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Rational Functions Associated To the White Noise Space and Related Topics

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Daniel Alpay and David Levanony

Abstract

Motivated by the hyper-holomorphic case we introduce and study rational functions in the setting of Hida's white noise space. The Fueter polynomials are replaced by a basis computed in terms of the Hermite functions, and the Cauchy-Kovalevskaya product is replaced by the Wick product.

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1 Introduction

Consider a K-vector space V spanned by a family of functions $(f_\alpha)_{\alpha\in\ell}$, where ℓ is a countable set of indices, and where K denotes either $\mathbb{R}$, $\mathbb{C}$, or the skew field of quaternions $\mathbb{H}$ (in this latter case we assume that V is a right vector space, to fix the ideas). Define on V an operation by

$$f_\alpha \circ f_\beta = f_{\alpha+\beta}, \quad \alpha,\beta\in\ell. \tag{1.1}$$

In general, such a product depends on the basis and need not carry any structure related to V. There are at least two cases we are aware of, where the multiplication-like law (1.1) carries much information. The first is the case of hyper-holomorphic functions. This case corresponds to $K=\mathbb{H}$ and $\ell=\mathbb{N}^3$. The (f_α) are the Fueter monomials and $\circ$ is the Cauchy-Kovalevskaya product; see [20], [33]. The second case, which is the topic of the present work, corresponds to $K = \mathbb{R}$, and ℓ the space of sequences $(\alpha_n)_{n\in\mathbb{N}}$ of integers for which $\alpha_n = 0$ for n sufficiently large; the space V is Kondratiev's space (which includes Hida's white noise space and Hida's space of distributions), with the (f_α) forming an orthonormal Hilbert space basis of the white noise space built in terms of the Hermite functions, and $\circ$ is the Wick product. All these notions will be reviewed in the sequel.

The notions of rational functions, de Branges Rovnyak spaces and Schur-Agler classes were introduced for the case of hyper-holomorphic functions in the papers [8], [9], [10], [11], [12]. The case of Clifford algebra valued functions has been considered in [2]. In the approach developed in these papers, important tools were the study of the Gleason problem and the introduction of counterparts of the Leibenzon operators for hyper-holomorphic functions. The purpose of this paper is to make a similar study within the white noise space setting. Results obtained in this paper are to be applied to problems in stochastic system theory, a work to be summarized in a future publication.

To provide further motivation, it is best to first take a detour via several complex variables, and discuss Gleason's problem and the Leibenzon operators. Recall that the backward shift operator

$$R_0 f(z) = \frac{f(z) - f(0)}{z}$$

plays an important role in operator theory and in the theory of linear systems; see [31], [17]. It has counterparts in several complex variables, as we now recall: Let f be a function of N complex variables, analytic in a neighborhood of the origin. Then (see e.g. [34, p. 151]), it holds that

$$f(z) - f(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{dt}} f(tz)dt = \sum_{j=1}^{N} z_j (\mathcal{R}_j f)(z) \tag{1.2}$$

where $\mathcal{R}_j$ denotes the Leibenzon's backward shift operator (see [30, p. 117-118])

$$\mathcal{R}_j f(z) = \int_0^1 \frac{\partial f(tz)}{\partial z_j} dt = \sum_{\alpha\in\mathbb{N}^N} \frac{\alpha_j}{|\alpha|} c_\alpha z^{\alpha - \mathbf{u}_j}, \tag{1.3}$$

with

$$f(z) = \sum_{\alpha\in\mathbb{N}^N} c_\alpha z^\alpha. \tag{1.4}$$

In (1.3)–(1.4), we have used the the multi-index notation, and, for $j = 1, \ldots, N$, the symbol $\mathbf{u}_j$ denotes the index with all components equal to 0, with the exception of the k-th, equal to 1. Moreover, $\alpha - \mathbf{u}_j$ is defined to be $(0, 0, \ldots, 0)$ when one of its entries is strictly negative.

Comparing the definition of R_0 with (1.2), suggests that the operators $\mathcal{R}_j$ are a generalization of the backward–shift operator R_0. This is indeed the case, but the situation is more complex: Gleason's problem for a space $\mathcal{M}$ of functions analytic in a neighborhood of the origin, asks the following: Can we write for $f \in \mathcal{M}$

$$f(z) - f(0) = \sum_{j=1}^{N} z_j f_j(z)$$

where $f_j \in \mathcal{M}$? The operators $\mathcal{R}_j$ allow to solve Gleason's problem in various spaces of power series. We refer to [4] for a study of these operators and to [3] for an extension of the Beurling–Lax theorem using Gleason's problem in the setting of the ball.

The paper is written with three different audiences in mind, namely researchers in multi-dimensional system theory, hyper-holomorphic functions and stochastic analysis. It consists of seven sections besides the introduction and is organized as follows: In Section 2 we briefly review the hyper-holomorphic case. This is important because we want to specify the similarities between the Cauchy-Kovalevskaya product and the Wick product. In Section 3 we review the main properties of Hida's white noise space. Rational functions in the white noise space are defined and studied in Section 4. In Sections 5 and 6 we study several counterparts of classical spaces in the stochastic setting. Section 5 is devoted to the Arveson space and its multipliers, while Section 6 is devoted to the counterpart of the Schur-Agler classes of the polydisk. In the last section we prove a uniqueness result related to the Leibenzon operators in certain spaces of power series in a countable number of variables.

A table showing the parallels between the hyper-holomorphic case and the case of the white noise space is provided at the end of the paper.

2 The hyper-holomorphic case: a short review

We review the main features of the hyper-holomorphic case relevant to the present paper. First recall that the skew-filed of quaternions consists of elements of the form

$$x = x_0\mathbf{e}_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3,$$

where the $x_i \in \mathbb{R}$ and the $\mathbf{e}_i$ satisfy the Cayley multiplication table

	$\mathbf{e_0}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$
$\mathbf{e_0}$	$\mathbf{e_0}$	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$
$\mathbf{e_1}$	$\mathbf{e_1}$	$-\mathbf{e_0}$	$\mathbf{e_3}$	$-\mathbf{e_2}$
$\mathbf{e_2}$	$\mathbf{e_2}$	$-\mathbf{e_3}$	$-\mathbf{e_0}$	$\mathbf{e_1}$
$\mathbf{e_3}$	$\mathbf{e_3}$	$\mathbf{e_2}$	$-\mathbf{e_1}$	$-\mathbf{e_0}$

We set $\mathbf{e}_0 = 1$.

