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A Note on Interpolation in the Generalized Schur Class. I. Applications of Realization theory

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A NOTE ON INTERPOLATION
IN THE GENERALIZED SCHUR CLASS

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To Harry Dym: teacher, colleague and friend, in appreciation and with best wishes for his 60-th birthday.

ABSTRACT. Realization theory for operator colligations on Pontryagin spaces is used to study interpolation and factorization in generalized Schur classes. Several criteria are derived which imply that a given function is almost the restriction of a generalized Schur function. The role of realization theory in coefficient problems is also discussed; a solution of an indefinite Carathéodory-Fejér problem is obtained, as well as a result that relates the number of negative (positive) squares of the reproducing kernels associated with the canonical coisometric, isometric, and unitary realizations of a generalized Schur function to the number of negative (positive) eigenvalues of matrices derived from their Taylor coefficients.

1. INTRODUCTION

Let $\mathfrak{F}$ and $\mathfrak{G}$ be Pontryagin spaces having the same negative index. For any integer $\kappa \geq 0$, the **generalized Schur class $\mathbf{S}_\kappa(\mathfrak{F}, \mathfrak{G})$** is the set of functions $S(z)$ with values in $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ which are holomorphic on some subregion Ω of the open unit disk $\mathbf{D}$ such that the kernel

$$(1.1) \qquad\qquad K_S(w, z) = \frac{1 - S(z)S(w)^*}{1 - z\bar{w}}$$

has κ negative squares. In the scalar case, that is, when $\mathfrak{F} = \mathfrak{G} = \mathbf{C}$ is the space of complex numbers in the Euclidean metric, we write simply $\mathbf{S}_\kappa$. Terminology and notation used here follow [3], where an account of the realization theory for the classes $\mathbf{S}_\kappa(\mathfrak{F}, \mathfrak{G})$ may be found (it should be noted that the definition of $\mathbf{S}_\kappa(\mathfrak{F}, \mathfrak{G})$ in [3] requires that functions are holomorphic at the origin, and we do not require this now). For example, we write $\mathrm{sq}_\pm K_S$ for the number of positive/negative squares of the kernel (1.1) and $\mathfrak{H}(S)$ for the associated reproducing kernel Pontryagin space. The generalized Schur classes were extensively studied by Kreĭn and Langer [16] when the coefficient spaces are Hilbert spaces, which is the main case of interest here; in particular, generalized Schur functions are meromorphic in $\mathbf{D}$. The literature on interpolation in such classes includes well-known works of Takagi [21], Adamjan, Arov, and Kreĭn [1], Kreĭn and Langer [17], Nudel′man [18], Ball and Helton [6], and others.

In interpolation theory, we consider kernels of the form (1.1) on $\Omega \times \Omega$, where Ω is any subset of $\mathbf{D}$ and may be finite. Nonnegativity of the kernel in this case implies that $S(z)$ is the restriction of a classical Schur function. Such a conclusion cannot quite be drawn when

(1.1) has κ negative squares: there are "nonanalytic" functions $S(z)$ such that (1.1) has a finite number of negative squares. The following result is a slightly more elaborate form of the standard example of this phenomenon [3, p. 82].

Proposition 1.1. *Let Ω be a subset of $\mathbf{D}$ containing the point w_0 and at least one other point. Let $S(z) = 1$ or 0 according as $z = w_0$ or $z \in \Omega \setminus \{w_0\}$. Then $\mathrm{sq}_- K_S = 1$, and the following statements are equivalent:*

    (1) *The function $S(z)$ is the restriction of a function in $\mathbf{S}_1$.*

    (2) *The set Ω is a Blaschke sequence.*

Recall that Ω is a Blaschke sequence if its points can be arranged in a finite or infinite sequence $z_1, z_2, \dots$ with $\sum(1 - |z_n|^2) < \infty$. The zero set of a classical Schur function is a Blaschke sequence [19, p. 64]. In the non-Blaschke case, $S(z)$ is the restriction to $\Omega \setminus \{w_0\}$ of a function in $S_{\kappa'}$, where $\kappa' = 0$, namely the function identically zero.

Proof. Let H^2 be the Hardy space on the unit disk, H_Ω^2 the space of restrictions of functions in H^2 to Ω. We view H_Ω^2 as a Hilbert space in the inner product such that the restriction mapping from H^2 onto H_Ω^2 is a partial isometry. The reproducing kernel for H^2 is the Szegő kernel $k(w, z) = 1/(1 - z\bar{w})$, and the reproducing kernel for H_Ω^2 is its restriction $k_\Omega(w, z)$ to $\Omega \times \Omega$. The identity

$$(1.2) \qquad K_S(w, z) = k_\Omega(w, z) - \frac{S(z)}{\sqrt{1 - |w_0|^2}} \frac{\overline{S(w)}}{\sqrt{1 - |w_0|^2}}, \qquad w, z \in \Omega,$$

shows that $\mathrm{sq}_- K_S \leq 1$ [5, Theorem 3.3]. When $z_1 = w_0$ and $z_2 \in \Omega \setminus \{w_0\}$, then

$$\left(K_S(z_i, z_j) \right)_{i,j=1}^2 = \begin{pmatrix} 0 & \dfrac{1}{1 - z_2\bar{w}_0} \\ \dfrac{1}{1 - w_0\bar{z}_2} & \dfrac{1}{1 - |z_2|^2} \end{pmatrix}.$$

The determinant of this matrix is negative, and so the matrix has one negative eigenvalue. Hence $\mathrm{sq}_- K_S = 1$.

If (1) holds, then $\Omega \setminus \{w_0\}$ is contained in the zero set of a nontrivial function in $\mathbf{S}_1$, and we obtain (2) because such functions are of bounded type in $\mathbf{D}$. Conversely, assume (2) and consider the function

$$S_1(z) = \left(\gamma \frac{z - \alpha}{1 - z\bar{\alpha}} \right)^{-1} B(z), \quad \gamma = \left(\frac{w_0 - \alpha}{1 - w_0\bar{\alpha}} \right)^{-1} B(w_0),$$

where $B(z)$ is a Blaschke product having simple zeros at the points of the Blaschke sequence $\Omega \setminus \{w_0\}$ (so that $0 \neq |B(w_0)| < 1$) and the number α is chosen in $\mathbf{D} \setminus \Omega$ such that $|\gamma| = 1$. Then $S_1(z)$ is the product of the inverse of a Blaschke factor and a classical Schur function which does not vanish at the zero of the Blaschke factor, and so it belongs to $\mathbf{S}_1$. Evidently, $S(z) = S_1(z)$, $z \in \Omega$. $\qquad\square$

Similar phenomena appear in Ball and Helton [6]. Again consider the scalar case. Let $S(z)$ be defined on a subset Ω of $\mathbf{D}$, and let H_Ω^2 be as in the proof of Proposition 1.1. If $S(z)$ is a multiplier for H_Ω^2 (that is, $Sh \in H_\Omega^2$ for each $h \in H_\Omega^2$) and (1.1) has κ negative squares, there is a classical Schur function $S_0(z)$ and a Blaschke product $B(z)$ of order κ such that $B(z)S(z) = S_0(z)$ on Ω. Thus in this case, there exists a function $\hat{S}(z)$ in $\mathbf{S}_{\kappa'}$ for

some $\kappa' \leq \kappa$ such that $S(z) = \hat{S}(z)$ for all but at must κ points of Ω. By Proposition 1.1, it may occur that this can only be satisfied with $\kappa' < \kappa$.

In Section 2 we use realization theory to obtain criteria which imply that a given function is the restriction of a generalized Schur function, provided that certain exceptional points are omitted. These results are related to factorization theorems for operator-valued functions of the Leech type [4]: that is, we are given partially defined operator-valued functions $A(z)$ and $B(z)$, and it is required to find a generalized Schur function $S(z)$ such that $B(z) = A(z)S(z)$.

In Section 3 we discuss coefficient problems, in which realization theory also plays a role in establishing analyticity (see Theorem 3.4). Here some of the results are restricted to the scalar case. Necessary conditions for the existence of solutions are derived by considering the three kernels associated with a generalized Schur function in its canonical coisometric, isometric, and unitary realizations [3]. First we show that these conditions are, in fact, equivalent (Theorem 3.3). Then we provide a complete solution to an indefinite form of the Carathéodory-Fejér problem in the scalar case. Key to this result is the equivalence of two matrix extension problems, one involving lower triangular Toeplitz matrices and the other Hermitian Toeplitz matrices.

