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## Matrix-J-Unitary Non-Commutative Rational Formal Power Series

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### Comments

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# Matrix- $J$ -unitary Non-commutative Rational Formal Power Series

D. Alpay and D. S. Kalyuzhnyi-Verbovetzkiĭ

**Abstract.** Formal power series in  $N$  non-commuting indeterminates can be considered as a counterpart of functions of one variable holomorphic at 0, and some of their properties are described in terms of coefficients. However, really fruitful analysis begins when one considers for them evaluations on  $N$ -tuples of  $n \times n$  matrices (with  $n = 1, 2, \dots$ ) or operators on an infinite-dimensional separable Hilbert space. Moreover, such evaluations appear in control, optimization and stabilization problems of modern system engineering.

In this paper, a theory of realization and minimal factorization of rational matrix-valued functions which are  $J$ -unitary on the imaginary line or on the unit circle is extended to the setting of non-commutative rational formal power series. The property of  $J$ -unitarity holds on  $N$ -tuples of  $n \times n$  skew-Hermitian versus unitary matrices ( $n = 1, 2, \dots$ ), and a rational formal power series is called *matrix- $J$ -unitary* in this case. The close relationship between minimal realizations and structured Hermitian solutions  $H$  of the Lyapunov or Stein equations is established. The results are specialized for the case of *matrix- $J$ -inner* rational formal power series. In this case  $H > 0$ , however the proof of that is more elaborated than in the one-variable case and involves a new technique. For the rational *matrix-inner* case, i.e., when  $J = I$ , the theorem of Ball, Groenewald and Malakorn on unitary realization of a formal power series from the non-commutative Schur–Agler class admits an improvement: the existence of a minimal (thus, finite-dimensional) such unitary realization and its uniqueness up to a unitary similarity is proved. A version of the theory for *matrix-selfadjoint* rational formal power series is also presented. The concept of non-commutative formal reproducing kernel Pontryagin spaces is introduced, and in this framework the backward shift realization of a matrix- $J$ -unitary rational formal power series in a finite-dimensional non-commutative de Branges–Rovnyak space is described.

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## 1. Introduction

In the present paper we study a non-commutative analogue of rational matrix-valued functions which are  $J$ -unitary on the imaginary line or on the unit circle

and, as a special case,  $J$ -inner ones. Let  $J \in \mathbb{C}^{q \times q}$  be a signature matrix, i.e., a matrix which is both self-adjoint and unitary. A  $\mathbb{C}^{q \times q}$ -valued rational function  $F$  is  $J$ -unitary on the imaginary line if

$$F(z)JF(z)^* = J \quad (1.1)$$

at every point of holomorphy of  $F$  on the imaginary line. It is called  $J$ -inner if moreover

$$F(z)JF(z)^* \leq J \quad (1.2)$$

at every point of holomorphy of  $F$  in the open right half-plane  $\Pi$ . Replacing the imaginary line by the unit circle  $\mathbb{T}$  in (1.1) and the open right half-plane  $\Pi$  by the open unit disk  $\mathbb{D}$  in (1.2), one defines  $J$ -unitary functions on the unit circle (resp.,  $J$ -inner functions in the open unit disk). These classes of rational functions were studied in [7] and [6] using the theory of realizations of rational matrix-valued functions, and in [4] using the theory of reproducing kernel Pontryagin spaces. The circle and line cases were studied in a unified way in [5]. We mention also the earlier papers [36, 23] that inspired much of investigation of these and other classes of rational matrix-valued functions with symmetries.

We now recall some of the arguments in [7], then explain the difficulties appearing in the several complex variables setting, and why the arguments of [7] extend to the non-commutative framework. So let  $F$  be a rational function which is  $J$ -unitary on the imaginary line, and assume that  $F$  is holomorphic in a neighbourhood of the origin. It then admits a minimal realization

$$F(z) = D + C(I_\gamma - zA)^{-1}zB$$

where  $D = F(0)$ , and  $A, B, C$  are matrices of appropriate sizes (the size  $\gamma \times \gamma$  of the square matrix  $A$  is minimal possible for such a realization). Rewrite (1.1) as

$$F(z) = JF(-\bar{z})^{-*}J, \quad (1.3)$$

where  $z$  is in the domain of holomorphy of both  $F(z)$  and  $F(-\bar{z})^{-*}$ . We can rewrite (1.3) as

$$D + C(I_\gamma - zA)^{-1}zB = J(D^{-*} + D^{-*}B^*(I_\gamma + z(A - BD^{-1}C)^*)^{-1}zC^*D^{-*})J.$$

The above equality gives two minimal realizations of a given rational matrix-valued function. These realizations are therefore similar, and there is a uniquely defined matrix (which, for convenience, we denote by  $-H$ ) such that

$$\begin{pmatrix} -H & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -(A^* - C^*D^{-*}B^*) & C^*D^{-*}J \\ JD^{-*}B^* & JD^{-*}J \end{pmatrix} \begin{pmatrix} -H & 0 \\ 0 & I_q \end{pmatrix}. \quad (1.4)$$

The matrix  $-H^*$  in the place of  $-H$  also satisfies (1.4), and by uniqueness of the similarity matrix we have  $H = H^*$ , which leads to the following theorem.

**Theorem 1.1.** *Let  $F$  be a rational matrix-valued function holomorphic in a neighbourhood of the origin and let  $F(z) = D + C(I_\gamma - zA)^{-1}zB$  be a minimal realization of  $F$ . Then  $F$  is  $J$ -unitary on the imaginary line if and only if the following conditions hold:*

- (1)  $D$  is  $J$ -unitary, that is,  $DJD^* = J$ ;  
(2) there exists an Hermitian invertible matrix  $H$  such that

$$A^*H + HA = -C^*JC, \quad (1.5)$$

$$B = -H^{-1}C^*JD. \quad (1.6)$$

The matrix  $H$  is uniquely determined by a given minimal realization (it is called the associated Hermitian matrix to this realization). It holds that

$$\frac{J - F(z)JF(z')^*}{z + \overline{z'}} = C(I_\gamma - zA)^{-1}H^{-1}(I_\gamma - z'A)^{-*}C^*. \quad (1.7)$$

In particular,  $F$  is  $J$ -inner if and only if  $H > 0$ .

The finite-dimensional reproducing kernel Pontryagin space  $\mathcal{K}(F)$  with reproducing kernel

$$K^F(z, z') = \frac{J - F(z)JF(z')^*}{(z + \overline{z'})}$$

provides a minimal state space realization for  $F$ : more precisely (see [4]),

$$F(z) = D + C(I_\gamma - zA)^{-1}zB,$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{K}(F) \\ \mathbb{C}^q \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{K}(F) \\ \mathbb{C}^q \end{pmatrix}$$

is defined by

$$\begin{aligned} (Af)(z) = (R_0f)(z) &:= \frac{f(z) - f(0)}{z}, & Bu &= \frac{F(z) - F(0)}{z}u, \\ Cf &= f(0), & Dx &= F(0)x. \end{aligned}$$

Another topic considered in [7] and [4] is  $J$ -unitary factorization. Given a matrix-valued function  $F$  which is  $J$ -unitary on the imaginary line one looks for all minimal factorizations of  $F$  (see [15]) into factors which are themselves  $J$ -unitary on the imaginary line. There are two equivalent characterizations of these factorizations: the first one uses the theory of realization and the second one uses the theory of reproducing kernel Pontryagin spaces.

**Theorem 1.2.** *Let  $F$  be a rational matrix-valued function which is  $J$ -unitary on the imaginary line and holomorphic in a neighbourhood of the origin, and let  $F(z) = D + C(I_\gamma - zA)^{-1}zB$  be a minimal realization of  $F$ , with the associated Hermitian matrix  $H$ . There is a one-to-one correspondence between minimal  $J$ -unitary factorizations of  $F$  (up to a multiplicative  $J$ -unitary constant) and  $A$ -invariant subspaces which are non-degenerate in the (possibly, indefinite) metric induced by  $H$ .*

In general,  $F$  may fail to have non-trivial  $J$ -unitary factorizations.

**Theorem 1.3.** *Let  $F$  be a rational matrix-valued function which is  $J$ -unitary on the imaginary line and holomorphic in a neighbourhood of the origin. There is a one-to-one correspondence between minimal  $J$ -unitary factorizations of  $F$  (up to a multiplicative  $J$ -unitary constant) and  $R_0$ -invariant non-degenerate subspaces of  $\mathcal{K}(F)$ .*

The arguments in the proof of Theorem 1.1 do not go through in the several complex variables context. Indeed, uniqueness, up to a similarity, of minimal realizations doesn't hold anymore (see, e.g., [27, 25, 33]). On the other hand, the notion of realization still makes sense in the non-commutative setting, namely for non-commutative rational *formal power series* (FPSs in short), and there is a uniqueness result for minimal realizations in this case (see [16, 39, 11]). The latter allows us to extend the notion and study of  $J$ -unitary matrix-valued functions to the non-commutative case. We introduce the notion of a *matrix- $J$ -unitary* rational FPS as a formal power series in  $N$  non-commuting indeterminates which is  $J \otimes I_n$ -unitary on  $N$ -tuples of  $n \times n$  skew-Hermitian versus unitary matrices for  $n = 1, 2, \dots$ . We extend to this case the theory of minimal realizations, minimal  $J$ -unitary factorizations, and backward shift models in finite-dimensional de Branges–Rovnyak spaces. We also introduce, in a similar way, the notion of matrix-selfadjoint rational formal power series, and show how to deduce the related theory for them from the theory of matrix- $J$ -unitary ones.

We now turn to the outline of this paper. It consists of eight sections. Section 1 is this introduction. In Section 2 we review various results in the theory of FPSs. Let us note that the theorem on null spaces for matrix substitutions and its corollary, from our paper [8], which are recollected in the end of Section 2, become an important tool in our present work on FPSs. In Section 3 we study the properties of observability, controllability and minimality of Givone-Roesser nodes in the non-commutative setting and give the corresponding criteria in terms of matrix evaluations for their “formal transfer functions”. We also formulate a theorem on minimal factorizations of a rational FPS. In Section 4 we define the non-commutative analogue of the imaginary line and study matrix- $J$ -unitary FPSs for this case. We in particular obtain a non-commutative version of Theorem 1.1. We obtain a counterpart of the Lyapunov equation (1.5) and of Theorem 1.2 on minimal  $J$ -unitary factorizations. The unique solution of the Lyapunov equation has in this case a block diagonal structure:  $H = \text{diag}(H_1, \dots, H_N)$ , and is said to be *the associated structured Hermitian matrix* (associated with a given minimal realization of a matrix- $J$ -unitary FPS). Section 5 contains the analogue of the previous section for the case of a non-commutative counterpart of the unit circle. These two sections do not take into account a counterpart of condition (1.2), which is considered in Section 6 where we study matrix- $J$ -inner rational FPSs. In particular, we show that the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$  is strictly positive in this case, which generalizes the statement in Theorem 1.1 on  $J$ -inner functions. We define non-commutative counterparts of the right halfplane and the unit disk, and formulate our results for both of these domains. The second

one is the disjoint union of the products of  $N$  copies of  $n \times n$  matrix unit disks,  $n = 1, 2, \dots$ , and plays a role of a “non-commutative polydisk”. In Theorem 6.6 we show that any (not necessarily rational) FPS with operator coefficients, which takes contractive values in this domain, belongs to the non-commutative Schur–Agler class, defined by J. A. Ball, G. Groenewald and T. Malakorn in [12]. (The opposite is trivial: any function from this class has the above-mentioned property.) In other words, the contractivity of values of a FPS on  $N$ -tuples of strictly contractive  $n \times n$  matrices,  $n = 1, 2, \dots$ , is sufficient for the contractivity of its values on  $N$ -tuples of strictly contractive operators in an infinite-dimensional separable Hilbert space. Thus, matrix-inner rational FPSs (i.e., matrix- $J$ -inner ones for the case  $J = I_q$ ) belong to the non-commutative Schur–Agler class. For this case, we recover the theorem on unitary realizations for FPSs from the latter class which was obtained in [12]. Moreover, our Theorem 6.4 establishes the existence of a *minimal*, thus *finite-dimensional*, unitary Givone–Roesser realization of a rational matrix-inner FPS and *the uniqueness of such a realization up to a unitary similarity*. This implies, in particular, non-commutative Lossless Bounded Real Lemma (see [41, 7] for its one-variable counterpart). A non-commutative version of standard Bounded Real Lemma (see [47]) has been presented recently in [13]. In Section 7 we study matrix-selfadjoint rational FPSs. In Section 8 we introduce non-commutative formal reproducing kernel Pontryagin spaces in a way which extends one that J. A. Ball and V. Vinnikov have introduced in [14] non-commutative formal reproducing kernel Hilbert spaces. We describe minimal backward shift realizations in non-commutative formal reproducing kernel Pontryagin spaces which serve as a counterpart of finite-dimensional de Branges–Rovnyak spaces. Let us note that we derive an explicit formula (8.12) for the corresponding reproducing kernels. In the last subsection of Section 8 we present examples of matrix-inner rational FPSs with scalar coefficients, in two non-commuting indeterminates, and the corresponding reproducing kernels computed by formula (8.12).

## 2. Preliminaries

In this section we introduce the notations which will be used throughout this paper and review some definitions from the theory of formal power series. The symbol  $\mathbb{C}^{p \times q}$  denotes the set of  $p \times q$  matrices with complex entries, and  $(\mathbb{C}^{r \times s})^{p \times q}$  is the space of  $p \times q$  block matrices with block entries in  $\mathbb{C}^{r \times s}$ . The tensor product  $A \otimes B$  of matrices  $A \in \mathbb{C}^{r \times s}$  and  $B \in \mathbb{C}^{p \times q}$  is the element of  $(\mathbb{C}^{r \times s})^{p \times q}$  with  $(i, j)$ -th block entry equal to  $Ab_{ij}$ . The tensor product  $\mathbb{C}^{r \times s} \otimes \mathbb{C}^{p \times q}$  is the linear span of finite sums of the form  $C = \sum_{k=1}^n A_k \otimes B_k$  where  $A_k \in \mathbb{C}^{r \times s}$  and  $B_k \in \mathbb{C}^{p \times q}$ . One identifies  $\mathbb{C}^{r \times s} \otimes \mathbb{C}^{p \times q}$  with  $(\mathbb{C}^{r \times s})^{p \times q}$ . Different representations for an element  $C \in \mathbb{C}^{r \times s} \otimes \mathbb{C}^{p \times q}$  can be reduced to a unique one:

$$C = \sum_{\mu=1}^r \sum_{\nu=1}^s \sum_{\tau=1}^p \sum_{\sigma=1}^q c_{\mu\nu\tau\sigma} E'_{\mu\nu} \otimes E''_{\tau\sigma},$$



where the matrices  $E'_{\mu\nu} \in \mathbb{C}^{r \times s}$  and  $E''_{\tau\sigma} \in \mathbb{C}^{p \times q}$  are given by

$$(E'_{\mu\nu})_{ij} = \begin{cases} 1 & \text{if } (i, j) = (\mu, \nu) \\ 0 & \text{if } (i, j) \neq (\mu, \nu) \end{cases}, \quad \mu, i = 1, \dots, r \quad \text{and} \quad \nu, j = 1, \dots, s,$$

$$(E''_{\tau\sigma})_{k\ell} = \begin{cases} 1 & \text{if } (k, \ell) = (\tau, \sigma) \\ 0 & \text{if } (k, \ell) \neq (\tau, \sigma) \end{cases}, \quad \tau, k = 1, \dots, p \quad \text{and} \quad \sigma, \ell = 1, \dots, q.$$

We denote by  $\mathcal{F}_N$  the free semigroup with  $N$  generators  $g_1, \dots, g_N$  and the identity element  $\emptyset$  with respect to the concatenation product. This means that the generic element of  $\mathcal{F}_N$  is a word  $w = g_{i_1} \cdots g_{i_n}$ , where  $i_\nu \in \{1, \dots, N\}$  for  $\nu = 1, \dots, n$ , the identity element  $\emptyset$  corresponds to the empty word, and for another word  $w' = g_{j_1} \cdots g_{j_m}$ , one defines the product as

$$ww' = g_{i_1} \cdots g_{i_n} g_{j_1} \cdots g_{j_m}, \quad w\emptyset = \emptyset w = w.$$

We denote by  $w^T = g_{i_n} \cdots g_{i_1} \in \mathcal{F}_N$  the *transpose* of  $w = g_{i_1} \cdots g_{i_n} \in \mathcal{F}_N$  and by  $|w| = n$  the *length* of the word  $w$ . Correspondingly,  $\emptyset^T = \emptyset$ , and  $|\emptyset| = 0$ .

A *formal power series* (FPS in short) in non-commuting indeterminates  $z_1, \dots, z_N$  with coefficients in a linear space  $\mathcal{E}$  is given by

$$f(z) = \sum_{w \in \mathcal{F}_N} f_w z^w, \quad f_w \in \mathcal{E}, \quad (2.1)$$

where for  $w = g_{i_1} \cdots g_{i_n}$  and  $z = (z_1, \dots, z_N)$  we set  $z^w = z_{i_1} \cdots z_{i_n}$ , and  $z^\emptyset = 1$ . We denote by  $\mathcal{E} \langle \langle z_1, \dots, z_N \rangle \rangle$  the linear space of FPSs in non-commuting indeterminates  $z_1, \dots, z_N$  with coefficients in  $\mathcal{E}$ . A series  $f \in \mathbb{C}^{p \times q} \langle \langle z_1, \dots, z_N \rangle \rangle$  of the form (2.1) can also be viewed as a  $p \times q$  matrix whose entries are formal power series with coefficients in  $\mathbb{C}$ , i.e., belong to the space  $\mathbb{C} \langle \langle z_1, \dots, z_N \rangle \rangle$ , which has an additional structure of non-commutative ring (we assume that the indeterminates  $z_j$  formally commute with the coefficients  $f_w$ ). The *support* of a FPS  $f$  given by (2.1) is the set

$$\text{supp } f = \{w \in \mathcal{F}_N : f_w \neq 0\}.$$

*Non-commutative polynomials* are formal power series with finite support. We denote by  $\mathcal{E} \langle z_1, \dots, z_N \rangle$  the subspace in the space  $\mathcal{E} \langle \langle z_1, \dots, z_N \rangle \rangle$  consisting of non-commutative polynomials. Clearly, a FPS is determined by its coefficients  $f_w$ . Sums and products of two FPSs  $f$  and  $g$  with matrix coefficients of compatible sizes (or with operator coefficients) are given by

$$(f + g)_w = f_w + g_w, \quad (fg)_w = \sum_{w'w''=w} f_{w'} g_{w''}. \quad (2.2)$$

A FPS  $f$  with coefficients in  $\mathbb{C}$  is invertible if and only if  $f_\emptyset \neq 0$ . Indeed, assume that  $f$  is invertible. From the definition of the product of two FPSs in (2.2) we get  $f_\emptyset(f^{-1})_\emptyset = 1$ , and hence  $f_\emptyset \neq 0$ . On the other hand, if  $f_\emptyset \neq 0$  then  $f^{-1}$  is given by

$$f^{-1}(z) = \sum_{k=0}^{\infty} (1 - f_\emptyset^{-1} f(z))^k f_\emptyset^{-1}.$$

The formal power series in the right-hand side is well defined since the expansion of  $(1 - f_\emptyset^{-1}f)^k$  contains words of length at least  $k$ , and thus the coefficients  $(f^{-1})_w$  are finite sums.

A FPS with coefficients in  $\mathbb{C}$  is called *rational* if it can be expressed as a finite number of sums, products and inversions of non-commutative polynomials. A formal power series with coefficients in  $\mathbb{C}^{p \times q}$  is called *rational* if it is a  $p \times q$  matrix whose all entries are rational FPSs with coefficients in  $\mathbb{C}$ . We will denote by  $\mathbb{C}^{p \times q} \langle\langle z_1, \dots, z_N \rangle\rangle_{\text{rat}}$  the linear space of rational FPSs with coefficients in  $\mathbb{C}^{p \times q}$ . Define the product of  $f \in \mathbb{C}^{p \times q} \langle\langle z_1, \dots, z_N \rangle\rangle_{\text{rat}}$  and  $p \in \mathbb{C} \langle z_1, \dots, z_N \rangle$  as follows:

1.  $f \cdot 1 = f$  for every  $f \in \mathbb{C}^{p \times q} \langle\langle z_1, \dots, z_N \rangle\rangle_{\text{rat}}$ ;
2. For every word  $w' \in \mathcal{F}_N$  and every  $f \in \mathbb{C}^{p \times q} \langle\langle z_1, \dots, z_N \rangle\rangle_{\text{rat}}$ ,

$$f \cdot z^{w'} = \sum_{w \in \mathcal{F}_N} f_w z^{ww'} = \sum_w f_v z^w$$

where the last sum is taken over all  $w$  which can be written as  $w = vw'$  for some  $v \in \mathcal{F}_N$ ;

3. For every  $f \in \mathbb{C}^{p \times q} \langle\langle z_1, \dots, z_N \rangle\rangle_{\text{rat}}$ ,  $p_1, p_2 \in \mathbb{C} \langle z_1, \dots, z_N \rangle$  and  $\alpha_1, \alpha_2 \in \mathbb{C}$ ,

$$f \cdot (\alpha_1 p_1 + \alpha_2 p_2) = \alpha_1 (f \cdot p_1) + \alpha_2 (f \cdot p_2).$$

The space  $\mathbb{C}^{p \times q} \langle\langle z_1, \dots, z_N \rangle\rangle_{\text{rat}}$  is a right module over the ring  $\mathbb{C} \langle z_1, \dots, z_N \rangle$  with respect to this product. A structure of left  $\mathbb{C} \langle z_1, \dots, z_N \rangle$ -module can be defined in a similar way since the indeterminates commute with coefficients.

Formal power series are used in various branches of mathematics, e.g., in abstract algebra, enumeration problems and combinatorics; rational formal power series have been extensively used in theoretical computer science, mostly in automata theory and language theory (see [18]). The Kleene–Schützenberger theorem [35, 44] (see also [24]) says that a FPS  $f$  with coefficients in  $\mathbb{C}^{p \times q}$  is rational if and only if it is *recognizable*, i.e., there exist  $r \in \mathbb{N}$  and matrices  $C \in \mathbb{C}^{p \times r}$ ,  $A_1, \dots, A_N \in \mathbb{C}^{r \times r}$  and  $B \in \mathbb{C}^{r \times q}$  such that for every word  $w = g_{i_1} \cdots g_{i_n} \in \mathcal{F}_N$  one has

$$f_w = CA^w B, \quad \text{where } A^w = A_{i_1} \cdots A_{i_n}. \quad (2.3)$$

Let  $\mathcal{H}_f$  be the *Hankel matrix* whose rows and columns are indexed by the words of  $\mathcal{F}_N$  and defined by

$$(\mathcal{H}_f)_{w, w'} = f_{ww'^T}, \quad w, w' \in \mathcal{F}_N.$$

It follows from (2.3) that if the FPS  $f$  is recognizable then  $(\mathcal{H}_f)_{w, w'} = CA^{ww'^T} B$  for all  $w, w' \in \mathcal{F}_N$ . M. Fliess has shown in [24] that a FPS  $f$  is rational (that is, recognizable) if and only if

$$\gamma := \text{rank } \mathcal{H}_f < \infty.$$

In this case the number  $\gamma$  is the smallest possible  $r$  for a representation (2.3).

In control theory, rational FPSs appear as the input/output mappings of linear systems with structured uncertainties. For instance, in [17] a system matrix

is given by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{(r+p) \times (r+q)},$$

and the uncertainty operator is given by

$$\Delta(\delta) = \text{diag}(\delta_1 I_{r_1}, \dots, \delta_N I_{r_N}),$$

where  $r_1 + \dots + r_N = r$ . The uncertainties  $\delta_k$  are linear operators on  $\ell^2$  representing disturbances or small perturbation parameters which enter the system at different locations. Mathematically, they can be interpreted as non-commuting indeterminates. The input/output map is a linear fractional transformation

$$LFT(M, \Delta(\delta)) = D + C(I_r - \Delta(\delta)A)^{-1} \Delta(\delta)B, \quad (2.4)$$

which can be interpreted as a non-commutative transfer function  $T_\alpha^{\text{nc}}$  of a linear system  $\alpha$  with evolution on  $\mathcal{F}_N$ :

$$\alpha : \begin{cases} x_j(g_j w) &= A_{j1}x_1(w) + \dots + A_{jN}x_N(w) + B_j u(w), \quad j = 1, \dots, N, \\ y(w) &= C_1 x_1(w) + \dots + C_N x_N(w) + D u(w), \end{cases} \quad (2.5)$$

where  $x_j(w) \in \mathbb{C}^{r_j}$  ( $j = 1, \dots, N$ ),  $u(w) \in \mathbb{C}^q$ ,  $y(w) \in \mathbb{C}^p$ , and the matrices  $A_{jk}$ ,  $B$  and  $C$  are of appropriate sizes along the decomposition  $\mathbb{C}^r = \mathbb{C}^{r_1} \oplus \dots \oplus \mathbb{C}^{r_N}$ . Such a system appears in [39, 11, 12, 13] and is known as the *non-commutative Givone–Roesser model* of multidimensional linear system; see [26, 27, 42] for its commutative counterpart.

In this paper we do not consider system evolutions (i.e., equations (2.5)). We will use the terminology  *$N$ -dimensional Givone–Roesser operator node* (for brevity, *GR-node*) for the collection of data

$$\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{j=1}^N \mathbb{C}^{r_j}, \mathbb{C}^q, \mathbb{C}^p). \quad (2.6)$$

Sometimes instead of spaces  $\mathbb{C}^r$ ,  $\mathbb{C}^{r_j}$  ( $j = 1, \dots, N$ ),  $\mathbb{C}^q$  and  $\mathbb{C}^p$  we shall consider abstract finite-dimensional linear spaces  $\mathcal{X}$  (the *state space*),  $\mathcal{X}_j$  ( $j = 1, \dots, N$ ),  $\mathcal{U}$  (the *input space*) and  $\mathcal{Y}$  (the *output space*), respectively, and a node

$$\alpha = (N; A, B, C, D; \mathcal{X} = \bigoplus_{j=1}^N \mathcal{X}_j, \mathcal{U}, \mathcal{Y}),$$

where  $A, B, C, D$  are linear operators in the corresponding pairs of spaces. The *non-commutative transfer function of a GR-node*  $\alpha$  is a rational FPS

$$T_\alpha^{\text{nc}}(z) = D + C(I_r - \Delta(z)A)^{-1} \Delta(z)B. \quad (2.7)$$

Minimal GR-realizations (2.6) of non-commutative rational FPSs, that is, representations of them in the form (2.7), with minimal possible  $r_k$  for  $k = 1, \dots, N$  were studied in [17, 16, 39, 11]. For  $k = 1, \dots, N$ , the  *$k$ -th observability matrix* is

$$\mathcal{O}_k = \text{col}(C_k, C_1 A_{1k}, \dots, C_N A_{Nk}, C_1 A_{11} A_{1k}, \dots, C_1 A_{1N} A_{Nk}, \dots)$$

and the  $k$ -th *controllability matrix* is

$$\mathcal{C}_k = \text{row}(B_k, A_{k1}B_1, \dots, A_{kN}B_N, A_{k1}A_{11}B_1, \dots, A_{kN}A_{N1}B_1, \dots)$$

(note that these are infinite block matrices). A GR-node  $\alpha$  is called *observable* (resp., *controllable*) if  $\text{rank } \mathcal{O}_k = r_k$  (resp.,  $\text{rank } \mathcal{C}_k = r_k$ ) for  $k = 1, \dots, N$ . A GR-node  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{j=1}^N \mathbb{C}^{r_j}, \mathbb{C}^q, \mathbb{C}^p)$  is observable if and only if its *adjoint GR-node*  $\alpha^* = (N; A^*, C^*, B^*, D^*; \mathbb{C}^r = \bigoplus_{j=1}^N \mathbb{C}^{r_j}, \mathbb{C}^p, \mathbb{C}^q)$  is controllable. (Clearly,  $(\alpha^*)^* = \alpha$ .)

