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Abstract

We give a new characterization of character-automorphic Hardy spaces of order 2 and of their contractive multipliers in terms of de Branges Rovnyak spaces. Keys tools in our arguments are analytic extension and a factorization result for matrix-valued analytic functions due to Leech.

Keywords. Character-automorphiuc functions, Hardy spaces, de Branges Rovnyak spaces, Schur multipliers.

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1 Introduction

Let Γ be a Fuchsian group of Möbius transformations of the unit disk $\mathbb{D} = \{z \in \mathbb{C}\,;\, |z| < 1\}$ onto itself. For $1 \leqslant p \leqslant \infty$ and for any character α of Γ, we consider the spaces

$$\mathcal{H}_p^\alpha = \{f \in \mathcal{H}_p \;\mid\; f \circ \gamma = \alpha(\gamma) f, \quad \forall\, \gamma \in \Gamma\}\,.$$

These spaces are called character-automorphic Hardy spaces. A characterization of such spaces in terms of Poincaré theta series may be found in [15], [18], [13], [9]. In particular, Pommerenke showed in [15] that the series

$$f(z) = \frac{b_0(z)}{b_0'(z)} \sum_{\gamma\in\Gamma} \overline{\alpha(\gamma)}\theta(\gamma(z))h(\gamma(z))\frac{\gamma'(z)}{\gamma(z)} \tag{1.1}$$

defines a bounded linear operator from the classical Hardy space $\mathcal{H}_p(\mathbb{D})$ into the subspace $\mathcal{H}_p^\alpha(\mathbb{D})$.

In the present paper we restrict ourselves to the case $p = 2$. We first give a characterization of the character-automorphic Hardy space $\mathcal{H}_2^\alpha(\mathbb{D})$ in terms of an associated de Branges Rovnyak space of functions analytic in the open unit disk; see Theorem 3.2. We also characterize the contractive multipliers between $\mathcal{H}_2^{\overline{\beta}\alpha}(\mathbb{D})$ and $\mathcal{H}_2^\alpha(\mathbb{D})$, where α and β two given characters; see Theorem 4.2. Our method is mainly based on analytic extension of positive kernels and factorization results from Nevanlinna-Pick interpolation theory.

2 A review on character-automorphic Hardy spaces

2.1 Fuchsian groups and automorphic functions

Let G be a group of linear transformations, $T(z) = \frac{az+b}{cz+d},\ ad - bc = 1$, in the complex plane and let ι denotes the identity transformation. Two points z and z' in $\mathbb{C}$ are said to be *congruent* with respect to G, if $z' = T(z)$ for some $T \in G$ and $T \neq \iota$. Two regions $R, R' \subset \mathbb{C}$ are said to be G-congruent or G-equivalent if there exists a transformation $T \neq \iota$ which sends R to R'. A region R which does not contain any two G-congruent points and such that the neighborhood of any point on the boundary contains G-congruent points of R is called a *fundamental region* for G. A *properly discontinuous* group is a group G having a fundamental region [10]. This amounts to saying that the identity transformation is isolated.

Definition 2.1 *A Fuchsian group is a properly discontinuous group each of whose transformation maps $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{C}\backslash\overline{\mathbb{D}}$ onto themselves.*

A Fuchsian group Γ is said to be of convergence type (see *e.g.* [15]) if

$$\sum_{\gamma\in\Gamma}\left(1 - |\gamma(z)|^2\right) = \left(1 - |z|^2\right)\sum_{\gamma\in\Gamma}|\gamma'(z)| < \infty \quad z \in \mathbb{D}.$$

Then, the Green's function [15] of Γ with respect to a point $\xi \in \mathbb{D}$ is defined as the Blaschke product

$$b_\xi(z) = \prod_{\gamma\in\Gamma}\frac{\gamma(\xi) - z}{1 - \overline{\gamma(\xi)}z}\frac{|\gamma(\xi)|}{\gamma(\xi)}. \tag{2.2}$$

It satisfies

$$b_\xi(\varphi(z)) = \mu_\xi(\varphi) b_\xi(z), \quad \forall \varphi \in \Gamma, \tag{2.3}$$

where μ_ξ is the character of Γ associated with $b_\xi(z)$. A function satisfying the relation (2.3) is said to be *character-automorphic* with respect to Γ while a Γ-periodic function, as for example $|b_\xi(z)| = |b_\xi(\varphi(z))|$, is called *automorphic* with respect to Γ.