The function $f : \Omega \subset \mathbb{R}^4 \to \mathbb{H}$ is called *left hyper-holomorphic* if

$$Df := \frac{\partial}{\partial x_0} f + \mathbf{e_1}\frac{\partial}{\partial x_1} f + \mathbf{e_2}\frac{\partial}{\partial x_2} f + \mathbf{e_3}\frac{\partial}{\partial x_3} f = 0. \tag{2.5}$$

Write $f = f_0 + \mathbf{e}_1 f_1 + \mathbf{e}_2 f_2 + \mathbf{e}_3 f_3$. The components f_j of f satisfy the system of equations

$$\begin{aligned}
\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} &= 0,\\
\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} &= 0,\\
\frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_1} &= 0,\\
\frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} &= 0.
\end{aligned} \tag{2.6}$$

One can apply to this system of partial differential equations the Cauchy–Kovalevskaya theorem: Let $\varphi(x_1, x_2, x_3)$ be a real analytic function from some open domain of $\mathbb{R}^3$ into $\mathbb{H}$, that is, φ is given by four coordinate real-analytic, real–valued functions

$$\varphi(x_1, x_2, x_3) = \varphi_0(x_1, x_2, x_3) + \sum_1^3 \mathbf{e}_i \varphi_i(x_1, x_2, x_3).$$

The Cauchy–Kovalevskaya theorem, see [28, Section 1.7], implies that the system of equations (2.6), with initial conditions

$$f_i(0, x_1, x_2, x_3) = \varphi_i(x_1, x_2, x_3)$$

admits a unique real analytic solution in a neighborhood of the origin in $\mathbb{R}^4$. This solution

$$f(x_0, x_1, x_2, x_3) = f_0(x_0, x_1, x_2, x_3) + \sum_1^3 \mathbf{e}_i f_i(x_0, x_1, x_2, x_3)$$

is hyper-holomorphic by definition and is called the Cauchy–Kovalevskaya extension of the function φ. We will use the notation

$$f = \mathbf{CK}(\varphi).$$

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ and let $\varphi(x) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \overset{\text{def.}}{=} x^\alpha$. The corresponding hyper-holomorphic function is the Fueter monomial ζ^α. The case where $\varphi(x) = x_\ell$ ($\ell = 1, 2, 3$) leads to the hyper-holomorphic variables

$$\zeta_\ell(x) = x_\ell - \mathbf{e}_\ell x_0, \quad \ell = 1, 2, 3,$$

that is $\zeta_\ell = \mathbf{CK}(x_\ell)$. The notations

$$\zeta(x) = \begin{pmatrix} \zeta_1(x) & \zeta_2(x) & \zeta_3(x) \end{pmatrix} \tag{2.7}$$

and

$$\zeta^{(N)}(x) = \begin{pmatrix} \zeta_1(x) I_N & \zeta_2(x) I_N & \zeta_3(x) I_N \end{pmatrix} \tag{2.8}$$

will prove useful.

The point-wise product of two hyper-holomorphic functions, say f and g, need not be hyper-holomorphic. Their Cauchy-Kovalevskaya product $f \circ g$

has been introduced in 1981 by F. Sommen in [33] and is defined as the Cauchy-Kovalevskaya extension of

$$f(0,x_1,x_2,x_3)g(0,x_1,x_2,x_3).$$

It is a hyper-holomorphic function. We set $\mathbf{R}$ to be the restriction of a hyper-holomorphic function to $x_0=0$. Then, with $x=\begin{pmatrix}x_1 & x_2 & x_3\end{pmatrix}$ we have

$$f\circ g=\mathbf{CK}(\mathbf{R}(f)\mathbf{R}(g)).$$

Hence, we have that $f\circ g=g\circ f$ if and only if the quaternionic-valued functions $\mathbf{R}(f)$ and $\mathbf{R}(g)$ commute.

Note that for the Fueter monomials we have

$$\zeta^\alpha\circ\zeta^\beta=\zeta^{\alpha+\beta},\quad \alpha,\beta\in\mathbb{N}^3.$$

Every function f hyper-holomorphic in a neighborhood of the origin can be written as a power series expansion using the Fueter monomials ζ^α

$$f=\sum_{\alpha\in\mathbb{N}^3}\zeta^\alpha f_\alpha,\quad f_\alpha\in\mathbb{H},$$

and the Cauchy-Kovalevskaya product has a nice interpretation in terms of these expansions: It is a convolution product, also called the Cauchy product:

$$f\circ g=\sum_{\alpha\in\mathbb{N}^3}\zeta^\alpha\left(\sum_{\beta\le\alpha}f_\beta g_{\alpha-\beta}\right).$$

To define rational functions we first need another definition and a result: If g is a $\mathbb{H}^{N\times N}$–valued hyper-holomorphic function, we denote

$$g^{n\circ}=g\circ\cdots\circ g\quad (n\quad\text{times}).$$

If moreover, $g(0)=0$, then the series

$$(I_N-g)^{-\circ}\stackrel{\text{def.}}{=}\sum_{n=0}^\infty g^{n\circ}\tag{2.9}$$

converges in a neighborhood of the origin to a hyper-holomorphic function.

Theorem 2.1 *The following conditions are equivalent:*

1. *The function $f(0, x_1, x_2, x_3)$ is a rational function of the three real variables x_1, x_2 and x_3 with values in $\mathbb{H}$ (that is, each of its real components is a rational function of the three real variables x_1, x_2 and x_3), and analytic at the origin.*

2. *We can write*

$$f(x) = D + C \circ (I_N - \zeta^{(N)}(x)A)^{-\circ} \circ \zeta^{(N)}(x)B \tag{2.10}$$

   where A, B, C and D are matrices with entries in $\mathbb{H}$ of appropriate dimensions.