In view of the sequel, we introduce some notations. We set:

$$\begin{aligned} A^{wg\nu} &= A_{j_1 j_2} A_{j_2 j_3} \cdots A_{j_{k-1} j_k} A_{j_k \nu}, \\ (C \flat A)^{g\nu w} &= C_\nu A_{\nu j_1} A_{j_1 j_2} \cdots A_{j_{k-1} j_k}, \\ (A \sharp B)^{wg\nu} &= A_{j_1 j_2} \cdots A_{j_{k-1} j_k} A_{j_k \nu} B_\nu, \\ (C \flat A \sharp B)^{g^\mu w g^\nu} &= C_\mu A_{\mu j_1} A_{j_1 j_2} \cdots A_{j_{k-1} j_k} A_{j_k \nu} B_\nu, \end{aligned}$$

where  $w = g_{j_1} \cdots g_{j_k} \in \mathcal{F}_N$  and  $\mu, \nu \in \{1, \dots, N\}$ . We also define:

$$\begin{aligned} A^{g\nu} &= A^\emptyset = I_\gamma \\ (C \flat A)^{g\nu} &= C_\nu, \\ (A \sharp B)^{g\nu} &= B_\nu, \\ (C \flat A \sharp B)^{g\nu} &= C_\nu B_\nu, \\ (C \flat A \sharp B)^{g^\mu g^\nu} &= C_\mu A_{\mu \nu} B_\nu, \end{aligned}$$

and hence, with the lexicographic order of words in  $\mathcal{F}_N$ ,

$$\mathcal{O}_k = \text{col}_{w \in \mathcal{F}_N} (C \flat A)^{wgk} \quad \text{and} \quad \mathcal{C}_k = \text{row}_{w \in \mathcal{F}_N} (A \sharp B)^{gk w^T},$$

and the coefficients of the FPS  $T_\alpha^{\text{nc}}$  (defined by (2.7)) are given by

$$(T_\alpha^{\text{nc}})_\emptyset = D, \quad (T_\alpha^{\text{nc}})_w = (C \flat A \sharp B)^w \quad \text{for} \quad w = g_{j_1} \cdots g_{j_n} \in \mathcal{F}_N.$$

The  $k$ -th *Hankel matrix associated with a FPS*  $f$  is defined in [39] (see also [11]) as

$$(\mathcal{H}_{f,k})_{w, w' g_k} = f_{w g_k w'^T} \quad \text{with} \quad w, w' \in \mathcal{F}_N,$$

that is, the rows of  $\mathcal{H}_{f,k}$  are indexed by all the words of  $\mathcal{F}_N$  and the columns of  $\mathcal{H}_{f,k}$  are indexed by all the words of  $\mathcal{F}_N$  ending by  $g_k$ , provided the lexicographic order is used. If a GR-node  $\alpha$  defines a realization of  $f$ , that is,  $f = T_\alpha^{\text{nc}}$ , then

$$(\mathcal{H}_{f,k})_{w, w' g_k} = (C \flat A \sharp B)^{w g_k w'^T} = (C \flat A)^{w g_k} (A \sharp B)^{g_k w'^T},$$

i.e.,  $\mathcal{H}_{f,k} = \mathcal{O}_k \mathcal{C}_k$ . Hence, the node  $\alpha$  is minimal if and only if  $\alpha$  is both observable and controllable, i.e.,

$$\gamma_k := \text{rank } \mathcal{H}_{f,k} = r_k \quad \text{for all} \quad k \in \{1, \dots, N\}.$$

This last set of conditions is an analogue of the above mentioned result of Fliess on minimal recognizable representations of rational formal power series. Every non-commutative rational FPS has a minimal GR-realization.

Finally, we note (see [17, 39]) that two minimal GR-realizations of a given rational FPS are *similar*: if  $\alpha^{(i)} = (N; A^{(i)}, B^{(i)}, C^{(i)}, D; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q, \mathbb{C}^p)$  ( $i=1,2$ ) are minimal GR-nodes such that  $T_{\alpha^{(1)}}^{\text{nc}} = T_{\alpha^{(2)}}^{\text{nc}}$  then there exists a block diagonal invertible matrix  $T = \text{diag}(T_1, \dots, T_N)$  (with  $T_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ) such that

$$A^{(1)} = T^{-1}A^{(2)}T, \quad B^{(1)} = T^{-1}B^{(2)}, \quad C^{(1)} = C^{(2)}T. \quad (2.8)$$

Of course, the converse is also true, moreover, any two similar (not necessarily minimal) GR-nodes have the same transfer functions.

Now we turn to the discussion on substitutions of matrices for indeterminates in formal power series. Many properties of non-commutative FPSs or non-commutative polynomials are described in terms of matrix substitutions, e.g., matrix-positivity of non-commutative polynomials (non-commutative Positivstellensatz) [29, 40, 31, 32], matrix-positivity of FPS kernels [34], matrix-convexity [21, 30]. The non-commutative Schur–Agler class, i.e., the class of FPSs with operator coefficients, which take contractive values on all  $N$ -tuples of strictly contractive operators on  $\ell^2$ , was studied in [12]<sup>1</sup>; we will show in Section 6 that in order that a FPS belongs to this class it suffices to check its contractivity on  $N$ -tuples of strictly contractive  $n \times n$  matrices, for all  $n \in \mathbb{N}$ . The notions of matrix- $J$ -unitary (in particular, matrix- $J$ -inner) and matrix-selfadjoint rational FPS, which will be introduced and studied in the present paper, are also defined in terms of substitutions of matrices (of a certain class) for indeterminates.

Let  $p(z) = \sum_{|w| \leq m} p_w z^w \in \mathbb{C}\langle z_1, \dots, z_N \rangle$ . For  $n \in \mathbb{N}$  and an  $N$ -tuple of matrices  $Z = (Z_1, \dots, Z_N) \in (\mathbb{C}^{n \times n})^N$ , set

$$p(Z) = \sum_{|w| \leq m} p_w Z^w,$$

where  $Z^w = Z_{i_1} \cdots Z_{i_{|w|}}$  for  $w = g_{i_1} \cdots g_{i_{|w|}} \in \mathcal{F}_N$ , and  $Z^\emptyset = I_n$ . Then for any rational expression for a FPS  $f \in \mathbb{C}\langle\langle z_1, \dots, z_N \rangle\rangle_{\text{rat}}$  its value at  $Z \in (\mathbb{C}^{n \times n})^N$  is well defined provided all of the inversions of polynomials  $p^{(j)} \in \mathbb{C}\langle z_1, \dots, z_N \rangle$  in this expression are well defined at  $Z$ . The latter is the case at least in some neighbourhood of  $Z = 0$ , since  $p_0^{(j)} \neq 0$ .

Now, if  $f \in \mathbb{C}^{p \times q}\langle\langle z_1, \dots, z_N \rangle\rangle_{\text{rat}}$  then the value  $f(Z)$  at some  $Z \in (\mathbb{C}^{n \times n})^N$  is well defined whenever the values of matrix entries  $(f_{ij}(Z))$  ( $i = 1, \dots, p; j = 1, \dots, q$ ) are well defined at  $Z$ . As a function of matrix entries  $(Z_k)_{ij}$  ( $k = 1, \dots, N; i, j = 1, \dots, n$ ),  $f(Z)$  is rational  $\mathbb{C}^{p \times q} \otimes \mathbb{C}^{n \times n}$ -valued function, which is holomorphic on an open and dense set in  $\mathbb{C}^{n \times n}$ . The latter set contains some neighbourhood

$$\Gamma_n(\varepsilon) := \{Z \in (\mathbb{C}^{n \times n})^N : \|Z_k\| < \varepsilon, k = 1, \dots, N\} \quad (2.9)$$

<sup>1</sup>In fact, a more general class was studied in [12], however for our purposes it is enough to consider here only the case mentioned above.

of  $Z = 0$ , where  $f(Z)$  is given by

$$f(Z) = \sum_{w \in \mathcal{F}_N} f_w \otimes Z^w.$$

The following results from [8] on matrix substitutions are used in the sequel.

**Theorem 2.1.** *Let  $f \in \mathbb{C}^{p \times q} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$ , and  $m \in \mathbb{Z}_+$  be such that*

$$\bigcap_{w \in \mathcal{F}_N: |w| \leq m} \ker f_w = \bigcap_{w \in \mathcal{F}_N} \ker f_w.$$

*Then there exists  $\varepsilon > 0$  such that for every  $n \in \mathbb{N} : n \geq m^m$  (in the case  $m = 0$ , for every  $n \in \mathbb{N}$ ),*

$$\bigcap_{Z \in \Gamma_n(\varepsilon)} \ker f(Z) = \left( \bigcap_{w \in \mathcal{F}_N: |w| \leq m} \ker f_w \right) \otimes \mathbb{C}^n, \quad (2.10)$$

*and moreover, there exist  $l \in \mathbb{N} : l \leq qn$ , and  $N$ -tuples of matrices  $Z^{(1)}, \dots, Z^{(l)}$  from  $\Gamma_n(\varepsilon)$  such that*

$$\bigcap_{j=1}^l \ker f(Z^{(j)}) = \left( \bigcap_{w \in \mathcal{F}_N: |w| \leq m} \ker f_w \right) \otimes \mathbb{C}^n.$$

**Corollary 2.2.** *In conditions of Theorem 2.1, if for some  $n \in \mathbb{N} : n \geq m^m$  (in the case  $m = 0$ , for some  $n \in \mathbb{N}$ ) one has  $f(Z) = 0, \forall Z \in \Gamma_n(\varepsilon)$ , then  $f = 0$ .*

### 3. More on observability, controllability, and minimality in the non-commutative setting

In this section we prove a number of results on observable, controllable and minimal GR-nodes in the multivariable non-commutative setting, which generalize some well known statements for one-variable nodes (see [15]).

Let us introduce the  $k$ -th truncated observability matrix  $\widetilde{\mathcal{O}}_k$  and the  $k$ -th truncated controllability matrix  $\widetilde{\mathcal{C}}_k$  of a GR-node (2.6) by

$$\widetilde{\mathcal{O}}_k = \text{col}_{|w| < pr} (C \flat A)^{wg_k}, \quad \widetilde{\mathcal{C}}_k = \text{row}_{|w| < rq} (A \sharp B)^{g_k w^T},$$

with the lexicographic order of words in  $\mathcal{F}_N$ .

**Theorem 3.1.** *For each  $k \in \{1, \dots, N\}$ :  $\text{rank } \widetilde{\mathcal{O}}_k = \text{rank } \mathcal{O}_k$  and  $\text{rank } \widetilde{\mathcal{C}}_k = \text{rank } \mathcal{C}_k$ .*

*Proof.* Let us show that for every fixed  $k \in \{1, \dots, N\}$  matrices of the form  $(C \flat A)^{wg_k}$  with  $|w| \geq pr$  are representable as linear combinations of matrices  $(C \flat A)^{\widetilde{w}g_k}$  with  $|\widetilde{w}| < pr$ . First we remark that if for each fixed  $k \in \{1, \dots, N\}$  and  $j \in \mathbb{N}$  all matrices of the form  $(C \flat A)^{wg_k}$  with  $|w| = j$  are representable as linear combinations of matrices of the form  $(C \flat A)^{w'g_k}$  with  $|w'| < j$  then the same holds for matrices of the form  $(C \flat A)^{wg_k}$  with  $|w| = j+1$ . Indeed, if  $w = i_1 \cdots i_j i_{j+1}$

then there exist words  $w'_1, \dots, w'_s$  with  $|w'_1| < j, \dots, |w'_s| < j$  and  $a_1, \dots, a_s \in \mathbb{C}$  such that

$$(C\flat A)^w = \sum_{\nu=1}^s a_\nu (C\flat A)^{w'_\nu g_{i_{j+1}}}.$$

Then for every  $k \in \{1, \dots, N\}$ ,

$$\begin{aligned} (C\flat A)^{wg_k} &= (C\flat A)^w A_{i_{j+1},k} = \sum_{\nu=1}^s a_\nu (C\flat A)^{w'_\nu g_{i_{j+1}}} A_{i_{j+1},k} \\ &= \sum_{\nu: |w'_\nu| < j-1} a_\nu (C\flat A)^{w'_\nu g_{i_{j+1}}} A_{i_{j+1},k} + \sum_{\nu: |w'_\nu| = j-1} a_\nu (C\flat A)^{w'_\nu g_{i_{j+1}}} A_{i_{j+1},k} \\ &= \sum_{\nu: |w'_\nu| < j-1} a_\nu (C\flat A)^{w'_\nu g_{i_{j+1}} g_k} + \sum_{\nu: |w'_\nu| = j-1} a_\nu (C\flat A)^{w'_\nu g_{i_{j+1}} g_k}. \end{aligned}$$

Consider these two sums separately. All the terms in the first sum are of the form  $a_\nu (C\flat A)^{(w'_\nu g_{i_{j+1}})g_k}$  with  $|w'_\nu g_{i_{j+1}}| < j$ . In the second sum, by the assumption, for each matrix  $(C\flat A)^{w'_\nu g_{i_{j+1}} g_k}$  there exist words  $w''_{1\nu}, \dots, w''_{t\nu}$  of length strictly less than  $j$  and complex numbers  $b_{1\nu}, \dots, b_{t\nu}$  such that

$$(C\flat A)^{w'_\nu g_{i_{j+1}} g_k} = \sum_{\mu=1}^t b_{\mu\nu} (C\flat A)^{w''_{\mu\nu} g_k}.$$

Hence  $(C\flat A)^{wg_k}$  is a linear combination of matrices of the form  $(C\flat A)^{\tilde{w}g_k}$  with  $|\tilde{w}| < j$ . Reiterating this argument we obtain that any matrix of the form  $(C\flat A)^{wg_k}$  with  $|w| \geq j$  and fixed  $k \in \{1, \dots, N\}$  can be represented as a linear combination of matrices of the form  $(C\flat A)^{\tilde{w}g_k}$  with  $|\tilde{w}| < j$ . In particular,

$$\text{rank col}_{|w| < j} (C\flat A)^{wg_k} = \text{rank } \mathcal{O}_k, \quad k = 1, \dots, N. \quad (3.1)$$

Since for any  $k \in \{1, \dots, N\}$  one has  $(C\flat A)^{wg_k} \in \mathbb{C}^{p \times r_k}$  and  $\dim \mathbb{C}^{p \times r_k} = pr_k$ , we obtain that for some  $j \leq pr$ , and moreover for  $j = pr$  (3.1) is true, i.e.,  $\text{rank } \widetilde{\mathcal{O}}_k = \text{rank } \mathcal{O}_k$ .

The second equality is proved analogously.  $\square$

*Remark 3.2.* The sizes of the truncated matrices  $\widetilde{\mathcal{O}}_k$  and  $\widetilde{\mathcal{C}}_k$  depend only on the sizes of matrices  $A, B$  and  $C$ , and do not depend on these matrices themselves. Our estimate for the size of  $\widetilde{\mathcal{O}}_k$  is rough, and one could probably improve it. For our present purposes, only the finiteness of the matrices  $\widetilde{\mathcal{O}}_k$  and  $\widetilde{\mathcal{C}}_k$  is important, and not their actual sizes.

**Corollary 3.3.** *A GR-node (2.6) is observable (resp., controllable) if and only if for every  $k \in \{1, \dots, N\}$ :*

$$\text{rank } \widetilde{\mathcal{O}}_k = r_k \quad (\text{resp.}, \text{rank } \widetilde{\mathcal{C}}_k = r_k),$$

*or equivalently, the matrix  $\mathcal{O}_k$  (resp.,  $\mathcal{C}_k$ ) is left (resp., right) invertible.*

*Remark 3.4.* Corollary 3.3 is comparable with Theorems 7.4 and 7.7 in [39], however we note again that the matrices  $\widetilde{\mathcal{O}}_k$  and  $\widetilde{\mathcal{C}}_k$  here are finite.

**Theorem 3.5.** *Let  $\alpha^{(i)} = (N; A^{(i)}, B^{(i)}, C^{(i)}, D, \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q, \mathbb{C}^p)$ ,  $i = 1, 2$ , be minimal GR-nodes with the same transfer function. Then they are similar, the similarity transform is unique and given by  $T = \text{diag}(T_1, \dots, T_N)$  where*

$$T_k = \left( \widetilde{\mathcal{O}}_k^{(2)} \right)^+ \widetilde{\mathcal{O}}_k^{(1)} = \widetilde{\mathcal{C}}_k^{(2)} \left( \widetilde{\mathcal{C}}_k^{(1)} \right)^\dagger \quad (3.2)$$

(here “+” denotes a left inverse, while “†” denotes a right inverse).

*Proof.* We already mentioned in Section 2 that two minimal nodes with the same transfer function are similar. Let  $T' = \text{diag}(T'_1, \dots, T'_N)$  and  $T'' = \text{diag}(T''_1, \dots, T''_N)$  be two similarity transforms. Let  $x \in \mathbb{C}^{\gamma_k}$ . Then, for every  $w \in \mathcal{F}_N$ ,

$$(C^{(2)} \mathfrak{b} A^{(2)})^{wg_k} (T''_k - T'_k) x = (C^{(1)} \mathfrak{b} A^{(1)})^{wg_k} x - (C^{(1)} \mathfrak{b} A^{(1)})^{wg_k} x = 0.$$

Since  $x$  is arbitrary, from the observability of  $\alpha^{(2)}$  we get  $T'_k = T''_k$  for  $k = 1, \dots, N$ , hence the similarity transform is unique. Comparing the coefficients in the two FPS representations of the transfer function, we obtain

$$(C^{(1)} \mathfrak{b} A^{(1)} \# B^{(1)})^w = (C^{(2)} \mathfrak{b} A^{(2)} \# B^{(2)})^w$$

for all of  $w \in \mathcal{F}_N \setminus \{\emptyset\}$ , and therefore

$$\widetilde{\mathcal{O}}_k^{(1)} \widetilde{\mathcal{C}}_k^{(1)} = \widetilde{\mathcal{O}}_k^{(2)} \widetilde{\mathcal{C}}_k^{(2)}, \quad k = 1, \dots, N.$$

Thus we obtain

$$\left( \widetilde{\mathcal{O}}_k^{(2)} \right)^+ \widetilde{\mathcal{O}}_k^{(1)} = \widetilde{\mathcal{C}}_k^{(2)} \left( \widetilde{\mathcal{C}}_k^{(1)} \right)^\dagger, \quad k = 1, \dots, N.$$

Denote the operators which appear in these equalities by  $T_k$ ,  $k = 1, \dots, N$ . A direct computation shows that  $T_k$  are invertible with

$$T_k^{-1} = \left( \widetilde{\mathcal{O}}_k^{(1)} \right)^+ \widetilde{\mathcal{O}}_k^{(2)} = \widetilde{\mathcal{C}}_k^{(1)} \left( \widetilde{\mathcal{C}}_k^{(2)} \right)^\dagger.$$

Let us verify that  $T = \text{diag}(T_1, \dots, T_N) \in \mathbb{C}^{\gamma \times \gamma}$  is a similarity transform between  $\alpha^{(1)}$  and  $\alpha^{(2)}$ . It follows from the controllability of  $\alpha^{(1)}$  that for arbitrary  $k \in \{1, \dots, N\}$  and  $x \in \mathbb{C}^{\gamma_k}$  there exist words  $w_j \in \mathcal{F}_N$ , with  $|w_j| < \gamma q$ , scalars  $a_j \in \mathbb{C}$  and vectors  $u_j \in \mathbb{C}^q$ ,  $j = 1, \dots, s$ , such that

$$x = \sum_{\nu=1}^s a_\nu (A^{(1)} \# B^{(1)})^{g_k w_\nu^T} u_\nu.$$



Then

$$\begin{aligned} T_k x &= \left( \widetilde{\mathcal{O}}_k^{(2)} \right)^+ \widetilde{\mathcal{O}}_k^{(1)} x = \sum_{\nu=1}^s a_\nu \left( \widetilde{\mathcal{O}}_k^{(2)} \right)^+ \widetilde{\mathcal{O}}_k^{(1)} (A^{(1)} \# B^{(1)})^{g_k w_\nu^T} u_\nu \\ &= \sum_{\nu=1}^s a_\nu \left( \widetilde{\mathcal{O}}_k^{(2)} \right)^+ \widetilde{\mathcal{O}}_k^{(2)} (A^{(2)} \# B^{(2)})^{g_k w_\nu^T} u_\nu = \sum_{\nu=1}^s a_\nu (A^{(2)} \# B^{(2)})^{g_k w_\nu^T} u_\nu. \end{aligned}$$

This explicit formula implies the set of equalities

$$T_k B_k^{(1)} = B_k^{(2)}, \quad T_k A_{kj}^{(1)} = A_{kj}^{(2)} T_j, \quad C_k^{(1)} = C_k^{(2)} T_k, \quad k, j = 1, \dots, N,$$

which is equivalent to (2.8).  $\square$

*Remark 3.6.* Theorem 3.5 is comparable with Theorem 7.9 in [39]. However, we establish in Theorem 3.5 the uniqueness and an explicit formula for the similarity transform  $T$ .

Using Theorem 2.1, we will prove now the following criteria of observability, controllability, and minimality for GR-nodes analogous to the ones proven in [8, Theorem 3.3] for recognizable FPS representations.

**Theorem 3.7.** *A GR node  $\alpha$  of the form (2.6) is observable (resp., controllable) if and only if for every  $k \in \{1, \dots, N\}$  and  $n \in \mathbb{N} : n \geq (pr - 1)^{pr-1}$  (resp.,  $n \geq (rq - 1)^{rq-1}$ ), which means in the case of  $pr = 1$  (resp.,  $rq = 1$ ): “for every  $n \in \mathbb{N}$ ”,*

$$\bigcap_{Z \in \Gamma_n(\varepsilon)} \ker \varphi_k(Z) = 0 \quad (3.3)$$

$$\left( \text{resp., } \bigvee_{Z \in \Gamma_n(\varepsilon)} \text{ran } \psi_k(Z) = \mathbb{C}^{r_k} \otimes \mathbb{C}^n \right), \quad (3.4)$$

where the rational FPSs  $\varphi_k$  and  $\psi_k$  are defined by

$$\varphi_k(z) = C(I_r - \Delta(z)A)^{-1} \Big|_{\mathbb{C}^{r_k}}, \quad (3.5)$$

$$\psi_k(z) = P_k(I_r - A\Delta(z))^{-1}B, \quad (3.6)$$

with  $P_k$  standing for the orthogonal projection onto  $\mathbb{C}^{r_k}$  (which is naturally identified here with the subspace in  $\mathbb{C}^r$ ), the symbol “ $\bigvee$ ” means linear span,  $\varepsilon = \|A\|^{-1}$  ( $\varepsilon > 0$  is arbitrary in the case  $A = 0$ ), and  $\Gamma_n(\varepsilon)$  is defined by (2.9). This GR-node is minimal if both of conditions (3.3) and (3.4) are fulfilled.

*Proof.* First, let us remark that for all  $k = 1, \dots, N$  the functions  $\varphi_k$  and  $\psi_k$  are well defined in  $\Gamma_n(\varepsilon)$ , and holomorphic as functions of matrix entries  $(Z_j)_{\mu\nu}$ ,  $j = 1, \dots, N$ ,  $\mu, \nu = 1, \dots, n$ . Second, Theorem 3.1 implies that in Theorem 2.1 applied to  $\varphi_k$  one can choose  $m = pr - 1$ , and then from (2.10) obtain that observability for a GR-node  $\alpha$  is equivalent to condition (3.3). Since  $\alpha$  is controllable if and only if  $\alpha^*$  is observable, controllability for  $\alpha$  is equivalent to condition (3.4). Since minimality for a GR-node  $\alpha$  is equivalent to controllability and observability together, it is in turn equivalent to conditions (3.3) and (3.4) together.  $\square$

Let  $\alpha' = (N; A', B', C', D'; \mathbb{C}^{r'} = \bigoplus_{j=1}^N \mathbb{C}^{r'_j}, \mathbb{C}^s, \mathbb{C}^p)$  and  $\alpha'' = (N; A'', B'', C'', D''; \mathbb{C}^{r''} = \bigoplus_{j=1}^N \mathbb{C}^{r''_j}, \mathbb{C}^q, \mathbb{C}^s)$  be GR-nodes. For  $k, j = 1, \dots, N$  set  $r_j = r'_j + r''_j$ , and

$$\begin{aligned} A_{kj} &= \begin{pmatrix} A'_{kj} & B'_k C''_j \\ 0 & A''_{kj} \end{pmatrix} \in \mathbb{C}^{r_k \times r_j}, & B_k &= \begin{pmatrix} B'_k D'' \\ B''_k \end{pmatrix} \in \mathbb{C}^{r_k \times q}, \\ C_j &= (C'_j \quad D' C''_j) \in \mathbb{C}^{p \times r_j}, & D &= D' D'' \in \mathbb{C}^{p \times q}. \end{aligned} \quad (3.7)$$

Then  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{j=1}^N \mathbb{C}^{r_j}, \mathbb{C}^q, \mathbb{C}^p)$  will be called the *product of GR-nodes*  $\alpha'$  and  $\alpha''$  and denoted by  $\alpha = \alpha' \alpha''$ . A straightforward calculation shows that

$$T_\alpha^{\text{nc}} = T_{\alpha'}^{\text{nc}} T_{\alpha''}^{\text{nc}}.$$

Consider a GR-node

$$\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{j=1}^N \mathbb{C}^{r_j}, \mathbb{C}^q) := (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{j=1}^N \mathbb{C}^{r_j}, \mathbb{C}^q, \mathbb{C}^q) \quad (3.8)$$

with invertible operator  $D$ . Then

$$\alpha^\times = (N; A^\times, B^\times, C^\times, D^\times; \mathbb{C}^r = \bigoplus_{j=1}^N \mathbb{C}^{r_j}, \mathbb{C}^q),$$

with

$$A^\times = A - BD^{-1}C, \quad B^\times = BD^{-1}, \quad C^\times = -D^{-1}C, \quad D^\times = D^{-1}, \quad (3.9)$$

will be called the *associated GR-node*, and  $A^\times$  the *associated main operator*, of  $\alpha$ . It is easy to see that, as well as in the one-variable case,  $(T_\alpha^{\text{nc}})^{-1} = T_{\alpha^\times}^{\text{nc}}$ . Moreover,  $(\alpha^\times)^\times = \alpha$  (in particular,  $(A^\times)^\times = A$ ), and  $(\alpha' \alpha'')^\times = \alpha''^\times \alpha'^\times$  up to the natural identification of  $\mathbb{C}^{r'_j} \oplus \mathbb{C}^{r''_j}$  with  $\mathbb{C}^{r'_j} \oplus \mathbb{C}^{r''_j}$ ,  $j = 1, \dots, N$ , which is a similarity transform.

**Theorem 3.8.** *A GR-node (3.8) with invertible operator  $D$  is minimal if and only if its associated GR-node  $\alpha^\times$  is minimal.*

*Proof.* Let a GR-node  $\alpha$  of the form (3.8) with invertible operator  $D$  be minimal, and  $x \in \ker \mathcal{O}_k^\times$  for some  $k \in \{1, \dots, N\}$ , where  $\mathcal{O}_k^\times$  is the  $k$ -th observability matrix for the GR-node  $\alpha^\times$ . Then  $x \in \ker (C^\times \flat A^\times)^{wg_k}$  for every  $w \in \mathcal{F}_N$ . Let us show that  $x \in \ker \mathcal{O}_k = \bigcap_{w \in \mathcal{F}_N} \ker (C \flat A)^{wg_k}$ , i.e.,  $x = 0$ .

For  $w = \emptyset$ ,  $C_k^\times x = 0$  means  $-D^{-1}C_k x = 0$  (see (3.9)), which is equivalent to  $C_k x = 0$ . For  $|w| > 0$ ,  $w = g_{i_1} \cdots g_{i_{|w|}}$ ,

$$\begin{aligned} (C \flat A)^{wg_k} &= C_{i_1} A_{i_1 i_2} \cdots A_{i_{|w|} k} \\ &= -DC_{i_1}^\times (A_{i_1 i_2}^\times + B_{i_1} D^{-1} C_{i_2}) \cdots (A_{i_{|w|} k}^\times + B_{i_{|w|}} D^{-1} C_k) \\ &= L_0 C_k^\times + \sum_{j=1}^{|w|} L_j C_{i_j}^\times A_{i_j i_{j+1}}^\times \cdots A_{i_{|w|} k}^\times, \end{aligned}$$

with some matrices  $L_j \in \mathbb{C}^{q \times q}$ ,  $j = 0, 1, \dots, |w|$ . Thus,  $x \in \ker(CbA)^{wg_k}$  for every  $w \in \mathcal{F}_N$ , i.e.,  $x = 0$ , which means that  $\alpha^\times$  is observable.

Since  $\alpha$  is controllable if and only if  $\alpha^*$  is observable (see Section 2), and  $D^*$  is invertible whenever  $D$  is invertible, the same is true for  $\alpha^\times$  and  $(\alpha^\times)^* = (\alpha^*)^\times$ . Thus, the controllability of  $\alpha^\times$  follows from the controllability of  $\alpha$ . Finally, the minimality of  $\alpha^\times$  follows from the minimality of  $\alpha$ . Since  $(\alpha^\times)^\times = \alpha$ , the minimality of  $\alpha$  follows from the minimality of  $\alpha^\times$ .  $\square$

Suppose that for a GR-node (3.8), projections  $\Pi_k$  on  $\mathbb{C}^{r_k}$  are defined such that

$$A_{kj} \ker \Pi_j \subset \ker \Pi_k, \quad (A^\times)_{kj} \text{ran } \Pi_j \subset \text{ran } \Pi_k, \quad k, j = 1, \dots, N.$$

We do not assume that  $\Pi_k$  are orthogonal. We shall call  $\Pi_k$  a  $k$ -th *supporting projection* for  $\alpha$ . Clearly, the map  $\Pi = \text{diag}(\Pi_1, \dots, \Pi_N) : \mathbb{C}^r \rightarrow \mathbb{C}^r$  satisfies

$$A \ker \Pi \subset \ker \Pi, \quad A^\times \text{ran } \Pi \subset \text{ran } \Pi,$$

i.e., it is a *supporting projection* for the one-variable node  $(1; A, B, C, D; \mathbb{C}^r, \mathbb{C}^q)$  in the sense of [15]. If  $\Pi$  is a supporting projection for  $\alpha$ , then  $I_r - \Pi$  is a supporting projection for  $\alpha^\times$ .