2.2 Spaces of character-automorphic functions

We now briefly mention the main properties pertaining to spaces of character-automorphic functions. The materials presented here are essentially borrowed from [15] and [19] (see also [11, 18]). Let $\widehat{\Gamma}$ be the dual group of Γ, *i.e.* the group of (unimodular) characters. For an arbitrary character $\alpha \in \widehat{\Gamma}$, associate the subspaces of the classical space $L_2(\mathbb{T})$

$$L_2^\alpha = \{f \in L_2 \,|\, f \circ \gamma = \alpha(\gamma) f, \, \forall \gamma \in \Gamma\}$$
$$\mathcal{H}_2^\alpha(\mathbb{D}) = L_2^\alpha \bigcap \mathcal{H}_2(\mathbb{D})$$

Let Γ be a Fuchsian group without elliptic and parabolic element. We say that Γ is of Widom type if, and only if, the derivative of $b_0(z)$ is of bounded characteristic. In this case, Widom [20] has shown that the space $\mathcal{H}_\infty^\alpha$ is not trivial for any character $\alpha \in \widehat{\Gamma}$ and we have

Theorem 2.2 (Pommerenke [15]) *Let Γ be of Widom type and let $\theta(z)$ be the inner factor of $b_0'(z)$. If α is any character of Γ and if $h(z)$ is in $\mathcal{H}_p(\mathbb{D}), 1 \leqslant p \leqslant \infty$, then the function defined by* (1.1) *is in $\mathcal{H}_p^\alpha(\mathbb{D})$ and*

$$\|f\|_p \leqslant \|h\|_p, \quad f(0) = \theta(0) h(0).$$

The Poincaré series [14] in (1.1) thus defines, in particular, a projection: $P^\alpha : \theta \mathcal{H}_2(\mathbb{D}) \to \mathcal{H}_2^\alpha(\mathbb{D})$. An important property of the space $\mathcal{H}_2^\alpha(\mathbb{D})$ that we will need is that, point evaluation $f \mapsto f(\xi), \xi \in \mathbb{D}$ is a bounded linear functional. The space therefore admits a reproducing kernel k^α:

$$\langle f(z), k^\alpha(z,\xi) \rangle_{\mathcal{H}_2^\alpha(\mathbb{D})} = f(\xi), \, \xi \in \mathbb{D} \quad \text{for all } f \in \mathcal{H}_2^\alpha(\mathbb{D})$$
$$\text{with } k^\alpha(z,\xi) \in \mathcal{H}_2^\alpha(\mathbb{D}), \quad \forall\, \xi \in \mathbb{D}$$

Since $\mathcal{H}_2^\alpha(\mathbb{D}) \neq \{const\}$, we have $k^\alpha(\xi,\xi) = \|k^\alpha(\cdot,\xi)\|^2_{\mathcal{H}_2^\alpha(\mathbb{D})} > 0$ for every $\xi \in \mathbb{D}$. In the sequel, the Green's function $b_0(z)$ with respect to 0, will be denoted by $b(z)$ for short.

Let Γ be a group of Widom type and let E be associated[1] to it in such a way that $\overline{\mathbb{C}}\backslash E$ be equivalent to the Riemann surface $\mathbb{D}/\Gamma$, obtained by identifying Γ-congruent points. Then, there exists a universal covering map $\mathfrak{z} : \mathbb{D} \to \overline{\mathbb{C}}\backslash E \simeq \mathbb{D}/\Gamma$ such that

- $\mathfrak{z}$ maps $\mathbb{D}$ conformally onto $\overline{\mathbb{C}}\backslash E$,
- $\mathfrak{z}$ is automorphic with respect to Γ: $\mathfrak{z} \circ \gamma = \mathfrak{z}, \quad \forall\, \gamma \in \Gamma$
- and $\mathfrak{z}(z_1) = \mathfrak{z}(z_2) \Rightarrow \exists \gamma \in \Gamma \mid z_1 = \gamma(z_2)$