2. Interpolation and factorization

Our approach is based on the use of characteristic functions of partially isometric operator colligations, and the interpolation and factorization criteria that we obtain are dictated by what is needed to construct the colligations.

A scalar example gives an idea of the nature of the conditions. Let $S(z)$ be a complex-valued function defined on a nonempty subset Ω of $\mathbf{D}$ such that the kernel (1.1) has κ negative squares, and let $\mathfrak{H}(S;\Omega)$ be the associated reproducing kernel Pontryagin space. Let e_w be the characteristic function of a point w of Ω ($e_w(z) = \delta_{wz}$ for all $z \in \Omega$, where δ is the Kronecker symbol).

(1) *If Ω is not a Blaschke sequence, a necessary condition for the interpolation of $S(z)$ by a function $\hat{S}(z)$ in $\mathbf{S}_\kappa$ is that $e_w \notin \mathfrak{H}(S;\Omega)$ for every $w \in \Omega$.*

For if interpolation is possible, $\mathfrak{H}(S;\Omega)$ is the set of restrictions of functions in $\mathfrak{H}(\hat{S})$ to Ω. If $e_w \in \mathfrak{H}(S;\Omega)$ for some $w \in \Omega$, then $\Omega \setminus \{w\}$ is contained in the zero set of a nontrivial function in $\mathfrak{H}(\hat{S})$, and hence Ω is a Blaschke sequence because functions in $\mathfrak{H}(\hat{S})$ are of bounded type (for example, see [3, Theorem 4.2.3(4)]).

(2) *The same necessary condition does not necessarily hold if Ω is a Blaschke sequence.*

For example, suppose that $\kappa = 0$ and $\Omega = \{w_1, w_2\}$ consists of two distinct points. If $S(z) = z$ on Ω, then $K_S(w, z) = 1$ identically on $\Omega \times \Omega$ and $\mathfrak{H}(S;\Omega)$ is a one-dimensional space consisting of constant functions on Ω; in this case, $e_w \notin \mathfrak{H}(S;\Omega)$ for all $w \in \Omega$. But if $S(z) = z^2$, then $K_S(w, z) = 1 + z\bar{w}$ on $\Omega \times \Omega$ and $\mathfrak{H}(S;\Omega)$ is two-dimensional; in this case $e_w \in \mathfrak{H}(S;\Omega)$ for all $w \in \Omega$. While the condition is not always necessary for interpolation, it turns out that such a condition can be sufficient.

The result below is stated in the form of a factorization problem and thus has a possible systems interpretation. We are given an "input" in the form of an operator-valued function $A(z)$ defined on some set Ω, which may be finite or infinite, and a target "output" function $B(z)$ on the same set. It is required to find a transfer function $S(z)$ for such a system. The functions $A(z)$ and $B(z)$ themselves need not be holomorphic.

3

Theorem 2.1. *Let $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{K}$ be Hilbert spaces, and let Ω be a subset of the unit disk containing the point w_0. Let $A(z)$ and $B(z)$ be functions on Ω with values in $\mathfrak{L}(\mathfrak{G}, \mathfrak{K})$ and $\mathfrak{L}(\mathfrak{F}, \mathfrak{K})$. Assume that the kernel*

$$(2.1) \qquad\qquad K(w, z) = \frac{A(z)A(w)^* - B(z)B(w)^*}{1 - \bar{w}z}$$

has κ negative squares on $\Omega \times \Omega$, and let $\mathfrak{H}_K$ be the associated reproducing kernel Pontryagin space. Let $\mathfrak{M}$ be the subspace of $\mathfrak{H}_K \oplus \mathfrak{G}$ consisting of all elements $k(z) \oplus g$ such that

$$A(w_0)g = 0 \quad and \quad \frac{z - w_0}{\sqrt{1 - |w_0|^2}} k(z) + [A(z) - A(w_0)]g \equiv 0 \quad on \quad \Omega.$$

Let $\mathfrak{N}$ be the subspace of $\mathfrak{H}_K \oplus \mathfrak{F}$ consisting of all elements $h(z) \oplus f$ such that

$$\frac{1 - z\bar{w}_0}{\sqrt{1 - |w_0|^2}} h(z) + B(z)f \equiv 0 \quad on \quad \Omega.$$

Assume that $\mathfrak{M}$ and $\mathfrak{N}$ are Hilbert spaces in the inner products of the larger spaces. Then there is a function $S(z) \in \mathbf{S}_{\kappa'}(\mathfrak{F}, \mathfrak{G})$ for some $\kappa' \leq \kappa$ such that $B(z) = A(z)S(z)$ for $z = w_0$ and for all but at most κ points z of $\Omega \setminus \{w_0\}$. In this case, $\kappa' = \kappa$ if and only if the elements h of $\mathfrak{H}(S)$ such that $A(z)h(z) \equiv 0$ on Ω form a Hilbert subspace of $\mathfrak{H}(S)$.

The function $S(z)$ which is constructed in the proof is holomorphic at w_0. The subspaces $\mathfrak{M}$ and $\mathfrak{N}$ defined in the statement of the theorem are automatically closed by the continuity of function values in a reproducing kernel space [3, Theorem 1.1.2].

Proof. It is sufficient to prove the result when $0 \in \Omega$ and $w_0 = 0$. For suppose that the result is known in this case, and consider the general situation. Let φ be the linear fractional mapping of $\mathbf{D}$ onto itself given by $\varphi(z) = (w_0 - z)/(1 - \bar{w}_0 z)$. Thus $\varphi(w_0) = 0$ and $\varphi^{-1} = \varphi$. Put $\Omega' = \varphi(\Omega)$, $w_0' = 0$, and

$$A'(z) = A(\varphi^{-1}(z)), \qquad z \in \Omega',$$
$$B'(z) = B(\varphi^{-1}(z)), \qquad z \in \Omega'.$$

Define $K'(w, z)$ on $\Omega' \times \Omega'$ by (2.7) using $A'(z)$ and $B'(z)$ in place of $A(z)$ and $B(z)$. A short calculation shows that

$$K'(w, z) = \frac{1 - |w_0|^2}{(1 - \bar{w}_0 z)(1 - w_0\bar{w})} K(\varphi^{-1}(w), \varphi^{-1}(z)), \qquad w, z \in \Omega',$$

and so $\mathrm{sq}_- K' = \kappa$; write $\mathfrak{H}_{K'}$ for the associated reproducing kernel Pontryagin space. The preceding reproducing kernel identity may be used to show that the mapping

$$V' \colon f(z) \to \frac{\sqrt{1 - |w_0|^2}}{1 - \bar{w}_0 z} f(\varphi^{-1}(z))$$

acts as an isometry from $\mathfrak{H}_K$ onto $\mathfrak{H}_{K'}$. Writing $\mathfrak{M}$ and $\mathfrak{N}$ for the subspaces defined in the theorem for the original functions $A(z)$ and $B(z)$ and point $w_0 \in \Omega$, and $\mathfrak{M}'$ and $\mathfrak{N}'$ for the corresponding subspaces relative to $A'(z)$ and $B'(z)$ and point $w_0' \in \Omega'$, we find that

$$(V' \oplus -1_{\mathfrak{G}})\,\mathfrak{M} = \mathfrak{M}' \quad \text{and} \quad (V' \oplus 1_{\mathfrak{F}})\,\mathfrak{N} = \mathfrak{N}.$$

Since we assume the result when $0 \in \Omega$ and $w_0 = 0$, we can find a function $S'(z) \in \mathbf{S}_{\kappa'}(\mathfrak{F}, \mathfrak{G})$ for some $\kappa' \leq \kappa$ such that $S'(z)$ is holomorphic at $w_0' = 0$ and $B'(z) = A'(z)S'(z)$ for $z = w_0'$

and for all but at most κ points z of $\Omega' \setminus \{w_0'\}$. Then $S(z) = S'(\varphi(z))$ has the required properties.

