lozenge u(t-s) ds \|_k \le M \left(\int_{\mathbb{R}} \|u(t)\|^2 dt \right)^{1/2}$$

for all $\mathcal{H}_k$-valued functions u which are moreover continuous in norm and for which the right handside of (9.4) is finite.

Theorem 9.2. *The system (7.2) is $\mathbf{L}_2$-$\mathbf{L}_\infty$ stable if and only if*

$$(9.5) \qquad \sup_{\substack{g \in \mathcal{H}_k \\ \|g\|_k \le 1}} \int_{\mathbb{R}} \|T^*_{h(t)} g\|_k^2 dt < \infty.$$

Proof: We first assume that (9.5) is in force. Let $g \in \mathcal{H}_k$ of norm less or equal to 1. Using the Cauchy-Schwarz inequality, we obtain

$$|\int_{\mathbb{R}} \langle h(s) \Diamond u(t-s), g \rangle_{\mathcal{H}_k} ds| = |\int_{\mathbb{R}} \langle u(t-s), T^*_{h(s)}g \rangle_{\mathcal{H}_k} ds|$$

$$\leq \int_{\mathbb{R}} \|T^*_{h(s)}g\|_k \|u(t-s)\|_k ds$$

$$\leq \left(\int_{\mathbb{R}} \|T^*_{h(s)}g\|^2_k ds \right)^{1/2} \left(\int_{\mathbb{R}} \|u(t-s)\|^2_k ds \right)^{1/2}$$

$$= \left(\int_{\mathbb{R}} \|T^*_{h(s)}g\|^2_k ds \right)^{1/2} \left(\int_{\mathbb{R}} \|u(s)\|^2_k ds \right)^{1/2},$$

and thus the system is $\mathbf{L}_2$-$\mathbf{L}_\infty$ stable since

$$\|\int_{\mathbb{R}} h(s) \Diamond u(t-s) ds\|_k = \sup_{\substack{g \in \mathcal{H}_k \\ \|g\|_k \leq 1}} \int_{\mathbb{R}} \langle h(s) \Diamond u(t-s), g \rangle_{\mathcal{H}_k}$$

Conversely, assume that the system (7.2) is $\mathbf{L}_2$-$\mathbf{L}_\infty$ stable. Then for every g as above,

(9.6)
$$|\int_{\mathbb{R}} \langle h(s) \Diamond u(t-s), g \rangle_{\mathcal{H}_k} ds| = |\int_{\mathbb{R}} \langle u(t-s), T^*_{h(s)}g \rangle_{\mathcal{H}_k} ds|$$

$$\leq M \left(\int_{\mathbb{R}} \|u(t)\|^2_k dt \right)^{1/2}$$

and so for every t, and in particular for $t = 0$, the functional

$$u \mapsto \int_{\mathbb{R}} \langle u(t-s), T^*_{h(s)}g \rangle_{\mathcal{H}_k} ds$$

is bounded. It follows from Riesz representation theorem for bounded functionals on Hilbert space that

$$\int_{\mathbb{R}} \|T^*_{h(s)}g\|^2_k ds < \infty.$$

This inequality is moreover uniform in g over all functions g such that $\|g\|_k \leq 1$, since g does not appear on the right handside of the last inequality in (9.6). □

We note that condition (9.5) will hold as soon as

$$\int_{\mathbb{R}} \|T_{h(s)}\|^2 ds < \infty.$$

In the nonrandom case, this means that h is in $\mathbf{L}_2(\mathbb{R})$, and we obtain the continuous time analog of Theorem 1.3. See [24, Table 2.2 p. 19

and pp. 21-25].