In particular, $\mathfrak{z}$ maps one-to-one the *normal fundamental domain* of Γ with respect to the origin,

$$\mathcal{F} = \{z \in \mathbb{D} \,:\, |\gamma'(z)| < 1 \quad \text{for all } \gamma \in \Gamma, \gamma \neq \iota\} \tag{2.4}$$

conformally onto some sub-domain of $\overline{\mathbb{C}}\backslash E$. We assume that $\mathfrak{z}$ is normalized so that $(\mathfrak{z}b)(0)$ is real and positive. In all the sequel, the character associated to the Green's function $b(z)$ will be denoted by μ. The starting point of the next section is the following result:

Lemma 2.3 ([12]) *The reproducing kernel for the space $\mathcal{H}_2^\alpha(\mathbb{D})$ has the form*

$$k^\alpha(z,\omega) = c(\alpha)\frac{\frac{k^{\alpha\mu}(z,0)}{b(z)}k^\alpha(\omega,0)^* - \left(\frac{k^{\alpha\mu}(\omega,0)}{b(\omega)}\right)^* k^\alpha(z,0)}{\mathfrak{z}(z) - \mathfrak{z}(\omega)^*} \tag{2.5}$$

where

$$c(\alpha) = \frac{\mathfrak{z}(0)b(0)}{k^{\alpha\mu}(0,0)} > 0. \tag{2.6}$$

3 An associated de Branges-Rovnyak space

In this section we give a characterization of the space $\mathcal{H}_2^\alpha(\mathbb{D})$ in terms of an associated de Branges Rovnyak space. To begin, let

$$\Omega_+ = \{z \in \mathbb{D}\,;\, \text{Im}\ \mathfrak{z}(z) > 0\}\,.$$

[1] See [21] for an example of a construction of a group Γ associated with a finite union of disjoint arcs of the unit circle.

Setting

$$A^\alpha(z) = \sqrt{\frac{c(\alpha)}{2}}\left(\frac{k^{\alpha\mu}(z,0)}{b(z)} + ik^\alpha(z,0)\right)$$
$$B^\alpha(z) = \sqrt{\frac{c(\alpha)}{2}}\left(\frac{k^{\alpha\mu}(z,0)}{b(z)} - ik^\alpha(z,0)\right)$$

we can rewrite the reproducing kernel k^α as

$$k^\alpha(z,\omega) = \frac{2A^\alpha(z)}{1-i\mathfrak{z}(z)}\frac{1-S_\alpha(z)S_\alpha(w)^*}{1-\sigma(z)\sigma(w)^*}\frac{2A^\alpha(w)^*}{1+i\mathfrak{z}(w)^*} \tag{3.1}$$

where $S_\alpha(z) = B^\alpha(z)/A^\alpha(z)$ and $\sigma(z) = \frac{1+i\mathfrak{z}(z)}{1-i\mathfrak{z}(z)}$. We note that the functions $A^\alpha(z)$ and $B^\alpha(z)$ are character-automorphic with the same character α while S_α and σ are automorphic functions. From now on, the notation $f^\nu(z)$ will means that the function $f^\nu(z)$ is character-automorphic with the superscript $\nu \in \widehat{\Gamma}$ being the associated character, and the notation $f_\nu(z)$ will stand for a function depending on the character ν (automorphic or not).

Proposition 3.1 *There exists a Schur function $\mathscr{S}_\alpha$ such that $S_\alpha(z) = \mathscr{S}_\alpha(\sigma(z))$.*

Proof: Since the kernel $k^\alpha(z,\omega)$ is positive in $\mathbb{D}$, and hence in Ω_+, it is clear that $\frac{1-S_\alpha(z)S_\alpha(w)^*}{1-\sigma(z)\sigma(w)^*}$ is also positive in Ω_+. Now, observe that the function σ maps $\Omega_+ \cap \mathcal{F}$ into some subset $\Delta \subset \mathbb{D}$ and this mapping is one-to-one. Let ς be given by: $(\varsigma \circ \sigma)(z) = z$, $\forall z \in \Omega_+ \cap \mathcal{F}$ (in particular this will also hold for any region congruent to $\Omega_+ \cap \mathcal{F}$) and let the function $\widetilde{\mathscr{S}_\alpha}$ be defined on Δ by:

$$\widetilde{\mathscr{S}_\alpha}(\lambda) = (S_\alpha \circ \varsigma)(\lambda),\ \forall \lambda \in \Delta.$$