Thus without loss of generality, we may assume that $0 \in \Omega$ and $w_0 = 0$. Define a linear relation $\mathbf{R}$ in $(\mathfrak{H}_K \oplus \mathfrak{G}) \times (\mathfrak{H}_K \oplus \mathfrak{F})$ as the span of all pairs

$$(2.2) \quad \left(\begin{pmatrix} K(\alpha, \cdot)u_1 \\ \dfrac{A(\alpha)^* - A(0)^*}{\bar{\alpha}} u_1 + A(0)^* u_2 \end{pmatrix}, \begin{pmatrix} \dfrac{K(\alpha, \cdot) - K(0, \cdot)}{\bar{\alpha}} u_1 + K(0, \cdot)u_2 \\ \dfrac{B(\alpha)^* - B(0)^*}{\bar{\alpha}} u_1 + B(0)^* u_2 \end{pmatrix} \right)$$

with $\alpha \in \Omega \setminus \{0\}$ and $u_1, u_2 \in \mathfrak{K}$. A direct calculation shows that $\mathbf{R}$ is isometric. In fact, consider a second pair with α replaced by β and u_1, u_2 replaced by v_1, v_2. Expand and simplify the inner products of the first members in $\mathfrak{H}_K \oplus \mathfrak{G}$ and second members in $\mathfrak{H}_K \oplus \mathfrak{F}$. After simplification, in both cases we obtain

$$(2.3) \quad \langle K(\alpha, \beta)u_1, v_1 \rangle_{\mathfrak{K}} + \left\langle \frac{A(\beta)A(\alpha)^* - A(\beta)A(0)^* - A(0)A(\alpha)^* + A(0)A(0)^*}{\bar{\alpha}\beta} u_1, v_1 \right\rangle_{\mathfrak{K}}$$

$$+ \left\langle \frac{A(0)A(\alpha)^* - A(0)A(0)^*}{\bar{\alpha}} u_1, v_2 \right\rangle_{\mathfrak{K}} + \left\langle \frac{A(\beta)A(0)^* - A(0)A(0)^*}{\beta} u_2, v_1 \right\rangle_{\mathfrak{K}}$$

$$+ \langle A(0)A(0)^* u_2, v_2 \rangle_{\mathfrak{K}},$$

and this verifies the assertion. The orthogonal complement of the domain of $\mathbf{R}$ is $\mathfrak{M}$, and the orthogonal complement of the range of $\mathbf{R}$ is $\mathfrak{N}$. Since these are Hilbert spaces, it follows from [3, Theorem 1.4.2] that there is a continuous partial isometry

$$V = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_K \\ \mathfrak{F} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}_K \\ \mathfrak{G} \end{pmatrix}$$

such that V^* has initial space $\overline{\mathrm{dom}\,\mathbf{R}}$ and final space $\overline{\mathrm{ran}\,\mathbf{R}}$ and

$$V^* : \begin{pmatrix} K(\alpha, \cdot)u_1 \\ \dfrac{A(\alpha)^* - A(0)^*}{\bar{\alpha}} u_1 + A(0)^* u_2 \end{pmatrix} \to \begin{pmatrix} \dfrac{K(\alpha, \cdot) - K(0, \cdot)}{\bar{\alpha}} u_1 + K(0, \cdot)u_2 \\ \dfrac{B(\alpha)^* - B(0)^*}{\bar{\alpha}} u_1 + B(0)^* u_2 \end{pmatrix}$$

for all $\alpha \in \Omega \setminus \{0\}$ and $u_1, u_2 \in \mathfrak{K}$. Calculating as in [3, p. 51], we find that

$$(Th)(z) = \frac{h(z) - A(z)Gh}{z}, \qquad z \in \Omega \setminus \{0\},$$

$$(Ff)(z) = \frac{B(z) - A(z)H}{z} f, \qquad z \in \Omega \setminus \{0\},$$

$$A(0)Gh = h(0),$$

$$A(0)Hf = B(0)f,$$

for all $h \in \mathfrak{H}_K$ and $f \in \mathfrak{F}$.

Since V is a partial isometry whose kernel is a Hilbert space, V is a contraction. The embedding mappings $E_{\mathfrak{F}}$ and $E_{\mathfrak{G}}$ from $\mathfrak{H}_K$ into $\mathfrak{H}_K \oplus \mathfrak{F}$ and $\mathfrak{H}_K \oplus \mathfrak{G}$ are contractions (in fact isometries), and their adjoints act as projections. The adjoints are also contractions because we assume that $\mathfrak{F}$ and $\mathfrak{G}$ are Hilbert spaces. Therefore

$$T = E_{\mathfrak{G}}^* V E_{\mathfrak{F}}$$

is a contraction on the Pontryagin space $\mathfrak{H}_K$. By [14, Lemma 11.1 (p. 75)], the part of the spectrum of T that lies in $|\lambda| > 1$ consists of normal eigenvalues. By [14, Theorem 11.2 (p. 84)], the span of root manifolds for eigenvalues in $|\lambda| > 1$ is contained in a nonpositive subspace, and hence the number of such eigenvalues is at most $\text{sq}\_\mathfrak{H}_K = \kappa$. It follows that $1 - zT$ is invertible for all but at most κ points in $\mathbf{D}$. Since these exceptional points obviously do not include 0, $1 - zT$ is invertible for all $z \in \Omega \setminus \{\lambda_1, \ldots, \lambda_q\}$ for some nonzero numbers $\lambda_1, \ldots, \lambda_q$ in $\mathbf{D}$; here $q \leq \kappa$ and possibly $q = 0$ when there are no exceptional points.

Claim 1: If $w \in \Omega \setminus \{\lambda_1, \ldots, \lambda_q\}$, $h \in \mathfrak{H}_K$, and $(1 - wT)^{-1}h = g$, then

$$(2.4) \qquad g(z) = \frac{zh(z) - wA(z)Gg}{z - w}, \qquad z \in \Omega \setminus \{w\},$$

and $h(w) = A(w)Gg$.

Since this is trivially true if $w = 0$, assume that $w \neq 0$. Then

$$(2.5) \qquad h(z) = g(z) - w\frac{g(z) - A(z)Gg}{z}, \qquad z \in \Omega \setminus \{0\}.$$

Since $w \neq 0$, we can take $z = w$ in (2.5) to get $h(w) = A(w)Gg$. Again by (2.5),

$$(z - w)g(z) = zh(z) - wA(z)Gg$$

for $z \in \Omega \setminus \{0\}$. Trivially the last identity holds for $z = 0$ as well, and we obtain (2.4).

Claim 2: Define $S(w) = H + wG(1 - wT)^{-1}F$ for all $w \in \mathbf{D} \setminus \{\lambda_1, \ldots, \lambda_q\}$. Then

$$B(w) = A(w)S(w)$$

for all $w \in \Omega \setminus \{\lambda_1, \ldots, \lambda_q\}$.

The case $w = 0$ is clear. Assume $w \in \Omega \setminus \{\lambda_1, \ldots, \lambda_q\}$ and $w \neq 0$. Fix $f \in \mathfrak{F}$. We use Claim 1 with $g = (1 - wT)^{-1}h$, $h = Ff$. Thus

$$wA(w)G(1 - wT)^{-1}Ff = wA(w)Gg = wh(w) = w\,(Ff)(w) = B(w)f - A(w)Hf.$$

Claim 2 follows.

Claim 3: $S \in \mathbf{S}_{\kappa'}$ for some $\kappa' \leq \kappa$.

It is clear from the definition of $S(z)$ that it is a holomorphic function on $\mathbf{D} \setminus \{\lambda_1, \ldots, \lambda_q\}$. For all $w, z \in \mathbf{D} \setminus \{\lambda_1, \ldots, \lambda_q\}$, by the identity [3, (1.2.9)],

$$1 - S(z)S(w)^* = \begin{pmatrix} G(1 - zT)^{-1} & 1 \end{pmatrix} \begin{pmatrix} (1 - \bar{w}T^*)^{-1}G^* \\ 1 \end{pmatrix}$$
$$- \begin{pmatrix} zG(1 - zT)^{-1} & 1 \end{pmatrix} VV^* \begin{pmatrix} \bar{w}(1 - \bar{w}T^*)^{-1}G^* \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} G(1 - zT)^{-1} & 1 \end{pmatrix} \begin{pmatrix} (1 - \bar{w}T^*)^{-1}G^* \\ 1 \end{pmatrix}$$
$$- \begin{pmatrix} zG(1 - zT)^{-1} & 1 \end{pmatrix} \begin{pmatrix} \bar{w}(1 - \bar{w}T^*)^{-1}G^* \\ 1 \end{pmatrix}$$
$$+ \begin{pmatrix} zG(1 - zT)^{-1} & 1 \end{pmatrix} (1 - VV^*) \begin{pmatrix} \bar{w}(1 - \bar{w}T^*)^{-1}G^* \\ 1 \end{pmatrix}$$

$$= (1 - z\bar{w})G(1 - zT)^{-1}(1 - \bar{w}T^*)^{-1}G^*$$

$$+ \left(zG(1 - zT)^{-1} \quad 1\right)(1 - VV^*)\begin{pmatrix}\bar{w}(1 - \bar{w}T^*)^{-1}G^* \\ 1\end{pmatrix}.$$