In connection with the preceding result, we recall the following inequalities: First, if p and q are positive real numbers such that $1/p+1/q = 1$, and $f \in \mathbf{L}_p(\mathbb{R})$ and $g \in \mathbf{L}_q(\mathbb{R})$, we have that the convolution $f * g \in \mathbf{L}_\infty(\mathbb{R})$ and

$$\|f * g\|_\infty \le \|f\|_p \cdot \|g\|_q,$$

where $\| \cdot \|_p$ denotes the norm in $\mathbf{L}_p(\mathbb{R})$; see [18, Théorème 2.3, p. 148], and [51, Corollary 1 p. 280] for a more general statement. The situation at hand here corresponds to $p = q = 2$. We also recall (see [16, Théorème IV.15, p. 66], [50, Exercice 4, p. 141]): If $f \in \mathbf{L}_1(\mathbb{R})$ and $g \in \mathbf{L}_p(\mathbb{R})$, with $p \in [1, \infty]$ (∞ included), then the convolution $f * g \in \mathbf{L}_p(\mathbb{R})$, and

$$\|f * g\|_p \le \|f\|_1 \cdot \|g\|_p.$$

These results should lead to other stability results in the random case. Such a line of research will not be developed here.

References

[1] D. Alpay. *The Schur algorithm, reproducing kernel spaces and system theory.* American Mathematical Society, Providence, RI, 2001. Translated from the 1998 French original by Stephen S. Wilson, Panoramas et Synthèses. [Panoramas and Syntheses].

[2] D. Alpay, J. Ball, and Y. Peretz. System theory, operator models and scattering: the time-varying case. *J. Operator Theory*, 47(2):245–286, 2002.

[3] D. Alpay and P. Dewilde. Time-varying signal approximation and estimation. In M.A. Kaashoek, J.H. van Schuppen, and A.C.M. Ran, editors, *Signal processing, scattering and operator theory, and numerical methods (Amsterdam, 1989)*, volume 5 of *Progress in systems and control theory*, pages 1–22. Birkhäuser Boston, Boston, MA, 1990.

[4] D. Alpay, P. Dewilde, and H. Dym. Lossless inverse scattering and reproducing kernels for upper triangular operators. In I. Gohberg, editor, *Extension and interpolation of linear operators and matrix functions*, volume 47 of *Oper. Theory Adv. Appl.*, pages 61–135. Birkhäuser, Basel, 1990.

[5] D. Alpay, A. Dijksma, and Y. Peretz. Nonstationary analogs of the Herglotz representation theorem: the discrete case. *J. Funct. Anal.*, 166(1):85–129, 1999.

[6] D. Alpay, A. Dijksma, J. Rovnyak, and H. de Snoo. *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, volume 96 of *Operator theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1997.

[7] D. Alpay and J. Leblond. Loewner type interpolation in matrix Hardy spaces. *Z. Anal. Anwendungen*, 14:225–233, 1995.

[8] D. Alpay and D. Levanony. Rational functions associated with the white noise space and related topics. *Potential Analysis*, 29:195–220, 2008.

[9] J. Ball, I. Gohberg, and M.A. Kaashoek. Nevanlinna–Pick interpolation for time–varying input–output maps: the discrete case. In I. Gohberg, editor, *Time–variant systems and interpolation*, volume 56 of *Oper. Theory Adv. Appl.*, pages 1–51. Birkhäuser, Basel, 1992.

[10] J. Ball, C. Sadosky, and V. Vinnikov. Scattering systems with several evolutions and multidimensional input/state/output systems. *Integral Equations Operator Theory*, 52(3):323–393, 2005.

[11] J. Ball and V. Vinnikov. Functional models for representations of the Cuntz algebra. In *Operator theory, systems theory and scattering theory: multidimensional generalizations*, volume 157 of *Oper. Theory Adv. Appl.*, pages 1–60. Birkhäuser, Basel, 2005.

[12] F. Biagini, B. Øksendal, A. Sulem, and N. Wallner. An introduction to white-noise theory and Malliavin calculus for fractional Brownian motion, stochastic analysis with applications to mathematical finance. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 460(2041):347–372, 2004.