The following theorem and corollary are analogous to, and are proved in the same way as Theorem 1.1 and its corollary in [15, pp. 7–9] (see also [43, Theorem 2.1]).

**Theorem 3.9.** *Let (3.8) be a GR-node with invertible operator  $D$ . Let  $\Pi_k$  be a projection on  $\mathbb{C}^{r_k}$ , and let*

$$A = \begin{pmatrix} A_{kj}^{(11)} & A_{kj}^{(12)} \\ A_{kj}^{(21)} & A_{kj}^{(22)} \end{pmatrix}, \quad B_j = \begin{pmatrix} B_j^{(1)} \\ B_j^{(2)} \end{pmatrix}, \quad C_k = \begin{pmatrix} C_k^{(1)} & C_k^{(2)} \end{pmatrix}$$

be the block matrix representations of the operators  $A_{kj}, B_j$  and  $C_k$  with respect to the decompositions  $\mathbb{C}^{r_k} = \ker \Pi_k \dot{+} \text{ran } \Pi_k$ , for  $k, j \in \{1, \dots, N\}$ . Assume that  $D = D' D''$ , where  $D'$  and  $D''$  are invertible operators on  $\mathbb{C}^q$ , and set

$$\alpha' = (N; A^{(11)}, B^{(1)}(D'')^{-1}, C^{(1)}, D'; \ker \Pi = \bigoplus_{k=1}^N \ker \Pi_k, \mathbb{C}^q),$$

$$\alpha'' = (N; A^{(22)}, B^{(2)}, (D')^{-1} C^{(2)}, D''; \text{ran } \Pi = \bigoplus_{k=1}^N \text{ran } \Pi_k, \mathbb{C}^q).$$

Then  $\alpha = \alpha' \alpha''$  (up to a similarity which maps  $\mathbb{C}^{r_k} = \ker \Pi_k \dot{+} \text{ran } \Pi_k$  onto  $\mathbb{C}^{\dim(\ker \Pi_k)} \oplus \mathbb{C}^{\dim(\text{ran } \Pi_k)}$  ( $k = 1, \dots, N$ ) such that  $\ker \Pi_k \dot{+} \{0\}$  is mapped onto  $\mathbb{C}^{\dim(\ker \Pi_k)} \oplus \{0\}$  and  $\{0\} \dot{+} \text{ran } \Pi_k$  is mapped onto  $\{0\} \oplus \mathbb{C}^{\dim(\text{ran } \Pi_k)}$ ) if and only if  $\Pi$  is a supporting projection for  $\alpha$ .

**Corollary 3.10.** *In the assumptions of Theorem 3.9,*

$$T_\alpha^{\text{nc}} = F' F'',$$

where

$$\begin{aligned} F'(z) &= D' + C(I_r - \Delta(z)A)^{-1}(I_r - \Pi)\Delta(z)B(D'')^{-1}, \\ F''(z) &= D'' + (D')^{-1}C\Pi(I_r - \Delta(z)A)^{-1}\Delta(z)B. \end{aligned}$$

We assume now that the external operator of the GR-node (3.8) is equal to  $D = I_q$  and that we also take  $D' = D'' = I_q$ . Then, the GR-nodes  $\alpha'$  and  $\alpha''$  of Theorem 3.9 are called *projections of  $\alpha$  with respect to the supporting projections  $I_r - \Pi$  and  $\Pi$* , respectively, and we use the notations

$$\begin{aligned} \alpha' &= \text{pr}_{I_r - \Pi}(\alpha) = \left( N; A^{(11)}, B^{(1)}, C^{(1)}, D'; \ker \Pi = \bigoplus_{k=1}^N \ker \Pi_k, \mathbb{C}^q \right), \\ \alpha'' &= \text{pr}_{\Pi}(\alpha) = \left( N; A^{(22)}, B^{(2)}, C^{(2)}, D''; \text{ran } \Pi = \bigoplus_{k=1}^N \text{ran } \Pi_k, \mathbb{C}^q \right). \end{aligned}$$

Let  $F', F''$  and  $F$  be rational FPSs with coefficients in  $\mathbb{C}^{q \times q}$  such that

$$F = F'F''. \quad (3.10)$$

The factorization (3.10) will be said to be *minimal* if whenever  $\alpha'$  and  $\alpha''$  are minimal GR-realizations of  $F'$  and  $F''$ , respectively,  $\alpha'\alpha''$  is a minimal GR-realization of  $F$ .

In the sequel, we will use the notation

$$\alpha = \left( N; A, B, C, D; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k \times \gamma_k}, \mathbb{C}^q \right) \quad (3.11)$$

for a minimal GR-realization (i.e.,  $r_k = \gamma_k$  for  $k = 1, \dots, N$ ) of a rational FPS  $F$  in the case when  $p = q$ .

The following theorem is the multivariable non-commutative version of [15, Theorem 4.8]. It gives a complete description of all minimal factorizations in terms of supporting projections.

**Theorem 3.11.** *Let  $F$  be a rational FPS with a minimal GR-realization (3.11). Then the following statements hold:*

- (i): *if  $\Pi = \text{diag}(\Pi_1, \dots, \Pi_N)$  is a supporting projection for  $\alpha$ , then  $F'$  is the transfer function of  $\text{pr}_{I_\gamma - \Pi}(\alpha)$ ,  $F''$  is the transfer function of  $\text{pr}_{\Pi}(\alpha)$ , and  $F = F'F''$  is a minimal factorization of  $F$ ;*
- (ii): *if  $F = F'F''$  is a minimal factorization of  $F$ , then there exists a uniquely defined supporting projection  $\Pi = \text{diag}(\Pi_1, \dots, \Pi_N)$  for the GR-node  $\alpha$  such that  $F'$  and  $F''$  are the transfer functions of  $\text{pr}_{I_\gamma - \Pi}(\alpha)$  and  $\text{pr}_{\Pi}(\alpha)$ , respectively.*

*Proof.* (i). Let  $\Pi$  be a supporting projection for  $\alpha$ . Then, by Theorem 3.9,

$$\alpha = \text{pr}_{I_\gamma - \Pi}(\alpha)\text{pr}_{\Pi}(\alpha).$$

By the assumption,  $\alpha$  is minimal. We now show that the GR-nodes  $\alpha' = \text{pr}_{I_\gamma - \Pi}(\alpha)$  and  $\alpha'' = \text{pr}_\Pi(\alpha)$  are also minimal. To this end, let  $x \in \text{ran } \Pi_k$ . Then

$$\left(C^{(2)} \flat A^{(22)}\right)^{wg_k} x = (C \flat A)^{wg_k} \Pi_k x = (C \flat A)^{wg_k} x.$$

Thus, if  $\mathcal{O}_k''$  denotes the  $k$ -th observability matrix of  $\alpha''$ , then  $x \in \ker \mathcal{O}_k''$  implies  $x \in \ker \mathcal{O}_k$ , and the observability of  $\alpha$  implies that  $\alpha''$  is also observable. Since

$$\left(A^{(22)} \sharp B^{(2)}\right)^{g_k w^T} = \Pi_k (A \sharp B)^{g_k w^T},$$

one has  $\mathcal{C}_k'' = \Pi_k \mathcal{C}_k$ , where  $\mathcal{C}_k''$  is the  $k$ -th controllability matrix of  $\alpha''$ . Thus, the controllability of  $\alpha$  implies the controllability of  $\alpha''$ . Hence, we have proved the minimality of  $\alpha''$ . Note that we have used that  $\ker \Pi = \text{ran}(I_\gamma - \Pi)$  is  $A$ -invariant. Since  $\text{ran } \Pi = \ker(I_\gamma - \Pi)$  is  $A^\times$ -invariant, by Theorem 3.8  $\alpha^\times$  is minimal. Using

$$\alpha^\times = (\alpha' \alpha'')^\times = (\alpha'')^\times (\alpha')^\times,$$

we prove the minimality of  $(\alpha')^\times$  in the same way as that of  $\alpha''$ . Applying once again Theorem 3.8, we obtain the minimality of  $\alpha'$ . The dimensions of the state spaces of the minimal GR-nodes  $\alpha', \alpha''$  and  $\alpha$  are related by

$$\gamma_k = \gamma_k' + \gamma_k'', \quad k = 1, \dots, N.$$

Therefore, given any minimal GR-realizations  $\beta'$  and  $\beta''$  of  $F'$  and  $F''$ , respectively, the same equalities hold for the state space dimensions of  $\beta', \beta''$  and  $\beta$ . Thus,  $\beta' \beta''$  is a minimal GR-node, and the factorization  $F = F' F''$  is minimal.

(ii). Assume that the factorization  $F = F' F''$  is minimal. Let  $\beta'$  and  $\beta''$  be minimal GR-realizations of  $F'$  and  $F''$  with  $k$ -th state space dimensions equal to  $\gamma_k'$  and  $\gamma_k''$ , respectively ( $k = 1, \dots, N$ ). Then  $\beta' \beta''$  is a minimal GR-realization of  $F$  and its  $k$ -th state space dimension is equal to  $\gamma_k = \gamma_k' + \gamma_k''$  ( $k = 1, \dots, N$ ). Hence  $\beta' \beta''$  is similar to  $\alpha$ . We denote the corresponding GR-node similarity by  $T = \text{diag}(T_1, \dots, T_N)$ , where

$$T_k : \mathbb{C}^{\gamma_k'} \oplus \mathbb{C}^{\gamma_k''} \rightarrow \mathbb{C}^{\gamma_k}, \quad k = 1, \dots, N,$$

is the canonical isomorphism. Let  $\Pi_k$  be the projection of  $\mathbb{C}^{\gamma_k}$  along  $T_k \mathbb{C}^{\gamma_k'}$  onto  $T_k \mathbb{C}^{\gamma_k''}$ ,  $k = 1, \dots, N$ , and set  $\Pi = \text{diag}(\Pi_1, \dots, \Pi_N)$ . Then  $\Pi$  is a supporting projection for  $\alpha$ . Moreover  $\text{pr}_{I_\gamma - \Pi}(\alpha)$  is similar to  $\beta'$ , and  $\text{pr}_\Pi(\alpha)$  is similar to  $\beta''$ . The uniqueness of  $\Pi$  is proved in the same way as in [15, Theorem 4.8]. The uniqueness of the GR-node similarity follows from Theorem 3.5.  $\square$

#### 4. Matrix- $J$ -unitary formal power series: A multivariable non-commutative analogue of the line case

In this section we study a multivariable non-commutative analogue of rational  $q \times q$  matrix-valued functions which are  $J$ -unitary on the imaginary line  $i\mathbb{R}$  of the complex plane  $\mathbb{C}$ .

#### 4.1. Minimal Givone–Roesser realizations and the Lyapunov equation

Denote by  $\mathbb{H}^{n \times n}$  the set of Hermitian  $n \times n$  matrices. Then  $(i\mathbb{H}^{n \times n})^N$  will denote the set of  $N$ -tuples of skew-Hermitian matrices. In our paper, the set

$$\mathcal{J}_N = \coprod_{n \in \mathbb{N}} (i\mathbb{H}^{n \times n})^N,$$

where “ $\coprod$ ” stands for a disjoint union, will be a counterpart of the imaginary line  $i\mathbb{R}$ .

Let  $J \in \mathbb{C}^{q \times q}$  be a signature matrix. We will call a rational FPS  $F \in \mathbb{C}^{q \times q} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$  *matrix- $J$ -unitary on  $\mathcal{J}_N$*  if for every  $n \in \mathbb{N}$ ,

$$F(Z)(J \otimes I_n)F(Z)^* = J \otimes I_n \quad (4.1)$$

at all points  $Z \in (i\mathbb{H}^{n \times n})^N$  where it is defined. For a fixed  $n \in \mathbb{N}$ ,  $F(Z)$  as a function of matrix entries is rational and holomorphic on some open neighbourhood  $\Gamma_n(\varepsilon)$  of  $Z = 0$ , e.g., of the form (2.9), and  $\Gamma_n(\varepsilon) \cap (i\mathbb{H}^{n \times n})^N$  is a uniqueness set in  $(\mathbb{C}^{n \times n})^N$  (see [45] for the uniqueness theorem in several complex variables). Thus, (4.1) implies that

$$F(Z)(J \otimes I_n)F(-Z^*)^* = J \otimes I_n \quad (4.2)$$

at all points  $Z \in (\mathbb{C}^{n \times n})^N$  where  $F(Z)$  is holomorphic and invertible (the set of such points is open and dense, since  $\det F(Z) \neq 0$ ).

The following theorem is a counterpart of Theorem 2.1 in [7].

**Theorem 4.1.** *Let  $F$  be a rational FPS with a minimal GR-realization (3.11). Then  $F$  is matrix- $J$ -unitary on  $\mathcal{J}_N$  if and only if the following conditions are fulfilled:*

- a)  $D$  is  $J$ -unitary, i.e.,  $DJD^* = J$ ;
- b) there exists an invertible Hermitian solution  $H = \text{diag}(H_1, \dots, H_N)$ , with  $H_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ , of the Lyapunov equation

$$A^*H + HA = -C^*JC, \quad (4.3)$$

and

$$B = -H^{-1}C^*JD. \quad (4.4)$$

The property b) is equivalent to

- b') there exists an invertible Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$ , with  $H_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ , such that

$$H^{-1}A^* + AH^{-1} = -BJB^*, \quad (4.5)$$

and

$$C = -DJB^*H. \quad (4.6)$$

*Proof.* Let  $F$  be matrix- $J$ -unitary. Then  $F$  is holomorphic at the point  $Z = 0$  in  $\mathbb{C}^N$ , hence  $D = F(0)$  is  $J$ -unitary (in particular, invertible). Equality (4.2) may be rewritten as

$$F(Z)^{-1} = (J \otimes I_n)F(-Z^*)^*(J \otimes I_n). \quad (4.7)$$

Since (4.7) holds for all  $n \in \mathbb{N}$ , it follows from Corollary 2.2 that the FPSs corresponding to the left and the right sides of equality (4.7) coincide. Due to Theorem 3.8,  $\alpha^\times = (N; A^\times, B^\times, C^\times, D^\times; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q)$  with  $A^\times, B^\times, C^\times, D^\times$  given by (3.9) is a minimal GR-realization of  $F^{-1}$ . Due to (4.7), another minimal GR-realization of  $F^{-1}$  is  $\tilde{\alpha} = (N; \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q)$ , where

$$\tilde{A} = -A^*, \quad \tilde{B} = C^*J, \quad \tilde{C} = -JB^*, \quad \tilde{D} = JD^*J.$$

By Theorem 3.5, there exists unique similarity transform  $T = \text{diag}(T_1, \dots, T_N)$  which relates  $\alpha^\times$  and  $\tilde{\alpha}$ , where  $T_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$  are invertible for  $k = 1, \dots, N$ , and

$$T(A - BD^{-1}C) = -A^*T, \quad TBD^{-1} = C^*J, \quad D^{-1}C = JB^*T. \quad (4.8)$$

Note that the relation  $D^{-1} = JD^*J$ , which means  $J$ -unitarity of  $D$ , has been already established above. It is easy to check that relations (4.8) are also valid for  $T^*$  in the place of  $T$ . Hence, by the uniqueness of similarity matrix,  $T = T^*$ . Setting  $H = -T$ , we obtain from (4.8) the equalities (4.3) and (4.4), as well as (4.5) and (4.6), by a straightforward calculation.

Let us prove now a slightly more general statement than the converse. Let  $\alpha$  be a (not necessarily minimal) GR-realization of  $F$  of the form (3.8), where  $D$  is  $J$ -unitary, and let  $H = \text{diag}(H_1, \dots, H_N)$  with  $H_k \in \mathbb{C}^{r_k \times r_k}$ ,  $k = 1, \dots, N$ , be an Hermitian invertible matrix satisfying (4.3) and (4.4). Then in the same way as in [7, Theorem 2.1] for the one-variable case, we obtain for  $Z, Z' \in \mathbb{C}^{n \times n}$ :

$$\begin{aligned} F(Z)(J \otimes I_n)F(Z')^* &= J \otimes I_n - (C \otimes I_n)(I_r \otimes I_n - \Delta(Z)(A \otimes I_n))^{-1} \\ &\quad \times \Delta(Z + Z'^*)(H^{-1} \otimes I_n)(I_r \otimes I_n - (A^* \otimes I_n)\Delta(Z'^*))^{-1}(C^* \otimes I_n) \end{aligned} \quad (4.9)$$

(note that  $\Delta(Z)$  commutes with  $H^{-1} \otimes I_n$ ). It follows from (4.9) that  $F(Z)$  is  $(J \otimes I_n)$ -unitary on  $(i\mathbb{H}^{n \times n})^N$  at all points  $Z$  where it is defined. Since  $n \in \mathbb{N}$  is arbitrary,  $F$  is matrix- $J$ -unitary on  $\mathcal{J}_N$ . Clearly, conditions a) and b') also imply the matrix- $J$ -unitarity of  $F$  on  $\mathcal{J}_N$ .  $\square$

Let us make some remarks. First, it follows from the proof of Theorem 4.1 that the structured solution  $H = \text{diag}(H_1, \dots, H_N)$  of the Lyapunov equation (4.3) is uniquely determined by a given minimal GR-realization of  $F$ . The matrix  $H = \text{diag}(H_1, \dots, H_N)$  is called the *associated structured Hermitian matrix* (associated with this minimal GR-realization of  $F$ ). The matrix  $H_k$  will be called the  *$k$ -th component of the associated Hermitian matrix* ( $k = 1, \dots, N$ ). The explicit formulas for  $H_k$  follow from (3.2):

$$\begin{aligned} H_k &= -[\text{col}_{|w| \leq qr-1} ((JB^*)b(-A^*))^{wg_k}]^+ \text{col}_{|w| \leq qr-1} ((D^{-1}C)bA^\times)^{wg_k} \\ &= -\text{row}_{|w| \leq qr-1} ((-A^*)\sharp(C^*J))^{g_k w^T} \left[ \text{row}_{|w| \leq qr-1} (A^\times \sharp(BD^{-1}))^{g_k w^T} \right]^\dagger. \end{aligned}$$

Second, let  $\alpha$  be a (not necessarily minimal) GR-realization of  $F$  of the form (3.8), where  $D$  is  $J$ -unitary, and let  $H = \text{diag}(H_1, \dots, H_N)$  with  $H_k \in \mathbb{C}^{r_k \times r_k}$ ,  $k = 1, \dots, N$ , be an Hermitian, not necessarily invertible, matrix satisfying (4.3) and

(4.6). Then in the same way as in [7, Theorem 2.1] for the one-variable case, we obtain for  $Z, Z' \in \mathbb{C}^{n \times n}$ :

$$\begin{aligned} F(Z')^*(J \otimes I_n)F(Z) &= J \otimes I_n - (B^* \otimes I_n)(I_r \otimes I_n - \Delta(Z'^*)(A^* \otimes I_n))^{-1} \\ &\quad \times (H \otimes I_n)\Delta(Z'^* + Z)(I_r \otimes I_n - (A \otimes I_n)\Delta(Z))^{-1}(B \otimes I_n) \end{aligned} \quad (4.10)$$

(note that  $\Delta(Z)$  commutes with  $H \otimes I_n$ ). It follows from (4.10) that  $F(Z)$  is  $(J \otimes I_n)$ -unitary on  $(i\mathbb{H}^{n \times n})^N$  at all points  $Z$  where it is defined. Since  $n \in \mathbb{N}$  is arbitrary,  $F$  is matrix- $J$ -unitary on  $\mathcal{J}_N$ .

Third, if  $\alpha$  is a (not necessarily minimal) GR-realization of  $F$  of the form (3.8), where  $D$  is  $J$ -unitary, and equalities (4.5) and (4.6) are valid with  $H^{-1}$  replaced by some, possibly not invertible, Hermitian matrix  $Y = \text{diag}(Y_1, \dots, Y_N)$  with  $Y_k \in \mathbb{C}^{r_k \times r_k}$ ,  $k = 1, \dots, N$ , then  $F$  is matrix- $J$ -unitary on  $\mathcal{J}_N$ . This follows from the fact that (4.9) is valid with  $H^{-1}$  replaced by  $Y$ .

**Theorem 4.2.** *Let  $(C, A)$  be an observable pair of matrices  $C \in \mathbb{C}^{q \times r}$ ,  $A \in \mathbb{C}^{r \times r}$  in the sense that  $\mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}$  and  $\mathcal{O}_k$  has full column rank for each  $k \in \{1, \dots, N\}$ , and let  $J \in \mathbb{C}^{q \times q}$  be a signature matrix. Then there exists a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS  $F$  with a minimal GR-realization  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$  if and only if the Lyapunov equation (4.3) has a structured solution  $H = \text{diag}(H_1, \dots, H_N)$  which is both Hermitian and invertible. If such a solution  $H$  exists, possible choices of  $D$  and  $B$  are*

$$D_0 = I_q, \quad B_0 = -H^{-1}C^*J. \quad (4.11)$$

Finally, for a given such  $H$ , all other choices of  $D$  and  $B$  differ from  $D_0$  and  $B_0$  by a right multiplicative  $J$ -unitary constant matrix.

*Proof.* Let  $H = \text{diag}(H_1, \dots, H_N)$  be a structured solution of the Lyapunov equation (4.3) which is both Hermitian and invertible. We first check that the pair  $(A, -H^{-1}C^*J)$  is controllable, or equivalently, that the pair  $(-JCH^{-1}, A^*)$  is observable. Using the Lyapunov equation (4.3), one can see that for any  $k \in \{1, \dots, N\}$  and  $w = g_{i_1} \cdots g_{i_{|w|}} \in \mathcal{F}_N$  there exist matrices  $K_0, \dots, K_{|w|-1}$  such that

$$\begin{aligned} (CbA)^{wg_k} &= (-1)^{|w|-1} J((-JCH^{-1})bA^*)^{wg_k} H_k \\ &\quad + K_0 J(-JC_{i_2} H_{i_2}^{-1} (A^*)_{i_2 i_3} \cdots (A^*)_{i_{|w|} k}) H_k + \cdots \\ &\quad + K_{|w|-2} J(-JC_{i_{|w|}} (A^*)_{i_{|w|} k}) H_k + K_{|w|-1} J(-JC_k H_k^{-1}) H_k. \end{aligned}$$

Thus, if  $x \in \ker((-JCH^{-1})bA^*)^{wg_k}$  for all  $w \in \mathcal{F}_N$  then  $H_k^{-1}x \in \ker \mathcal{O}_k$ , and the observability of the pair  $(C, A)$  implies that  $x = 0$ . Therefore, the pair  $(-JCH^{-1}, A^*)$  is observable, and the pair  $(A, -H^{-1}C^*J)$  is controllable. By Theorem 4.1 we obtain that

$$F_0(z) = I_q - C(I_r - \Delta(z)A)^{-1}\Delta(z)H^{-1}C^*J \quad (4.12)$$

is a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS, which has a minimal GR-realization  $\alpha_0 = (N : A, -H^{-1}C^*J, C, I_q; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$  with the associated structured Hermitian matrix  $H$ .



Conversely, let  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$  be a minimal GR-node. Then by Theorem 4.1 there exists an Hermitian and invertible matrix  $H = \text{diag}(H_1, \dots, H_N)$  which solves (4.3).

Given  $H = \text{diag}(H_1, \dots, H_N)$ , let  $B, D$  be any solution of the inverse problem, i.e.,  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$  is a minimal GR-node with the associated structured Hermitian matrix  $H$ . Then for  $F_0 = T_{\alpha_0}^{\text{nc}}$  and  $F = T_{\alpha}^{\text{nc}}$  we obtain from (4.9) that

$$F(Z)(J \otimes I_n)F(Z')^* = F_0(Z)(J \otimes I_n)F_0(Z')^*$$

for any  $n \in \mathbb{N}$ , at all points  $Z, Z' \in (\mathbb{C}^{n \times n})^N$  where both  $F$  and  $F_0$  are defined. By the uniqueness theorem in several complex variables (matrix entries for  $Z_k$ 's and  $Z_k'^*$ 's,  $k = 1, \dots, N$ ), we obtain that  $F(Z)$  and  $F_0(Z)$  differ by a right multiplicative  $(J \otimes I_n)$ -unitary constant, which clearly has to be  $D \otimes I_n$ , i.e.,

$$F(Z) = F_0(Z)(D \otimes I_n).$$

Since  $n \in \mathbb{N}$  is arbitrary, by Corollary 2.2 we obtain

$$F(z) = F_0(z)D.$$

Equating the coefficients of these two FPSs, we easily deduce using the observability of the pair  $(C, A)$  that  $B = -H^{-1}C^*JD$ .  $\square$

The following dual theorem is proved analogously.

**Theorem 4.3.** *Let  $(A, B)$  be a controllable pair of matrices  $A \in \mathbb{C}^{r \times r}, B \in \mathbb{C}^{r \times q}$  in the sense that  $\mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}$  and  $C_k$  has full row rank for each  $k \in \{1, \dots, N\}$ , and let  $J \in \mathbb{C}^{q \times q}$  be a signature matrix. Then there exists a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS  $F$  with a minimal GR-realization  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$  if and only if the Lyapunov equation*

$$GA^* + AG = -BJB^*$$

*has a structured solution  $G = \text{diag}(G_1, \dots, G_N)$  which is both Hermitian and invertible. If such a solution  $G$  exists, possible choices of  $D$  and  $C$  are*

$$D_0 = I_q, \quad C_0 = -JB^*G^{-1}. \quad (4.13)$$

*Finally, for a given such  $G$ , all other choices of  $D$  and  $C$  differ from  $D_0$  and  $C_0$  by a left multiplicative  $J$ -unitary constant matrix.*

**Theorem 4.4.** *Let  $F$  be a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS, and  $\alpha$  be its GR-realization. Let  $H = \text{diag}(H_1, \dots, H_N)$  with  $H_k \in \mathbb{C}^{r_k \times r_k}$ ,  $k = 1, \dots, N$ , be an Hermitian invertible matrix satisfying (4.3) and (4.4), or equivalently, (4.5) and (4.6). Then  $\alpha$  is observable if and only if  $\alpha$  is controllable.*

*Proof.* Suppose that  $\alpha$  is observable. Since by Theorem 4.1  $D = F_\emptyset$  is  $J$ -unitary, by Theorem 4.2  $\alpha$  is a minimal GR-node. In particular,  $\alpha$  is controllable.

Suppose that  $\alpha$  is controllable. Then by Theorem 4.3  $\alpha$  is minimal, and in particular, observable.  $\square$

#### 4.2. The associated structured Hermitian matrix

**Lemma 4.5.** *Let  $F$  be a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS, and let  $\alpha^{(i)} = (N; A^{(i)}, B^{(i)}, C^{(i)}, D; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q)$  be minimal GR-realizations of  $F$ , with the associated structured Hermitian matrices  $H^{(i)} = \text{diag}(H_1^{(i)}, \dots, H_N^{(i)})$ ,  $i = 1, 2$ . Then  $\alpha^{(1)}$  and  $\alpha^{(2)}$  are similar, i.e., (2.8) holds with a uniquely defined invertible matrix  $T = \text{diag}(T_1, \dots, T_N)$ , and*

$$H_k^{(1)} = T_k^* H_k^{(2)} T_k, \quad k = 1, \dots, N. \quad (4.14)$$

*In particular, the matrices  $H_k^{(1)}$  and  $H_k^{(2)}$  have the same signature.*

The proof is easy and analogous to the proof of Lemma 2.1 in [7].

*Remark 4.6.* The similarity matrix  $T = \text{diag}(T_1, \dots, T_N)$  is a unitary mapping from  $\mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}$  endowed with the inner product  $[\cdot, \cdot]_{H^{(1)}}$  onto  $\mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}$  endowed with the inner product  $[\cdot, \cdot]_{H^{(2)}}$ , where

$$[x, y]_{H^{(i)}} = \langle H^{(i)} x, y \rangle_{\mathbb{C}^\gamma}, \quad x, y \in \mathbb{C}^\gamma, \quad i = 1, 2,$$

that is,

$$[x, y]_{H^{(i)}} = \sum_{k=1}^N [x_k, y_k]_{H_k^{(i)}}, \quad i = 1, 2,$$

where  $x_k, y_k \in \mathbb{C}^{\gamma_k}$ ,  $x = \text{col}_{k=1, \dots, N}(x_k)$ ,  $y = \text{col}_{k=1, \dots, N}(y_k)$ , and

$$[x_k, y_k]_{H_k^{(i)}} = \langle H_k^{(i)} x_k, y_k \rangle_{\mathbb{C}^{\gamma_k}}, \quad k = 1, \dots, N, \quad i = 1, 2.$$

Recall the following definition [37]. Let  $K_{w, w'}$  be a  $\mathbb{C}^{q \times q}$ -valued function defined for  $w$  and  $w'$  in some set  $E$  and such that  $(K_{w, w'})^* = K_{w', w}$ . Then  $K_{w, w'}$  is called a *kernel with  $\kappa$  negative squares* if for any  $m \in \mathbb{N}$ , any points  $w_1, \dots, w_m$  in  $E$ , and any vectors  $c_1, \dots, c_m$  in  $\mathbb{C}^q$  the matrix  $(c_j^* K_{w_j, w_i} c_i)_{i, j=1, \dots, m} \in \mathbb{H}^{m \times m}$  has at most  $\kappa$  negative eigenvalues, and has exactly  $\kappa$  negative eigenvalues for some choice of  $m, w_1, \dots, w_m, c_1, \dots, c_m$ . We will use this definition to give a characterization of the number of negative eigenvalues of the  $k$ -th component  $H_k$ ,  $k = 1, \dots, N$ , of the associated structured Hermitian matrix  $H$ .