Since $1 - VV^* \geq 0$ in the partial ordering of selfadjoint operators, $1 - VV^* = MM^*$ for some operator $M \in \mathfrak{L}(\mathfrak{D}, \mathfrak{H}_K \oplus \mathfrak{G})$, where $\mathfrak{D}$ is a Hilbert space (see, for example, [11, Theorem 2.1]; we can choose M so that it has zero kernel, but this property is not needed. Therefore

$$(2.6) \quad K_S(w, z) = G(1 - zT)^{-1}(1 - \bar{w}T^*)^{-1}G^* + \frac{\Phi(z)\Phi(w)^*}{1 - z\bar{w}}, \qquad w, z \in \mathbf{D} \setminus \{\lambda_1, \ldots, \lambda_q\},$$

where

$$\Phi(z) = \left(zG(1 - zT)^{-1} \quad 1\right)M, \qquad z \in \mathbf{D} \setminus \{\lambda_1, \ldots, \lambda_q\},$$

is a holomorphic function with values in $\mathfrak{L}(\mathfrak{D}, \mathfrak{G})$. The first summand on the right of (2.6) has κ'' negative squares for some $\kappa'' \leq \kappa$ by [3, Lemma 1.1.1′], and the second summand is nonnegative because $\mathfrak{D}$ is a Hilbert space. Thus by [3, Theorem 1.5.5] the kernel (2.6) has κ' negative squares, where $\kappa' \leq \kappa'' \leq \kappa$. Hence $S \in \mathbf{S}_{\kappa'}$, which proves Claim 3.

The function $S(z)$ has the required properties by Claims 2 and 3. The last statement, which gives the condition for $\kappa' = \kappa$, follows from [3, Theorem 1.5.7]. $\square$

The next result identifies a case in which the conditions in Theorem 2.1 can be verified. Namely, we assume that the values of $A(z)$ are "square" in the sense that $\mathfrak{K} = \mathfrak{G}$ and so the values of $A(z)$ are in $\mathfrak{L}(\mathfrak{G})$. We also assume that one of these values is invertible, and we take this to be $1_{\mathfrak{G}}$.

Theorem 2.2. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be Hilbert spaces, and let $A(z)$ and $B(z)$ be functions which are defined on a subset Ω of $\mathbf{D}$ with values in $\mathfrak{L}(\mathfrak{G})$ and $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$. Assume that the kernel*

$$(2.7) \qquad\qquad K(w, z) = \frac{A(z)A(w)^* - B(z)B(w)^*}{1 - \bar{w}z}$$

has κ negative squares on $\Omega \times \Omega$, and let $\mathfrak{H}_K$ be the associated reproducing kernel Pontryagin space. Assume that there is a point $w_0 \in \Omega$ such that

(1) $A(w_0) = 1_{\mathfrak{G}}$, *and*
(2) *the set of elements of $\mathfrak{H}_K$ which vanish on $\Omega \setminus \{w_0\}$ is a Hilbert subspace of $\mathfrak{H}_K$.*

Then there is a function $S(z) \in \mathbf{S}_{\kappa'}(\mathfrak{F}, \mathfrak{G})$ for some $\kappa' \leq \kappa$ such that $B(z) = A(z)S(z)$ for $z = w_0$ and for all but at most κ points z of $\Omega \setminus \{w_0\}$. In this case, $\kappa' = \kappa$ if and only if the elements h of $\mathfrak{H}(S)$ such that $A(z)h(z) \equiv 0$ on Ω form a Hilbert subspace of $\mathfrak{H}(S)$.

The function $S(z)$ constructed in the proof is holomorphic at w_0.

Proof. The last statement follows from [3, Theorem 1.5.7]. It is convenient to assume that $0 \in \Omega$ and $w_0 = 0$. If the result is known in this case, then as in the proof of Theorem 2.1, define $A'(z)$ and $B'(z)$ on $\Omega' = \varphi(\Omega)$, where $\varphi(z) = (w_0 - z)/(1 - \bar{w}_0 z)$. As in the same proof, introduce the kernel $K'(w, z)$ and isomorphism V' from $\mathfrak{H}_K$ onto $\mathfrak{H}_{K'}$. Under V', the functions in $\mathfrak{H}_K$ which vanish on $\Omega \setminus \{w_0\}$ correspond to the functions in $\mathfrak{H}_{K'}$ which vanish on $\Omega' \setminus \{w_0'\}$, where $w_0' = 0$. Then as before, the special case implies the general result.

In what follows, we assume that $0 \in \Omega$ and $w_0 = 0$. We apply Theorem 2.1 in this situation and also with $\mathfrak{K} = \mathfrak{G}$. It is easy to see that the subspace $\mathfrak{M}$ in Theorem 2.1 coincides with the the set of elements of $\mathfrak{H}_K$ which vanish on $\Omega \setminus \{0\}$ and is thus a Hilbert space by hypothesis.

7

We show that the subspace $\mathfrak{N}$ in Theorem 2.1 is a Hilbert space. By the first part of the proof of Theorem 2.1, $\mathfrak{N}$ is the orthogonal complement of the range of the relation $\mathbf{R}$ in $\mathfrak{H}_K \oplus \mathfrak{F}$, and therefore it is the same thing to show that the range of $\mathbf{R}$ contains a strictly negative subspace of dimension κ. By [3, Lemma 1.1.1'], it is sufficient to show that some Gram matrix of elements of the range of $\mathbf{R}$ has κ negative eigenvalues. In fact, consider two of the second members of the pairs (2.2) that define $\mathbf{R}$, say

$$\begin{pmatrix} \dfrac{K(\alpha,\cdot) - K(0,\cdot)}{\bar{\alpha}} u_1 + K(0,\cdot)u_2 \\[2mm] \dfrac{B(\alpha)^* - B(0)^*}{\bar{\alpha}} u_1 + B(0)^* u_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dfrac{K(\beta,\cdot) - K(0,\cdot)}{\bar{\beta}} v_1 + K(0,\cdot)v_2 \\[2mm] \dfrac{B(\beta)^* - B(0)^*}{\bar{\beta}} v_1 + B(0)^* v_2 \end{pmatrix}.$$

By (2.3), since now $A(0) = 1_{\mathfrak{G}}$, the inner product of these elements in $\mathfrak{H}_K \oplus \mathfrak{F}$ is equal to

$$\langle K(\alpha,\beta)u_1, v_1 \rangle_{\mathfrak{G}} + \left\langle \frac{A(\beta)A(\alpha)^* - A(\beta) - A(\alpha)^* + 1_{\mathfrak{G}}}{\bar{\alpha}\beta} u_1, v_1 \right\rangle_{\mathfrak{G}}$$

$$+ \left\langle \frac{A(\alpha)^* - 1_{\mathfrak{G}}}{\bar{\alpha}} u_1, v_2 \right\rangle_{\mathfrak{G}} + \left\langle \frac{A(\beta) - 1_{\mathfrak{G}}}{\beta} u_2, v_1 \right\rangle_{\mathfrak{G}} + \langle u_2, v_2 \rangle_{\mathfrak{G}}$$

$$= \langle K(\alpha,\beta)u_1, v_1 \rangle_{\mathfrak{G}} + \left\langle \frac{A(\alpha)^* - 1_{\mathfrak{G}}}{\bar{\alpha}} u_1 + u_2, \frac{A(\beta)^* - 1_{\mathfrak{G}}}{\bar{\beta}} v_1 + v_2 \right\rangle_{\mathfrak{G}}.$$

Here we can choose α, β and u_1, u_2 arbitrarily, and then choose v_1, v_2 so that

$$\frac{A(\alpha)^* - 1_{\mathfrak{G}}}{\bar{\alpha}} u_1 + u_2 = \frac{A(\beta)^* - 1_{\mathfrak{G}}}{\bar{\beta}} v_1 + v_2 = 0.$$

Since we assume that $\mathrm{sq}_- K = \kappa$, it follows that some Gram matrix of elements of the range of $\mathbf{R}$ has κ negative eigenvalues, as was to be shown. This completes the proof that $\mathfrak{N}$ is a Hilbert space.