3. *f is obtained from the Fueter monomials and the quaternions after a finite number of additions, Cauchy-Kovalevskaya multiplications and inversions (the latter defined by (2.9)).*

Expression (2.10) is called a *realization of f centered at the origin*, and comes from system theory; see [17], [24], [25]. Using the abuse of notation

$$\zeta^{(N)}(x) = \zeta,$$

we will write this expression as

$$f(x) = D + C \circ (I - \zeta A)^{-\circ} \circ \zeta B.$$

More explicitly, one can also write (2.10) as

$$f(x) = D + C(I_N - \zeta_1 A_1 - \zeta_2 A_2 - \zeta_3 A_3)^{-\circ} \circ (\zeta_1 B_1 + \zeta_2 B_2 + \zeta_3 B_3)$$

for matrices $A_1, B_1, \cdots$ of appropriate dimensions. Note also that

$$\mathbf{R}(D + C \circ (I - \zeta A)^{-\circ} \circ \zeta B) = D + C(I - xA)^{-1}xB, \tag{2.11}$$

and

$$\mathbf{CK}(D + C(I - xA)^{-1}xB) = D + C \circ (I - \zeta A)^{-\circ} \circ \zeta B. \tag{2.12}$$

Definition 2.2 *A $\mathbb{H}^{p\times q}$-valued function f hyper-holomorphic in a neighborhood of the origin, is called* rational *if any of the equivalent conditions in the previous theorem is in force.*

Theorem 2.1 is proved in the above mentioned papers. Here we will give a short and slightly different proof, both for completeness and because similar arguments will be used in Section 4.

Proof of Theorem 2.1: We begin by proving the equivalence between the first two conditions. We recall that any rational $\mathbb{C}^{p\times q}$-valued function $W(z)$ of N complex variables $z_1, \ldots, z_M$ can be written as

$$W(z) = D + C(I_N - zA)^{-1}zB. \tag{2.13}$$

In this expression, $D = W(0) \in \mathbb{C}^{p\times q}$, $N \in \mathbb{N}$ and $C \in \mathbb{C}^{p\times N}$. Furthermore, $zA = z_1A_1 + \cdots + z_MA_M$ and $zB = z_1B_1 + \cdots z_MB_M$, where the $A_j \in \mathbb{C}^{N\times N}$ and the $B_j \in \mathbb{C}^{N\times q}$. This result originated with [19, Theorem 5, p. 107]. A different proof, based on Gleason's problem, has been recently given in [5]. The realization (2.13) still holds for real-valued functions of three real variables. Identifying the quaternions with $\mathbb{R}^4$, it is also seen that the realization result still holds for quaternionic-valued rational functions of three real variables. Equation (2.10) then follows directly from (2.12). The converse follows from (2.11).

To study the equivalence with the third condition we note the following: Since the first condition is invariant under summation, pointwise multiplication and pointwise inversion, the sum and product of two hyper-holomorphic functions of compatible dimensions and the inverse (in the sense of (2.9)), are still rational. The equivalence with the third condition follows then from the fact that a rational function of three real variables with quaternionic entries is obtained after a finite number of sums, products and divisions of monomials x^α. $\square$

3 The white noise space and Kondratiev's space

In the previous section the underlying space was the space of functions hyper-holomorphic in a neighborhood of the origin. In the setting we will now review, the situation is more complex (at least in the present stage of the theory). The first step is to build the white noise space, and then to go beyond, to a space of distributions.

To define Hida's white noise space first set $\mathcal{S}$ to be the Schwartz space of rapidly decreasing functions, and $\mathcal{S}'$ its topological dual (the space of tempered distributions). We denote by $\mathcal{F}$ the σ-algebra of the Borel sets of $\mathcal{S}'$. Hida's white noise space is constructed as follows, using the Bochner-Minlos theorem. First note that the function

$$K(s_1 - s_2) = \exp(-\|s_1 - s_2\|^2_{\mathbf{L}_2(\mathbb{R})}/2),$$

is positive in the sense of reproducing kernels in $\mathcal{S}$. Since $\mathcal{S}$ is nuclear, there exits a probability measure P on $(\mathcal{S}', \mathcal{F})$ such that, for all $s \in \mathcal{S}$,

$$E(e^{iQ_s}) = e^{-\frac{\|s\|^2_{\mathbf{L}_2(\mathbb{R})}}{2}}, \tag{3.14}$$

where Q_s denotes the linear functional $Q_s(s') = \langle s', s\rangle_{\mathcal{S}',\mathcal{S}}$. See for instance [26, Théorème 2, p. 342]. Equation (3.14) implies in particular that

$$E(Q_s) = 0 \quad \text{and} \quad E(Q_s^2) = \|s\|^2_{\mathbf{L}_2(\mathbb{R})}. \tag{3.15}$$

$\mathcal{W} \stackrel{\text{def.}}{=} \mathbf{L}_2(\mathcal{S}', \mathcal{F}, P)$ is the white noise probability space. In accordance with the notation standard in probability theory, we set $\Omega = \mathcal{S}'$. Thus,

$$\mathcal{W} = \mathbf{L}_2(\Omega, \mathcal{F}, P).$$

The white noise space $\mathcal{W}$ admits a special orthonormal basis (H_α), indexed by the set ℓ of finite sequences of $\mathbb{N}^{\mathbb{N}}$, and is built in terms of the Hermite functions (which themselves, are constructed by the Hermite polynomials). We refer the reader to [27, Chapter 2] and to the papers [18], [22, p. 305], where the main features of the theory are reviewed. Because of the forthcoming definition of the Wick product (see (3.17) below), it suffices to briefly recall the definition of the H_α when $\alpha = e_k$, $k = 1, 2, \ldots$. Here, we have denoted by e_k the element of ℓ with all entries equal to 0, with the exception of the k-th, being equal to 1. To define H_{e_k}, let ζ_k be the k-th Hermite function (which itself is computed in terms of the k-th Hermite polynomial h_k). Then, $\zeta_k \in \mathcal{S}$ and

$$H_{e_k}(\omega) = h_k(\langle \omega, \zeta_k\rangle_{\Omega,\mathcal{S}}).$$

Every $F \in \mathbf{L}_2(\Omega, \mathcal{F}, P)$ admits a representation

$$F = \sum_{\alpha \in \ell} c_\alpha H_\alpha, \tag{3.16}$$

with

$$\sum_{\alpha \in \ell} c_\alpha^2 \alpha! < \infty,$$

called Wiener-Itô chaos expansion; see [27, Theorem 2.2.4 p. 23].