**Theorem 4.7.** *Let  $F$  be a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS, and let  $\alpha$  be its minimal GR-realization of the form (3.11), with the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$ . Then for  $k = 1, \dots, N$  the number of negative eigenvalues of the matrix  $H_k$  is equal to the number of negative squares of each of the kernels*

$$K_{w, w'}^{F, k} = (C^{\flat} b A)^{w g_k} H_k^{-1} (A^* \sharp C^*)^{g_k w'^T}, \quad w, w' \in \mathcal{F}_N, \quad (4.15)$$

$$K_{w, w'}^{F^*, k} = (B^* \flat A^*)^{w g_k} H_k (A \sharp B)^{g_k w'^T}, \quad w, w' \in \mathcal{F}_N, \quad (4.16)$$

For  $k = 1, \dots, N$ , denote by  $\mathcal{K}_k(F)$  (resp.,  $\mathcal{K}_k(F^*)$ ) the linear span of the functions  $w \mapsto K_{w, w'}^{F, k} c$  (resp.,  $w \mapsto K_{w, w'}^{F^*, k} c$ ) where  $w' \in \mathcal{F}_N$  and  $c \in \mathbb{C}^q$ . Then

$$\dim \mathcal{K}_k(F) = \dim \mathcal{K}_k(F^*) = \gamma_k.$$

*Proof.* Let  $m \in \mathbb{N}$ ,  $w_1, \dots, w_m \in \mathcal{F}_N$ , and  $c_1, \dots, c_m \in \mathbb{C}^q$ . Then the matrix equality

$$(c_j^* K_{w_j, w_i}^{F, k} c_i)_{i, j=1, \dots, m} = X^* H_k^{-1} X,$$

with

$$X = \text{row}_{1 \leq i \leq m} \left( (A^* \sharp C^*)^{g_k w_i^T} c_i \right),$$

implies that the kernel  $K_{w, w'}^{F, k}$  has at most  $\kappa_k$  negative squares, where  $\kappa_k$  denotes the number of negative eigenvalues of  $H_k$ . The pair  $(C, A)$  is observable, hence we can choose a basis of  $\mathbb{C}^q$  of the form  $x_i = (A^* \sharp C^*)^{g_k w_i^T} c_i$ ,  $i = 1, \dots, q$ . Since the matrix  $X = \text{row}_{i=1, \dots, q}(x_i)$  is non-degenerate, and therefore the matrix  $X^* H_k^{-1} X$  has exactly  $\kappa_k$  negative eigenvalues, the kernel  $K_{w, w'}^{F, k}$  has  $\kappa_k$  negative squares. Analogously, from the controllability of the pair  $(A, B)$  one can obtain that the kernel  $\mathcal{K}_k(F^*)$  has  $\kappa_k$  negative squares.

Since  $\mathcal{K}_k(F)$  is the span of functions (of variable  $w \in \mathcal{F}_N$ ) of the form  $(CbA)^{wg^k} y$ ,  $y \in \mathbb{C}^{\gamma_k}$ , it follows that  $\dim \mathcal{K}_k(F) \leq \gamma_k$ . From the observability of the pair  $(C, A)$  we obtain that  $(CbA)^{wg^k} y \equiv 0$  implies  $y = 0$ , thus  $\dim \mathcal{K}_k(F) = \gamma_k$ . In the same way we obtain that the controllability of the pair  $(A, B)$  implies that  $\dim \mathcal{K}_k(F^*) = \gamma_k$ .  $\square$

We will denote by  $\nu_k(F)$  the number of negative squares of either the kernel  $K_{w, w'}^{F, k}$  or the kernel  $K_{w, w'}^{F^*, k}$  defined by (4.15) and (4.16), respectively.

**Theorem 4.8.** *Let  $F^{(i)}$  be matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPSs, with minimal GR-realizations  $\alpha^{(i)} = (N; A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}; \mathbb{C}^{\gamma^{(i)}} = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k^{(i)}}, \mathbb{C}^q)$  and the associated structured Hermitian matrices  $H^{(i)} = \text{diag}(H_1^{(i)}, \dots, H_N^{(i)})$ , respectively,  $i = 1, 2$ . Suppose that the product  $\alpha = \alpha^{(1)}\alpha^{(2)}$  is a minimal GR-node. Then the matrix  $H = \text{diag}(H_1, \dots, H_N)$ , with*

$$H_k = \begin{pmatrix} H_k^{(1)} & 0 \\ 0 & H_k^{(2)} \end{pmatrix} \in \mathbb{C}^{(\gamma_k^{(1)} + \gamma_k^{(2)}) \times (\gamma_k^{(1)} + \gamma_k^{(2)})}, \quad k = 1, \dots, N, \quad (4.17)$$

is the associated structured Hermitian matrix for  $\alpha = \alpha^{(1)}\alpha^{(2)}$ .

*Proof.* It suffices to check that (4.3) and (4.4) hold for the matrices  $A, B, C, D$  defined as in (3.7), and  $H = \text{diag}(H_1, \dots, H_N)$  where  $H_k$ ,  $k = 1, \dots, N$ , are defined in (4.17). This is an easy computation which is omitted.  $\square$

**Corollary 4.9.** *Let  $F_1$  and  $F_2$  be matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPSs, and suppose that the factorization  $F = F_1 F_2$  is minimal. Then*

$$\nu_k(F_1 F_2) = \nu_k(F_1) + \nu_k(F_2), \quad k = 1, \dots, N.$$

### 4.3. Minimal matrix- $J$ -unitary factorizations

In this subsection we consider minimal factorizations of rational formal power series which are matrix- $J$ -unitary on  $\mathcal{J}_N$  into factors both of which are also matrix- $J$ -unitary on  $\mathcal{J}_N$ . Such factorizations will be called *minimal matrix- $J$ -unitary factorizations*.

Let  $H \in \mathbb{C}^{r \times r}$  be an invertible Hermitian matrix. We denote by  $[\cdot, \cdot]_H$  the Hermitian sesquilinear form

$$[x, y]_H = \langle Hx, y \rangle$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product of  $\mathbb{C}^r$ . Two vectors  $x$  and  $y$  in  $\mathbb{C}^r$  are called  $H$ -orthogonal if  $[x, y]_H = 0$ . For any subspace  $M \subset \mathbb{C}^r$  denote

$$M^{[\perp]} = \{y \in \mathbb{C}^r : \langle y, m \rangle_H = 0 \quad \forall m \in M\}.$$

The subspace  $M \subset \mathbb{C}^r$  is called *non-degenerate* if  $M \cap M^{[\perp]} = \{0\}$ . In this case,

$$M[+]M^{[\perp]} = \mathbb{C}^r$$

where  $[+]$  denotes the  *$H$ -orthogonal direct sum*.

In the case when  $H = \text{diag}(H_1, \dots, H_N)$  is the structured Hermitian matrix associated with a given minimal GR-realization of a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS  $F$ , we will call  $[\cdot, \cdot]_H$  the *associated inner product* (associated with the given minimal GR-realization of  $F$ ). In more details,

$$[x, y]_H = \sum_{k=1}^N [x_k, y_k]_{H_k},$$

where  $x_k, y_k \in \mathbb{C}^{\gamma_k}$  and  $x = \text{col}_{k=1, \dots, N}(x_k)$ ,  $y = \text{col}_{k=1, \dots, N}(y_k)$ , and

$$[x_k, y_k]_{H_k} = \langle H_k x_k, y_k \rangle_{\mathbb{C}^{\gamma_k}}, \quad k = 1, \dots, N.$$

The following theorem (as well as its proof) is analogous to its one-variable counterpart, Theorem 2.6 from [7] (see also [43, Chapter II]).

**Theorem 4.10.** *Let  $F$  be a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS, and let  $\alpha$  be its minimal GR-realization of the form (3.11), with the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$ . Let  $\mathcal{M} = \bigoplus_{k=1}^N \mathcal{M}_k$  be an  $A$ -invariant subspace such that  $\mathcal{M}_k \subset \mathbb{C}^{\gamma_k}$ ,  $k = 1, \dots, N$ , and  $\mathcal{M}$  is non-degenerate in the associated inner product  $[\cdot, \cdot]_H$ . Let  $\Pi = \text{diag}(\Pi_1, \dots, \Pi_N)$  be the projection defined by*

$$\ker \Pi = \mathcal{M}, \quad \text{ran } \Pi = \mathcal{M}^{[\perp]},$$

or in more details,

$$\ker \Pi_k = \mathcal{M}_k, \quad \text{ran } \Pi_k = \mathcal{M}_k^{[\perp]}, \quad k = 1, \dots, N.$$

Let  $D = D_1 D_2$  be a factorization of  $D$  into two  $J$ -unitary factors. Then the factorization  $F = F_1 F_2$  where

$$\begin{aligned} F_1(z) &= D_1 + C(I_\gamma - \Delta(z)A)^{-1} \Delta(z)(I_\gamma - \Pi) B D_2^{-1}, \\ F_2(z) &= D_2 + D_1^{-1} C \Pi (I_\gamma - \Delta(z)A)^{-1} \Delta(z) B, \end{aligned}$$

is a minimal matrix- $J$ -unitary factorization of  $F$ .

Conversely, any minimal matrix- $J$ -unitary factorization of  $F$  can be obtained in such a way. For a fixed  $J$ -unitary decomposition  $D = D_1 D_2$ , the correspondence between minimal matrix- $J$ -unitary factorizations of  $F$  and non-degenerate  $A$ -invariant subspaces of the form  $\mathcal{M} = \bigoplus_{k=1}^N \mathcal{M}_k$ , where  $\mathcal{M}_k \subset \mathbb{C}^{\gamma_k}$  for  $k = 1, \dots, N$ , is one-to-one.

*Remark 4.11.* We omit here the proof, which can be easily restored, with making use of Theorem 3.9 and Corollary 3.10.

*Remark 4.12.* Minimal matrix- $J$ -unitary factorizations do not always exist, even for  $N = 1$ . Examples of  $J$ -unitary on  $i\mathbb{R}$  rational functions which have non-trivial minimal factorizations but lack minimal  $J$ -unitary factorizations can be found in [4] and [7].

#### 4.4. Matrix-unitary rational formal power series

In this subsection we specialize some of the preceding results to the case  $J = I_q$ . We call the corresponding rational formal power series *matrix-unitary* on  $\mathcal{J}_N$ .

**Theorem 4.13.** *Let  $F$  be a rational FPS and  $\alpha$  be its minimal GR-realization of the form (3.11). Then  $F$  is matrix-unitary on  $\mathcal{J}_N$  if and only if the following conditions are fulfilled:*

- a)  $D$  is a unitary matrix, i.e.,  $DD^* = I_q$ ;
- b) there exists an Hermitian solution  $H = \text{diag}(H_1, \dots, H_N)$ , with  $H_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ , of the Lyapunov equation

$$A^* H + H A = -C^* C, \quad (4.18)$$

and

$$C = -D^{-1} B^* H. \quad (4.19)$$

The property b) is equivalent to

- b') there exists an Hermitian solution  $G = \text{diag}(G_1, \dots, G_N)$ , with  $G_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ , of the Lyapunov equation

$$G A^* + A G = -B B^*, \quad (4.20)$$

and

$$B = -G C^* D^{-1}. \quad (4.21)$$

*Proof.* To obtain Theorem 4.13 from Theorem 4.1 it suffices to show that any structured Hermitian solution to the Lyapunov equation (4.18) (resp., (4.20)) is invertible. Let  $H = \text{diag}(H_1, \dots, H_N)$  be a structured Hermitian solution to (4.18), and  $x \in \ker H$ , i.e.,  $x = \text{col}_{1 \leq k \leq N}(x_k)$  and  $x_k \in \ker H_k$ ,  $k = 1, \dots, N$ . Then

$\langle HAx, x \rangle = \langle Ax, Hx \rangle = 0$ , and equation (4.18) implies  $Cx = 0$ . In particular, for every  $k \in \{1, \dots, N\}$  one can define  $\tilde{x} = \text{col}(0, \dots, 0, x_k, 0, \dots, 0)$  where  $x_k \in \ker H_k$  is on the  $k$ -th block entry of  $\tilde{x}$ , and from  $C\tilde{x} = 0$  get  $C_k x_k = 0$ . Thus,  $\ker H_k \subset \ker C_k$ ,  $k = 1, \dots, N$ . Consider the following block representations with respect to the decompositions  $\mathbb{C}^{r_k} = \ker H_k \oplus \text{ran } H_k$ :

$$A_{ij} = \begin{pmatrix} A_{ij}^{(11)} & A_{ij}^{(12)} \\ A_{ij}^{(21)} & A_{ij}^{(22)} \end{pmatrix}, \quad C_k = \begin{pmatrix} 0 & C_k^{(2)} \end{pmatrix}, \quad H_k = \begin{pmatrix} 0 & 0 \\ 0 & H_k^{(22)} \end{pmatrix},$$

where  $i, j, k = 1, \dots, N$ . Then (4.18) implies

$$(A^*H + HA)_{ij}^{(12)} = (A_{ji}^* H_j + H_i A_{ij})^{(12)} = (A_{ji}^{(21)})^* H_j^{(22)} = 0,$$

and  $A_{ji}^{(21)} = 0$ ,  $i, j = 1, \dots, N$ . Therefore, for any  $w \in \mathcal{F}_N$  we have

$$(CbA)^{wg_k} = \begin{pmatrix} 0 & (C^{(2)}bA^{(22)})^{wg_k} \end{pmatrix}, \quad k = 1, \dots, N,$$

where  $C^{(2)} = \text{row}_{1 \leq k \leq N}(C_k^{(2)})$ ,  $A^{(22)} = (A_{ij}^{(22)})_{i,j=1,\dots,N}$ . If there exists  $k \in \{1, \dots, N\}$  such that  $\ker H_k \neq \{0\}$ , then the pair  $(C, A)$  is not observable, which contradicts to the assumption on  $\alpha$ . Thus,  $H$  is invertible.

In a similar way one can show that any structured Hermitian solution  $G = \text{diag}(G_1, \dots, G_N)$  of the Lyapunov equation (4.20) is invertible.  $\square$

A counterpart of Theorem 4.2 in the present case is the following theorem.

**Theorem 4.14.** *Let  $(C, A)$  be an observable pair of matrices  $C \in \mathbb{C}^{q \times r}$ ,  $A \in \mathbb{C}^{r \times r}$  in the sense that  $\mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}$  and  $\mathcal{O}_k$  has full column rank for each  $k \in \{1, \dots, N\}$ . Then there exists a matrix-unitary on  $\mathcal{J}_N$  rational FPS  $F$  with a minimal GR-realization  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$  if and only if the Lyapunov equation (4.18) has a structured Hermitian solution  $H = \text{diag}(H_1, \dots, H_N)$ . If such a solution  $H$  exists, it is invertible, and possible choices of  $D$  and  $B$  are*

$$D_0 = I_q, \quad B_0 = -H^{-1}C^*. \quad (4.22)$$

Finally, for a given such  $H$ , all other choices of  $D$  and  $B$  differ from  $D_0$  and  $B_0$  by a right multiplicative unitary constant matrix.

The proof of Theorem 4.14 is a direct application of Theorem 4.2 and Theorem 4.13. One can prove analogously the following theorem which is a counterpart of Theorem 4.3.

**Theorem 4.15.** *Let  $(A, B)$  be a controllable pair of matrices  $A \in \mathbb{C}^{r \times r}$ ,  $B \in \mathbb{C}^{r \times q}$  in the sense that  $\mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}$  and  $\mathcal{C}_k$  has full row rank for each  $k \in \{1, \dots, N\}$ . Then there exists a matrix-unitary on  $\mathcal{J}_N$  rational FPS  $F$  with a minimal GR-realization  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$  if and only if the Lyapunov equation (4.20) has a structured Hermitian solution  $G = \text{diag}(G_1, \dots, G_N)$ . If such a solution  $G$  exists, it is invertible, and possible choices of  $D$  and  $C$  are*

$$D_0 = I_q, \quad C_0 = -B^*G^{-1}. \quad (4.23)$$

Finally, for a given such  $G$ , all other choices of  $D$  and  $C$  differ from  $D_0$  and  $C_0$  by a left multiplicative unitary constant matrix.

Let  $\overline{A} = (\overline{A_1}, \dots, \overline{A_N})$  be an  $N$ -tuple of  $r \times r$  matrices. A non-zero vector  $x \in \mathbb{C}^r$  is called a *common eigenvector* for  $\overline{A}$  if there exists  $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$  (which is called a *common eigenvalue* for  $\overline{A}$ ) such that

$$\overline{A_k}x = \lambda_k x, \quad k = 1, \dots, N.$$

The following theorem, which is a multivariable non-commutative counterpart of statements a) and b) of Theorem 2.10 in [7], gives a necessary condition on a minimal GR-realization of a matrix-unitary on  $\mathcal{J}_N$  rational FPS.

**Theorem 4.16.** *Let  $F$  be a matrix-unitary on  $\mathcal{J}_N$  rational FPS and  $\alpha$  be its minimal GR-realization, with the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$  and the associated inner products  $[\cdot, \cdot]_{H_k}$ ,  $k = 1, \dots, N$ . Let  $P_k$  denote the orthogonal projection in  $\mathbb{C}^\gamma$  onto the subspace  $\{0\} \oplus \dots \oplus \{0\} \oplus \mathbb{C}^{\gamma_k} \oplus \{0\} \oplus \dots \oplus \{0\}$ , and  $\overline{A_k} = AP_k$ ,  $k = 1, \dots, N$ . If  $x \in \mathbb{C}^\gamma$  is a common eigenvector for  $\overline{A}$  corresponding to a common eigenvalue  $\lambda \in \mathbb{C}^N$  then there exists  $j \in \{1, \dots, N\}$  such that  $\text{Re } \lambda_j \neq 0$  and  $[P_j x, P_j x]_{H_j} \neq 0$ . In particular,  $\overline{A}$  has no common eigenvalues on  $(i\mathbb{R})^N$ .*

*Proof.* By (4.18), we have for every  $k \in \{1, \dots, N\}$ ,

$$(\overline{\lambda_k} + \lambda_k)[P_k x, P_k x]_{H_k} = -\langle CP_k x, CP_k x \rangle.$$

Suppose that for all  $k \in \{1, \dots, N\}$  the left-hand side of this equality is zero, then  $CP_k x = 0$ . Since for  $\emptyset \neq w = g_{i_1} \dots g_{i_{|w|}} \in \mathcal{F}_N$ ,

$$(C_b A)^{wg_k} P_k x = CP_{i_1} \overline{A_{i_2}} \dots \overline{A_{i_{|w|}}} \cdot \overline{A_k} x = \lambda_{i_2} \dots \lambda_{i_{|w|}} \lambda_k CP_{i_1} x = 0,$$

the observability of the pair  $(C, A)$  implies  $P_k x = 0$ ,  $k = 1, \dots, N$ , i.e.,  $x = 0$  which contradicts to the assumption that  $x$  is a common eigenvector for  $\overline{A}$ . Thus, there exists  $j \in \{1, \dots, N\}$  such that  $(\overline{\lambda_j} + \lambda_j)[P_j x, P_j x]_{H_j} \neq 0$ , as desired.  $\square$

## 5. Matrix- $J$ -unitary formal power series: A multivariable non-commutative analogue of the circle case

In this section we study a multivariable non-commutative analogue of rational  $\mathbb{C}^{q \times q}$ -valued functions which are  $J$ -unitary on the unit circle  $\mathbb{T}$ .

### 5.1. Minimal Givone–Roesser realizations and the Stein equation

Let  $n \in \mathbb{N}$ . We denote by  $\mathbb{T}^{n \times n}$  the *matrix unit circle*

$$\mathbb{T}^{n \times n} = \{W \in \mathbb{C}^{n \times n} : WW^* = I_n\},$$

i.e., the family of unitary  $n \times n$  complex matrices. We will call the set  $(\mathbb{T}^{n \times n})^N$  the *matrix unit torus*. The set

$$\mathcal{T}_N = \prod_{n \in \mathbb{N}} (\mathbb{T}^{n \times n})^N$$

serves as a multivariable non-commutative counterpart of the unit circle. Let  $J = J^{-1} = J^* \in \mathbb{C}^{q \times q}$ . We will say that a rational FPS  $f$  is *matrix- $J$ -unitary* on  $\mathcal{T}_N$  if for every  $n \in \mathbb{N}$ ,

$$f(W)(J \otimes I_n)f(W)^* = J \otimes I_n$$

at all points  $W = (W_1, \dots, W_N) \in (\mathbb{T}^{n \times n})^N$  where it is defined. In the following theorem we establish the relationship between matrix- $J$ -unitary rational FPSs on  $\mathcal{J}_N$  and on  $\mathcal{T}_N$ , their minimal GR-realizations, and the structured Hermitian solutions of the corresponding Lyapunov and Stein equations.

**Theorem 5.1.** *Let  $f$  be a matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPS, with a minimal GR-realization  $\alpha$  of the form (3.11), and let  $a \in \mathbb{T}$  be such that  $-\bar{a} \notin \sigma(A)$ . Then*

$$F(z) = f(a(z_1 - 1)(z_1 + 1)^{-1}, \dots, a(z_N - 1)(z_N + 1)^{-1}) \quad (5.1)$$

is well defined as a rational FPS which is matrix- $J$ -unitary on  $\mathcal{J}_N$ , and  $F = T_\beta^{\text{nc}}$ , where  $\beta = (N; A_a, B_a, C_a, D_a; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q)$ , with

$$\begin{aligned} A_a &= (aA - I_\gamma)(aA + I_\gamma)^{-1}, & B_a &= \sqrt{2}(aA + I_\gamma)^{-1}aB, \\ C_a &= \sqrt{2}C(aA + I_\gamma)^{-1}, & D_a &= D - C(aA + I_\gamma)^{-1}aB. \end{aligned} \quad (5.2)$$

A GR-node  $\beta$  is minimal, and its associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$  is the unique invertible structured Hermitian solution of

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix}. \quad (5.3)$$

*Proof.* For any  $a \in \mathbb{T}$  and  $n \in \mathbb{N}$  the Cayley transform

$$Z_0 \mapsto W_0 = a(Z_0 - I_n)(Z_0 + I_n)^{-1}$$

maps  $i\mathbb{H}^{n \times n}$  onto  $\mathbb{T}^{n \times n}$ , thus its simultaneous application to each matrix variable maps  $(i\mathbb{H}^{n \times n})^N$  onto  $(\mathbb{T}^{n \times n})^N$ . Since the simultaneous application of the Cayley transform to each formal variable in a rational FPS gives a rational FPS, (5.1) defines a rational FPS  $F$ . Since  $f$  is matrix- $J$ -unitary on  $\mathcal{T}_N$ ,  $F$  is matrix- $J$ -unitary on  $\mathcal{J}_N$ . Moreover,

$$\begin{aligned} F(z) &= D + C(I_\gamma - a(\Delta(z) - I_\gamma)(\Delta(z) + I_\gamma)^{-1}A)^{-1} \\ &\times a(\Delta(z) - I_\gamma)(\Delta(z) + I_\gamma)^{-1}B \\ &= D + C(\Delta(z) + I_\gamma - a(\Delta(z) - I_\gamma)A)^{-1}a(\Delta(z) - I_\gamma)B \\ &= D + C(aA + I_\gamma - \Delta(z)(aA - I_\gamma))^{-1}a(\Delta(z) - I_\gamma)B \\ &= D + C(aA + I_\gamma)^{-1}(I_\gamma - \Delta(z)(aA - I_\gamma)(aA + I_\gamma)^{-1})^{-1}\Delta(z)aB \\ &\quad - C(aA + I_\gamma)^{-1}(I_\gamma - \Delta(z)(aA - I_\gamma)(aA + I_\gamma)^{-1})^{-1}aB \\ &= D - C(aA + I_\gamma)^{-1}aB + C(aA + I_\gamma)^{-1} \\ &\times (I_\gamma - \Delta(z)(aA - I_\gamma)(aA + I_\gamma)^{-1})^{-1} \\ &\times \Delta(z)(I_\gamma - (aA - I_\gamma)(aA + I_\gamma)^{-1})aB \\ &= D_a + C_a(I_\gamma - \Delta(z)A_a)^{-1}\Delta(z)B_a. \end{aligned}$$



Thus,  $F = T_\beta^{\text{nc}}$ . Let us remark that the FPS

$$\varphi_k^a(z) = C_a(I_\gamma - \Delta(z)A_a)^{-1}|_{\mathbb{C}^{\gamma_k}}$$

(c.f. (3.5)) has the coefficients

$$(\varphi_k^a)_w = (C_a \flat A_a)^{wg_k}, \quad w \in \mathcal{F}_N.$$

Remark also that

$$\begin{aligned} \tilde{\varphi}_k(z) &:= \varphi_k(a(z_1 - 1)(z_1 + 1)^{-1}, \dots, a(z_N - 1)(z_N + 1)^{-1}) \\ &= C(I_\gamma - a(\Delta(z) - I_\gamma)(\Delta(z) + I_\gamma)^{-1}A)^{-1}|_{\mathbb{C}^{\gamma_k}} \\ &= C((\Delta(z) + I_\gamma) - a(\Delta(z) - I_\gamma)A)^{-1}(\Delta(z) + I_\gamma)|_{\mathbb{C}^{\gamma_k}} \\ &= C((aA + I_\gamma) - \Delta(z)(aA - I_\gamma))^{-1}(\Delta(z) + I_\gamma)|_{\mathbb{C}^{\gamma_k}} \\ &= C(aA + I_\gamma)^{-1}(I_\gamma - \Delta(z)(aA - I_\gamma)(aA + I_\gamma)^{-1})^{-1}(\Delta(z) + I_\gamma)|_{\mathbb{C}^{\gamma_k}} \\ &= \frac{1}{\sqrt{2}}(C_a(I_\gamma - \Delta(z)A_a)^{-1}|_{\mathbb{C}^{\gamma_k}})(z_k + 1) \\ &= \frac{1}{\sqrt{2}}(\varphi_k^a(z) \cdot z_k + \varphi_k^a(z)). \end{aligned}$$

Let  $k \in \{1, \dots, N\}$  be fixed. Suppose that  $n \in \mathbb{N}$ ,  $n \geq (q\gamma - 1)^{q\gamma - 1}$  (for  $q\gamma - 1 = 0$  choose arbitrary  $n \in \mathbb{N}$ ), and  $x \in \bigcap_{Z \in \Gamma_n(\varepsilon)} \ker \varphi_k^a(Z)$ , where  $\Gamma_n(\varepsilon)$  is a neighborhood of the origin of  $\mathbb{C}^{n \times n}$  where  $\varphi_k^a(Z)$  is well defined, e.g., of the form (2.9) with  $\varepsilon = \|A_a\|^{-1}$ . Then, by Theorem 3.1 and Theorem 2.1, one has

$$\begin{aligned} \bigcap_{Z \in \Gamma_n(\varepsilon)} \ker \varphi_k^a(Z) &= \left( \bigcap_{w \in \mathcal{F}_N: |w| \leq q\gamma - 1} \ker (\varphi_k^a)_w \right) \otimes \mathbb{C}^n \\ &= \left( \bigcap_{w \in \mathcal{F}_N: |w| \leq q\gamma - 1} \ker (C_a \flat A_a)^{wg_k} \right) \otimes \mathbb{C}^n = \left( \ker \tilde{O}_k(\beta) \right) \otimes \mathbb{C}^n. \end{aligned}$$

Thus, there exist  $l \in \mathbb{N}$ ,  $\{u^{(\mu)}\}_{\mu=1}^l \subset \ker \tilde{O}_k(\beta)$ ,  $\{y^{(\mu)}\}_{\mu=1}^l \subset \mathbb{C}^n$  such that

$$x = \sum_{\mu=1}^l u^{(\mu)} \otimes y^{(\mu)}. \quad (5.4)$$

Since  $(\varphi_k^a(z) \cdot z_k)_{wg_k} = (C_a \flat A_a)^{wg_k}$  for  $w \in \mathcal{F}_N$ , and  $(\varphi_k^a(z) \cdot z_k)_{w'} = 0$  for  $w' \neq wg_k$  with any  $w \in \mathcal{F}_N$ , (5.4) implies that  $\varphi_k^a(Z)(I_{\gamma_k} \otimes Z_k)x \equiv 0$ . Thus,

$$\tilde{\varphi}_k(Z)x = \frac{1}{\sqrt{2}}(\varphi_k^a(Z)(I_{\gamma_k} \otimes Z_k) + \varphi_k^a(Z))x \equiv 0.$$

Since the Cayley transform  $a(\Delta(z) - I_\gamma)(\Delta(z) + I_\gamma)^{-1}$  maps an open and dense subset of the set of matrices of the form  $\Delta(Z) = \text{diag}(Z_1, \dots, Z_N)$ ,  $Z_j \in \mathbb{C}^{\gamma_j \times \gamma_j}$ ,  $j = 1, \dots, N$ , onto an open and dense subset of the same set,

$$\varphi_k(Z)x = (C \otimes I_n)(I_\gamma - \Delta(Z)(A \otimes I_n))^{-1}x \equiv 0.$$

Since the GR-node  $\alpha$  is observable, by Theorem 3.7 we get  $x = 0$ . Therefore,

$$\bigcap_{Z \in \Gamma_n(\varepsilon)} \ker \varphi_k^a(Z) = 0, \quad k = 1, \dots, N.$$

Applying Theorem 3.7 once again, we obtain the observability of the GR-node  $\beta$ . In the same way one can prove the controllability of  $\beta$ . Thus,  $\beta$  is minimal.