The hypotheses of Theorem 2.1 are thus met, and Theorem 2.1 yields a function $S(z) \in \mathbf{S}_{\kappa'}(\mathfrak{F}, \mathfrak{G})$, $\kappa' \leq \kappa$, such that $B(z) = A(z)S(z)$ for $z = 0$ and for all but at most κ points z of $\Omega \setminus \{0\}$. $\qquad\square$

We give another condition for interpolation. Suppose that $S(z)$ belongs to $\mathbf{S}_\kappa$ and is holomorphic at the origin. Then $zS(z)$ also belongs to $\mathbf{S}_\kappa$, and thus both kernels

$$\frac{1 - S(z)\overline{S(w)}}{1 - z\bar{w}} \quad \text{and} \quad \frac{1 - z\bar{w}S(z)\overline{S(w)}}{1 - z\bar{w}}$$

have κ negative squares (see [3, Example 1 on p. 132]). In the other direction, a condition on two kernels is sufficient for interpolation from an arbitrary set Ω with at most a finite number of exceptional points.

Theorem 2.3. *Let $A(z)$ and $B(z)$ be functions defined on a subset Ω of the unit disk $\mathbf{D}$ with values in $\mathfrak{L}(\mathfrak{G}, \mathfrak{K})$ and $\mathfrak{L}(\mathfrak{F}, \mathfrak{K})$, where $\mathfrak{F}, \mathfrak{G}, \mathfrak{K}$ are Hilbert spaces. Assume that both*

$$K_1(w, z) = \frac{A(z)A(w)^* - B(z)B(w)^*}{1 - z\bar{w}}$$

and

$$K_2(w, z) = \frac{A(z)A(w)^* - z\bar{w}B(z)B(w)^*}{1 - z\bar{w}}$$

have κ negative squares on $\Omega \times \Omega$. Then there is a function $S(z)$ in $\mathbf{S}_{\kappa'}(\mathfrak{F}, \mathfrak{G})$, $\kappa' \leq \kappa$, such that $B(z) = A(z)S(z)$ for all but at most κ points z of Ω. In this case, $\kappa' = \kappa$ if and only if the elements h of $\mathfrak{H}(S)$ such that $A(z)h(z) \equiv 0$ on Ω form a Hilbert subspace of $\mathfrak{H}(S)$.

The proof uses a different colligation from that of Theorem 2.1. It is adapted from the work of V. E. Katsnelson, A. Kheifets, and P. M. Yuditskiĭ; see Kheifets [15] for an account and references to earlier works. The idea is used by Ball and Trent [7], who extend it to a several variable setting and apply it in a form for reproducing kernel functions that is close to our situation.

Theorem 2.3 is a non-holomorphic analog of [4, Theorem 11]: there the coefficient spaces are indefinite, but we have the stronger hypothesis that Ω is a neighborhood of the origin and $A(z)$ and $B(z)$ are holomorphic. Now the functions $A(z)$ and $B(z)$ are not assumed to be holomorphic, but in compensation $\mathfrak{F}$ and $\mathfrak{G}$ are required to be Hilbert spaces (for simplicity we have taken $\mathfrak{K}$ to be a Hilbert space also, but this plays no role in the argument). The proof of Theorem 2.3 runs along the same lines.

Proof. Write $\mathfrak{H}(K_1)$ and $\mathfrak{H}(K_2)$ for the Pontryagin spaces with reproducing kernels $K_1(w, z)$ and $K_2(w, z)$. Define a relation

$$\mathbf{R} = \mathrm{span} \left\{ \left(\begin{pmatrix} K_1(w, \cdot)k \\ B(w)^*k \end{pmatrix}, \begin{pmatrix} \bar{w}K_1(w, \cdot)k \\ A(w)^*k \end{pmatrix} \right) : w \in \Omega, \ k \in \mathfrak{K} \right\} \subseteq \begin{pmatrix} \mathfrak{H}(K_1) \\ \mathfrak{F} \end{pmatrix} \times \begin{pmatrix} \mathfrak{H}(K_1) \\ \mathfrak{G} \end{pmatrix}.$$

It is easy to see that $\mathbf{R}$ is isometric. We show that the domain $\mathfrak{M}$ of $\mathbf{R}$ contains a maximal uniformly negative subspace of $\mathfrak{H}(K_1) \oplus \mathfrak{F}$. To this end, consider a Gram matrix of the form

$$M = \left(\left\langle \begin{pmatrix} K_1(w_j, \cdot)k_j \\ B(w_j)^*k_j \end{pmatrix}, \begin{pmatrix} K_1(w_i, \cdot)k_i \\ B(w_i)^*k_i \end{pmatrix} \right\rangle_{\mathfrak{H}(K_1) \oplus \mathfrak{F}} \right)_{i,j=1}^{n},$$

where $w_1, \ldots, w_n$ are any points in Ω and $k_1, \ldots, k_n$ are arbitrary vectors in $\mathfrak{K}$. Thus

$$M = \left(\left\langle [K_1(w_j, w_i) + B(w_i)B(w_j)^*] k_j, k_i \right\rangle_{\mathfrak{K}} \right)_{i,j=1}^{n} = \left(\left\langle K_2(w_j, w_i)k_j, k_i \right\rangle_{\mathfrak{K}} \right)_{i,j=1}^{n}.$$

Since we assume that $K_2(w, z)$ has κ negative squares, M has at most κ negative eigenvalues no matter how $w_1, \ldots, w_n$ and $k_1, \ldots, k_n$ are chosen, and some such Gram matrix has exactly κ negative eigenvalues. By [3, Lemma 1.1.1'], $\mathfrak{M}$ contains a κ-dimensional subspace which is the antispace of a Hilbert space in the inner product of $\mathfrak{H}(K_1) \oplus \mathfrak{F}$. Since $\mathrm{sq}_-(\mathfrak{H}(K_1) \oplus \mathfrak{F}) = \kappa$, this verifies the assertion. It follows that the closure of $\mathfrak{M}$ in $\mathfrak{H}(K_1) \oplus \mathfrak{F}$ is a regular subspace whose orthogonal complement $\mathfrak{M}^\perp$ is a Hilbert space.

By [3, Theorem 1.4.2], the closure of the range of $\mathbf{R}$ is likewise a regular subspace $\mathfrak{N}$ of $\mathfrak{H}(K_1) \oplus \mathfrak{G}$, and we can construct a partial isometry

$$V = \begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{H}(K_1) \\ \mathfrak{F} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}(K_1) \\ \mathfrak{G} \end{pmatrix}$$

with initial space $\mathfrak{M}$ and final space $\mathfrak{N}$ such that

$$V^* = \begin{pmatrix} T^* & G^* \\ F^* & H^* \end{pmatrix} : \begin{pmatrix} \bar{w}K_1(w, \cdot)k \\ A(w)^*k \end{pmatrix} \to \begin{pmatrix} K_1(w, \cdot)k \\ B(w)^*k \end{pmatrix}$$

for all $k \in \mathfrak{K}$ and all $w \in \Omega$. Thus for $w \in \Omega$,

(2.8) $$T^* \{\bar{w}K_1(w, \cdot)k\} + G^* \{A(w)^*k\} = K_1(w, \cdot)k,$$

9

and

(2.9) $$F^* \{\bar{w} K_1(w, \cdot)k\} + H^* \{A(w)^* k\} = B(w)^* k.$$

Hence

(2.10) $$(1 - \bar{w} T^*) \{K_1(w, \cdot) A(w)^* k\} = G^* \{A(w)^* k\}.$$

Since $\ker V$ is a Hilbert space, V is a contraction. As in the proof of Theorem 2.1, because we assume that $\mathfrak{F}$ and $\mathfrak{G}$ are Hilbert spaces, T is a contraction, and the part of the spectrum of T that lies in $|\lambda| > 1$ consists of at most κ normal eigenvalues.