The Wick product is defined by

$$H_\alpha \lozenge H_\beta = H_{\alpha+\beta}, \tag{3.17}$$

which is reminiscent of the Cauchy–Kovalevskaya product, used in [8], [10] and [11] to define rational hyper-holomorphic functions, as we have discussed in the previous section. The Wick product of two elements in the white noise space need not be in the white noise space; for an example due to Gjessing, see [27, Example 2.4.8 p. 45]. There is therefore a need to go beyond the white noise space. Appropriate settings are Kondratiev's space S_{-1} and Hida's space of distributions. These spaces are defined below (see also [27, pp. 35-36]), where the notation

$$(2\mathbb{N})^{-q\alpha} = \prod_j (2j)^{-q\alpha_j}$$

is used.

Definition 3.1 *The Kondratiev space S_{-1} consists of all formal power series* (3.16) *such that*

$$\sum_\alpha c_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty \tag{3.18}$$

for some $q \in \mathbb{N}$.
The Hida space S^* consists of formal power series (3.16) *such that*

$$\sup_\alpha c_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty \tag{3.19}$$

for some $q \in \mathbb{N}$.

We now introduce two spaces which are the counterparts of the Hardy space of the polydisk and of the Arveson space, respectively. The case of the Arveson space is considered in Section 6 and the polydisk case in Section 7.

Definition 3.2 *The space $\mathcal{P}$ consists of all formal power series* (3.16) *for which*

$$\sum_{\alpha\in\ell} c_\alpha^2 < \infty. \tag{3.20}$$

The Arveson space $\mathcal{A}$ consists of all formal power series (3.16) *for which*

$$\sum_{\alpha\in\ell} \frac{\alpha!}{|\alpha|!} c_\alpha^2 < \infty. \tag{3.21}$$

It is clear that $\mathcal{P}$ is included in the Hida space. We will show in Theorem 6.1 below that $\mathcal{A}$ is included in the Kondratiev space.

The notion of the Wick product has been used in the development of an Ito-like calculus for the fractional Brownian; see [18], [21]. The main properties of the Wick product are listed in [27, p. 43]. We note in particular the following:

1. The Wick product is independent of the basis; see [27, Appendix D, p. 209].

2. The Wick product differs in general from the point-wise product. They coincide when at least one of the factors is deterministic (see [27, Example 2.4.6, p.43]).

3. The Wick product is not local. See [27, 2.4.10, p.45].

A key property of the basis (H_α) is the following: define a map $\mathbf{I}$ such that

$$\mathbf{I}(H_\alpha) = z^\alpha,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots) \in \ell$, where $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ and where we use the classical multi-index notation

$$z^\alpha = z_1^{\alpha_1} \cdots$$

Then,

$$\mathbf{I}(H_\alpha \lozenge H_\beta) = \mathbf{I}(H_\alpha)\mathbf{I}(H_\beta) = z^{\alpha+\beta}$$

The map $\mathbf{I}$ is called the Hermite transform; it exhibits an isomorphism between the white noise probability space and a certain reproducing kernel Hilbert space, namely the space of powers series

$$f(z) = \sum_\alpha z^\alpha f_\alpha, \quad \text{with norm} \quad \|f\|^2 = \sum_{\alpha\in\ell} \frac{|f_\alpha|^2}{\alpha!}.$$

This space, called the Fock space, has been studied for a long time, see for instance the 1962 paper of V. Bargmann [16]. It is the reproducing kernel Hilbert space with reproducing kernel

$$K(z,w) = \sum_{\alpha\in\ell} \frac{z^\alpha \overline{w}^\alpha}{\alpha!} = e^{\langle z,w\rangle_{\ell_2}}, \quad z,w \in \ell_2. \tag{3.22}$$

See [16, (10), p. 201].

The analogue of the Cauchy-Kovalesvkaya extension theorem is the following result; see [27, Theorem 2.6.11, p. 62]. In the statement, $(\mathbb{C}^{\mathbb{N}})_c$ denotes the space of finite sequences of complex numbers indexed by the integers, and (see [27, Definition 2.6.4 p. 59])

$$K_q(\delta) = \left\{ z \in \mathbb{C}^{\mathbb{N}} \ : \ \sum_{\alpha\neq 0} |z|^\alpha (2\mathbb{N})^{q\alpha} < \delta^2 \right\}.$$

Theorem 3.3 *[27] Let $g(z) = \sum_{\alpha\in\ell} g_\alpha z^\alpha$ be a power series defined in $(\mathbb{C}^{\mathbb{N}})_c$, and assume that g is absolutely convergent in a domain K_q^δ for some $q < \infty$ and $\delta > 0$. Then g is the Hermite transform of the element $G = \sum_{\alpha\in\ell} g_\alpha H_\alpha$, which belongs to S_{-1}.*

For $F = (F_{\ell j}) \in (S_{-1})^{p\times q}$ we define $\mathbf{I}(F) = (\mathbf{I}(F_{\ell j}))$. The following propositions will be used in the next section. The first claim is [27, Proposition 2.2.6 p. 59], when F is a row-valued function and G is a column-valued function. The result still holds for matrices of appropriate dimensions, as is seen by taking component by component. A similar remark holds for the second claim, which is a consequence of Kondratiev's theorem (Theorem 3.3 above), and corresponds to the function $f(x) = 1 - x$ in [27, Definition 2.6.14 p. 65].

Proposition 3.4 *[27] Let F and G be two matrix-valued functions with entries in the Kondratiev space. Then*

$$\mathbf{I}(FG) = \mathbf{I}(F)\mathbf{I}(G) \quad \text{and} \quad \mathbf{I}(F+G) = \mathbf{I}(F) + \mathbf{I}(G),$$

where in each case, F and G are assumed to be of compatible dimensions.