Note that

$$\begin{aligned} & \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix} = \\ & = \begin{pmatrix} A^*HA + C^*JC - H & A^*HB + C^*JD \\ B^*HA + D^*JC & B^*HB + D^*JD - J \end{pmatrix}. \end{aligned} \quad (5.5)$$

Since  $-\bar{a} \notin \sigma(A)$ , the matrix  $(aA + I_\gamma)^{-1}$  is well defined, as well as  $A_a = (aA - I_\gamma)(aA + I_\gamma)^{-1}$ , and  $I_\gamma - A_a = 2(aA + I_\gamma)^{-1}$  is invertible. Having this in mind, one can deduce from (5.2) the following relations:

$$A^*HA + C^*JC - H = 2(I_\gamma - A_a^*)^{-1}(A_a^*H + HA_a + C_a^*JC_a)(I_\gamma - A_a)^{-1}$$

$$\begin{aligned} B^*HA + D^*JC &= \sqrt{2}(B_a^*H + D_a^*JC_a)(I_\gamma - A_a)^{-1} \\ &+ \sqrt{2}B_a^*(I_\gamma - A_a^*)^{-1}(A_a^*H + HA_a + C_a^*JC_a)(I_\gamma - A_a)^{-1} \end{aligned}$$

$$\begin{aligned} & B^*HB + D^*JD - J \\ &= B_a^*(I_\gamma - A_a^*)^{-1}(A_a^*H + HA_a + C_a^*JC_a)(I_\gamma - A_a)^{-1}B_a \\ &+ (B_a^*H + D_a^*JC_a)(I_\gamma - A_a)^{-1}B_a + B_a^*(I_\gamma - A_a^*)^{-1}(C_a^*JD_a + HB_a). \end{aligned}$$

Thus,  $A, B, C, D, H$  satisfy (5.3) if and only if  $A_a, B_a, C_a, D_a, H$  satisfy (4.3) and (4.4) (in the place of  $A, B, C, D, H$  therein), which completes the proof.  $\square$

We will call the invertible Hermitian solution  $H = \text{diag}(H_1, \dots, H_N)$  of (5.3), which is determined uniquely by a minimal GR-realization  $\alpha$  of a matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPS  $f$ , the *associated structured Hermitian matrix* (associated with a minimal GR-realization  $\alpha$  of  $f$ ). Let us note also that since for the GR-node  $\beta$  from Theorem 5.1 a pair of the equalities (4.3) and (4.4) is equivalent to a pair of the equalities (4.5) and (4.6), the equality (5.3) is equivalent to

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} H^{-1} & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} H^{-1} & 0 \\ 0 & J \end{pmatrix}. \quad (5.6)$$

*Remark 5.2.* Equality (5.3) can be replaced by the following three equalities:

$$H - A^*HA = C^*JC, \quad (5.7)$$

$$D^*JC = -B^*HA, \quad (5.8)$$

$$J - D^*JD = B^*HB, \quad (5.9)$$

and equality (5.6) can be replaced by

$$H^{-1} - AH^{-1}A^* = BJB^*, \quad (5.10)$$

$$DJB^* = -CH^{-1}A^*, \quad (5.11)$$

$$J - DJD^* = CH^{-1}C^*. \quad (5.12)$$

Theorem 5.1 allows to obtain a counterpart of the results from Section 4 in the setting of rational FPSs which are matrix- $J$ -unitary on  $\mathcal{T}_N$ . We will skip the proofs when it is clear how to get them.

**Theorem 5.3.** *Let  $f$  be a rational FPS and  $\alpha$  be its minimal GR-realization of the form (3.11). Then  $f$  is matrix- $J$ -unitary on  $\mathcal{T}_N$  if and only if there exists an invertible Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$ , with  $H_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ , which satisfies (5.3), or equivalently, (5.6).*

*Remark 5.4.* In the same way as in [7, Theorem 3.1] one can show that if a rational FPS  $f$  has a (not necessarily minimal) GR-realization (3.8) which satisfies (5.3) (resp., (5.6)), with an Hermitian invertible matrix  $H = \text{diag}(H_1, \dots, H_N)$ , then for any  $n \in \mathbb{N}$ ,

$$\begin{aligned} f(Z')^*(J \otimes I_n)f(Z) &= J \otimes I_n - (B^* \otimes I_n)(I_\gamma \otimes I_n - \Delta(Z')^*(A^* \otimes I_n))^{-1} \\ &\times (H \otimes I_n)(I_\gamma \otimes I_n - \Delta(Z')^*\Delta(Z)) \\ &\times (I_\gamma \otimes I_n - (A \otimes I_n)\Delta(Z))^{-1}(B \otimes I_n) \end{aligned} \quad (5.13)$$

and respectively,

$$\begin{aligned} f(Z)(J \otimes I_n)f(Z')^* &= J \otimes I_n - (C \otimes I_n)(I_\gamma \otimes I_n - \Delta(Z)(A \otimes I_n))^{-1} \\ &\times (I_\gamma \otimes I_n - \Delta(Z)\Delta(Z')^*)(H^{-1} \otimes I_n) \\ &\times (I_\gamma \otimes I_n - (A^* \otimes I_n)\Delta(Z')^*)^{-1}(C^* \otimes I_n), \end{aligned} \quad (5.14)$$

at all the points  $Z, Z' \in (\mathbb{C}^{n \times n})^N$  where it is defined, which implies that  $f$  is matrix- $J$ -unitary on  $\mathcal{T}_N$ . Moreover, the same statement holds true if  $H = \text{diag}(H_1, \dots, H_N)$  in (5.3) and (5.13) is not supposed to be invertible, and if  $H^{-1} = \text{diag}(H_1^{-1}, \dots, H_N^{-1})$  in (5.6) and (5.14) is replaced by any Hermitian, not necessarily invertible matrix  $Y = \text{diag}(Y_1, \dots, Y_N)$ .

**Theorem 5.5.** *Let  $f$  be a matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPS, and  $\alpha$  be its GR-realization. Let  $H = \text{diag}(H_1, \dots, H_N)$  with  $H_k \in \mathbb{C}^{r_k \times r_k}$ ,  $k = 1, \dots, N$ , be an Hermitian invertible matrix satisfying (5.3) or, equivalently, (5.6). Then  $\alpha$  is observable if and only if  $\alpha$  is controllable.*

*Proof.* Let  $a \in \mathbb{T}$ ,  $-\bar{a} \notin \sigma(A)$ . Then  $F$  defined by (5.1) is a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS, and (5.2) is its GR-realization. As shown in the proof of Theorem 5.1,  $\alpha$  is observable (resp., controllable) if and only if so is  $\beta$ . Since by Theorem 5.1 the GR-node  $\beta$  satisfies (4.3) and (4.4) (equivalently, (4.5) and (4.6)), Theorem 4.4 implies the statement of the present theorem.  $\square$

**Theorem 5.6.** *Let  $f$  be a matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPS and  $\alpha$  be its minimal GR-realization of the form (3.11), with the associated structured Hermitian matrix  $H$ . If  $D = f_0$  is invertible then so is  $A$ , and*

$$A^{-1} = H^{-1}(A^\times)^*H. \quad (5.15)$$

*Proof.* It follows from (5.8) that  $C = -JD^{-*}B^*HA$ . Then (5.7) turns into

$$H - A^*HA = C^*J(-JD^{-*}B^*HA) = -C^*D^{-*}B^*HA,$$

which implies that  $H = (A^\times)^*HA$ , and (5.15) follows.  $\square$

The following two lemmas, which are used in the sequel, can be found in [7].

**Lemma 5.7.** *Let  $A \in \mathbb{C}^{r \times r}$ ,  $C \in \mathbb{C}^{q \times r}$ , where  $A$  is invertible. Let  $H$  be an invertible Hermitian matrix and  $J$  be a signature matrix such that*

$$H - A^*HA = C^*JC.$$

Let  $a \in \mathbb{T}$ ,  $a \notin \sigma(A)$ . Define

$$D_a = I_q - CH^{-1}(I_r - aA^*)^{-1}C^*J, \quad (5.16)$$

$$B_a = -H^{-1}A^{-*}C^*JD_a. \quad (5.17)$$

Then

$$\begin{pmatrix} A & B_a \\ C & D_a \end{pmatrix}^* \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} A & B_a \\ C & D_a \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & J \end{pmatrix}.$$

**Lemma 5.8.** *Let  $A \in \mathbb{C}^{r \times r}$ ,  $B \in \mathbb{C}^{r \times q}$ , where  $A$  is invertible. Let  $H$  be an invertible Hermitian matrix and  $J$  be a signature matrix such that*

$$H^{-1} - AH^{-1}A^* = BJB^*.$$

Let  $a \in \mathbb{T}$ ,  $a \notin \sigma(A)$ . Define

$$D'_a = I_q - JB^*(I_r - aA^*)^{-1}HB, \quad (5.18)$$

$$C'_a = -D'_aJB^*A^{-*}H. \quad (5.19)$$

Then

$$\begin{pmatrix} A & B \\ C'_a & D'_a \end{pmatrix} \begin{pmatrix} H^{-1} & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} A & B \\ C'_a & D'_a \end{pmatrix}^* = \begin{pmatrix} H^{-1} & 0 \\ 0 & J \end{pmatrix}.$$

**Theorem 5.9.** *Let  $(C, A)$  be an observable pair of matrices  $C \in \mathbb{C}^{q \times r}$ ,  $A \in \mathbb{C}^{r \times r}$  in the sense that  $\mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}$  and  $\mathcal{O}_k$  has full column rank for each  $k \in \{1, \dots, N\}$ . Let  $A$  be invertible and  $J \in \mathbb{C}^{q \times q}$  be a signature matrix. Then there exists a matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPS  $f$  with a minimal GR-realization  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$  if and only if the Stein equation (5.7) has a structured solution  $H = \text{diag}(H_1, \dots, H_N)$  which is both Hermitian and invertible. If such a solution  $H$  exists, possible choices of  $D$  and  $B$  are  $D_a$  and  $B_a$  defined in (5.16) and (5.17), respectively. For a given such  $H$ , all other choices of  $D$  and  $B$  differ from  $D_a$  and  $B_a$  by a right multiplicative  $J$ -unitary constant matrix.*

*Proof.* Let  $H = \text{diag}(H_1, \dots, H_N)$  be a structured solution of the Stein equation (5.7) which is both Hermitian and invertible,  $D_a$  and  $B_a$  are defined as in (5.16) and (5.17), respectively, where  $a \in \mathbb{T}$ ,  $a \notin \sigma(A)$ . Set  $\alpha_a = (N; A, B_a, C, D_a; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$ . By Lemma 5.7 and due to Remark 5.4, the transfer function  $T_\alpha^{\text{nc}}$  of  $\alpha_a$  is a matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPS. Since  $\alpha_a$  is observable, by Theorem 5.5  $\alpha_a$  is controllable, and thus, minimal.

Conversely, if  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$  is a minimal GR-node whose transfer function is matrix- $J$ -unitary on  $\mathcal{T}_N$  then by Theorem 5.3 there exists a solution  $H = \text{diag}(H_1, \dots, H_N)$  of the Stein equation (5.7) which is both Hermitian and invertible. The rest of the proof is analogous to the one of Theorem 4.2.  $\square$

Analogously, one can obtain the following.

**Theorem 5.10.** *Let  $(A, B)$  be a controllable pair of matrices  $A \in \mathbb{C}^{r \times r}$ ,  $B \in \mathbb{C}^{r \times q}$  in the sense that  $\mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}$  and  $C_k$  has full row rank for each  $k \in \{1, \dots, N\}$ . Let  $A$  be invertible and  $J \in \mathbb{C}^{q \times q}$  be a signature matrix. Then there exists a matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPS  $f$  with a minimal GR-realization  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$  if and only if the Stein equation*

$$G - AGA^* = BJB^* \quad (5.20)$$

*has a structured solution  $G = \text{diag}(G_1, \dots, G_N)$  which is both Hermitian and invertible. If such a solution  $G$  exists, possible choices of  $D$  and  $C$  are  $D'_a$  and  $C'_a$  defined in (5.16) and (5.17), respectively, where  $H = G^{-1}$ . For a given such  $G$ , all other choices of  $D$  and  $C$  differ from  $D'_a$  and  $C'_a$  by a left multiplicative  $J$ -unitary constant matrix.*

## 5.2. The associated structured Hermitian matrix

In this subsection we give the analogue of the results of Section 4.2. The proofs are similar and will be omitted.

**Lemma 5.11.** *Let  $f$  be a matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPS and  $\alpha^{(i)} = (N; A^{(i)}, B^{(i)}, C^{(i)}, D; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q)$  be its minimal GR-realizations, with the associated structured Hermitian matrices  $H^{(i)} = \text{diag}(H_1^{(i)}, \dots, H_N^{(i)})$ ,  $i = 1, 2$ . Then  $\alpha^{(1)}$  and  $\alpha^{(2)}$  are similar, that is*

$$C^{(1)} = C^{(2)}T, \quad TA^{(1)} = A^{(2)}T, \quad \text{and} \quad TB^{(1)} = B^{(2)},$$

*for a uniquely defined invertible matrix  $T = \text{diag}(T_1, \dots, T_N) \in \mathbb{C}^{\gamma \times \gamma}$  and*

$$H_k^{(1)} = T_k^* H_k^{(2)} T_k, \quad k = 1, \dots, N.$$

*In particular, the matrices  $H_k^{(1)}$  and  $H_k^{(2)}$  have the same signature.*

**Theorem 5.12.** *Let  $f$  be a matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPS, and let  $\alpha$  be its minimal GR-realization of the form (3.11), with the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$ . Then for each  $k \in \{1, \dots, N\}$  the number of*

negative eigenvalues of the matrix  $H_k$  is equal to the number of negative squares of each of the kernels (on  $\mathcal{F}_N$ ):

$$\begin{aligned} K_{w,w'}^{f,k} &= (C \flat A)^{wg_k} H_k^{-1} (A^* \sharp C^*)^{g_k w'^T}, \\ K_{w,w'}^{f^*,k} &= (B^* \flat A^*)^{wg_k} H_k (A \sharp B)^{g_k w'^T}. \end{aligned} \quad (5.21)$$

Finally, for  $k \in \{1, \dots, N\}$  let  $\mathcal{K}_k(f)$  (resp.,  $\mathcal{K}_k(f^*)$ ) be the span of the functions  $w \mapsto K_{w,w'}^{f,k} c$  (resp.,  $w \mapsto K_{w,w'}^{f^*,k} c$ ) where  $w' \in \mathcal{F}_N$  and  $c \in \mathbb{C}^q$ . Then

$$\dim \mathcal{K}_k(f) = \dim \mathcal{K}_k(f^*) = \gamma_k.$$

We will denote by  $\nu_k(f)$  the number of negative squares of either of the functions defined in (5.21).

**Theorem 5.13.** *Let  $f_i$ ,  $i = 1, 2$ , be two matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPSs, with minimal GR-realizations*

$$\alpha^{(i)} = \left( N; A^{(i)}, B^{(i)}, C^{(i)}, D; \mathbb{C}^{\gamma^{(i)}} = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k^{(i)}}, \mathbb{C}^q \right)$$

and the associated structured Hermitian matrices  $H^{(i)} = \text{diag}(H_1^{(i)}, \dots, H_N^{(i)})$ . Assume that the product  $\alpha = \alpha^{(1)} \alpha^{(2)}$  is a minimal GR-node. Then, for each  $k \in \{1, \dots, N\}$  the matrix

$$H_k = \begin{pmatrix} H_k^{(1)} & 0 \\ 0 & H_k^{(2)} \end{pmatrix} \in \mathbb{C}^{(\gamma_k^{(1)} + \gamma_k^{(2)}) \times (\gamma_k^{(1)} + \gamma_k^{(2)})} \quad (5.22)$$

is the associated  $k$ -th Hermitian matrix for  $\alpha = \alpha^{(1)} \alpha^{(2)}$ .

**Corollary 5.14.** *Let  $f_1$  and  $f_2$  be two matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPSs, and assume that the factorization  $f = f_1 f_2$  is minimal. Then,*

$$\nu(f_1 f_2) = \nu(f_1) + \nu(f_2).$$

### 5.3. Minimal matrix- $J$ -unitary factorizations

In this subsection we consider minimal factorizations of matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPSs into two factors, both of which are also matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPSs. Such factorizations will be called *minimal matrix- $J$ -unitary factorizations*.

The following theorem is analogous to its one-variable counterpart [7, Theorem 3.7] and proved in the same way.

**Theorem 5.15.** *Let  $f$  be a matrix- $J$ -unitary on  $\mathcal{T}_N$  rational FPS and  $\alpha$  be its minimal GR-realization of the form (3.11), with the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$ , and assume that  $D$  is invertible. Let  $\mathcal{M} = \bigoplus_{k=1}^N \mathcal{M}_k$  be an  $A$ -invariant subspace of  $\mathbb{C}^\gamma$ , which is non-degenerate in the associated inner product  $[\cdot, \cdot]_H$  and such that  $\mathcal{M}_k \subset \mathbb{C}^{\gamma_k}$ ,  $k = 1, \dots, N$ . Let  $\Pi = \text{diag}(\Pi_1, \dots, \Pi_N)$  be a projection defined by*

$$\ker \Pi = M, \quad \text{and} \quad \text{ran } \Pi = M^{\perp},$$

that is

$$\ker \Pi_k = M_k, \quad \text{and} \quad \text{ran} \Pi_k = M_k^{\perp} \quad \text{for} \quad k = 1, \dots, N.$$

Then  $f(z) = f_1(z)f_2(z)$ , where

$$f_1(z) = [I_q + C(I_\gamma - \Delta(z)A)^{-1}\Delta(z)(I_\gamma - \Pi)BD^{-1}] D_1, \quad (5.23)$$

$$f_2(z) = D_2 [I_q + D^{-1}C\Pi(I_\gamma - \Delta(z)A)^{-1}\Delta(z)B], \quad (5.24)$$

with

$$D_1 = I_q - CH^{-1}(I_\gamma - aA^*)^{-1}C^*J, \quad D = D_1D_2,$$

where  $a \in \mathbb{T}$  belongs to the resolvent set of  $A_1$ , and where

$$C_1 = C|_{\mathcal{M}}, \quad A_1 = A|_{\mathcal{M}}, \quad H_1 = P_{\mathcal{M}}H|_{\mathcal{M}}$$

(with  $P_{\mathcal{M}}$  being the orthogonal projection onto  $\mathcal{M}$  in the standard metric of  $\mathbb{C}^\gamma$ ), is a minimal matrix- $J$ -unitary factorization of  $f$ .

Conversely, any minimal matrix- $J$ -unitary factorization of  $f$  can be obtained in such a way, and the correspondence between minimal matrix- $J$ -unitary factorizations of  $f$  with  $f_1(\bar{a}, \dots, \bar{a}) = I_q$  and non-degenerate subspaces of  $A$  of the form  $\mathcal{M} = \bigoplus_{k=1}^N \mathcal{M}_k$ , with  $\mathcal{M}_k \subset \mathbb{C}^{\gamma_k}$ ,  $k = 1, \dots, N$ , is one-to-one.

*Remark 5.16.* In the proof of Theorem 5.15, as well as of Theorem 4.10, we make use of Theorem 3.9 and Corollary 3.10.

*Remark 5.17.* Minimal matrix- $J$ -unitary factorizations do not always exist, even in the case  $N = 1$ . See [7] for examples in that case.

#### 5.4. Matrix-unitary rational formal power series

In this subsection we specialize some of the results in the present section to the case  $J = I_q$ . We shall call corresponding rational FPSs *matrix-unitary on  $\mathcal{T}_N$* .

**Theorem 5.18.** *Let  $f$  be a rational FPS and  $\alpha$  be its minimal GR-realization of the form (3.11). Then  $f$  is matrix-unitary on  $\mathcal{T}_N$  if and only if:*

(a) *There exists an Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$  (with  $H_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ ) such that*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} H & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & I_q \end{pmatrix}. \quad (5.25)$$

*Condition (a) is equivalent to:*

(a') *There exists an Hermitian matrix  $G = \text{diag}(G_1, \dots, G_N)$  (with  $G_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ ) such that*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} G & 0 \\ 0 & I_q \end{pmatrix}. \quad (5.26)$$

*Proof.* The necessity follows from Theorem 5.1. To prove the sufficiency, suppose that the Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$  satisfies (5.25) and let  $a \in \mathbb{T}$  be such that  $-\bar{a} \notin \sigma(A)$ . Then,  $H$  satisfies conditions (4.18) and (4.19) for the GR-node  $\beta = (N; A_a, B_a, C_a, D_a; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q)$  defined by (5.2) (this follows

from the proof of Theorem 5.1). Thus, from Theorem 4.13 and Theorem 5.1 we obtain that  $f$  is matrix-unitary on  $\mathcal{T}_N$ . Analogously, condition (a') implies that the FPS  $f$  is matrix-unitary on  $\mathcal{T}_N$ .  $\square$

A counterpart of Theorem 4.14 in the present case is the following theorem:

**Theorem 5.19.** *Let  $(C, A)$  be an observable pair of matrices in the sense that  $\mathcal{O}_k$  has full column rank for each  $k = 1, \dots, N$ . Assume that  $A \in \mathbb{C}^{r \times r}$  is invertible. Then there exists a matrix-unitary on  $\mathcal{T}_N$  rational FPS  $f$  with a minimal GR-realization  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$  if and only if the Stein equation*

$$H - A^* H A = C^* C \quad (5.27)$$

*has an Hermitian solution  $H = \text{diag}(H_1, \dots, H_N)$ , with  $H_k \in \mathbb{C}^{r_k \times r_k}$ ,  $k = 1, \dots, N$ . If such a matrix  $H$  exists, it is invertible, and possible choices of  $D$  and  $B$  are  $D_a$  and  $B_a$  given by (5.16) and (5.17) with  $J = I_q$ . Finally, for a given  $H = \text{diag}(H_1, \dots, H_N)$ , all other choices of  $D$  and  $B$  differ from  $D_a$  and  $B_a$  by a right multiplicative unitary constant.*

A counterpart of Theorem 4.15 is the following theorem:

**Theorem 5.20.** *Let  $(A, B)$  be a controllable pair of matrices, in the sense that  $\mathcal{C}_k$  has full row rank for each  $k = 1, \dots, N$ . Assume that  $A \in \mathbb{C}^{r \times r}$  is invertible. Then there exists a matrix-unitary on  $\mathcal{T}_N$  rational FPS  $f$  with a minimal GR-realization  $\alpha = (N; A, B, C, D; \mathbb{C}^r = \bigoplus_{k=1}^N \mathbb{C}^{r_k}, \mathbb{C}^q)$  if and only if the Stein equation*

$$G - A G A^* = B B^* \quad (5.28)$$

*has an Hermitian solution  $G = \text{diag}(G_1, \dots, G_N)$  with  $G_k \in \mathbb{G}^{r_k \times r_k}$ ,  $k = 1, \dots, N$ . If such a matrix  $G$  exists, it is invertible, and possible choices of  $D$  and  $C$  are  $D'_a$  and  $C'_a$  given by (5.18) and (5.19) with  $H = G^{-1}$  and  $J = I_q$ . Finally, for a given  $G = \text{diag}(G_1, \dots, G_N)$ , all other choices of  $D$  and  $C$  differ from  $D'_a$  and  $C'_a$  by a left multiplicative unitary constant.*

A counterpart of Theorem 4.16 in the present case is the following:

**Theorem 5.21.** *Let  $f$  be a matrix-unitary on  $\mathcal{T}_N$  rational FPS and  $\alpha$  be its minimal GR-realization of the form (3.11), with the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$  and the associated  $k$ -th inner products  $[\cdot, \cdot]_{H_k}$ ,  $k = 1, \dots, N$ . Let  $P_k$  denote the orthogonal projection in  $\mathbb{C}^\gamma$  onto the subspace  $\{0\} \oplus \dots \oplus \{0\} \oplus \mathbb{C}^{\gamma_k} \oplus \{0\} \oplus \dots \oplus \{0\}$ , and set  $\overline{A}_k = A P_k$  for  $k = 1, \dots, N$ . If  $x \in \mathbb{C}^\gamma$  is a common eigenvector for  $\overline{A} = \{\overline{A}_1, \dots, \overline{A}_N\}$  corresponding to a common eigenvalue  $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ , then there exists  $j \in \{1, \dots, N\}$  such that  $|\lambda_j| \neq 1$  and  $[P_j x, P_j x]_{H_j} \neq 0$ . In particular  $\overline{A}$  has no common eigenvalues on  $\mathbb{T}^N$ .*

The proof of this theorem relies on the equality

$$(1 - |\lambda_k|^2) [P_k x, P_k x]_{H_k} = \langle C P_k x, C P_k x \rangle, \quad k = 1, \dots, N,$$

and follows the same argument as the proof of Theorem 4.16.



## 6. Matrix- $J$ -inner rational formal power series

### 6.1. A multivariable non-commutative analogue of the halfplane case

Let  $n \in \mathbb{N}$ . We define the *matrix open right poly-halfplane* as the set

$$(\Pi^{n \times n})^N = \left\{ Z = (Z_1, \dots, Z_N) \in (\mathbb{C}^{n \times n})^N : Z_k + Z_k^* > 0, k = 1, \dots, N \right\},$$

and the *matrix closed right poly-halfplane* as the set

$$\begin{aligned} \text{clos } (\Pi^{n \times n})^N &= (\text{clos } \Pi^{n \times n})^N \\ &= \left\{ Z = (Z_1, \dots, Z_N) \in (\mathbb{C}^{n \times n})^N : Z_k + Z_k^* \geq 0, k = 1, \dots, N \right\}. \end{aligned}$$

We also introduce

$$\mathcal{P}_N = \prod_{n \in \mathbb{N}} (\Pi^{n \times n})^N \quad \text{and} \quad \text{clos } \mathcal{P}_N = \prod_{n \in \mathbb{N}} \text{clos } (\Pi^{n \times n})^N.$$

It is clear that

$$(i\mathbb{H}^{n \times n})^N \subset \text{clos } (\Pi^{n \times n})^N$$

is the *essential* (or *Shilov*) *boundary* of the matrix poly-halfplane  $(\Pi^{n \times n})^N$  (see [45]) and that  $\mathcal{J}_N \subset \text{clos } \mathcal{P}_N$  (recall that  $\mathcal{J}_N = \prod_{n \in \mathbb{N}} (i\mathbb{H}^{n \times n})^N$ ).

Let  $J = J^{-1} = J^* \in \mathbb{C}^{q \times q}$ . A matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS  $F$  is called *matrix- $J$ -inner* (in  $\mathcal{P}_N$ ) if for each  $n \in \mathbb{N}$ :

$$F(Z)(J \otimes I_n)F(Z)^* \leq J \otimes I_n \quad (6.1)$$

at those points  $Z \in \text{clos } (\Pi^{n \times n})^N$  where it is defined (the set of such points is open and dense, in the relative topology, in  $\text{clos } (\Pi^{n \times n})^N$  since  $F(Z)$  is a rational matrix-valued function of the complex variables  $(Z_k)_{ij}$ ,  $k = 1, \dots, N$ ,  $i, j = 1, \dots, n$ ).

The following theorem is a counterpart of part a) of Theorem 2.16 of [7].