Let $\Omega' = \Omega \setminus \{\lambda_1, \ldots, \lambda_q\}$, where $\lambda_1, \ldots, \lambda_q$ are the points λ of the unit disk at which $1 - \lambda T$ is not invertible ($q \leq \kappa$). For all $w \in \Omega'$ and all $k \in \mathfrak{K}$,

$$K_1(w, \cdot)k = (1 - \bar{w} T^*)^{-1} G^* \{A(w)^* k\}$$

by (2.10). Define

$$S(z) = H + zG(1 - zT)^{-1} F, \qquad z \in \mathbf{D} \setminus \{\lambda_1, \ldots, \lambda_q\}.$$

Then $B(w) = A(w)S(w)$, $w \in \Omega'$, by (2.8) and (2.9). The proof that $\hat{S} \in \mathbf{S}_{\kappa'}$ for some $\kappa' \leq \kappa$ is the same as in the proof of Theorem 2.1. The last statement follows from [3, Theorem 1.5.7]. $\square$

In the next theorem, we allow $\mathfrak{F}, \mathfrak{G}, \mathfrak{K}$ to be indefinite, but the functions $A(z)$ and $B(z)$ are required to be holomorphic. This yields a new result of Leech type factorization theorems as a companion to those of [4].

Theorem 2.4. *Let $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{K}$ be Kreĭn spaces with $\mathrm{sq}_-\mathfrak{F} = \mathrm{sq}_-\mathfrak{G} < \infty$. Let Ω be a subregion of the unit disk containing the origin. Let $A(z)$ and $B(z)$ be holomorphic functions on Ω with values in $\mathfrak{L}(\mathfrak{G}, \mathfrak{K})$ and $\mathfrak{L}(\mathfrak{F}, \mathfrak{K})$. Assume that the kernel*

(2.11) $$K(w, z) = \frac{A(z)A(w)^* - B(z)B(w)^*}{1 - \bar{w}z}$$

has κ negative squares on $\Omega \times \Omega$, and let $\mathfrak{H}_K$ be the associated reproducing kernel Pontryagin space. Let $\mathfrak{M}$ be the subspace of $\mathfrak{H}_K \oplus \mathfrak{G}$ consisting of all elements $k(z) \oplus g$ such that

$$A(0)g = 0 \quad \text{and} \quad zk(z) + [A(z) - A(0)]g \equiv 0 \quad \text{on} \quad \Omega.$$

Let $\mathfrak{N}$ be the subspace of $\mathfrak{H}_K \oplus \mathfrak{F}$ consisting of all elements $h(z) \oplus f$ such that

$$h(z) + B(z)f \equiv 0 \quad \text{on} \quad \Omega.$$

Assume that $\mathfrak{M}$ and $\mathfrak{N}$ are Hilbert spaces in the inner products of the larger spaces. Then there is a function $S(z) \in \mathbf{S}_{\kappa'}(\mathfrak{F}, \mathfrak{G})$ for some $\kappa' \leq \kappa$ which is holomorphic at the origin and such that $B(z) = A(z)S(z)$ for all but at most κ points z of Ω. In this case, $\kappa' = \kappa$ if and only if the elements h of $\mathfrak{H}(S)$ such that $A(z)h(z) \equiv 0$ on Ω form a Hilbert subspace of $\mathfrak{H}(S)$.

Proof. We repeat the constructions in the proof of Theorem 2.1. The partial isometry V is again a contraction in the present situation. In general, the operator T is not a contraction, but it is a bounded operator and so $(1 - wT)^{-1}$ is defined for $|w|$ sufficiently small. The argument goes through if we restrict attention to a suitable neighborhood of the origin. At the end, the identity $B(z) = A(z)S(z)$ extends to all but at most κ points of Ω by analytic continuation. $\square$

3. Coefficient and moment problems

Let $z_1, \ldots, z_n$ be points in the unit disk, and let $w_1, \ldots, w_n$ be any complex numbers. If we specialize Section 2 to the scalar case and set $\Omega = \{z_1, \ldots, z_n\}$, $A(z_j) = 1$, and $B(z_j) = w_j$ for all $j = 1, \ldots, n$, then the interpolation problem in Section 2 reduces to the Nevanlinna-Pick problem. The indefinite form of interpolation was introduced by Takagi [21], and it has been studied by Adamjan, Arov, and Kreĭn [1], Kreĭn and Langer [17], and others. A rather complete picture of the solution of the indefinite Nevanlinna-Pick problem emerged from this work. A remaining issue concerning the degenerate case was recently settled. Namely, one can ask, for which nonnegative integers κ can the Nevanlinna-Pick problem be solved in $\mathbf{S}_\kappa$ for given data $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$? A more precise question can be posed. Define $\mathbf{S}_{\nu,\pi}$ as the class of all meromorphic functions $S(z)$ on the unit disk for which the kernel $K_S(w, z)$ has ν negative squares and π positive squares (thus $\mathbf{S}_{\nu,\pi}$ is a subclass of $\mathbf{S}_\nu$). For which nonnegative integers ν and π can the Nevanlinna-Pick problem be solved in $\mathbf{S}_{\nu,\pi}$ for given data $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$? These questions were answered by Woracek [22, 23] (with the disk replaced by the upper half-plane), yielding a complete solution of the Nevanlinna-Pick problem in the scalar case.

We consider analogous questions for the indefinite Carathéodory-Fejér problem and obtain a complete solution in the scalar case. The solution depends on results of Iokhvidov [12] on a related trigonometric moment problem. In the positive definite case this connection is well-known. We refer to [12] for references to the original papers (some jointly with M. G. Kreĭn) pertaining to this problem. A key step involves another application of the characteristic function of a partially isometric operator colligation, which was the principal tool in Section 2.

Problem I (Carathéodory-Fejér problem). *Let $a_0, a_1, \ldots, a_{n-1}$ be n complex numbers. For which nonnegative integers κ is there a function $S(z)$ in $\mathbf{S}_\kappa$ which is holomorphic at the origin and such that $S(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + \mathcal{O}(z^n)$ in a neighborhood of the origin? For which ν and π do there exist solutions in $\mathbf{S}_{\nu,\pi}$?*

Necessary conditions on coefficients are obtained from the series expansions of standard kernel functions. Suppose that $S(z)$ is a holomorphic (scalar-valued) function defined in a neighborhood of the origin. Let $S(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ be its Taylor series expansion, and write

$$
(3.1) \qquad T_r = \begin{pmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & \cdots & 0 \\ & & \cdots & & \\ a_{r-1} & a_{r-2} & a_{r-3} & \cdots & a_0 \end{pmatrix}, \qquad \tilde{T}_r = \begin{pmatrix} \bar{a}_0 & 0 & 0 & \cdots & 0 \\ \bar{a}_1 & \bar{a}_0 & 0 & \cdots & 0 \\ & & \cdots & & \\ \bar{a}_{r-1} & \bar{a}_{r-2} & \bar{a}_{r-3} & \cdots & \bar{a}_0 \end{pmatrix},
$$

$$
(3.2) \qquad Q_r = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_r \\ a_2 & a_3 & a_4 & \cdots & a_{r+1} \\ & & \cdots & & \\ a_r & a_{r+1} & a_{r+2} & \cdots & a_{2r-1} \end{pmatrix},
$$

$r = 1, 2, \ldots$. Set $\tilde{S}(z) = \bar{S}(\bar{z})$. Straightforward calculations yield the expansions

$$
K_S(w, z) = \frac{1 - S(z)\overline{S(w)}}{1 - z\bar{w}} = \sum_{p,q=0}^{\infty} C_{pq} z^p \bar{w}^q,
$$

11

$$K_{\tilde{S}}(w,z) = \frac{1 - \tilde{S}(z)\overline{\tilde{S}(w)}}{1 - z\bar{w}} = \sum_{p,q=0}^{\infty} \tilde{C}_{pq} z^p \bar{w}^q,$$

$$D_S(w,z) = \begin{pmatrix} K_S(w,z) & \dfrac{S(z) - S(\bar{w})}{z - \bar{w}} \\ \dfrac{\tilde{S}(z) - \tilde{S}(\bar{w})}{z - \bar{w}} & K_{\tilde{S}}(w,z) \end{pmatrix} = \sum_{p,q=0}^{\infty} D_{pq} z^p \bar{w}^q,$$

where

$$\left[C_{pq}\right]_{p,q=0}^{n-1} = I_n - T_n T_n^*, \qquad \left[\tilde{C}_{pq}\right]_{p,q=0}^{n-1} = I_n - \tilde{T}_n \tilde{T}_n^*,$$

$$\left[D_{pq}\right]_{p,q=0}^{r-1} = \begin{pmatrix} I_r - T_r T_r^* & Q_r \\ Q_r^* & I_r - \tilde{T}_r \tilde{T}_r^* \end{pmatrix}, \qquad 1 \leq r \leq n/2.$$

Thus the coefficients $a_0, a_1, \ldots$ of $S(z)$ give rise to three families of matrices:

$$(3.3) \qquad I_n - T_n T_n^*, \qquad I_n - \tilde{T}_n \tilde{T}_n^*, \qquad \begin{pmatrix} I_r - T_r T_r^* & Q_r \\ Q_r^* & I_r - \tilde{T}_r \tilde{T}_r^* \end{pmatrix}, \qquad 1 \leq r \leq n/2,$$

$n = 0, 1, \ldots$. For fixed n, the matrices (3.3) depend only on $a_0, \ldots, a_{n-1}$.