Then it comes that the kernel

$$\frac{1-\widetilde{\mathscr{S}_\alpha}(\lambda)\widetilde{\mathscr{S}_\alpha}(\mu)^*}{1-\lambda\mu^*}$$

is positive on Δ. Now, this implies (see for instance [1, Theorem 2.6.5]) the existence of a unique extension of $\widetilde{\mathscr{S}_\alpha}(\lambda)$, analytic and contractive in all $\mathbb{D}$. We subsequently call $\mathscr{S}_\alpha(\lambda)$ this extension, and denote by $\mathcal{H}(\mathscr{S}_\alpha)$ the reproducing kernel Hilbert space with reproducing kernel

$$K_{\mathscr{S}_\alpha}(\lambda,\mu) = \frac{1-\mathscr{S}_\alpha(\lambda)\mathscr{S}_\alpha(\mu)^*}{1-\lambda\mu^*}$$

By construction, the equality

$$S_\alpha(z) = \mathscr{S}_\alpha(\sigma(z))$$

holds for all $z \in \Omega_+ \cap \mathcal{F}$. Since $S_\alpha(z)$ is analytic in $\mathbb{D}$, it must also hold for all $\mathbb{D}$. $\square$

In connection with the previous proposition and the next theorem, we recall that reproducing kernel Hilbert spaces $\mathcal{H}(\mathscr{S})$ of functions analytic in the open unit disk and with a reproducing kernel of the form

$$\frac{1 - \mathscr{S}(\lambda)\mathscr{S}(\mu)^*}{1 - \lambda\mu^*}$$

were introduced and studied by de Branges and Rovnyak; see [5, Appendix], [6]. We also refer the reader to [8] and [1] for more information on these and on related spaces.

Theorem 3.2 *The character-automorphic Hardy space $\mathcal{H}_2^\alpha(\mathbb{D})$ can be described as*

$$\mathcal{H}_2^\alpha(\mathbb{D}) = \left\{ F(z) = \frac{\sqrt{2}A^\alpha(z)}{1 - i\mathfrak{z}(z)} f(\sigma(z)) \ ; f \in \mathcal{H}(\mathscr{S}_\alpha) \right\} \tag{3.2}$$

with the norm

$$\|F\|_{\mathcal{H}_2^\alpha(\mathbb{D})} = \|f\|_{\mathcal{H}(\mathscr{S}_\alpha)}.$$

Proof: Recall that the map which to $F \in \mathcal{H}_2^\alpha(\mathbb{D})$ associates its restriction $F|_{\Omega_+}$ to Ω_+ is an isometry from $\mathcal{H}_2^\alpha(\mathbb{D})$ onto the reproducing kernel Hilbert space with reproducing $k^\alpha(z, \omega)$ defined by (2.5), where z, ω are now restricted to Ω_+. We denote this last space by $\mathcal{H}_2^\alpha(\mathbb{D})\big|_{\Omega_+}$. By Proposition 3.1 and using (3.1) we see that the operator of multiplication by $\frac{2A^\alpha(z)}{1-i\mathfrak{z}(z)}$ is an isometry from the reproducing kernel Hilbert space $\mathcal{H}$ with reproducing kernel

$$\frac{1 - \mathscr{S}_\alpha(\sigma(z))\mathscr{S}_\alpha(\sigma(w)^*}{1 - \sigma(z)\sigma(w)^*}$$

onto $\mathcal{H}_2^\alpha(\mathbb{D})\big|_{\Omega_+}$. Furthermore, the composition map by σ is an isometry from the de Branges Rovnyak space $\mathcal{H}(\mathscr{S}_\alpha)$ onto $\mathcal{H}$. We have that

$$\mathcal{H} = \{ f \circ \sigma \ ; \ f \in \mathcal{H}(\mathscr{S}_\alpha) \},$$

with norm $\|f\circ\sigma\|_{\mathcal{H}}=\|f\|_{\mathcal{H}(\mathscr{S}_\alpha)}$, as follows from the equalities

$$f(\sigma(\omega))=\langle f(\cdot),K_{\mathscr{S}_\alpha}(\cdot,\sigma(\omega))\rangle_{\mathcal{H}(\mathscr{S}_\alpha)}=\langle f\circ\sigma(\cdot),K_{\mathscr{S}_\alpha}(\sigma(\cdot),\sigma(\omega))\rangle_{\mathcal{H}}$$

Thus the restrictions of the elements of $\mathcal{H}_2^\alpha(\mathbb{D})$ to Ω_+ are of the form as in (3.2) for z restricted to Ω_+. By analytic extension, the elements of $\mathcal{H}_2^\alpha(\mathbb{D})$ have the same form in the whole of $\mathbb{D}$. □

4 Schur multipliers

A Schur function is a function analytic and contractive in the open unit disk. Equivalently, it is a function s such that the operator of multiplication by s is a contraction from the classical Hardy of the open unit disk into itself. This last definition is our starting point to define character-automorphic Schur multipliers.