**Theorem 6.1.** *Let  $F$  be a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS and  $\alpha$  be its minimal GR-realization of the form (3.11). Then  $F$  is matrix- $J$ -inner in  $\mathcal{P}_N$  if and only if the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$  is strictly positive.*

*Proof.* Let  $n \in \mathbb{N}$ . Equality (4.9) can be rewritten as

$$J \otimes I_n - F(Z)(J \otimes I_n)F(Z')^* = \varphi(Z)\Delta(Z + Z'^*)(H^{-1} \otimes I_n)\varphi(Z')^* \quad (6.2)$$

where  $\varphi$  is a FPS defined by

$$\varphi(z) := C(I_\gamma - \Delta(z)A)^{-1} \in \mathbb{C}^{q \times \gamma} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}},$$

and (6.2) is well defined at all points  $Z, Z' \in (\mathbb{C}^{n \times n})^N$  for which

$$1 \notin \sigma(\Delta(Z)(A \otimes I_n)), \quad 1 \notin \sigma(\Delta(Z')(A \otimes I_n)).$$

Set  $\varphi_k(z) := C(I_\gamma - \Delta(z)A)^{-1}|_{\mathbb{C}^{\gamma_k}} \in \mathbb{C}^{q \times \gamma_k} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$ ,  $k = 1, \dots, N$ . Then (6.2) becomes:

$$J \otimes I_n - F(Z)(J \otimes I_n)F(Z')^* = \sum_{k=1}^N \varphi_k(Z)(H_k^{-1} \otimes (Z_k + Z_k'^*))\varphi_k(Z')^*. \quad (6.3)$$

Let  $X \in \mathbb{H}^{n \times n}$  be some positive semidefinite matrix, let  $Y \in (\mathbb{H}^{n \times n})^N$  be such that  $1 \notin \sigma(\Delta(iY)(A \otimes I_n))$ , and set for  $k = 1, \dots, N$ :

$$e_k := (0, 0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{C}^N$$

with 1 at the  $k$ -th place. Then for  $\lambda \in \mathbb{C}$  set

$$Z_{X,Y}^{(k)}(\lambda) := \lambda X \otimes e_k + iY = (iY_1, \dots, iY_{k-1}, \lambda X + iY_k, iY_{k+1}, \dots, iY_N).$$

Now, (6.3) implies that

$$\begin{aligned} J \otimes I_n - F(Z_{X,Y}^{(k)}(\lambda))(J \otimes I_n)F(Z_{X,Y}^{(k)}(\lambda'))^* \\ = (\lambda + \overline{\lambda'})\varphi_k(Z_{X,Y}^{(k)}(\lambda))(H_k^{-1} \otimes X)\varphi_k(Z_{X,Y}^{(k)}(\lambda'))^*. \end{aligned} \quad (6.4)$$

The function  $h(\lambda) = F(Z_{X,Y}^{(k)}(\lambda))$  is a rational function of  $\lambda \in \mathbb{C}$ . It is easily seen from (6.4) that  $h$  is  $(J \otimes I_n)$ -inner in the open right halfplane. In particular, it is  $(J \otimes I_n)$ -contractive in the closed right halfplane (this also follows directly from (6.1)). Therefore (see e.g. [22]) the function

$$\Psi(\lambda, \lambda') = \frac{J \otimes I_n - F(Z_{X,Y}^{(k)}(\lambda))(J \otimes I_n)F(Z_{X,Y}^{(k)}(\lambda'))^*}{\lambda + \overline{\lambda'}} \quad (6.5)$$

is a positive semidefinite kernel on  $\mathbb{C}$ : for every choice of  $r \in \mathbb{N}$ , of points  $\lambda_1, \dots, \lambda_r \in \mathbb{C}$  for which the matrices  $\Psi(\lambda_j, \lambda_i)$  are well defined, and vectors  $c_1, \dots, c_r \in \mathbb{C}^q \otimes \mathbb{C}^n$  one has

$$\sum_{i,j=1}^r c_j^* \Psi(\lambda_j, \lambda_i) c_i \geq 0,$$

i.e., the matrix  $(\Psi(\lambda_j, \lambda_i))_{i,j=1,\dots,r}$  is positive semidefinite. Since  $\varphi_k(Z_{X,Y}^{(k)}(0)) = \varphi_k(iY)$  is well-defined, we obtain from (6.4) that  $\Psi(0,0)$  is also well-defined and

$$\Psi(0,0) = \varphi_k(iY)(H_k^{-1} \otimes X)\varphi_k(iY)^* \geq 0.$$

This inequality holds for every  $n \in \mathbb{N}$ , every positive semidefinite  $X \in \mathbb{H}^{n \times n}$  and every  $Y \in (\mathbb{H}^{n \times n})^N$ . Thus, for an arbitrary  $r \in \mathbb{N}$  we can define  $\tilde{n} = nr$ ,  $\tilde{Y} = (\tilde{Y}_1, \dots, \tilde{Y}_N) \in (\mathbb{H}^{\tilde{n} \times \tilde{n}})^N$ , where  $\tilde{Y}_k = \text{diag}(Y_k^{(1)}, \dots, Y_k^{(r)})$  and  $Y_k^{(j)} \in \mathbb{H}^{n \times n}$ ,  $k = 1, \dots, N$ ,  $j = 1, \dots, r$ , such that  $\varphi_k(i\tilde{Y})$  is well defined,

$$\tilde{X} = \begin{pmatrix} I_n & \cdots & I_n \\ \vdots & & \vdots \\ I_n & \cdots & I_n \end{pmatrix} \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^{r \times r} \cong \mathbb{C}^{\tilde{n} \times \tilde{n}}$$

and get

$$\begin{aligned}
0 &\leq \varphi_k(i\tilde{Y})(H_k^{-1} \otimes \tilde{X})\varphi_k(i\tilde{Y})^* \\
&= \text{diag}(\varphi_k(iY^{(1)}), \dots, \varphi_k(iY^{(r)})) \times \\
&\times \left( H_k^{-1} \otimes \begin{pmatrix} I_n & & \\ & \ddots & \\ & & I_n \end{pmatrix} \begin{pmatrix} I_n & \cdots & I_n \end{pmatrix} \right) \text{diag}(\varphi_k(iY^{(1)})^*, \dots, \varphi_k(iY^{(r)})^*) \\
&= \begin{pmatrix} \varphi_k(iY^{(1)}) \\ \vdots \\ \varphi_k(iY^{(r)}) \end{pmatrix} (H_k^{-1} \otimes I_n) \begin{pmatrix} \varphi_k(iY^{(1)})^* & \cdots & \varphi_k(iY^{(r)})^* \end{pmatrix} \\
&= \left( \varphi_k(iY^{(\mu)})(H_k^{-1} \otimes I_n)\varphi_k(iY^{(\nu)})^* \right)_{\mu, \nu=1, \dots, r}.
\end{aligned}$$

Therefore, the function

$$K_k(iY, iY') = \varphi_k(iY)(H_k^{-1} \otimes I_n)\varphi_k(iY')^*$$

is a positive semidefinite kernel on any subset of  $(i\mathbb{H}^{n \times n})^N$  where it is defined, and in particular in some neighbourhood of the origin. One can extend this function to

$$K_k(Z, Z') = \varphi_k(Z)(H_k^{-1} \otimes I_n)\varphi_k(Z')^* \quad (6.6)$$

at those points  $Z, Z' \in (\mathbb{C}^{n \times n})^N \times (\mathbb{C}^{n \times n})^N$  where  $\varphi_k$  is defined. Thus, on some neighbourhood  $\Gamma$  of the origin in  $(\mathbb{C}^{n \times n})^N \times (\mathbb{C}^{n \times n})^N$ , the function  $K_k(Z, Z')$  is holomorphic in  $Z$  and anti-holomorphic in  $Z'$ . On the other hand, it is well-known (see e.g. [9]) that one can construct a reproducing kernel Hilbert space (which we will denote by  $\mathcal{H}(K_k)$ ) with reproducing kernel  $K_k(iY, iY')$ , which is obtained as the completion of

$$\mathcal{H}_0 = \text{span} \{ K_k(\cdot, iY)x; iY \in (i\mathbb{H}^{n \times n})^N \cap \Gamma, x \in \mathbb{C}^q \otimes \mathbb{C}^n \}$$

with respect to the inner product

$$\begin{aligned}
&\left\langle \sum_{\mu=1}^r K_k(\cdot, iY^{(\mu)})x_\mu, \sum_{\nu=0}^{\ell} K_k(\cdot, iY^{(\nu)})x_\nu \right\rangle_{\mathcal{H}_0} \\
&= \sum_{\mu=1}^r \sum_{\nu=1}^{\ell} \left\langle K_k(iY^{(\nu)}, iY^{(\mu)})x_\mu, x_\nu \right\rangle_{\mathbb{C}^q \otimes \mathbb{C}^n}.
\end{aligned}$$

The reproducing kernel property reads:

$$\langle f(\cdot), K_k(\cdot, iY)x \rangle_{\mathcal{H}(K_k)} = \langle f(iY), x \rangle_{\mathbb{C}^q \otimes \mathbb{C}^n},$$

and thus  $K_k(iY, iY') = \Phi(iY)\Phi(iY')^*$  where

$$\Phi(iY) : f(\cdot) \mapsto f(iY)$$

is the evaluation map. In view of (6.6), the kernel  $K_k(\cdot, \cdot)$  is extendable on  $\Gamma \times \Gamma$  to the function  $K(Z, Z')$  which is holomorphic in  $Z$  and antiholomorphic in  $Z'$ ,

all the elements of  $\mathcal{H}(K_k)$  have holomorphic continuations to  $\Gamma$ , and so has the function  $\Phi(\cdot)$ . Thus,

$$K_k(Z, Z') = \Phi(Z)\Phi(Z')^*$$

and so  $K_k(Z, Z')$  is a positive semidefinite kernel on  $\Gamma$ . (We could also use [3, Theorem 1.1.4, p.10] to obtain this conclusion.) Therefore, for any choice of  $\ell \in \mathbb{N}$  and  $Z^{(1)}, \dots, Z^{(\ell)} \in \Gamma$  the matrix

$$\begin{aligned} & \left( \varphi_k(Z^{(\mu)})(H_k^{-1} \otimes I_n) \varphi_k(Z^{(\nu)})^* \right)_{\mu, \nu=1, \dots, \ell} \\ &= \begin{pmatrix} \varphi_k(Z^{(1)}) \\ \vdots \\ \varphi_k(Z^{(\ell)}) \end{pmatrix} \cdot (H_k^{-1} \otimes I_n) \cdot (\varphi_k(Z^{(1)})^* \quad \dots \quad \varphi_k(Z^{(\ell)})^*) \end{aligned} \quad (6.7)$$

is positive semidefinite. Since the coefficients of the FPS  $\varphi_k$  are  $(\varphi_k)_w = (C \flat A)^{wg_k}$ ,  $w \in \mathcal{F}_N$ , and since  $\alpha$  is an observable GR-node, we have

$$\bigcap_{w \in \mathcal{F}_N} \ker(C \flat A)^{wg_k} = \{0\}.$$

Hence, by Theorem 2.1 we can chose  $n, \ell \in \mathbb{N}$  and  $Z^{(1)}, \dots, Z^{(\ell)} \in \Gamma$  such that

$$\bigcap_{j=1}^{\ell} \ker \varphi_k(Z^{(j)}) = \{0\}.$$

Thus the matrix  $\text{col}_{j=1, \dots, \ell}(\varphi_k(Z^{(j)}))$  has full column rank. (We could also use Theorem 3.7.) From (6.7) it then follows that  $H_k^{-1} > 0$ . Since this holds for all  $k \in \{1, \dots, N\}$ , we get  $H > 0$ .

Conversely, if  $H > 0$  then it follows from (6.2) that for every  $n \in \mathbb{N}$  and  $Z \in (\Pi^{n \times n})^N$  for which  $1 \notin \sigma(\Delta(Z)(A \otimes I_n))$ , one has

$$J \otimes I_n - F(Z)(J \otimes I_n)F(Z)^* \geq 0.$$

Therefore  $F$  is matrix- $J$ -inner in  $\mathcal{P}_N$ , and the proof is complete.  $\square$

**Theorem 6.2.** *Let  $F \in \mathbb{C}^{q \times q} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$  be matrix- $J$ -inner in  $\mathcal{P}_N$ . Then  $F$  has a minimal GR-realization of the form (3.11) with the associated structured Hermitian matrix  $H = I_\gamma$ . This realization is unique up to a unitary similarity.*

*Proof.* Let

$$\alpha^\circ = (N; A^\circ, B^\circ, C^\circ, D; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q)$$

be a minimal GR-realization of  $F$ , with the associated structured Hermitian matrix  $H^\circ = \text{diag}(H_1^\circ, \dots, H_N^\circ)$ . By Theorem 6.1 the matrix  $H^\circ$  is strictly positive. Therefore,  $(H^\circ)^{1/2} = \text{diag}((H_1^\circ)^{1/2}, \dots, (H_N^\circ)^{1/2})$  is well defined and strictly positive, and

$$\alpha = (N; A, B, C, D; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q),$$

where

$$A = (H^\circ)^{1/2} A^\circ (H^\circ)^{-1/2}, \quad B = (H^\circ)^{1/2} B^\circ, \quad C = C^\circ (H^\circ)^{-1/2}, \quad (6.8)$$

is a minimal GR-realization of  $F$  satisfying

$$A^* + A = -C^* J C, \quad (6.9)$$

$$B = -C^* J D, \quad (6.10)$$

or equivalently,

$$A^* + A = -B J B^*, \quad (6.11)$$

$$C = -D J B^*, \quad (6.12)$$

and thus having the associated structured Hermitian matrix  $H = I_\gamma$ . Since in this case the inner product  $[\cdot, \cdot]_H$  coincides with the standard inner product  $\langle \cdot, \cdot \rangle$  of  $\mathbb{C}^\gamma$ , by Remark 4.6 this minimal GR-realization with the property  $H = I_\gamma$  is unique up to unitary similarity.  $\square$

We remark that a one-variable counterpart of the latter result is essentially contained in [20], [38] (see also [10, Section 4.2]).

## 6.2. A multivariable non-commutative analogue of the disk case

Let  $n \in \mathbb{N}$ . We define the *matrix open unit polydisk* as

$$(\mathbb{D}^{n \times n})^N = \left\{ W = (W_1, \dots, W_N) \in (\mathbb{C}^{n \times n})^N : W_k W_k^* < I_n, k = 1, \dots, N \right\},$$

and the *matrix closed unit polydisk* as

$$\begin{aligned} \text{clos}(\mathbb{D}^{n \times n})^N &= (\text{clos} \mathbb{D}^{n \times n})^N \\ &= \left\{ W = (W_1, \dots, W_N) \in (\mathbb{C}^{n \times n})^N : W_k W_k^* \leq I_n, k = 1, \dots, N \right\}. \end{aligned}$$

The matrix unit torus  $(\mathbb{T}^{n \times n})^N$  is the essential (or Shilov) boundary of  $(\mathbb{D}^{n \times n})^N$  (see [45]). In our setting, the set

$$\mathcal{D}_N = \prod_{n \in \mathbb{N}} (\mathbb{D}^{n \times n})^N \quad \left( \text{resp.,} \quad \text{clos} \mathcal{D}_N = \prod_{n \in \mathbb{N}} \text{clos}(\mathbb{D}^{n \times n})^N \right)$$

is a multivariable non-commutative counterpart of the open (resp., closed) unit disk.

Let  $J = J^{-1} = J^* \in \mathbb{C}^{q \times q}$ . A rational FPS  $f$  which is matrix- $J$ -unitary on  $\mathcal{T}_N$  is called *matrix- $J$ -inner in  $\mathcal{D}_N$*  if for every  $n \in \mathbb{N}$ :

$$f(W)(J \otimes I_n) f(W)^* \leq J \otimes I_n \quad (6.13)$$

at those points  $W \in \text{clos}(\mathbb{D}^{n \times n})^N$  where it is defined. We note that the set of such points is open and dense (in the relative topology) in  $\text{clos}(\mathbb{D}^{n \times n})^N$  since  $f(W)$  is a rational matrix-valued function of the complex variables  $(W_k)_{ij}$ ,  $k = 1, \dots, N$ ,  $i, j = 1, \dots, n$ .

**Theorem 6.3.** *Let  $f$  be a rational FPS which is matrix- $J$ -unitary on  $\mathcal{T}_N$ , and let  $\alpha$  be its minimal GR-realization of the form (3.11). Then  $f$  is matrix- $J$ -inner in  $\mathcal{D}_N$  if and only if the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$  is strictly positive.*

*Proof.* The statement of this theorem follows from Theorem 6.1 and Theorem 5.1, since the Cayley transform defined in Theorem 5.1 maps each open matrix unit polydisk  $(\mathbb{D}^{n \times n})^N$  onto the open right matrix poly-halfplane  $(\Pi^{n \times n})^N$ , and the inequality (6.13) turns into (6.1) for the function  $F$  defined in (5.1).  $\square$

The following theorem is an analogue of Theorem 6.2.

**Theorem 6.4.** *Let  $f$  be a rational FPS which is matrix- $J$ -inner in  $\mathcal{D}_N$ . Then there exists its minimal GR-realization  $\alpha$  of the form (3.11), with the associated structured Hermitian matrix  $H = I_\gamma$ . Such a realization is unique up to a unitary similarity.*

In the special case of Theorem 6.4 where  $J = I_q$  the FPS  $f$  is called *matrix-inner*, and the GR-node  $\alpha$  satisfies

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} A & B \\ C & D \end{pmatrix} = I_{\gamma+q},$$

i.e.,  $\alpha$  is a *unitary GR-node*, which has been considered first by J. Agler in [1]. In what follows we will show that Theorem 6.4 for  $J = I_q$  is a special case of the theorem of J. A. Ball, G. Groenewald and T. Malakorn on unitary GR-realizations of FPSs from the non-commutative Schur–Agler class [12], which becomes in several aspects stronger in this special case.

Let  $\mathcal{U}$  and  $\mathcal{Y}$  be Hilbert spaces. Denote by  $L(\mathcal{U}, \mathcal{Y})$  the Banach space of bounded linear operators from  $\mathcal{U}$  into  $\mathcal{Y}$ . A GR-node in the general setting of Hilbert spaces is

$$\alpha = (N; A, B, C, D; \mathcal{X} = \bigoplus_{k=1}^N \mathcal{X}_k, \mathcal{U}, \mathcal{Y}),$$

i.e., a collection of Hilbert spaces  $\mathcal{X}, \mathcal{X}_1, \dots, \mathcal{X}_N, \mathcal{U}, \mathcal{Y}$  and operators  $A \in L(\mathcal{X}) = L(\mathcal{X}, \mathcal{X})$ ,  $B \in L(\mathcal{U}, \mathcal{X})$ ,  $C \in L(\mathcal{X}, \mathcal{Y})$ , and  $D \in L(\mathcal{U}, \mathcal{Y})$ . Such a GR-node  $\alpha$  is called *unitary* if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} A & B \\ C & D \end{pmatrix} = I_{\mathcal{X} \oplus \mathcal{U}}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = I_{\mathcal{X} \oplus \mathcal{Y}},$$

i.e.,  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$  is a unitary operator from  $\mathcal{X} \oplus \mathcal{U}$  onto  $\mathcal{X} \oplus \mathcal{Y}$ . The *non-commutative transfer function* of  $\alpha$  is

$$T_\alpha^{\text{nc}}(z) = D + C(I - \Delta(z)A)^{-1}\Delta(z)B, \quad (6.14)$$

where the expression (6.14) is understood as a FPS from  $L(\mathcal{U}, \mathcal{Y}) \langle \langle z_1, \dots, z_N \rangle \rangle$  given by

$$T_\alpha^{\text{nc}}(z) = D + \sum_{w \in \mathcal{F}_N \setminus \{\emptyset\}} (C \flat A \sharp B)^w z^w = D + \sum_{k=0}^{\infty} C (\Delta(z)A)^k \Delta(z)B. \quad (6.15)$$

The non-commutative Schur–Agler class  $\mathcal{SA}_N^{\text{nc}}(\mathcal{U}, \mathcal{Y})$  consists of all FPSs  $f \in L(\mathcal{U}, \mathcal{Y}) \langle \langle z_1, \dots, z_N \rangle \rangle$  such that for any separable Hilbert space  $\mathcal{K}$  and any  $N$ -tuple  $\delta = (\delta_1, \dots, \delta_N)$  of strict contractions in  $\mathcal{K}$  the limit in the operator norm topology

$$f(\delta) = \lim_{m \rightarrow \infty} \sum_{w \in \mathcal{F}_N: |w| \leq m} f_w \otimes \delta^w$$

exists and defines a contractive operator  $f(\delta) \in L(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K})$ . We note that the non-commutative Schur–Agler class was defined in [12] also for a more general class of operator  $N$ -tuples  $\delta$ .

Consider another set of non-commuting indeterminates  $z' = (z'_1, \dots, z'_N)$ . For  $f(z) \in L(\mathcal{V}, \mathcal{Y}) \langle \langle z_1, \dots, z_N \rangle \rangle$  and  $f'(z') \in L(\mathcal{V}, \mathcal{U}) \langle \langle z'_1, \dots, z'_N \rangle \rangle$  we define a FPS

$$f(z)f'(z')^* \in L(\mathcal{U}, \mathcal{Y}) \langle \langle z_1, \dots, z_N, z'_1, \dots, z'_N \rangle \rangle$$

by

$$f(z)f'(z')^* = \sum_{w, w' \in \mathcal{F}_N} f_w (f'_{w'})^* z^w z'^{w'T}. \quad (6.16)$$

In [12] the class  $\mathcal{SA}_N^{\text{nc}}(\mathcal{U}, \mathcal{Y})$  was characterized as follows:

**Theorem 6.5.** *Let  $f \in L(\mathcal{U}, \mathcal{Y}) \langle \langle z_1, \dots, z_N \rangle \rangle$ . The following statements are equivalent:*

- (1)  $f \in \mathcal{SA}_N^{\text{nc}}(\mathcal{U}, \mathcal{Y})$ ;
- (2) *there exist auxiliary Hilbert spaces  $\mathcal{H}, \mathcal{H}_1, \dots, \mathcal{H}_N$  which are related by  $\mathcal{H} = \bigoplus_{k=1}^N \mathcal{H}_k$ , and a FPS  $\varphi \in L(\mathcal{H}, \mathcal{Y}) \langle \langle z_1, \dots, z_N \rangle \rangle$  such that*

$$I_{\mathcal{Y}} - f(z)f'(z')^* = \varphi(z)(I_{\mathcal{H}} - \Delta(z)\Delta(z')^*)\varphi(z')^*; \quad (6.17)$$

- (3) *there exists a unitary GR-node  $\alpha = (N; A, B, C, D; \mathcal{X} = \bigoplus_{k=1}^N \mathcal{X}_k, \mathcal{U}, \mathcal{Y})$  such that  $f = T_\alpha^{\text{nc}}$ .*

We now give another characterization of the Schur–Agler class  $\mathcal{SA}_N^{\text{nc}}(\mathcal{U}, \mathcal{Y})$ .

**Theorem 6.6.** *A FPS  $f$  belongs to  $\mathcal{SA}_N^{\text{nc}}(\mathcal{U}, \mathcal{Y})$  if and only if for every  $n \in \mathbb{N}$  and  $W \in (\mathbb{D}^{n \times n})^N$  the limit in the operator norm topology*

$$f(W) = \lim_{m \rightarrow \infty} \sum_{w \in \mathcal{F}_N: |w| \leq m} f_w \otimes W^w \quad (6.18)$$

*exists and  $\|f(W)\| \leq 1$ .*

*Proof.* The necessity is clear. We prove the sufficiency. We set

$$f_k(z) = \sum_{w \in \mathcal{F}_N: |w|=k} f_w z^w, \quad k = 0, 1, \dots$$

Then for every  $n \in \mathbb{N}$  and  $W \in (\mathbb{D}^{n \times n})^N$ , (6.18) becomes

$$f(W) = \lim_{m \rightarrow \infty} \sum_{k=0}^m f_k(W), \quad (6.19)$$

where the limit is taken in the operator norm topology. Let  $r \in (0, 1)$  and choose  $\tau > 0$  such that  $r + \tau < 1$ . Let  $W \in (\mathbb{D}^{n \times n})^N$  be such that  $\|W_j\| \leq r$ ,  $j = 1, \dots, N$ . Then, for every  $x \in \mathcal{U} \otimes \mathbb{C}^n$  the series

$$f\left(\frac{r+\tau}{r}\lambda W\right)x = \sum_{k=0}^{\infty} \lambda^k f_k\left(\frac{r+\tau}{r}W\right)x$$

converges uniformly in  $\lambda \in \text{clos } \mathbb{D}$  to a  $\mathcal{Y} \otimes \mathbb{C}^n$ -valued function holomorphic on  $\text{clos } \mathbb{D}$ . Furthermore,

$$\left\| f_k\left(\frac{r+\tau}{r}W\right)x \right\| = \left\| \frac{1}{2\pi i} \int_{\mathbb{T}} f\left(\frac{r+\tau}{r}\lambda W\right)x \lambda^{-k-1} d\lambda \right\| \leq \|x\|,$$

and therefore

$$\|f_k(W)\| = \left\| f_k\left(\frac{r+\tau}{r}W\right) \left(\frac{r}{r+\tau}\right)^k \right\| \leq \left(\frac{r}{r+\tau}\right)^k. \quad (6.20)$$

Thus we have

$$\left\| f(W) - \sum_{k=0}^m f_k(W) \right\| \leq \sum_{k=m+1}^{\infty} \|f_k(W)\| \leq \sum_{k=m+1}^{\infty} \left(\frac{r}{r+\tau}\right)^k < \infty.$$

We observe that the limit in (6.19) is uniform in  $n \in \mathbb{N}$  and  $W \in (\mathbb{D}^{n \times n})^N$  such that  $\|W_j\| \leq r$ ,  $j = 1, \dots, N$ . Without loss of generality we may assume that in the definition of the Schur–Agler class the space  $\mathcal{K}$  is taken to be the space  $\ell_2$  of square summable sequences  $s = (s_j)_{j=1}^{\infty}$  of complex numbers indexed by  $\mathbb{N}$ :  $\sum_{j=1}^{\infty} |s_j|^2 < \infty$ . We denote by  $P_n$  the orthogonal projection from  $\ell_2$  onto the subspace of sequences for which  $s_j = 0$  for  $j > n$ . This subspace is isomorphic to  $\mathbb{C}^n$ , and thus for every  $\delta = (\delta_1, \dots, \delta_N) \in L(\ell_2)^N$  such that  $\|\delta_j\| \leq r$ ,  $j = 1, \dots, N$ , we may use (6.20) and write

$$\|f_k(P_n \delta_1 P_n, \dots, P_n \delta_N P_n)\| \leq \left(\frac{r}{r+\tau}\right)^k. \quad (6.21)$$

Since the sequence  $P_n$  converges to  $I_{\ell_2}$  in the strong operator topology (see, e.g., [2]), and since strong limits of finite sums and products of operator sequences are equal to the corresponding sums and products of strong limits of these sequences, we obtain that

$$s - \lim_{n \rightarrow \infty} f_k(P_n \delta_1 P_n, \dots, P_n \delta_N P_n) = f_k(\delta).$$



Thus from (6.21) we obtain

$$\|f_k(\delta)\| \leq \left(\frac{r}{r+\tau}\right)^k.$$

Therefore, the limit in the operator norm topology

$$f(\delta) = \lim_{m \rightarrow \infty} \sum_{k=0}^m f_k(\delta)$$

does exist, and

$$\|f(\delta)\| \leq \sum_{k=0}^{\infty} \|f_k(\delta)\| \leq \sum_{k=0}^{\infty} \left(\frac{r}{r+\tau}\right)^k < \infty.$$

Moreover, since the limit in (6.19) is uniform in  $n \in \mathbb{N}$  and  $W \in (\mathbb{D}^{n \times n})^N$  such that  $\|W_j\| \leq r < 1$ ,  $j = 1, \dots, N$ , the rearrangement of limits in the following chain of equalities is justified:

$$\begin{aligned} \lim_{n \rightarrow \infty} f(P_n \delta_1 P_n, \dots, P_n \delta_N P_n) h &= \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{k=0}^m f_k(P_n \delta_1 P_n, \dots, P_n \delta_N P_n) h \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{k=0}^m f_k(P_n \delta_1 P_n, \dots, P_n \delta_N P_n) h = \lim_{m \rightarrow \infty} \sum_{k=0}^m f_k(\delta) h = f(\delta) h \end{aligned}$$

(here  $h$  is an arbitrary vector in  $\mathcal{U} \otimes \ell_2$  and  $\delta \in L(\ell_2)^N$  such that  $\|\delta_j\| \leq r$ ,  $j = 1, \dots, N$ ). Thus for every  $\delta \in L(\ell_2)^N$  such that  $\|\delta_j\| < 1$ ,  $j = 1, \dots, N$ , we obtain  $\|f(\delta)\| \leq 1$ , i.e.,  $f \in \mathcal{SA}_N^{\text{nc}}(\mathcal{U}, \mathcal{Y})$ .  $\square$

*Remark 6.7.* One can see from the proof of Theorem 6.6 that for arbitrary  $f \in \mathcal{SA}_N^{\text{nc}}(\mathcal{U}, \mathcal{Y})$  and  $r : 0 < r < 1$ , the series

$$f(\delta) = \sum_{k=0}^{\infty} f_k(\delta)$$

converges uniformly and absolutely in  $\delta \in L(\mathcal{K})^N$  such that  $\|\delta_j\| \leq r$ ,  $j = 1, \dots, N$ , where  $\mathcal{K}$  is any separable Hilbert space.