Proposition 3.5 *For $F \in {S_{-1}}^{p\times p}$*

$$F = \sum_{\alpha} H_\alpha F_\alpha, \quad F_\alpha \in \mathbb{R}^{p\times p}$$

such that the constant coefficient is $F_0 = 0_{p\times p}$, the von Neumann series

$$(I - F)^{-\lozenge} = \sum_{k=0}^{\infty} F^{k\lozenge}$$

converges in the Kontratiev space and

$$\mathbf{I}((I_{p\times p} - F)^{-\lozenge}) = (I_{p\times p} - \mathbf{I}(F))^{-1}.$$

Finally we recall that the Hermite transform allows to reduce convergence in the Kondratiev space into convergence in terms of power series. The following result holds (see [27, Theorem 2.8.1 p. 74]).

Theorem 3.6 *[27] A sequence of elements $F^{(n)}$ in the Kondratiev space S_{-1} converges to $F \in S_{-1}$ if there exists $\delta > 0$ and $q < \infty$ such that $\mathbf{I}(F^{(n)})$ converges to $\mathbf{I}(F)$ pointwise boundedly, or equivalently, uniformly, in $K_q(\delta)$.*

As a corollary we give a direct application of this theorem. Recall first that a Hilbert space $\mathcal{H}$ is called a sub-Hilbert space of a topological vector space $\mathcal{V}$ if it is included in $\mathcal{V}$ and if, moreover, the inclusion is continuous. See [32].

Corollary 3.7 *Let $\mathcal{H}$ be a reproducing kernel Hilbert space of functions $K(z,w)$ defined in a neighborhood $K_q(\delta)$ and assume that $K(z,z)$ is uniformly bounded there. Then, $\mathcal{H}$ is a subHilbert space of S_{-1}.*

Proof: It suffices to show that the inclusion map is continuous. Let (f_k) be a sequence of elements of $\mathcal{H}$ converging to f in the topology of $\mathcal{H}$. The reproducing kernel property and the hypothesis on $K(z,z)$ implies that the pointwise convergence is uniform in $K_q(\delta)$. □

4 Rational functions

We now define and characterize rational functions in the white noise space. We follow the methodology of the previous section. As we have already mentioned, in the hyper-holomorphic case, the Cauchy-Kovalevskaya theorem ensures that every $\mathcal{H}$–valued function real analytic in a neighborhood of the origin, leads to a hyper-holomorphic function. This result applies in particular when the function is rational. In the stochastic setting, we will define rational functions in terms of Hermite transforms. The Cauchy-Kovalevskaya theorem is now replaced by Kondratiev's theorem (Theorem 3.3 above), and we need to show that a rational function satisfies the hypothesis of Theorem 3.3. This is done in the first theorem of this section. In the statement we have set

$$H_k = H_{e_k} \tag{4.1}$$

where $e_k = (0, 0, \ldots, 1, 0, 0, \ldots)$ is the element of ℓ with all entries 0, at the exception of the k-th entry, which is equal to 1.

Theorem 4.1 *Let f be a $\mathbb{C}^{p\times q}$-valued rational function, analytic in a neighborhood of the origin. Then f is the image under the Hermite transform of an element $F \in (S_{-1})^{p\times q}$. If*

$$f(z) = D + C(I_N - \sum_{k=1}^{M} z_k A_k)^{-1}(\sum_{k=1}^{M} z_k B_k)$$

is a realization of f, then

$$F = D + C(I_N - \sum_{k=1}^{M} H_k A_k)^{-\lozenge}\lozenge(\sum_{k=1}^{M} H_k B_k). \tag{4.2}$$

Proof: The function f is analytic in a neighborhood of the closed ball

$$B_{k,\epsilon} = \left\{ (z_1, \ldots, z_M) \in \mathbb{C}^M \ ; \ \sum_{k=1}^{M} |z_k|^2 \le \epsilon^2 \right\}$$

for some $\epsilon > 0$. Fix $r_0 > 0$. We claim that for q large enough, it holds that

$$K_q(R_0) \subset B_{k,\epsilon}. \tag{4.3}$$

Indeed, let $z \in K_q(r_0)$. Then

$$\sum_{\alpha \neq 0} |z|^\alpha (2\mathbb{N}^{q\alpha}) < r_0^2.$$

In particular,

$$\sum_{k=1}^{M} |z_k|^2 (2k)^{4qk} < r_0^2,$$

and so

$$\sum_{k=1}^{M} |z_k|^2 < \frac{r_0^2}{2^{4q}}.$$

For q large enough, $\frac{r_0^2}{2^{4q}} \leq \epsilon^2$. The first claim of the theorem follows then from Theorem 3.3, and the second claim follows from Proposition 3.5. □

We now present the counterpart of Theorem 2.1. First a remark: The range of the Hermite functions consists of functions which depend on a countable number of variables. We say that such a function is *rational* if it depends only on a finite number of these variables and is, moreover, a rational function of these variables.

Theorem 4.2 *Let $F \in (S_{-1})^{p\times q}$. The following are equivalent:*

1. $\mathbf{I}(F)$ *is a rational function, analytic at the origin.*

2. *There are $N, M \in \mathbb{N}$ and matrices $D \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{p\times N}$, $A_1, \ldots, A_M \in \mathbb{R}^{N\times N}$ and $B_1, \ldots, B_M \in \mathbb{R}^{N\times q}$ such that F is of the form* (4.2)*.*

3. *F is obtained from the (H_α) after a finite number of additions, Wick multiplications and inversions (the latter defined as in Proposition 3.5).*

Proof: Assume that $\mathbf{I}(F)$ is rational and analytic in a neighborhood of the origin. By the previous theorem, F is then of the asserted form. The converse follows by applying the Hermite transform on F. The equivalence with the third statement follows from the fact that a rational function is obtained after a finite number of sums, products and divisions of monomials □

Definition 4.3 *A function in the white noise space will be called* rational *if any of the equivalent conditions in Theorem 4.2 holds.*

Theorem 4.4 *The Wick product and sum of two rational functions of compatible dimensions are rational. If F is rational and invertible at the origin, then F^{-1} is also rational.*

Proof: This follows from the first definition, since the corresponding properties for rational functions hold. $\square$

5 Realizable functions

We now widen the class of rational functions.