Definition 4.1 *A character-automorphic function $s^\beta(z)$, with character β, will be called a Schur multiplier if the operator of multiplication by $s^\beta(z)$ is a contraction from $\mathcal{H}_2^{\overline{\beta}\alpha}(\mathbb{D})$ into $\mathcal{H}_2^\alpha(\mathbb{D})$.*

Equivalently, the character-automorphic function $s^\beta(z)$ is a Schur multiplier if and only if the kernel

$$K^\alpha_{s^\beta}(z,w)=k^\alpha(z,w)-s^\beta(z)s^\beta(w)^*k^{\overline{\beta}\alpha}(z,w) \tag{4.1}$$

is positive in $\mathbb{D}$. The kernel $K_{s^\beta}(z,w)$ is in particular positive in Ω_+, and we will consider it in Ω_+. We note that in view of [7, Lemma 2, p. 142], [4, Theorem 1.1.4, p. 10], the positivity of the analytic kernel K_{s^β} on Ω_+ implies its positivity on $\mathbb{D}$.

Theorem 4.2 *A character-automorphic function s^β is a Schur multiplier if and only if there exists a $\mathbb{C}^{2\times2}$–matrix valued Schur function $\Sigma(z)$ such that*

$$s^\beta(z)=\frac{A^\alpha(z)}{A^{\overline{\beta}\alpha}(z)}\frac{\Sigma_{12}(\sigma(z))}{1-S_{\overline{\beta}\alpha}(z)\Sigma_{22}(\sigma(z))} \tag{4.2}$$

$$S_\alpha(z)=\Sigma_{11}(\sigma(z))+\frac{\Sigma_{12}(\sigma(z))S_{\overline{\beta}\alpha}(z)\Sigma_{21}(\sigma(z))}{1-S_{\overline{\beta}\alpha}(z)\Sigma_{22}(\sigma(z))} \tag{4.3}$$

Proof: The positivity of the kernel (4.1) in Ω_+ is equivalent to the positivity in Ω_+ of the kernel

$$K_{\mathscr{S}_\alpha}(\sigma(z),\sigma(\omega)) - T(z)T(\omega)^* K_{\mathscr{S}_{\overline{\beta}\alpha}}(\sigma(z),\sigma(\omega)),$$

where

$$T(z) = \frac{A^{\overline{\beta}\alpha}(z)}{A^\alpha(z)} s^\beta(z).$$

As in the proof of Proposition 3.1, we note that the function σ maps $\Omega_+ \cap \mathcal{F}$ into some subset $\Delta \subset \mathbb{D}$ and this mapping is one-to-one, and consider again the function ς defined by: $(\varsigma \circ \sigma)(z) = z$, $\forall z \in \Omega_+ \cap \mathcal{F}$. The kernel

$$K_{\mathscr{S}_\alpha}(\lambda,\mu) - T(\varsigma(\lambda))T(\varsigma(\mu))^* K_{\mathscr{S}_{\overline{\beta}\alpha}}(\lambda,\mu), \tag{4.4}$$

is positive in Δ. We now show that $T \circ \varsigma$ admits an analytic extension to $\mathbb{D}$. To that purpose we consider the linear relation in $\mathcal{H}(\mathscr{S}_\alpha) \times \mathcal{H}(\mathscr{S}_{\overline{\beta}\alpha})$ spanned by the elements of the form

$$(K_{\mathscr{S}_\alpha}(\cdot,\omega), T(\varsigma(\omega))^* K_{\mathscr{S}_{\overline{\beta}\alpha}}(\cdot,\omega)), \quad \omega \in \Delta.$$