If $S(z)$ belongs to $\mathbf{S}_\kappa$, then the three kernels each have κ negative squares [3, Theorem 2.5.2]. It follows that the number of negative eigenvalues of each of the matrices in (3.3) is a nondecreasing function of the order of the matrix, and this number is ultimately equal to κ in each case (see the result in Section 4).

If $S(z)$ belongs to $\mathbf{S}_{\nu,\pi}$, similar remarks apply not only to the number of negative squares but also to the number of positive squares. For simplicity, suppose that $S(0) \neq 0$, and note the identities

$$K_S(w,z) = -S(z) K_{1/S}(w,z) \overline{S(w)},$$

$$K_{\tilde{S}}(w,z) = -\tilde{S}(z) K_{1/\tilde{S}}(w,z) \overline{\tilde{S}(w)},$$

$$D_S(w,z) = -\begin{pmatrix} S(z) & 0 \\ 0 & \tilde{S}(z) \end{pmatrix} D_{1/S}(w,z) \begin{pmatrix} \overline{S(w)} & 0 \\ 0 & \overline{\tilde{S}(w)} \end{pmatrix}.$$

The numbers of positive squares of $K_S(w,z)$, $K_{\tilde{S}}(w,z)$, and $D_S(w,z)$ thus coincide, with the numbers of negative squares of $K_{1/S}(w,z)$, $K_{1/\tilde{S}}(w,z)$, and $D_{1/S}(w,z)$, respectively. Thus if one of the three kernels has π positive squares, then all do. In this case, applying the previous assertions concerning negative squares, we see that the number of positive eigenvalues of each of the matrices in (3.3) is a nondecreasing function of the order of the matrix, and this number is ultimately equal to π in each case.

This raises questions concerning the general behavior of the numbers of negative and positive eigenvalues for the matrices (3.3) whenever (3.1) and (3.2) are defined for any complex numbers $a_0, a_1, \ldots$, whether these numbers are the Taylor coefficients of a holomorphic function or not. We show that the behavior is indeed always similar to the special cases noted above: the numbers of negative (positive) eigenvalues for the three types are nondecreasing functions of the order, and if one eventually has some constant value, then all have the same constant value eventually. These questions are purely algebraic. There is a separate convergence question, namely, under what conditions are the given numbers $a_0, a_1, \ldots$ the

Taylor coefficients of a holomorphic function $S(z)$ in $\mathbf{S}_\kappa$ or $\mathbf{S}_{\nu,\pi}$? Finally, if we only define (3.1), (3.2), and (3.3) as far as we can go with a finite sequence $a_0, \ldots, a_{n-1}$, what are the possible extensions to an infinite sequence $a_0, a_1, \ldots$?

To answer such questions, we relate given complex numbers $a_0, \ldots, a_{n-1}$ to a trigonometric moment problem. Define $c_0 = 1, c_1, \ldots, c_n$ by

(3.4)
$$\begin{cases} c_0 = 1, \\ c_1 = c_0 a_0, \\ c_2 = c_0 a_1 + c_1 a_0, \\ \quad \ldots \\ c_n = c_0 a_{n-1} + c_1 a_{n-2} + \cdots + c_{n-1} a_0, \end{cases}$$

This correspondence is one-to-one and has the property that if $a_0, \ldots, a_{n-1}$ corresponds to $c_0 = 1, c_1, \ldots, c_n$ then for each $1 \le k \le n$, $a_0, \ldots, a_{k-1}$ corresponds to $c_0 = 1, c_1, \ldots, c_k$ also via (3.4) with n replaced by k. We consider the associated matrix

(3.5)
$$M_n = \begin{pmatrix} c_0 & \bar{c}_1 & \bar{c}_2 & \ldots & \bar{c}_n \\ c_1 & c_0 & \bar{c}_1 & \ldots & \bar{c}_{n-1} \\ c_2 & c_1 & c_0 & \ldots & \bar{c}_{n-2} \\ & & \ldots & & \\ c_n & c_{n-1} & c_{n-2} & \ldots & c_0 \end{pmatrix}$$

In the sequel J_n stands for the selfadjoint and unitary $n \times n$ matrix

$$J_n = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ & & \ldots & & \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Also define

$$B_r = \begin{pmatrix} c_0 & 0 & \ldots & 0 \\ c_1 & c_0 & \ldots & 0 \\ & & \ldots & \\ c_r & c_{r-1} & \ldots & c_0 \end{pmatrix}, \quad C_r = \begin{pmatrix} I_r & 0 \\ 0 & B_r \end{pmatrix} \begin{pmatrix} 0 & B_r^* J_{r+1} \\ I_r & 0 \end{pmatrix}.$$

Theorem 3.1. *Let $a_0, a_1, \ldots, a_{n-1}$ be complex numbers and define $c_0 = 1, c_1, \ldots, c_n$ by (3.4). The following equalities hold:*

(3.6)
$$M_r = B_r \begin{pmatrix} 1 & 0 \\ 0 & I_r - T_r T_r^* \end{pmatrix} B_r^* = B_r^* \begin{pmatrix} I_r - T_r^* T_r & 0 \\ 0 & 1 \end{pmatrix} B_r, \quad 1 \le r \le n,$$

(3.7)
$$\overline{M}_r = \tilde{B}_r^* \begin{pmatrix} I_r - \tilde{T}_r^* \tilde{T}_r & 0 \\ 0 & 1 \end{pmatrix} \tilde{B}_r, \quad 1 \le r \le n,$$

and

(3.8)
$$M_{2r} = C_r \begin{pmatrix} I_r - T_r T_r^* & 0 & Q_r \\ 0 & 1 & 0 \\ Q_r^* & 0 & I_r - \tilde{T}_r \tilde{T}_r^* \end{pmatrix} C_r^*, \quad 1 \le r \le n/2.$$

The bar in $\overline{M}_r$ in (3.7) indicates that all entries in the matrix M_r have been replaced by their complex conjugates.

13

Proof. The first equality in (3.6) can be shown by induction. The second equality follows from the first. To see this, use the identities $J_{r+1}M_rJ_{r+1} = \overline{M}_r$, $J_{r+1}B_rJ_{r+1} = \tilde{B}_r^*$, $J_rT_rJ_r = \tilde{T}_r^*$, and

$$\begin{pmatrix} 0 & J_r \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & I_r - T_rT_r^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ J_r & 0 \end{pmatrix} = \begin{pmatrix} I_r - \tilde{T}_r^*\tilde{T}_r & 0 \\ 0 & 1 \end{pmatrix},$$

to obtain

$$M_r = J_{r+1}\overline{M}_rJ_{r+1} = J_{r+1}\overline{B}_rJ_{r+1} \begin{pmatrix} I_r - T_r^*T_r & 0 \\ 0 & 1 \end{pmatrix} J_{r+1}\overline{B}_r^*J_{r+1} = B_r^* \begin{pmatrix} I_r - T_r^*T_r & 0 \\ 0 & 1 \end{pmatrix} B_r,$$

which is the second equality in (3.6). We get (3.7) on replacing the entries of the matrices by their complex conjugates.