[13] S. Bochner and K. Chandrasekharan. *Fourier transforms*. Princeton University Press, 1949. Reprinted with permission of the original publishers, Kraus Reprint Corporation, New-York, 1965.

[14] L. de Branges and J. Rovnyak. Canonical models in quantum scattering theory. In C. Wilcox, editor, *Perturbation theory and its applications in quantum mechanics*, pages 295–392. Wiley, New York, 1966.

[15] L. de Branges and J. Rovnyak. *Square summable power series*. Holt, Rinehart and Winston, New York, 1966.

[16] H. Brezis. *Analyse fonctionnelle*. Masson, Paris, 1987.

[17] E. Deprettere and P. Dewilde. *The generalized Schur algorithm*, volume 29 of *Operator Theory: Advances and Applications*, pages 97–115. Birkhäuser Verlag, Basel, 1988.

[18] R. Descombes. *Intégration*, volume 15 of *Enseigement des sciences*. Hermann. 293 rue Lecourbe, 75015 Paris, 1972.

[19] P. Dewilde and E. Deprettere. Approximative inversion of positive matrices with applications to modelling. In R. Curtain, editor, *Modelling, robustness and sensitivity reduction in control systems*, volume F34 of *NATO ASI Series*, pages 312–238. Springer–Verlag, 1987.

[20] P. Dewilde and H. Dym. Schur recursions, error formulas and convergence of rational estimators for stationary stochastic processes. *IEEE Trans. Inform. Theory*, 27:446–461, 1981.

[21] P. Dewilde and H. Dym. Interpolation for upper triangular operators. In I. Gohberg, editor, *Time–variant systems and interpolation*, volume 56 of *Oper. Theory Adv. Appl.*, pages 153–260. Birkhäuser Verlag, Basel, 1992.

[22] P. Dewilde and A.-J. van der Veen. *Time-varying systems and computations*. Kluwer Academic Publishers, Boston, MA, 1998.

[23] J. Dieudonné. *Eléments d'analyse, Volume 1: fondements de l'analyse moderne.* Gauthier–Villars, Paris, 1969.

[24] J. Doyle, B. Francis, and A. Tannenbaum. *Feedback control theory.* Macmillan Publishing Company, New York, 1992.

[25] T.E. Duncan. Some applications of fractional Brownian motion to linear systems. In *System theory: modeling, analysis and control (Cambridge, MA, 1999)*, volume 518 of *Kluwer Internat. Ser. Engrg. Comput. Sci.*, pages 97–105. Kluwer Acad. Publ., Boston, MA, 2000.

[26] T.E. Duncan, Y. Hu, and B. Pasik-Duncan. Stochastic calculus for fractional Brownian motion. I. Theory. *SIAM J. Control Optim.*, 38(2):582–612 (electronic), 2000.

[27] T.E. Duncan, B. Maslowski, and B. Pasik-Duncan. Adaptive control for semi-linear stochastic systems. *SIAM J. Control Optim.*, 38(6):1683–1706 (electronic), 2000.

[28] R.J. Elliott and J. van der Hoek. A general fractional white noise theory and applications to finance. *Math. Finance*, 13(2):301–330, 2003.

[29] Bernd Fritzsche, Bernd Kirstein, and Andreas Lasarow. The matricial Carathéodory problem in both nondegenerate and degenerate cases. In *Interpolation, Schur functions and moment problems*, volume 165 of *Oper. Theory Adv. Appl.*, pages 251–290. Birkhäuser, Basel, 2006.

[30] I. Gohberg, editor. *I. Schur methods in operator theory and signal processing*, volume 18 of *Operator theory: Advances and Applications.* Birkhäuser Verlag, Basel, 1986.

[31] I. Gohberg, S. Goldberg, and M.A. Kaashoek. *Classes of linear operators. Vol. II*, volume 63 of *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel, 1993.