It is densely defined and it is contractive because of the positivity of the kernel (4.4) in Δ. It is therefore the graph of a densely defined contraction $\widetilde{X}$. We note its extension to $\mathcal{H}(\mathscr{S}_\alpha)$ by X. For $\omega \in \Delta$ and $f \in \mathcal{H}(\mathscr{S}_\alpha)$ we have

$$\begin{aligned}(X^*f)(\omega) &= \langle X^*f, K_{\mathscr{S}_\alpha}(\cdot,\omega)\rangle_{\mathcal{H}(\mathscr{S}_\alpha)} \\ &= \langle f, T(\varsigma(\omega))^* K_{\mathscr{S}_{\overline{\beta}\alpha}}(\cdot,\omega)\rangle_{\mathcal{H}(\mathscr{S}_\alpha)} \\ &= T(\varsigma(\omega)) f(\omega).\end{aligned}$$

Let $f_0(\lambda) = K_{\mathscr{S}_{\overline{\beta}\alpha}}(\lambda,\omega_0)$ where $\omega_0 \in \mathbb{D}$. We have

$$T(\varsigma(\lambda)) = \frac{(X^*f_0)(\lambda)}{f_0(\lambda)}, \quad \lambda \in \Delta.$$

It follows that $T \circ \varsigma$ has an analytic extension to $\mathbb{D}$, which we will denote by $\mathscr{R}$. Thus the kernel

$$K_{\mathscr{S}_\alpha}(\lambda,\mu) - \mathscr{R}(\lambda)\mathscr{R}(\mu)^* K_{\mathscr{S}_{\overline{\beta}\alpha}}(\lambda,\mu), \tag{4.5}$$

is analytic in λ and μ^* in $\mathbb{D}$. Therefore it is still positive in $\mathbb{D}$; see [7, Lemma 2, p. 142], [4, Theorem 1.1.4, p. 10]. By [2, Theorem 11.1, p. 61], a necessary and sufficient condition for the kernel (4.5) to be positive is that there exists a $\mathbb{C}^{2\times 2}$–matrix valued Schur function $\Sigma(\lambda)$ such that

$$\mathscr{R}(\lambda) = \frac{\Sigma_{12}(\lambda)}{1 - \mathscr{S}_{\overline{\beta}\alpha}(\lambda)\Sigma_{22}(\lambda)} \tag{4.6}$$

$$\mathscr{S}_\alpha(\lambda) = \Sigma_{11}(\lambda) + \frac{\Sigma_{12}(\lambda)\mathscr{S}_{\overline{\beta}\alpha}(\lambda)\Sigma_{21}(\lambda)}{1 - \mathscr{S}_{\overline{\beta}\alpha}(\lambda)\Sigma_{22}(\lambda)}. \tag{4.7}$$

The above mentioned result from [2] stems from rewriting the positive kernel (4.5) as

$$\frac{\mathcal{A}(\lambda)\mathcal{A}(\mu)^* - \mathcal{B}(\lambda)\mathcal{B}(\mu)^*}{1 - \lambda\mu^*}$$

where

$$\begin{aligned}\mathcal{A}(\lambda) &= \begin{pmatrix} 1 & \mathscr{R}(\lambda)\mathscr{S}_{\overline{\beta}\alpha}(\lambda)\end{pmatrix}\\ \mathcal{B}(\lambda) &= \begin{pmatrix} \mathscr{S}_\alpha(\lambda) & \mathscr{R}(\lambda)\end{pmatrix},\end{aligned}$$

and using a factorization result known as Leech's theorem, which insures the existence of a $\mathbb{C}^{2\times 2}$-valued Schur function Σ such that

$$\mathcal{B}(z) = \mathcal{A}(z)\Sigma(z),$$

from which (4.6) follows.

This unpublished result of R.B. Leech has been proved using the commutant lifting theorem by M. Rosenblum; see [16, Theorem 2, p. 134] and [17, Example 1, p. 107]. Further discussions and applications can also be found in [3]. It can also be proved using tangential Nevanlinna-Pick interpolation and Montel's theorem.

Finally we replace in (4.6) λ by $\sigma(z)$ where $z \in \Omega_+ \cap \mathcal{F}$. We obtain the formulas in the statement of the theorem for $z \in \Omega_+ \cap \mathcal{F}$, and hence for $z \in \mathbb{D}$ by analytic extension. □

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