Definition 5.1 *An element $F = \sum_\alpha c_\alpha H_\alpha$ in the white noise space will be called* realizable *if its Hermite transform can be written as*

$$\mathbf{I}(F)(z) = D + C(I - zA)^{-1} zB$$

where $D = \mathbf{I}(F)(0)$, $z = (z_1, z_2, \ldots)$ and

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \end{pmatrix},$$

and where, moreover, the A_j are operators acting on a common Hilbert space $\mathcal{H}$ and the B_j are bounded operators from $\mathbb{C}$ into $\mathcal{H}$. We will say that F is finitely realizable *if $\mathcal{H}$ is finite dimensional.*

Example 5.2 *Let $a \in \ell_2$ of norm strictly less than one, the function*

$$b_a(z) = \frac{(1 - |a|^2_{\ell_2})^{1/2}}{1 - \langle z, a\rangle_{\ell_2}} (z - a)(I_{\ell_2} - a^* a)^{-1/2} \tag{5.4}$$

is realizable.

Indeed,

$$b_a(z) = -(1-|a|^2_{\ell_2})^{1/2} a(I_{\ell_2} - a^*a)^{-1/2} + (1-|a|^2_{\ell_2})^{1/2}(1-\langle z, a\rangle_{\ell_2})^{-1} z(I_{\ell_2} - a^*a)^{1/2}.$$

The function (5.4) is the infinite dimensional version of the Blaschke factors in the ball; it satisfies

$$\frac{1-\langle b_a(z), b_a(w)\rangle_{\ell_2}}{1-\langle z, w\rangle_{\ell_2}} = \frac{1-|a|^2_{\ell_2}}{(1-\langle z, a\rangle_{\ell_2})(1-\langle a, w\rangle_{\ell_2})}.$$

See [30, p. 26], [7, pp. 11-13] for a proof in the unit ball of $\mathbb{C}^N$. The proof presented in this last reference still holds in ℓ_2.

Theorem 5.3 *The product and sum of two realizable functions of compatible dimensions are realizable. If F is realizable and invertible at the origin, then F^{-1} is also realizable.*

Proof: For non square matrices, addition is a special case of multiplication since

$$M + N = \begin{pmatrix} M & I_p \end{pmatrix} \begin{pmatrix} I_q \\ N \end{pmatrix},$$

where M and N are $p \times q$ matrices. To check that the product of two realizable functions is still realizable we proceed as follows: let

$$\mathbf{I}(F^{(k)}) = D^{(k)} + C^{(k)}(I - zA^{(k)})^{-1} zB^{(k)}, \quad k = 1, 2$$

be two realizable functions such that the product $\mathbf{I}(F^{(1)})\mathbf{I}(F^{(2)})$ makes sense. Then,

$$\begin{aligned}
\mathbf{I}(F^{(1)})\mathbf{I}(F^{(2)}) &= (D^{(1)} + C^{(1)}(I - zA^{(1)})^{-1} zB^{(1)})(D^{(2)} + C^{(2)}(I - zA^{(2)})^{-1} zB^{(2)}) \\
&= D^{(1)}D^{(2)} + \begin{pmatrix} C^{(1)} & D^{(1)}C^{(2)} \end{pmatrix} \begin{pmatrix} I - zA^{(1)} & -zB^{(1)}C^{(2)} \\ 0 & I - zA^{(2)} \end{pmatrix}^{-1} \begin{pmatrix} zB^{(1)}D^{(2)} \\ zB^{(2)} \end{pmatrix} \\
&= D + C(I - zA)^{-1} zB,
\end{aligned}$$

where A, B, C and D are defined as follows: $A = \begin{pmatrix} A_1 & A_2 & \cdots \end{pmatrix}$ with

$$A_\ell = \begin{pmatrix} A_\ell^{(1)} & B_\ell^{(1)} C^{(2)} \\ 0 & A_\ell^{(2)} \end{pmatrix}, \quad \ell = 1, 2, \ldots,$$

and

$$B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \end{pmatrix}, \quad \text{where} \quad B_\ell = \begin{pmatrix} B_\ell^{(1)} D^{(2)} \\ B_\ell^{(2)} \end{pmatrix}, \quad \ell = 1, 2, \ldots$$

These formulas follow from

$$\begin{pmatrix} zA^{(1)} & zB^{(1)}C^{(2)} \\ 0 & zA^{(2)} \end{pmatrix} = \begin{pmatrix} z_1A_1^{(1)} + z_2A_2^{(1)} + \cdots & z_1B_1^{(1)}C^{(2)} + z_2B_2^{(1)}C^{(2)} + \cdots \\ 0 & z_1A_1^{(2)} + z_2A_2^{(2)} + \cdots \end{pmatrix}$$
$$= z_1A_1 + z_2A_2 + \cdots,$$

with the A_ℓ as above. A similar argument holds for B. Moreover, C and D are given by the formula

$$C = \begin{pmatrix} C^{(1)} & D^{(1)}C^{(2)} \end{pmatrix}, \quad D = D^{(1)}D^{(2)}.$$

The claim on the inverse follows from the formula

$$(D + C(I - zA)^{-1}zB)^{-1} = D^{-1} - D^{-1}C(I - z(A - BD^{-1}C))^{-1}zBD^{-1}.$$

This formula is easily verifiable through direct computation. □

At present we do not have necessary and sufficient conditions characterizing realizable functions. Two examples are provided in the next two sections.

6 The Arveson space and multipliers

We study here the Arveson space (see Definition 3.2). Its classical counterpart plays an important role in multi-dimensional system theory. See [15], [29].

Theorem 6.1 *The Arveson space is a subHilbert space of the Kondratiev space.*

Proof: The Hermite transform of $\mathcal{A}$ is the classical Arveson space, the reproducing kernel Hilbert space with reproducing kernel

$$k_w(z) = (1 - \langle z, w \rangle_{\ell_2})^{-1} = \sum_{\alpha \in \ell} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{w}^\alpha, \tag{6.5}$$

where z and w are in the unit ball of ℓ_2

$$\mathbb{B} = \left\{ z \in \ell_2 \ ; \ \sum_{k=1}^{\infty} |z_k|^2 < 1 \right\}.$$

At this stage we wish to apply Corollary 3.7. We note that $K(z,z)$ is uniformly bounded in the closed ball of radius $1/\sqrt{2}$

$$\left\{ z \in \ell_2 \ ; \ \sum_{k=1}^{\infty} |z_k|^2 \leq 1/2 \right\}.$$

But this closed ball is in turn included in $K_q(1/\sqrt{2})$ for any $q \geq 1$, and the proof is easily concluded. $\square$