**Corollary 6.8.** *A matrix-inner in  $\mathcal{D}_N$  rational FPS  $f$  belongs to the class  $\mathcal{SA}_N^{\text{nc}}(\mathbb{C}^q) = \mathcal{SA}_N^{\text{nc}}(\mathbb{C}^q, \mathbb{C}^q)$ .*

Thus, for the case  $J = I_q$ , Theorem 6.4 establishes the existence of a unitary GR-realization for an arbitrary matrix-inner rational FPS, i.e., recovers Theorem 6.5 for the case of a *matrix-inner rational* FPS. However, it says even more than Theorem 6.5 in this case, namely that such a unitary realization can be found minimal, thus finite-dimensional, and that this minimal unitary realization is unique up to a unitary similarity. The representation (6.17) with the rational FPS  $\varphi \in \mathbb{C}^{q \times \gamma} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$  given by

$$\varphi(z) = C(I_\gamma - \Delta(z)A)^{-1}$$

is obtained from (5.14) by making use of Corollary 2.2.

## 7. Matrix-selfadjoint rational formal power series

### 7.1. A multivariable non-commutative analogue of the line case

A rational FPS  $\Phi \in \mathbb{C}^{q \times q} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$  will be called *matrix-selfadjoint on  $\mathcal{J}_N$*  if for every  $n \in \mathbb{N}$ :

$$\Phi(Z) = \Phi(Z)^*$$

at all points  $Z \in (i\mathbb{H}^{n \times n})^N$  where it is defined.

The following theorem is a multivariable non-commutative counterpart of Theorem 4.1 from [7] which was originally proved in [28].

**Theorem 7.1.** *Let  $\Phi \in \mathbb{C}^{q \times q} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$ , and let  $\alpha$  be a minimal GR-realization of  $\Phi$  of the form (3.11). Then  $\Phi$  is matrix-selfadjoint on  $\mathcal{J}_N$  if and only if the following conditions hold:*

(a) *the matrix  $D$  is Hermitian, that is,  $D = D^*$ ;*

(b) *there exists an invertible Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$  with  $H_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ , and such that*

$$A^*H + HA = 0, \quad (7.1)$$

$$C = iB^*H. \quad (7.2)$$

*Proof.* We first observe that  $\Phi$  is matrix-selfadjoint on  $\mathcal{J}_N$  if and only if the FPS  $F \in \mathbb{C}^{2q \times 2q} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$  given by

$$F(z) = \begin{pmatrix} I_q & i\Phi(z) \\ 0 & I_q \end{pmatrix} \quad (7.3)$$

is matrix- $J_1$ -unitary on  $\mathcal{J}_N$ , where

$$J_1 = \begin{pmatrix} 0 & I_q \\ I_q & 0 \end{pmatrix}. \quad (7.4)$$

Moreover,  $F$  admits the GR-realization

$$\beta = (N; A, \begin{pmatrix} 0 & B \end{pmatrix}, \begin{pmatrix} iC \\ 0 \end{pmatrix}, \begin{pmatrix} I_q & iD \\ 0 & I_q \end{pmatrix}; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^{2q}).$$

This realization is minimal. Indeed, the  $k$ -th truncated observability (resp., controllability) matrix of  $\beta$  is equal to

$$\widetilde{\mathcal{O}}_k(\beta) = \begin{pmatrix} i\widetilde{\mathcal{O}}_k(\alpha) \\ 0 \end{pmatrix} \quad (7.5)$$

and, resp.,

$$\widetilde{\mathcal{C}}_k(\beta) = \begin{pmatrix} 0 & \widetilde{\mathcal{C}}_k(\alpha) \end{pmatrix}, \quad (7.6)$$

and therefore has full column (resp., row) rank. Using Theorem 4.1 of the present paper we see that  $\Phi$  is matrix-selfadjoint on  $\mathcal{J}_N$  if and only if:

- (1) the matrix  $\begin{pmatrix} I_q & iD \\ 0 & I_q \end{pmatrix}$  is  $J_1$ -unitary;
- (2) there exists an invertible Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$ , with  $H_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ , such that

$$\begin{aligned} A^*H + HA &= - \begin{pmatrix} iC \\ 0 \end{pmatrix}^* J_1 \begin{pmatrix} iC \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & B \end{pmatrix} &= -H^{-1} \begin{pmatrix} iC \\ 0 \end{pmatrix}^* J_1 \begin{pmatrix} I_q & iD \\ 0 & I_q \end{pmatrix}. \end{aligned}$$

These conditions are in turn readily seen to be equivalent to conditions (a) and (b) in the statement of the theorem.  $\square$

From Theorem 4.1 it follows that the matrix  $H = \text{diag}(H_1, \dots, H_N)$  is uniquely determined by the given minimal GR-realization of  $\Phi$ . In a similar way as in Section 4, it can be shown that  $H_k$ ,  $k = 1, \dots, N$ , are given by the formulas

$$\begin{aligned} H_k &= - \left( \text{col}_{w \in \mathbb{F}_N: |w| \leq q\gamma-1} (B^* \flat(-A^*))^{wg_k} \right)^+ \left( \text{col}_{w \in \mathbb{F}_N: |w| \leq q\gamma-1} (C \flat A)^{wg_k} \right) \\ &= \left( \text{row}_{w \in \mathcal{F}_N: |w| \leq q\gamma-1} ((-A^*) \sharp C^*)^{g_k w^T} \right) \left( \text{row}_{w \in \mathcal{F}_N: |w| \leq q\gamma-1} (A \sharp B)^{g_k w^T} \right)^\dagger. \end{aligned}$$

The matrix  $H = \text{diag}(H_1, \dots, H_N)$  is called in this case *the associated structured Hermitian matrix* (associated with a minimal GR-realization of the FPS  $\Phi$ ).

It follows from (7.1) and (7.2) that for  $n \in \mathbb{N}$  and  $Z, Z' \in (i\mathbb{H}^{n \times n})^N$  we have:

$$\begin{aligned} \Phi(Z) - \Phi(Z')^* &= i(C \otimes I_n) (I_\gamma \otimes I_n - \Delta(Z)(A \otimes I_n))^{-1} \\ &\quad \times \Delta(Z + Z'^*) (H^{-1} \otimes I_n) (I_\gamma \otimes I_n - (A^* \otimes I_n)\Delta(Z'^*))^{-1} (C^* \otimes I_n), \end{aligned} \quad (7.7)$$

$$\begin{aligned} \Phi(Z) - \Phi(Z')^* &= i(B^* \otimes I_n) (I_\gamma \otimes I_n - \Delta(Z'^*)(A^* \otimes I_n))^{-1} \\ &\quad \times \Delta(Z + Z'^*) (H \otimes I_n) (I_\gamma \otimes I_n - (A \otimes I_n)\Delta(Z))^{-1} (B \otimes I_n). \end{aligned} \quad (7.8)$$

Note that if  $A, B$  and  $C$  are matrices which satisfy (7.1) and (7.2) for some (not necessarily invertible) Hermitian matrix  $H$ , and if  $D$  is Hermitian, then

$$\Phi(z) = D + C(I - \Delta(z)A)^{-1}\Delta(z)B$$

is a rational FPS which is matrix-selfadjoint on  $\mathcal{J}_N$ . This follows from the fact that (7.8) is still valid in this case (the corresponding GR-realization of  $\Phi$  is, in general, not minimal).

If  $A, B$  and  $C$  satisfy the equalities

$$GA^* + AG = 0, \quad (7.9)$$

$$B = iGC^* \quad (7.10)$$

for some (not necessarily invertible) Hermitian matrix  $G = \text{diag}(G_1, \dots, G_N)$  then (7.7) is valid with  $H^{-1}$  replaced by  $G$  (the diagonal structures of  $G$ ,  $\Delta(Z)$  and  $\Delta(Z')$  are compatible), and hence  $\Phi$  is matrix-selfadjoint on  $\mathcal{J}_N$ .

As in Section 4, we can solve inverse problems using Theorem 7.1. The proofs are easy and omitted.

**Theorem 7.2.** *Let  $(C, A)$  be an observable pair of matrices, in the sense that  $\mathcal{O}_k$  has a full column rank for all  $k \in \{1, \dots, N\}$ . Then there exists a rational FPS which is matrix-selfadjoint on  $\mathcal{J}_N$  with a minimal GR-realization  $\alpha$  of the form (3.11) if and only if the equation*

$$A^*H + HA = 0$$

*has a solution  $H = \text{diag}(H_1, \dots, H_N)$  (with  $H_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ ) which is both Hermitian and invertible. When such a solution exists,  $D$  can be any Hermitian matrix and  $B = iH^{-1}C^*$ .*

**Theorem 7.3.** *Let  $(A, B)$  be a controllable pair of matrices, in the sense that  $\mathcal{C}_k$  has a full row rank for all  $k \in \{1, \dots, N\}$ . Then there exists a rational FPS which is matrix-selfadjoint on  $\mathcal{J}_N$  with a minimal GR-realization  $\alpha$  of the form (3.11) if and only if the equation*

$$GA^* + AG = 0$$

*has a solution  $G = \text{diag}(G_1, \dots, G_N)$  (with  $G_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ ) which is both Hermitian and invertible. When such a solution exists,  $D$  can be any Hermitian matrix and  $C = iB^*G^{-1}$ .*

From (7.5) and (7.6) obtained in Theorem 7.1, and from Theorem 4.4 we obtain the following result:

**Theorem 7.4.** *Let  $\Phi$  be a matrix-selfadjoint on  $\mathcal{J}_N$  rational FPS with a GR-realization  $\alpha$  of the form (3.8). Let  $H = \text{diag}(H_1, \dots, H_N)$  (with  $H_k \in \mathbb{C}^{r_k \times r_k}$ ,  $k = 1, \dots, N$ ) be both Hermitian and invertible and satisfy (7.1) and (7.2). Then the GR-node  $\alpha$  is observable if and only if it is controllable.*

The following Lemma is an analogue of Lemma 4.5. It is easily proved by applying Lemma 4.5 to the matrix- $J_1$ -unitary on  $\mathcal{J}_N$  function  $F$  defined in (7.3).

**Lemma 7.5.** *Let  $\Phi \in \mathbb{C}^{q \times q} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$  be matrix-selfadjoint on  $\mathcal{J}_N$ , and let  $\alpha^{(i)} = (N; A^{(i)}, B^{(i)}, C^{(i)}, D; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q)$  be two minimal GR-realizations of  $\Phi$ , with the associated structured Hermitian matrices  $H^{(i)} = \text{diag}(H_1^{(i)}, \dots, H_N^{(i)})$ ,  $i = 1, 2$ . Then these two realizations and associated matrices  $H^{(i)}$  are linked by (2.8) and (4.14). In particular, for each  $k \in \{1, \dots, N\}$  the matrices  $H_k^{(1)}$  and  $H_k^{(2)}$  have the same signature.*

For  $n \in \mathbb{N}$ , points  $Z, Z' \in (\mathbb{C}^{n \times n})^N$  where  $\Phi(Z)$  and  $\Phi(Z')$  are well-defined,  $F$  given by (7.3), and  $J_1$  defined by (7.4) we have:

$$J_1 \otimes I_n - F(Z)(J_1 \otimes I_n)F(Z')^* = \begin{pmatrix} \frac{\Phi(Z) - \Phi(Z')^*}{i} & 0 \\ 0 & 0 \end{pmatrix} \quad (7.11)$$

and

$$J_1 \otimes I_n - F(Z')^*(J_1 \otimes I_n)F(Z) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\Phi(Z) - \Phi(Z')^*}{i} \end{pmatrix}. \quad (7.12)$$

Combining these equalities with (7.7) and (7.8) and using Corollary 2.2 we obtain the following analogue of Theorem 4.7.

**Theorem 7.6.** *Let  $\Phi$  be a matrix-selfadjoint on  $\mathcal{J}_N$  rational FPS, and let  $\alpha$  be its minimal GR-realization of the form (3.11), with the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$ . Then for each  $k \in \{1, \dots, N\}$  the number of negative eigenvalues of the matrix  $H_k$  is equal to the number of negative squares of the kernels*

$$\begin{aligned} K_{w,w'}^{\Phi,k} &= (C \flat A)^{wg_k} H_k^{-1} (A^* \sharp C^*)^{g_k w'^T} \\ K_{w,w'}^{\Phi^*,k} &= (B^* \flat A^*)^{wg_k} H_k (A \sharp B)^{g_k w'^T}, \end{aligned} \quad w, w' \in \mathcal{F}_N. \quad (7.13)$$

Finally, for  $k \in \{1, \dots, N\}$ , let  $\mathcal{K}_k(\Phi)$  (resp.,  $\mathcal{K}_k(\Phi^*)$ ) denote the span of the functions  $w \mapsto K_{w,w'}^{\Phi,k}$  (resp.,  $w \mapsto K_{w,w'}^{\Phi^*,k}$ ) where  $w' \in \mathcal{F}_N$  and  $c \in \mathbb{C}^q$ . Then,

$$\dim \mathcal{K}_k(\Phi) = \dim \mathcal{K}_k(\Phi^*) = \gamma_k.$$

Let  $\Phi_1$  and  $\Phi_2$  be two FPSs from  $\mathbb{C}^{q \times q} \langle \langle z_1, \dots, z_N \rangle \rangle_{\text{rat}}$ . The additive decomposition

$$\Phi = \Phi_1 + \Phi_2$$

is called *minimal* if

$$\gamma_k(\Phi) = \gamma_k(\Phi_1) + \gamma_k(\Phi_2), \quad k = 1, \dots, N,$$

where  $\gamma_k(\Phi)$ ,  $\gamma_k(\Phi_1)$  and  $\gamma_k(\Phi_2)$  denote the dimensions of the  $k$ -th component of the state space of a minimal GR-realization of  $\Phi$ ,  $\Phi_1$  and  $\Phi_2$ , respectively. The following theorem is an analogue of Theorem 4.8.

**Theorem 7.7.** *Let  $\Phi_i$ ,  $i = 1, 2$ , be matrix-selfadjoint on  $\mathcal{J}_N$  rational FPSs, with minimal GR-realizations  $\alpha^{(i)} = (N; A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}; \mathbb{C}^{\gamma^{(i)}} = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k^{(i)}}, \mathbb{C}^q)$  and the associated structured Hermitian matrices  $H^{(i)} = \text{diag}(H_1^{(i)}, \dots, H_N^{(i)})$ . Assume that the additive decomposition  $\Phi = \Phi_1 + \Phi_2$  is minimal. Then the GR-node  $\alpha = (N; A, B, C, D; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q)$  defined by*

$$D = D^{(1)} + D^{(2)}, \quad \gamma_k = \gamma_k^{(1)} + \gamma_k^{(2)}, \quad k = 1, \dots, N,$$

and with respect to the decomposition  $\mathbb{C}^\gamma = \mathbb{C}^{\gamma^{(1)}} \oplus \mathbb{C}^{\gamma^{(2)}}$ ,

$$A = \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix}, \quad B = \begin{pmatrix} B^{(1)} \\ B^{(2)} \end{pmatrix}, \quad C = (C^{(1)} \quad C^{(2)}), \quad (7.14)$$

is a minimal GR-realization of  $\Phi$ , with the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$  such that for each  $k \in \{1, \dots, N\}$ :

$$H_k = \begin{pmatrix} H_k^{(1)} & 0 \\ 0 & H_k^{(2)} \end{pmatrix}.$$

Let  $\nu_k(\Phi)$  denote the number of negative squares of either of the functions defined in (7.13). In view of Theorem 7.6 and Theorem 7.1 these numbers are uniquely determined by  $\Phi$ .

**Corollary 7.8.** *Let  $\Phi_1$  and  $\Phi_2$  be matrix-selfadjoint on  $\mathcal{J}_N$  rational FPSs, and assume that the additive decomposition  $\Phi = \Phi_1 + \Phi_2$  is minimal. Then*

$$\nu_k(\Phi) = \nu_k(\Phi_1) + \nu_k(\Phi_2), \quad k = 1, 2, \dots, N.$$

An additive decomposition of a matrix-selfadjoint on  $\mathcal{J}_N$  rational FPS  $\Phi$  is called a *minimal matrix-selfadjoint decomposition* if it is minimal and both  $\Phi_1$  and  $\Phi_2$  are matrix-selfadjoint on  $\mathcal{J}_N$  rational FPSs. The set of all minimal matrix-selfadjoint decompositions of a matrix-selfadjoint on  $\mathcal{J}_N$  rational FPS is given by the following theorem, which is a multivariable non-commutative counterpart of [7, Theorem 4.6]. The proof uses Theorem 4.10 applied to the FPS  $F$  defined by (7.3), and follows the same argument as one in the proof of Theorem 4.6 in [7].

**Theorem 7.9.** *Let  $\Phi$  be a matrix-selfadjoint on  $\mathcal{J}_N$  rational FPS, with a minimal GR-realization  $\alpha$  of the form (3.11) and the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$ . Let  $\mathcal{M} = \bigoplus_{k=1}^N \mathcal{M}_k$  be an  $A$ -invariant subspace, with  $\mathcal{M}_k \subset \mathbb{C}^{\gamma_k}$ ,  $k = 1, \dots, N$ , and assume that  $\mathcal{M}$  is non-degenerate in the associated inner product  $[\cdot, \cdot]_H$ . Let  $\Pi = \text{diag}(\Pi_1, \dots, \Pi_N)$  be the projection defined by*

$$\ker \Pi = \mathcal{M}, \quad \text{ran } \Pi = \mathcal{M}^{\perp},$$

that is,

$$\ker \Pi_k = \mathcal{M}_k, \quad \text{ran } \Pi_k = \mathcal{M}_k^{\perp}, \quad k = 1, \dots, N.$$

Let  $D = D_1 + D_2$  be a decomposition of  $D$  into two Hermitian matrices. Then the decomposition  $\Phi = \Phi_1 + \Phi_2$ , where

$$\begin{aligned} \Phi_1(z) &= D_1 + C(I_\gamma - \Delta(z)A)^{-1}\Delta(z)(I_\gamma - \Pi)B, \\ \Phi_2(z) &= D_2 + C\Pi(I_\gamma - \Delta(z)A)^{-1}\Delta(z)B, \end{aligned}$$

is a minimal matrix-selfadjoint decomposition of  $\Phi$ .

Conversely, any minimal matrix-selfadjoint decomposition of  $\Phi$  can be obtained in such a way, and with a fixed decomposition  $D = D_1 + D_2$ , the correspondence between minimal matrix-selfadjoint decompositions of  $\Phi$  and non-degenerate  $A$ -invariant subspaces of the form  $\mathcal{M} = \bigoplus_{k=1}^N \mathcal{M}_k$ , where  $\mathcal{M}_k \subset \mathbb{C}^{\gamma_k}$ ,  $k = 1, \dots, N$ , is one-to-one.

*Remark 7.10.* Minimal matrix-selfadjoint decompositions do not always exist, even in the case  $N = 1$ . For counterexamples see [7].

## 7.2. A multivariable non-commutative analogue of the circle case

In this subsection we briefly review some analogues of the theorems presented in Section 7.1.

**Theorem 7.11.** *Let  $\Psi$  be a rational FPS and  $\alpha$  be its minimal GR-realization of the form (3.11). Then  $\Psi$  is matrix-selfadjoint on  $\mathcal{T}_N$  (that is, for all  $n \in \mathbb{N}$  one has  $\Psi(Z) = \Psi(Z)^*$  at all points  $Z \in (\mathbb{T}^{n \times n})^N$  where  $\Psi$  is defined) if and only if there exists an invertible Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$ , with  $H_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ , such that*

$$A^*HA = H, \quad D - D^* = iB^*HB, \quad C = iB^*HA. \quad (7.15)$$

*Proof.* Consider the FPS  $f \in \mathbb{C}^{2q \times 2q} \langle\langle z_1, \dots, z_N \rangle\rangle_{\text{rat}}$  defined by

$$f(z) = \begin{pmatrix} I_q & i\Psi(z) \\ 0 & I_q \end{pmatrix}. \quad (7.16)$$

Using Theorem 5.3, we see that  $f$  is matrix- $J_1$ -unitary on  $\mathcal{T}_N$ , with

$$J_1 = \begin{pmatrix} 0 & I_q \\ I_q & 0 \end{pmatrix}, \quad (7.17)$$

if and only if its GR-realization

$$\beta = (N; A, \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} iC \\ 0 \end{pmatrix}, \begin{pmatrix} I_q & iD \\ 0 & I_q \end{pmatrix}; \mathbb{C}^\gamma = \bigoplus_{j=1}^N \mathbb{C}^{\gamma_j}, \mathbb{C}^{2q})$$

(which turns out to be minimal, as can be shown in the same way as in Theorem 7.1) satisfies the following condition: there exists an Hermitian invertible matrix  $H = \text{diag}(H_1, \dots, H_N)$ , with  $H_k \in \mathbb{C}^{\gamma_k \times \gamma_k}$ ,  $k = 1, \dots, N$ , such that

$$\begin{pmatrix} A & 0 & B \\ iC & I_q & iD \\ 0 & 0 & I_q \end{pmatrix}^* \begin{pmatrix} H & 0 & 0 \\ 0 & 0 & I_q \\ 0 & I_q & 0 \end{pmatrix} \begin{pmatrix} A & 0 & B \\ iC & I_q & iD \\ 0 & 0 & I_q \end{pmatrix} = \begin{pmatrix} H & 0 & 0 \\ 0 & 0 & I_q \\ 0 & I_q & 0 \end{pmatrix},$$

which is equivalent to the condition stated in the theorem.  $\square$

For a given minimal GR-realization of  $\Psi$  the matrix  $H$  is unique, as follows from Theorem 5.1. It is called *the associated structured Hermitian matrix* of  $\Psi$ .

The set of all minimal matrix-selfadjoint additive decompositions of a given matrix-selfadjoint on  $\mathcal{T}_N$  rational FPS is described by the following theorem, which is a multivariable non-commutative counterpart of [7, Theorem 5.2], and is proved by applying Theorem 5.15 to the matrix- $J_1$ -unitary on  $\mathcal{T}_N$  FPS  $f$  defined by (7.16), where  $J_1$  is defined by (7.17). (We omit the proof.)

**Theorem 7.12.** *Let  $\Psi$  be a matrix-selfadjoint on  $\mathcal{T}_N$  rational FPS and  $\alpha$  be its minimal GR-realization of the form (3.11), with the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$ . Let  $\mathcal{M} = \bigoplus_{k=1}^N \mathcal{M}_k$  be an  $A$ -invariant subspace, with  $\mathcal{M}_k \subset \mathbb{C}^{\gamma_k}$ ,  $k = 1, \dots, N$ , and assume that  $\mathcal{M}$  is non-degenerate in the associated inner product  $[\cdot, \cdot]_H$ . Let  $\Pi = \text{diag}(\Pi_1, \dots, \Pi_N)$  be the projection defined by*

$$\ker \Pi = \mathcal{M}, \quad \text{ran } \Pi = \mathcal{M}^{[\perp]},$$

that is,

$$\ker \Pi_k = \mathcal{M}_k, \quad \text{ran } \Pi_k = \mathcal{M}_k^{[\perp]}, \quad k = 1, \dots, N.$$

Then the decomposition  $\Psi = \Psi_1 + \Psi_2$ , where

$$\Psi_1(z) = D_1 + C(I_\gamma - \Delta(z)A)^{-1} \Delta(z)(I_\gamma - \Pi)B,$$

$$\Psi_2(z) = D_2 + C\Pi(I_\gamma - \Delta(z)A)^{-1} \Delta(z)B,$$

with  $D_1 = \frac{i}{2} B_1^* H^{(1)} B_1 + S$ , the matrix  $S$  being an arbitrary Hermitian matrix, and

$$B_1 = P_{\mathcal{M}} B, \quad H^{(1)} = P_{\mathcal{M}} H|_{\mathcal{M}},$$

is a minimal matrix-selfadjoint additive decomposition of  $\Psi$  (here  $P_{\mathcal{M}}$  denotes the orthogonal projection onto  $\mathcal{M}$  in the standard metric of  $\mathbb{C}^\gamma$ ).

Conversely, any minimal matrix-selfadjoint additive decomposition of  $\Psi$  is obtained in such a way, and for a fixed  $S$ , the correspondence between minimal matrix-selfadjoint additive decompositions of  $\Psi$  and non-degenerate  $A$ -invariant subspaces of the form  $\mathcal{M} = \bigoplus_{k=1}^N \mathcal{M}_k$ , where  $\mathcal{M}_k \subset \mathbb{C}^{\gamma_k}$ ,  $k = 1, \dots, N$ , is one-to-one.

## 8. Finite-dimensional de Branges–Rovnyak spaces and backward shift realizations: The multivariable non-commutative setting

In this section we describe certain model realizations of matrix- $J$ -unitary rational FPSs. We restrict ourselves to the case of FPSs which are matrix- $J$ -unitary on  $\mathcal{J}_N$ . Analogous realizations can be constructed for rational FPSs which are matrix- $J$ -unitary on  $\mathcal{T}_N$  or matrix-selfadjoint either on  $\mathcal{J}_N$  or  $\mathcal{T}_N$ .

### 8.1. Non-commutative formal reproducing kernel Pontryagin spaces

Let  $F$  be a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS and  $\alpha$  be its minimal GR-realization of the form (3.11), with the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$ . Then by Theorem 4.7, for each  $k \in \{1, \dots, N\}$  the kernel (4.15) has the number  $\nu_k(F)$  of negative eigenvalues equal to the number of negative squares of  $H_k$ . Lemma 4.5 implies that the kernel  $K_{w,w'}^{F,k}$ , from (4.15) does not depend on the choice of a minimal realization of  $F$ . Theorem 4.7 also asserts that the span of the functions

$$w \mapsto K_{w,w'}^{F,k} c, \quad \text{where } w' \in \mathcal{F}_N \text{ and } c \in \mathbb{C}^q,$$

is the space  $\mathcal{K}_k(F)$  with  $\dim \mathcal{K}_k(F) = \gamma_k$ ,  $k = 1, \dots, N$ . One can introduce a new metric on each of the spaces  $\mathcal{K}_k(F)$  as follows. First, define an Hermitian form  $[\cdot, \cdot]_{F,k}$  by:

$$[K_{\cdot,w'}^{F,k} c', K_{\cdot,w}^{F,k} c]_{F,k} = c^* K_{w,w'}^{F,k} c'.$$

This form is easily seen to be well defined on the whole space  $\mathcal{K}_k(F)$ , that is, if  $f$  and  $h$  belong to  $\mathcal{K}_k(F)$  and

$$f_w = \sum_j K_{w,w_j}^{F,k} c_j = \sum_\ell K_{w,w'_\ell}^{F,k} c'_\ell$$

and

$$h_w = \sum_s K_{w,v_s}^{F,k} d_s = \sum_t K_{w,v'_t}^{F,k} d'_t,$$

where all the sums are finite, then

$$[f, h]_{F,k} = \left[ \sum_j K_{\cdot,w_j}^{F,k} c_j, \sum_s K_{\cdot,v_s}^{F,k} d_s \right]_{F,k} = \left[ \sum_\ell K_{\cdot,w'_\ell}^{F,k} c'_\ell, \sum_t K_{\cdot,v'_t}^{F,k} d'_t \right]_{F,k}.$$



Thus, the space  $\mathcal{K}_k(F)$  endowed with this new (indefinite) metric is a finite dimensional reproducing kernel Pontryagin space (RKPS) of functions on  $\mathcal{F}_N$  with the reproducing kernel  $K_{w,w'}^{F,k}$ . We refer to [46, 4, 3] for more information on the theory of reproducing kernel Pontryagin spaces. In a similar way, the space  $\mathcal{K}(F) = \bigoplus_{k=1}^N \mathcal{K}_k(F)$  endowed with the indefinite inner product

$$[f, h]_F = \sum_{k=1}^N [f_k, h_k]_{F,k}.$$

where  $f = \text{col}(f_1, \dots, f_N)$  and  $h = \text{col}(h_1, \dots, h_N)$ , becomes a reproducing kernel Pontryagin space with the reproducing kernel

$$K_{w,w'}^F = \text{diag}(K_{w,w'}^{F,1}, \dots, K_{w,w'}^{F,N}), \quad w, w' \in \mathcal{F}^N.$$

Rather than the kernels  $K_{w,w'}^{F,k}$ ,  $k = 1, \dots, N$ , and  $K_{w,w'}^F$  we prefer to use the FPS kernels

$$K^{F,k}(z, z') = \sum_{w, w' \in \mathcal{F}^N} K_{w,w'}^{F,k} z^w z'^{w'^T}, \quad k = 1, \dots, N, \quad (8.1)$$

$$K^F(z, z') = \sum_{w, w' \in \mathcal{F}^N} K_{w,w'}^F z^w z'^{w'^T}, \quad (8.2)$$

and instead of the reproducing kernel Pontryagin spaces  $\mathcal{K}_k(F)$  and  $\mathcal{K}(F)$  we will use the notion of *non-commutative formal reproducing kernel Pontryagin spaces* (NFRKPS for short; we will use the same notations for these spaces) which we introduce below in a way analogous to the way J. A. Ball and V. Vinnikov introduce non-commutative formal reproducing kernel Hilbert spaces (NFRKHS for short) in [14].