[32] I.M. Guelfand and N.Y. Vilenkin. *Les distributions. Tome 4: Applications de l'analyse harmonique.* Collection Universitaire de Mathématiques, No. 23. Dunod, Paris, 1967.

[33] J.W. Helton. *Operator theory, analytic functions, matrices and electrical engineering*, volume 68 of *CBMS Lecture Notes.* Amer. Math. Soc., Rhodes Island, 1987.

[34] T. Hida. *Analysis of Brownian functionals.* Carleton Univ., Ottawa, Ont., 1975. Carleton Mathematical Lecture Notes, No. 13.

[35] T. Hida, H. Kuo, J. Potthoff, and L. Streit. *White noise*, volume 253 of *Mathematics and its Applications.* Kluwer Academic Publishers Group, Dordrecht, 1993. An infinite-dimensional calculus.

[36] H. Holden, B. Øksendal, J. Ubøe, and T. Zhang. *Stochastic partial differential equations.* Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1996.

[37] J. Horváth. *Topological vector spaces and distributions. Vol. I.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.

[38] T. Kailath. *Linear systems.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1980. Prentice-Hall Information and System Sciences Series.

[39] R. E. Kalman. Mathematical description of linear dynamical systems. *J. SIAM Control Ser. A*, 1:152–192 (1963), 1963.

[40] P. R. Kumar. Optimal adaptive control of linear-quadratic-Gaussian systems. *SIAM J. Control Optim.*, 21(2):163–178, 1983.

[41] Hui-Hsiung Kuo. *White noise distribution theory*. Probability and Stochastics Series. CRC Press, Boca Raton, FL, 1996.

[42] M.T. Lacroix-Sonrier. *Distributions, Espaces de Sobolev, Applications*. Mathématiques 2e cycle. Ellipses, Éditions Marketing S.A., 32 rue Bargue, Paris 15e, 1998.

[43] P.D. Lax. Translation invariant spaces. In *Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960)*, pages 299–306. Jerusalem Academic Press, Jerusalem, 1961.

[44] D. Levanony and P. Caines. Stochastic Lagrangian adaptive LQG control. In *Stochastic theory and control (Lawrence, KS, 2001)*, volume 280 of *Lecture Notes in Control and Inform. Sci.*, pages 283–300. Springer, Berlin, 2002.

[45] M. S. Livšic. On the theory of isometric operators with equal deficiency indices. *Doklady Akad. Nauk SSSR (N.S.)*, 58:13–15, 1947.

[46] M.S. Livšic. On a class of linear operators in Hilbert spaces. *Math. USSR-Sb.*, 61:239–262, 1946. English translation in: American mathematical society translations (2), vol. 13, p. 61-84 (1960).

[47] M.S. Livšic. *Operator colligations, waves, open systems*. Transl. Math. Monog. AMS. AMS, Providence, R.I., 1973.

[48] L. Padulo and M.A. Arbib. *System theory*. W. B. Saunders Co., Philadelphia, Pa., 1974. A unified state-space approach to continuous and discrete systems.

[49] M. Reed and B. Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York, 1972.

[50] W. Rudin. *Analyse réelle et complexe*. Masson, Paris, 1980.

[51] F. Trèves. *Topological vector spaces, distributions and kernels*. Academic Press, New York, 1967.

[52] Tu Sheng Zhang. Characterizations of the white noise test functionals and Hida distributions. *Stochastics Stochastics Rep.*, 41(1-2):71–87, 1992.

(DA) DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY OF THE NEGEV, P.O. BOX 653, BEER-SHEVA 84105, ISRAEL
E-mail address: `dany@math.bgu.ac.il`

(DL) DEPARTMENT OF ELECTRICAL ENGINEERING, BEN-GURION UNIVERSITY OF THE NEGEV, P.O. BOX 653, BEER-SHEVA 84105, ISRAEL
E-mail address: `levanony@ee.bgu.ac.il`