Definition 6.2 *A function s, defined in the open unit ball of ℓ_2, is a* Schur multiplier *if, by definition, the operator of multiplication by s is a contraction from the classical Arveson space into itself.*

Equivalently, s is a Schur multiplier if and only if the kernel

$$K_s(z,w) = \frac{1 - s(z)s(w)^*}{1 - \langle z, w \rangle_{\ell_2}}$$

is positive in the unit ball of ℓ_2. We will denote by $\mathcal{H}(s)$ the corresponding reproducing kernel Hilbert space and $\Gamma_s = I - M_s M_s^*$, where M_s is the operator of multiplication in the Arveson space. We have

$$K_s(z,w) = (\Gamma_s(k_w))(w),$$

and the range of Γ_s is dense in $\mathcal{H}(s)$, as follows from general results on operator ranges and reproducing kernel spaces; see for instance [23].

Theorem 6.3 *A Schur multiplier is the image under the Hermite transform of a realizable function in the Kondratiev space.*

Proof: A number of related proofs hold for this result in the case where only a finite number of variables are involved; see for instance [15], [3]. Here we briefly outline the method of [10, Théorème 2.1], suitably adapted to the case of a countable number of variables. We consider the case of a scalar-valued Schur multiplier (the general case is treated in the same way). We proceed with the following steps:

STEP 1: *The operators of multiplication by the variables M_{z_k} are bounded in $\mathcal{A}$ and it holds that*

$$\sum_{k=0}^{\infty} M_{z_k} M_{z_k}^* = I - C^* C, \tag{6.6}$$

where $Cf = f(0)$.

One first verifies that $M_{z_k}^* = \mathcal{R}_k$. The identity (6.6) is then obvious.

STEP 2: *Let $\mathcal{H}(s)_\infty$ be the closure in $\mathcal{H}(s) \oplus \mathcal{H}(s) \oplus \cdots$ of the functions*

$$w_y = \begin{pmatrix} \Gamma_s M_{z_0}^* k_y \\ \Gamma_s M_{z_1}^* k_y \\ \vdots \end{pmatrix}$$

where y runs through the open unit ball of ℓ_2, and where k_y is defined by (6.5)*. The formulas*

$$\widetilde{G}(1) = K_s(z, 0), \quad \widetilde{H}(1) = s(0)^*$$

and

$$\widetilde{T}(w_y) = K_s(z, y) - K_s(z, 0), \quad \widetilde{F}(w_y) = s(y)^* - s(0)^*$$

define an isometric relation on $\mathcal{H}(s)_\infty \oplus \mathbb{C}$ into $\mathcal{H}(s) \oplus \mathbb{C}$ with a dense domain.

The proof is as in [10], and uses equality (6.6).

Since the linear relation in the previous step is isometric and densely defined, it is the graph of an isometric operator. We denote by

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix}$$

its adjoint.

STEP 3: *It holds that*

$$s(z) = H + G(I - \sum_{k=0}^{\infty} z_k T_k)^{-1}(\sum_{k=0}^{\infty} z_k F_k).$$

As in [10], one first proves that

$$\sum_{k=0}^{\infty} z_k (T_k f)(z) = f(z) - f(0)$$
$$\sum_{k=0}^{\infty} z_k (F_k f)(z) = s(z) - s(0).$$

The formula for s is then a direct consequence of these equations. □

7 The space $\mathcal{P}$ and Schur-Agler classes

We now present another class of realizable functions, related to the space $\mathcal{P}$. Recall that this space was defined Definition 3.2. As in the previous section, we consider the scalar case to simplify notation.

Definition 7.1 *A function s is in the Schur-Agler class if there exist a family $k_1(z,w), k_2(z,w), \ldots$ of functions positive in $\prod_{k=1}^{\infty} \mathbb{D}$, where $\mathbb{D}$ denotes the open unit disk, and such that*

$$1 - s(z)s(w)^* = \sum_{\ell=1}^{\infty} (1 - z_\ell w_\ell^*) k_\ell(z,w) \tag{7.1}$$

there.

In the case of the finite polydisk, these classes originate with the work of Agler and have been much studied; see [1], [13], [14].

Theorem 7.2 *A Schur-Agler multiplier is the Hermite transform of a realizable element in the Kondratiev space.*

Proof: We follows the proof given in [12, Section 4.2]. Let $\mathcal{H}(k_\ell)$ be the reproducing kernel Hilbert space of functions with reproducing kernel k_ℓ and let

$$\mathcal{H} = \oplus_{\ell=1}^{\infty}\mathcal{H}(k_\ell),$$

with norm $(\sum_{\ell=1}^{\infty}\|f_\ell\|^2_{\mathcal{H}(k_\ell)})^{1/2}$. One defines a linear relation on $\mathcal{H}\oplus\mathbb{C}\times\mathcal{H}\oplus\mathbb{C}$ via the formula

$$\widehat{A}\begin{pmatrix} w_1^*k(\cdot,w_1)\\ w_2^*k(\cdot,w_2)\\ \vdots\end{pmatrix} = \begin{pmatrix} w_1^*(k(\cdot,w_1)-k(\cdot,0))\\ w_2^*(k(\cdot,w_2)-k_2(\cdot,0))\\ \vdots\end{pmatrix},\quad \widehat{B}\begin{pmatrix} w_1^*k(\cdot,w_1)\\ w_2^*k(\cdot,w_2)\\ \vdots\end{pmatrix} = s(w)^*-s(0)^*,$$

and

$$\widehat{C}(1) = \begin{pmatrix} k_1(\cdot,0)\\ k_2(\cdot,0)\\ \vdots\end{pmatrix},\quad \widehat{D} = s(0)^*.$$

Then

$$\begin{pmatrix}\widehat{A} & \widehat{C}\\ \widehat{B} & \widehat{D}\end{pmatrix}$$

extends to an isometric relation, which is the graph of an operator. We denote by