Consider a FPS

$$K(z, z') = \sum_{w, w' \in \mathcal{F}_N} K_{w,w'} z^w z'^{w'^T} \in L(\mathcal{C}) \langle \langle z_1, \dots, z_N, z'_1, \dots, z'_N \rangle \rangle_{\text{rat}},$$

where  $\mathcal{C}$  is a Hilbert space. Suppose that

$$K(z', z) = K(z, z')^* = \sum_{w, w' \in \mathcal{F}_N} K_{w,w'}^* z'^w z^{w'^T}.$$

Then  $K_{w,w'}^* = K_{w',w}$  for all  $w, w' \in \mathcal{F}_N$ . Let  $\kappa \in \mathbb{N}$ . We will say that the FPS  $K(z, z')$  is a *kernel with  $\kappa$  negative squares* if  $K_{w,w'}$  is a kernel on  $\mathcal{F}_N$  with  $\kappa$  negative squares, i.e. for every integer  $\ell$  and every choice of  $w_1, \dots, w_\ell \in \mathcal{F}_N$  and  $c_1, \dots, c_\ell \in \mathcal{C}$  the  $\ell \times \ell$  Hermitian matrix with  $(i, j)$ -th entry equal to  $c_i^* K_{w_i, w_j} c_j$  has at most  $\kappa$  strictly negative eigenvalues, and exactly  $\kappa$  such eigenvalues for some choice of  $\ell, w_1, \dots, w_\ell, c_1, \dots, c_\ell$ .

Define on the space  $\mathcal{G}$  of finite sums of FPSs of the form

$$K_{w'}(z)c = \sum_{w \in \mathcal{F}_N} K_{w,w'} z^w c,$$

where  $w' \in \mathcal{F}_N$  and  $c \in \mathcal{C}$ , the inner product as follows:

$$\left[ \sum_i K_{w_i}(z)c_i, \sum_j K_{w'_j}(z)c'_j \right]_{\mathcal{G}} = \sum_{i,j} \langle K_{w'_j, w_i} c_i, c'_j \rangle_{\mathcal{C}}.$$

It is easily seen to be well defined. The space  $\mathcal{G}$  endowed with this inner product can be completed in a unique way to a Pontryagin space  $\mathcal{P}(K)$  of FPSs, and in  $\mathcal{P}(K)$  the reproducing kernel property is

$$[f, K_w(\cdot)c]_{\mathcal{P}(K)} = \langle f_w, c \rangle_{\mathcal{C}}. \quad (8.3)$$

See [4, Theorem 6.4] for more details on such completions.

Define the pairings  $[\cdot, \cdot]_{\mathcal{P}(K) \times \mathcal{P}(K) \langle \langle z_1, \dots, z_N \rangle \rangle}$  and  $\langle \cdot, \cdot \rangle_{\mathcal{C} \langle \langle z_1, \dots, z_N \rangle \rangle \times \mathcal{C}}$  as mappings  $\mathcal{P}(K) \times \mathcal{P}(K) \langle \langle z_1, \dots, z_N \rangle \rangle \rightarrow \mathbb{C} \langle \langle z_1, \dots, z_N \rangle \rangle$  and  $\mathcal{C} \langle \langle z_1, \dots, z_N \rangle \rangle \times \mathcal{C} \rightarrow \mathbb{C} \langle \langle z_1, \dots, z_N \rangle \rangle$  by

$$\begin{aligned} \left[ f, \sum_{w \in \mathcal{F}_N} g_w z^w \right]_{\mathcal{P}(K) \times \mathcal{P}(K) \langle \langle z_1, \dots, z_N \rangle \rangle} &= \sum_{w \in \mathcal{F}_N} [f, g_w]_{\mathcal{P}(K)} z^{w^T}, \\ \left\langle \sum_{w \in \mathcal{F}_N} f_w z^w, c \right\rangle_{\mathcal{C} \langle \langle z_1, \dots, z_N \rangle \rangle \times \mathcal{C}} &= \sum_{w \in \mathcal{F}_N} \langle f_w, c \rangle_{\mathcal{C}} z^w. \end{aligned}$$

Then the reproducing kernel property (8.3) can be rewritten as

$$[f, K(\cdot, z)c]_{\mathcal{P}(K) \times \mathcal{P}(K) \langle \langle z_1, \dots, z_N \rangle \rangle} = \langle f(z), c \rangle_{\mathcal{C} \langle \langle z_1, \dots, z_N \rangle \rangle \times \mathcal{C}}. \quad (8.4)$$

The space  $\mathcal{P}(K)$  endowed with the metric  $[\cdot, \cdot]_{\mathcal{P}(K)}$  will be said to be a *NFRKPS* associated with the FPS kernel  $K(z, z')$ . It is clear that this space is isomorphic to the RKPS associated with the kernel  $K_{w, w'}$  on  $\mathcal{F}_N$ , and this isomorphism is well defined by

$$K_{w'}(\cdot)c \mapsto K_{\cdot, w'}c, \quad w' \in \mathcal{F}_N, c \in \mathcal{C}.$$

Let us now come back to the kernels (8.1) and (8.2) (see also (4.15)). Clearly, they can be rewritten as

$$K^{F,k}(z, z') = \varphi_k(z) H_k^{-1} \varphi_k(z')^*, \quad k = 1, \dots, N, \quad (8.5)$$

$$K^F(z, z') = \varphi(z) H^{-1} \varphi(z')^*, \quad (8.6)$$

where rational FPSs  $\varphi_k$ ,  $k = 1, \dots, N$ , and  $\varphi$  are determined by a given minimal GR-realization  $\alpha$  of the FPS  $F$  as

$$\begin{aligned} \varphi(z) &= C(I_\gamma - \Delta(z)A)^{-1}, \\ \varphi_k(z) &= \varphi(z)|_{\mathbb{C}^{\gamma_k}}, \quad k = 1, \dots, N. \end{aligned}$$

For a model minimal GR-realization of  $F$ , we will start, conversely, with establishing an explicit formula for the kernels (8.1) and (8.2) in terms of  $F$  and then define a minimal GR-realization via these kernels.

Suppose that for a fixed  $k \in \{1, \dots, N\}$ , (8.5) holds with some rational FPS  $\varphi_k$ . Recall that

$$J - F(z)JF(z')^* = \sum_{k=1}^N \varphi_k(z)H_k^{-1}(z_k + (z'_k)^*)\varphi_k(z')^* \quad (8.7)$$

(note that  $(z'_k)^* = z'_k$ ). Then for any  $n \in \mathbb{N}$  and  $Z, Z' \in \mathbb{C}^{n \times n}$ :

$$J \otimes I_n - F(Z)(J \otimes I_n)F(Z')^* = \sum_{k=1}^N \varphi_k(Z)(H_k^{-1} \otimes (Z_k + (Z'_k)^*))\varphi_k(Z')^*. \quad (8.8)$$

Therefore, for  $\lambda \in \mathbb{C}$ :

$$\begin{aligned} & J \otimes I_{2n} - F(\Lambda_{Z,Z'}(\lambda))(J \otimes I_{2n})F(\text{diag}(-Z^*, Z'))^* \\ &= \lambda \varphi_k(\Lambda_{Z,Z'}(\lambda)) \left\{ H_k^{-1} \otimes \begin{pmatrix} I_n & I_n \\ I_n & I_n \end{pmatrix} \right\} \varphi_k(\text{diag}(-Z^*, Z'))^*, \end{aligned} \quad (8.9)$$

where

$$\begin{aligned} \Lambda_{Z,Z'}(\lambda) &:= \lambda \begin{pmatrix} I_n & I_n \\ I_n & I_n \end{pmatrix} \otimes e_k + \begin{pmatrix} Z & 0 \\ 0 & -Z'^* \end{pmatrix} \\ &= \left( \begin{pmatrix} Z_1 & 0 \\ 0 & -(Z'_1)^* \end{pmatrix}, \dots, \begin{pmatrix} Z_{k-1} & 0 \\ 0 & -(Z'_{k-1})^* \end{pmatrix}, \begin{pmatrix} \lambda I_n + Z_k & \lambda I_n \\ \lambda I_n & \lambda I_n - (Z'_k)^* \end{pmatrix}, \right. \\ &\quad \left. \begin{pmatrix} Z_{k+1} & 0 \\ 0 & -(Z'_{k+1})^* \end{pmatrix}, \dots, \begin{pmatrix} Z_N & 0 \\ 0 & -(Z'_N)^* \end{pmatrix} \right), \\ \text{diag}(-Z^*, Z') &:= \left( \begin{pmatrix} -Z_1^* & 0 \\ 0 & Z'_1 \end{pmatrix}, \dots, \begin{pmatrix} -Z_N^* & 0 \\ 0 & Z'_N \end{pmatrix} \right), \end{aligned}$$

and, in particular,

$$\Lambda_{Z,Z'}(0) = \text{diag}(Z, -Z'^*).$$

For  $Z$  and  $Z'$  where both  $F$  and  $\varphi_k$  are holomorphic,  $\varphi_k(\Lambda_{Z,Z'}(\lambda))$  is continuous in  $\lambda$ , and  $F(\Lambda_{Z,Z'}(\lambda))$  is holomorphic in  $\lambda$  at  $\lambda = 0$ . Thus, dividing by  $\lambda$  the expressions in both sides of (8.9) and passing to the limit as  $\lambda \rightarrow 0$ , we get

$$\begin{aligned} & -\frac{d}{d\lambda} \{F(\Lambda_{Z,Z'}(\lambda))\} \Big|_{\lambda=0} (J \otimes I_{2n})F(\text{diag}(-Z^*, Z'))^* \\ &= \varphi_k(\text{diag}(Z, -Z'^*)) \left\{ H_k^{-1} \otimes \begin{pmatrix} I_n & I_n \\ I_n & I_n \end{pmatrix} \right\} \varphi_k(\text{diag}(-Z^*, Z'))^* \\ &= \begin{pmatrix} \varphi_k(Z) \\ \varphi_k(-Z'^*) \end{pmatrix} (H_k^{-1} \otimes I_n) \begin{pmatrix} \varphi_k(-Z^*)^* & \varphi_k(Z')^* \end{pmatrix}. \end{aligned}$$

Taking the (1,2)-th entry of the  $2 \times 2$  block matrices in this equality, we get:

$$K^{F,k}(Z, Z') = -\frac{d}{d\lambda} \{F(\Lambda_{Z,Z'}(\lambda))_{12}\} \Big|_{\lambda=0} (J \otimes I_n)F(Z')^*. \quad (8.10)$$

Using the FPS representation for  $F$  we obtain from (8.10) the representation

$$K^{F,k}(Z, Z') = \sum_{w, w' \in \mathcal{F}_N} \left( \sum_{v, v' \in \mathcal{F}_N: vv'=w'} (-1)^{|v'|+1} F_{wg_k v'^T} JF_v \right) \otimes Z^w (Z'^*)^{w'^T}.$$

From Corollary 2.2 we get the expression for a FPS  $K^{F,k}(z, z')$ , namely:

$$K^{F,k}(z, z') = \sum_{w, w' \in \mathcal{F}_N} \left( \sum_{v, v' \in \mathcal{F}_N: vv'=w'} (-1)^{|v'|+1} F_{wg_k v'^T} JF_v \right) z^w z'^{w'^T}. \quad (8.11)$$

Using formal differentiation with respect to  $\lambda$  we can also represent this kernel as

$$K^{F,k}(z, z') = -\frac{d}{d\lambda} \{F(\Lambda_{z, z'}(\lambda))_{12}\} \Big|_{\lambda=0} JF(z')^*. \quad (8.12)$$

We note that one gets (8.11) and (8.12) from (8.7) using the same argument applied to FPSs.

Let us now consider the NFRKPSs  $\mathcal{K}_k(F)$ ,  $k = 1, \dots, N$ , and  $\mathcal{K}(F) = \bigoplus_{k=1}^N \mathcal{K}_k(F)$ . They are finite dimensional and isomorphic to the reproducing kernel Pontryagin spaces on  $\mathcal{F}_N$  which were denoted above with the same notation. Thus

$$\begin{aligned} \dim \mathcal{K}_k(F) &= \gamma_k, & k &= 1, \dots, N, \\ \dim \mathcal{K}(F) &= \gamma. \end{aligned} \quad (8.13)$$

The space  $\mathcal{K}(F)$  is a multivariable non-commutative analogue of a certain de Branges–Rovnyak space (see [19, p. 24], [4, Section 6.3], and [7, p. 217]).

## 8.2. Minimal realizations in non-commutative de Branges–Rovnyak spaces

Let us define for every  $k \in \{1, \dots, N\}$  the backward shift operator

$$R_k : \mathbb{C}^q \langle\langle z_1, \dots, z_N \rangle\rangle_{\text{rat}} \longrightarrow \mathbb{C}^q \langle\langle z_1, \dots, z_N \rangle\rangle_{\text{rat}}$$

by

$$R_k : \sum_{w \in \mathcal{F}_N} f_w z^w \longmapsto \sum_{w \in \mathcal{F}_N} f_{wg_k} z^w.$$

(Compare with the one-variable backward shift operator  $R_0$  considered in Section 1.)

**Lemma 8.1.** *Let  $F$  be a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS. Then for every  $k \in \{1, \dots, N\}$  the following is true:*

1.  $R_k F(z)c \in \mathcal{K}_k(F)$  for every  $c \in \mathbb{C}^q$ ;
2.  $R_k \mathcal{K}_j(F) \subset \mathcal{K}_k(F)$  for every  $j \in \{1, \dots, N\}$ .

*Proof.* From (8.7) and the  $J$ -unitarity of  $F_\emptyset$  we get

$$\begin{aligned} J - F(z)JF_\emptyset^* &= (F_\emptyset - F(z))JF_\emptyset^* = -\sum_{k=1}^N R_k F(z)z_k JF_\emptyset^* \\ &= \sum_{k=1}^N \varphi_k(z)H_k^{-1}z_k(\varphi_k)_\emptyset^*, \end{aligned}$$

and therefore for every  $k \in \{1, \dots, N\}$  and every  $c \in \mathbb{C}^q$  we get

$$R_k F(z)c = -\varphi_k(z)H_k^{-1}(\varphi_k)_\emptyset^* JF_\emptyset c = K_\emptyset^{F,k}(z)(-JF_\emptyset c) \in \mathcal{K}_k(F).$$

Thus, the first statement of this Lemma is true. To prove the second statement we start again from (8.7) and get for a fixed  $j \in \{1, \dots, N\}$  and  $w \in \mathcal{F}_N$ :

$$-F(z)JF_{wg_j}^* = \varphi_j(z)H_j^{-1}(\varphi_j)_w^* + \sum_{k=1}^N \varphi_k(z)H_k^{-1}z_k(\varphi_k)_{wg_j}^*,$$

and therefore for any  $c \in \mathbb{C}^q$ :

$$-\sum_{k=1}^N \left( R_k F(z)JF_{wg_j}^* c \right) z_k = \sum_{k=1}^N \left( R_k K_w^{F,j}(z)c \right) z_k + \sum_{k=1}^N \left( K_{wg_j}^{F,k}(z)c \right) z_k.$$

Hence, one has for every  $k \in \{1, \dots, N\}$ :

$$R_k K_w^{F,j}(z)c = -R_k F(z)JF_{wg_j}^* c - K_{wg_j}^{F,k}(z)c, \quad (8.14)$$

and from the first statement of this Lemma we obtain that the right-hand side of this equality belongs to  $\mathcal{K}_k(F)$ . Thus, the second statement is true, too.  $\square$

We now define operators  $A_{kj} : \mathcal{K}_j(F) \rightarrow \mathcal{K}_k(F)$ ,  $A : \mathcal{K}(F) \rightarrow \mathcal{K}(F)$ ,  $B : \mathbb{C}^q \rightarrow \mathcal{K}(F)$ ,  $C : \mathcal{K}(F) \rightarrow \mathbb{C}^q$ ,  $D : \mathbb{C}^q \rightarrow \mathbb{C}^q$  by

$$A_{kj} = R_k|_{\mathcal{K}_j(F)}, \quad k, j = 1, \dots, N, \quad (8.15)$$

$$A = (A_{kj})_{k,j=1,\dots,N}, \quad (8.16)$$

$$B : c \mapsto \begin{pmatrix} R_1 F(z)c \\ \vdots \\ R_N F(z)c \end{pmatrix}, \quad (8.17)$$

$$C : \begin{pmatrix} f_1(z) \\ \vdots \\ f_N(z) \end{pmatrix} \mapsto \sum_{k=1}^N (f_k)_\emptyset, \quad (8.18)$$

$$D = F_\emptyset. \quad (8.19)$$

These definitions make sense in view of Lemma 8.1.

**Theorem 8.2.** *Let  $F$  be a matrix- $J$ -unitary on  $\mathcal{J}_N$  rational FPS. Then the GR-node  $\alpha = (N; A, B, C, D; \mathcal{K}(F) = \bigoplus_{k=1}^N \mathcal{K}_k(F), \mathbb{C}^q)$ , with operators defined by (8.15)–(8.19), is a minimal GR-realization of  $F$ .*

*Proof.* We first check that for every  $w \in \mathcal{F}_N : w \neq \emptyset$  we have

$$F_w = (C^b A \sharp B)^w. \quad (8.20)$$

Let  $w = g_k$  for some  $k \in \{1, \dots, N\}$ . Then for  $c \in \mathbb{C}^q$ :

$$(C^b A \sharp B)^w c = C_k B_k c = (R_k F(z) c)_\emptyset = \left( \sum_{w \in \mathcal{F}_N} F_w g_k z^w c \right)_\emptyset = F_{g_k} c.$$

Assume now that  $|w| > 1$ ,  $w = g_{j_1} \dots g_{j_{|w|}}$ . Then for  $c \in \mathbb{C}^q$ :

$$\begin{aligned} (C^b A \sharp B)^w c &= C_{j_1} A_{j_1, j_2} \dots A_{j_{|w|-1}, j_{|w|}} B_{j_{|w|}} c = (R_{j_1} \dots R_{j_{|w|}} F(z) c)_\emptyset \\ &= \left( \sum_{w' \in \mathcal{F}_N} F_{w' g_{j_1} \dots g_{j_{|w|}}} z^{w'} c \right)_\emptyset = F_{g_{j_1} \dots g_{j_{|w|}}} c = F_w c. \end{aligned}$$

Since  $F_\emptyset = D$ , we obtain that

$$F(z) = D + C(I - \Delta(z)A)^{-1} \Delta(z)B,$$

that is,  $\alpha$  is a GR-realization of  $F$ . The minimality of  $\alpha$  follows from (8.13).  $\square$

Let us now show how the associated structured Hermitian matrix  $H = \text{diag}(H_1, \dots, H_N)$  arises from this special realization. Let

$$h = \text{col}_{1 \leq j \leq N} (K_{w_j}^{F,j}(\cdot) c_j) \quad \text{and} \quad h' = \text{col}_{1 \leq j \leq N} (K_{w'_j}^{F,j}(\cdot) c'_j).$$

Using (8.14), we obtain

$$\begin{aligned} & [A_{kj} h_j, h'_k]_{F,k} + [h_j, A_{jk} h'_k]_{F,j} \\ &= [R_k K_{w_j}^{F,j}(\cdot) c_j, K_{w'_k}^{F,k}(\cdot) c'_k]_{F,k} + [K_{w_j}^{F,j}(\cdot) c_j, R_j K_{w'_k}^{F,k}(\cdot) c'_k]_{F,j} \\ &= (c'_k)^* \left( K_{w'_k g_k, w_j}^{F,j} + K_{w'_k, w_j g_j}^{F,k} \right) c_j. \end{aligned} \quad (8.21)$$

Let  $\overset{\circ}{\alpha} = (N; \overset{\circ}{A}, \overset{\circ}{B}, \overset{\circ}{C}, \overset{\circ}{D}; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q)$  be any minimal GR-realization of  $F$ , with the associated structured Hermitian matrix  $\overset{\circ}{H} = \text{diag}(\overset{\circ}{H}_1, \dots, \overset{\circ}{H}_N)$ . Then the

right-hand side of (8.21) can be rewritten as

$$\begin{aligned}
& (c'_k)^* \left( K_{w'_k g_k, w_j}^{F,j} + K_{w'_k, w_j g_j}^{F,k} \right) c_j \\
&= (c'_k)^* \left( \left( \overset{\circ}{C} \flat \overset{\circ}{A} \right)^{w'_k g_k g_j} \left( \overset{\circ}{H}_j \right)^{-1} \left( \overset{\circ}{A}^* \# \overset{\circ}{C}^* \right)^{g_j w_j^T} \right. \\
&+ \left. \left( \overset{\circ}{C} \flat \overset{\circ}{A} \right)^{w'_k g_k} \left( \overset{\circ}{H}_k \right)^{-1} \left( \overset{\circ}{A}^* \# \overset{\circ}{C}^* \right)^{g_k g_j w_j^T} \right) c_j \\
&= (c'_k)^* \left( \overset{\circ}{C} \flat \overset{\circ}{A} \right)^{w'_k g_k} \left( A_{kj} \left( \overset{\circ}{H}_j \right)^{-1} + \left( \overset{\circ}{H}_k \right)^{-1} \left( A_{kj} \right)^* \right) \left( \overset{\circ}{A}^* \# \overset{\circ}{C}^* \right)^{g_j w_j^T} c_j \\
&= -(c'_k)^* \left( \overset{\circ}{C} \flat \overset{\circ}{A} \right)^{w'_k g_k} \overset{\circ}{B}_k J \left( \overset{\circ}{B}_j \right)^* \left( \overset{\circ}{A}^* \# \overset{\circ}{C}^* \right)^{g_j w_j^T} c_j \\
&= -(c'_k)^* \left( \overset{\circ}{C} \flat \overset{\circ}{A} \right)^{w'_k g_k} \left( \overset{\circ}{H}_k \right)^{-1} \left( \overset{\circ}{C}_k \right)^* J \overset{\circ}{C}_j \left( \overset{\circ}{H}_j \right)^{-1} \left( \overset{\circ}{A}^* \# \overset{\circ}{C}^* \right)^{g_j w_j^T} c_j \\
&= -(c'_k)^* K_{w'_k, \emptyset}^{F,k} J K_{\emptyset, w_j}^{F,j} c_j \\
&= -(c'_k)^* \left( K_{\emptyset, w'_k}^{F,k} \right)^* J K_{\emptyset, w_j}^{F,j} c_j \\
&= -(h'_k)_{\emptyset}^* J(h_j)_{\emptyset}.
\end{aligned}$$

In this chain of equalities we have exploited the relationship between  $\overset{\circ}{A}$ ,  $\overset{\circ}{B}$ ,  $\overset{\circ}{C}$ ,  $\overset{\circ}{D}$ ,  $J$  and  $\overset{\circ}{H}$  from Theorem 4.1 applied to a GR-node  $\overset{\circ}{\alpha}$ . Thus we have for all  $k, j \in \{1, \dots, N\}$ :

$$[A_{kj} h_j, h'_k]_{F,k} + [h_j, A_{jk} h'_k]_{F,j} = -(h'_k)^* C_k^* J C_j h_j. \quad (8.22)$$

Since this equality holds for generating elements of the spaces  $\mathcal{K}_k(F)$ ,  $k = 1, \dots, N$ ) it extends by linearity to arbitrary elements  $h = \text{col}(h_1, \dots, h_N)$  and  $h' = \text{col}(h'_1, \dots, h'_N)$  in  $\mathcal{K}(F)$ . For  $k = 1, \dots, N$ , let  $\langle \cdot, \cdot \rangle_{F,k}$  be any inner product for which  $\mathcal{K}_k(F)$  is a Hilbert space. Thus,  $\mathcal{K}(F)$  is a Hilbert space with respect to the inner product

$$\langle h, h' \rangle_F := \sum_{k=1}^N \langle h_k, h'_k \rangle_{F,k}.$$

Then there exist uniquely defined linear operators  $H_k : \mathcal{K}_k(F) \rightarrow \mathcal{K}_k(F)$  such that:

$$[h_k, h'_k]_{F,k} = \langle H_k h_k, h'_k \rangle_{F,k}, \quad k = 1, \dots, N,$$

and so with  $H := \text{diag}(H_1, \dots, H_N) : \mathcal{K}(F) \rightarrow \mathcal{K}(F)$  we have:

$$[h, h']_F = \langle H h, h' \rangle_F.$$

Since the spaces  $\mathcal{K}_k(F)$  are non-degenerate (see [4]), the operators  $H_k$  are invertible and (8.22) can be rewritten as:

$$(A^*)_{kj}H_j + H_kA_{kj} = -C_k^*JC_j, \quad k, j = 1, \dots, N,$$

which is equivalent to (4.3).

Now, for arbitrary  $c, c' \in \mathbb{C}^q$  and  $w \in \mathcal{F}_N$  we have:

$$\langle H_kB_kc, K_{w'}^{F,k}(\cdot)c' \rangle_{F,k} = [R_kF(\cdot)c, K_{w'}^{F,k}(\cdot)c']_{F,k} = c'^*F_{w'g_k}c.$$

On the other hand,

$$\begin{aligned} -\langle C_k^*JDC, K_{w'}^{F,k}(\cdot)c' \rangle_{F,k} &= -\langle JF_\emptyset c, C_kK_{w'}^{F,k}(\cdot)c' \rangle_{F,k} = -\langle JF_\emptyset c, K_{\emptyset, w'}^{F,k}c' \rangle_{\mathbb{C}^q} \\ &= -c'^*K_{w', \emptyset}^{F,k}JF_\emptyset c = -c'^* \begin{pmatrix} \circ C \\ \circ b \end{pmatrix} \begin{pmatrix} \circ A \end{pmatrix}^{w'g_k} \begin{pmatrix} \circ H_k \end{pmatrix}^{-1} \begin{pmatrix} \circ C_k \end{pmatrix}^* J \begin{pmatrix} \circ D \end{pmatrix} c \\ &= c'^* \begin{pmatrix} \circ C \\ \circ b \end{pmatrix} \begin{pmatrix} \circ A \end{pmatrix}^{w'g_k} \begin{pmatrix} \circ B_k \end{pmatrix} c = c'^* \begin{pmatrix} \circ C \\ \circ b \end{pmatrix} \begin{pmatrix} \circ A \\ \circ \sharp \end{pmatrix} \begin{pmatrix} \circ B \end{pmatrix}^{w'g_k} c = c'^*F_{w'g_k}c. \end{aligned}$$

Here we have used the relation (4.4) for an arbitrary minimal GR-realization  $\overset{\circ}{\alpha} = (N; \overset{\circ}{A}, \overset{\circ}{B}, \overset{\circ}{C}, \overset{\circ}{D}; \mathbb{C}^\gamma = \bigoplus_{k=1}^N \mathbb{C}^{\gamma_k}, \mathbb{C}^q)$  of  $F$ , with the associated structured Hermitian matrix  $\overset{\circ}{H} = \text{diag}(\overset{\circ}{H}_1, \dots, \overset{\circ}{H}_N)$ . Thus,  $H_kB_k = -C_k^*JD$ ,  $k = 1, \dots, N$ , that is,  $B = -H^{-1}C^*JD$ , and (4.4) holds for the GR-node  $\alpha$ . Finally, by Theorem 4.1, we may conclude that  $H = \text{diag}(H_1, \dots, H_N)$  is the associated structured Hermitian matrix of the special GR-realization  $\alpha$ .

### 8.3. Examples

In this subsection we give certain examples of matrix-inner rational FPSs on  $\mathcal{J}_2$  with scalar coefficients (i.e.,  $N = 2$ ,  $q = 1$ , and  $J = 1$ ). We also present the corresponding non-commutative positive kernels  $K^{F,1}(z, z')$  and  $K^{F,2}(z, z')$  computed using formula (8.12).

*Example 1.*  $F(z) = (z_1 + 1)^{-1}(z_1 - 1)(z_2 + 1)^{-1}(z_2 - 1)$ .

$$K^{F,1}(z, z') = 2(z_1 + 1)^{-1}(z'_1 + 1)^{-1},$$

$$K^{F,2}(z, z') = 2(z_1 + 1)^{-1}(z_1 - 1)(z_2 + 1)^{-1}(z'_2 + 1)^{-1}(z'_1 - 1)(z'_1 + 1)^{-1}.$$

*Example 2.*  $F(z) = (z_1 + z_2 + 1)^{-1}(z_1 + z_2 - 1)$ .

$$K^{F,1}(z, z') = K^{F,2}(z, z') = 2(z_1 + z_2 + 1)^{-1}(z'_1 + z'_2 + 1)^{-1}.$$

*Example 3.*

$$\begin{aligned} F(z) &= (z_1 + (z_2 + i)^{-1} + 1)^{-1} (z_1 + (z_2 + i)^{-1} - 1) \\ &= ((z_2 + i)(z_1 + 1) + 1)^{-1} ((z_2 + i)(z_1 - 1) + 1). \end{aligned}$$

$$K^{F,1}(z, z') = 2((z_2 + i)(z_1 + 1) + 1)^{-1} (z_2 + i)(z'_2 - i) ((z'_1 + 1)(z'_2 - i) + 1)^{-1},$$

$$K^{F,2}(z, z') = 2((z_2 + i)(z_1 + 1) + 1)^{-1} ((z'_1 + 1)(z'_2 - i) + 1)^{-1}.$$



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