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}$$

its adjoint. We have the formula

$$z(Af)(z) = \sum_{\ell=0}^{\infty}(f_\ell(z)-f_\ell(0)),\quad z(B(z)) = s(z)-s(0),$$

and

$$Ch = \sum_{\ell=0}^{\infty} f_\ell(0),\quad D = s(0).$$

We note that the infinite sums above converge because of the Cauchy-Schwartz inequality. For instance

$$\begin{aligned}
|\sum_{\ell=0}^{\infty} f_\ell(0)| &= |\sum_{\ell=0}^{\infty}\langle f_\ell, k_\ell(\cdot,0)\rangle_{\mathcal{H}(k_\ell)}\\
&\le \sum_{\ell=0}^{\infty} \|f_\ell\|_{\mathcal{H}(k_\ell)}\sqrt{k_\ell(0,0)}\\
&\le (\sum_{\ell=0}^{\infty} \|f_\ell\|^2_{\mathcal{H}(k_\ell)})^{1/2}(\sum_{\ell=0}^{\infty} k_\ell(0,0))^{1/2}.
\end{aligned}$$

Finally, one has the realization

$$s(z) = D + C(I - \sum_{\ell=1}^{\infty} z_\ell A_\ell)^{-1}(\sum_{\ell=1}^{\infty} z_\ell B_\ell),$$

where $A_\ell = \pi_\ell A$ and $B_\ell = \pi_\ell B$, with π_ℓ being the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_\ell$. □

8 The Gleason problem in certain spaces of power series

Consider a power series in a countable number of variables $f(z) = \sum_{\alpha\in\ell} z^\alpha f_\alpha$, convergent in a neighborhood of the origin. The Leibenzon's operators $\mathcal{R}_j$ can still be defined as power series as in (1.3), and we have:

$$f(z) - f(0) = \sum_{j=0}^{\infty} z_j \mathcal{R}_j f(z). \tag{8.1}$$

We will say that a space of power series in the z_j is *resolvent-invariant* if Gleason's problem is solvable with bounded operators A_j:

$$f(z) - f(0) = \sum_{j=0}^{\infty} z_j (A_j f)(z).$$

The space will be called *backward-shift invariant* if the A_j commute.

We now prove a uniqueness theorem in a collection of spaces of functions depending on a countable number of variables, which includes in particular the Fock space, the Arveson space and the infinite polydisk space. The argument follows the one given in the hyper-holomorphic setting in [12]. For an earlier result in the setting of power series in finite number of variables, see [6]. Note that in the theorem we do not assume the operator of multiplication by z_k to be bounded. The theorem holds in particular in the Arveson space and in the Fock space. In the Fock space the operators of multiplication by the variables are not bounded.

Theorem 8.1 *Let $\mathcal{H}$ be a reproducing kernel Hilbert space of functions depending on a countable number of variables $z_1, \ldots$, defined in a neighborhood of the origin, and assume that the Leibenzon operators $\mathcal{R}_j$ are uniformly bounded in $\mathcal{H}$. Any other uniformly commuting solution to Gleason's problem, coincides with the $\mathcal{R}_j$.*

Proof: Let $A_1, \ldots$ be a family of commuting and uniformly bounded operators in $\mathcal{H}$ such that

$$f(z) - f(0) = \sum_{n=0}^{\infty} z_n (A_n f)(z).$$

Since the A_j commute and are uniformly bounded, we can reiterate this equation and obtain

$$f(z) = \sum_{\alpha \in \ell} z^\alpha C A^\alpha f = C(I - zA)^{-1} f,$$

where C is the operator of evaluation at the origin. In particular,

$$C(I - zA)^{-1} f \equiv 0 \Longrightarrow f = 0, \tag{8.2}$$

and replacing f by $A_k f$ we have:

$$(A_k f)(z) = C(I - zA)^{-1} A_k f.$$

On the other hand by definition of $\mathcal{R}_k$,

$$(\mathcal{R}_k f)(z) = \sum_{\alpha \geq e_k} z^{\alpha - e_k} \frac{\alpha_k}{|\alpha|} C A^\alpha f = C(I - zA)^{-1} A_k f,$$

and so $A_k = \mathcal{R}_k$ in view of (8.2). $\square$

As a corollary of the preceding theorem we have:

Theorem 8.2 *The Leibenzon operators are bounded in the Fock space, in the Arveson space and in the infinite polydisk space. In particular, they are the only commutative solution of Gleason's problem in these spaces.*

Proof: Let $f(z) = \sum_{\alpha\in\ell} z^\alpha f_\alpha$ be in the Fock space $\mathcal{F}$. By definition of $\mathcal{R}_j$ and of the norm in the Fock space, we have:

$$\begin{aligned}
\|\mathcal{R}_j f\|_{\mathcal{F}} &= \sum_{\alpha\geq e_j} \frac{\alpha_j^2}{|\alpha|^2} f_\alpha^2 (\alpha - e_j)! \\
&\leq \sum_{\alpha\geq e_j} f_\alpha^2 \alpha! \\
&\leq \|f\|_{\mathcal{F}}^2.
\end{aligned}$$

$\square$

We now give the table presenting the parallels between the hyper-holomorphic case and the stochastic case.

The setting	Hyper-holomorphic case	Stochastic case
The underlying space	Functions hyper-holomorphic at the origin	The Kondratiev's space S_{-1}
The building blocks	The hyperholomorphic variables ζ_j, $j = 1, 2, 3$	The functions H_{e_k}, $k \in \mathbb{N}$
Power series expansions in terms of	the Fueter polynomials	the functions H_α
The product	Cauchy-Kovalesvkaya	Wick product
Commutativity of the product	When the restrictions to to $x_0 = 0$ commute	Always commutes
Convolution on "power series" expansions	$f \circ g = \sum_\alpha \zeta^\alpha (\sum_{\beta \le \alpha} f_\beta g_{\alpha-\beta})$	$f \lozenge g = \sum_{\alpha \in \ell} H_\alpha (\sum_{\beta \le \alpha} f_\beta g_{\alpha-\beta})$
Going to the classical case from the hyper-holomorphic/stochastic case	Restriction to the hyperplane $x_0 = 0$	The Hermite transform
From the classical case to the hyper-holomorphic/stochastic case	CK extension	Kontradiev's theorem (see Theorem 3.3)
Uniqueness theorem for the Leibenzon's operators	In particular in the Arveson space	In particular in the Fock space

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