We prove (3.8). Assume $1 \le r \le n/2$. Then

$$(3.9) \qquad M_{2r} = \begin{pmatrix} M_{r-1} & S_r^* \\ S_r & M_r \end{pmatrix}, \qquad S_r = \begin{pmatrix} c_r & c_{r-1} & \ldots & c_2 & c_1 \\ c_{r+1} & c_r & \ldots & c_3 & c_2 \\ & & \ldots & & \\ c_{2r} & c_{2r-1} & \ldots & c_{r+2} & c_{r+1} \end{pmatrix}.$$

In (3.9) we use the first equality in (3.6) to obtain

$$M_{2r} = \begin{pmatrix} M_{r-1} & S_r^* \\ S_r & B_r \begin{pmatrix} 1 & 0 \\ 0 & I_r - T_rT_r^* \end{pmatrix} B_r^* \end{pmatrix}$$

$$= \begin{pmatrix} I_r & 0 \\ 0 & B_r \end{pmatrix} \begin{pmatrix} M_{r-1} & S_r^*B_r^{*-1} \\ B_r^{-1}S_r & \begin{pmatrix} 1 & 0 \\ 0 & I_r - T_rT_r^* \end{pmatrix} \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & B_r^* \end{pmatrix}.$$

Due to the lower triangular form of B_r, we get

$$B_r^{-1}S_r = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ * & 1 & \ldots & 0 & 0 \\ & & \ldots & & \\ * & * & \ldots & * & 1 \end{pmatrix} \begin{pmatrix} c_r & c_{r-1} & \ldots & c_1 \\ & & & \\ & & * & \\ & & & \end{pmatrix} = \begin{pmatrix} c_r & c_{r-1} & \ldots & c_1 \\ & & & \\ & & Z_r & \\ & & & \end{pmatrix}.$$

With this definition of Z_r and (3.6), we obtain

$$M_{2r} = \begin{pmatrix} I_r & 0 \\ 0 & B_r \end{pmatrix} \begin{pmatrix} M_{r-1} & \begin{pmatrix} \bar{c}_r \\ \vdots \\ \bar{c}_1 \end{pmatrix} & Z_r^* \\ \begin{pmatrix} c_r & c_{r-1} & \ldots & c_1 \\ & & & \\ & & Z_r & \\ & & & \end{pmatrix} & \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & I_r - T_rT_r^* & \\ 0 & & & \end{pmatrix} \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & B_r^* \end{pmatrix}$$

14

$$
= \begin{pmatrix} I_r & 0 \\ 0 & B_r \end{pmatrix} \begin{pmatrix} M_r & \begin{pmatrix} Z_r^* \\ 0 \ \cdots \ 0 \end{pmatrix} \\ \begin{pmatrix} & & 0 \\ Z_r & \vdots \\ & & 0 \end{pmatrix} & I_r - T_r T_r^* \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & B_r^* \end{pmatrix}
$$

$$
= \begin{pmatrix} I_r & 0 \\ 0 & B_r \end{pmatrix} \begin{pmatrix} B_r^* \begin{pmatrix} I_r - T_r^* T_r & 0 \\ 0 & 1 \end{pmatrix} B_r & \begin{pmatrix} Z_r^* \\ 0 \ \cdots \ 0 \end{pmatrix} \\ \begin{pmatrix} & & 0 \\ Z_r & \vdots \\ & & 0 \end{pmatrix} & I_r - T_r T_r^* \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & B_r^* \end{pmatrix}
$$

$$
= \begin{pmatrix} I_r & 0 \\ 0 & B_r \end{pmatrix} \begin{pmatrix} B_r^* & 0 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} \begin{pmatrix} I_r - T_r^* T_r & 0 \\ 0 & 1 \end{pmatrix} & B_r^{*-1} \begin{pmatrix} Z_r^* \\ 0 \ \cdots \ 0 \end{pmatrix} \\ \begin{pmatrix} & & 0 \\ Z_r & \vdots \\ & & 0 \end{pmatrix} B_r^{-1} & I_r - T_r T_r^* \end{pmatrix} .
$$

$$
\cdot \begin{pmatrix} B_r & 0 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} I_r & 0 \\ 0 & B_r^* \end{pmatrix}
$$

Here the matrix

$$
C_r' = \begin{pmatrix} I_r & 0 \\ 0 & B_r \end{pmatrix} \begin{pmatrix} B_r^* & 0 \\ 0 & I_r \end{pmatrix}
$$

is invertible. Note also that

$$
\begin{pmatrix} & & 0 \\ Z_r & \vdots \\ & & 0 \end{pmatrix} B_r^{-1} = \begin{pmatrix} & & 0 \\ Z_r & \vdots \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ & & \cdots & \\ * & * & \cdots & 1 \end{pmatrix} = \begin{pmatrix} & & 0 \\ Y_r & \vdots \\ & & 0 \end{pmatrix} ,
$$

so that with Y_r defined in this way, we have

$$
(3.10) \qquad M_{2r} = C_r' \begin{pmatrix} I_r - T_r^* T_r & 0 & Y_r^* \\ 0 & 1 & 0 \\ Y_r & 0 & I_r - T_r T_r^* \end{pmatrix} C_r'^*, \quad 1 \le r \le n/2.
$$

15

We now identify Y_r as

$$(3.11) \qquad Y_r = \begin{pmatrix} a_r & a_{r-1} & \dots & a_1 \\ a_{r+1} & a_r & \dots & a_2 \\ & & \dots & \\ a_{2r-1} & a_{2r-2} & \dots & a_r \end{pmatrix}.$$

¿From the definition of B_r we find that

$$B_r^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ -a_0 & 1 & \dots & 0 \\ & & \dots & \\ -a_{r-1} & -a_{r-2} & \dots & 1 \end{pmatrix}.$$

It follows that

$$B_r^{-1} S_r = \left(\begin{pmatrix} \begin{pmatrix} c_r & c_{r-1} & \dots & c_1 \end{pmatrix} \\ \begin{pmatrix} a_r & a_{r-1} & \dots & a_1 \\ a_{r+1} & a_r & \dots & a_2 \\ & & \dots & \\ a_{2r-1} & a_{2r-2} & \dots & a_r \end{pmatrix} B_{r-1} \end{pmatrix} \right) \quad \text{and} \quad Z_r = \begin{pmatrix} a_r & a_{r-1} & \dots & a_1 \\ a_{r+1} & a_r & \dots & a_2 \\ & & \dots & \\ a_{2r-1} & a_{2r-2} & \dots & a_r \end{pmatrix} B_{r-1}.$$

Finally, we obtain

$$\begin{pmatrix} Y_r & 0 \end{pmatrix} = \begin{pmatrix} Z_r & 0 \end{pmatrix} B_r^{-1} = \left(\begin{pmatrix} a_r & a_{r-1} & \dots & a_1 \\ a_{r+1} & a_r & \dots & a_2 \\ & & \dots & \\ a_{2r-1} & a_{2r-2} & \dots & a_r \end{pmatrix} B_{r-1} \quad 0 \right) \begin{pmatrix} B_{r-1}^{-1} & 0 \\ * & 1 \end{pmatrix}$$

$$= \left(\begin{pmatrix} a_r & a_{r-1} & \dots & a_1 \\ a_{r+1} & a_r & \dots & a_2 \\ & & \dots & \\ a_{2r-1} & a_{2r-2} & \dots & a_r \end{pmatrix} \quad 0 \right),$$

proving (3.11). Evidently, $Q_r = Y_r J_r$ and

$$C_r = C_r' \begin{pmatrix} 0 & J_{r+1} \\ I_r & 0 \end{pmatrix} = C_r' \begin{pmatrix} 0 & 0 & J_r \\ 0 & 1 & 0 \\ I_r & 0 & 0 \end{pmatrix}.$$

Substituting this in (3.10) we obtain (3.8). $\qquad \square$

A number of consequences follow. For any Hermitian matrix A we write $\pi(A)$ and $\nu(A)$ for the numbers of positive and negative eigenvalues of A counting multiplicity.

Corollary 3.2. *Let $a_0, a_1, \dots, a_{n-1}$ be complex numbers and define $c_0 = 1, c_1, \dots, c_n$ by (3.4).*

(1) *Each of the four quantities*

$$\nu(I_r - T_r T_r^*), \quad \pi(I_r - T_r T_r^*), \qquad 1 \le r \le n,$$

$$\nu \begin{pmatrix} I_r - T_r T_r^* & Q_r \\ Q_r^* & I_r - \tilde{T}_r \tilde{T}_r^* \end{pmatrix}, \quad \pi \begin{pmatrix} I_r - T_r T_r^* & Q_r \\ Q_r^* & I_r - \tilde{T}_r \tilde{T}_r^* \end{pmatrix}, \qquad 1 \le r \le n/2,$$

is a nondecreasing function of r.

(2) *For $0 \leq r \leq n$,*

$$\nu(I_r - T_r T_r^*) = \nu(I_r - T_r^* T_r) = \nu(I_r - \tilde{T}_r \tilde{T}_r^*) = \nu(I_r - \tilde{T}_r^* \tilde{T}_r)$$

and

$$\pi(I_r - T_r T_r^*) = \pi(I_r - T_r^* T_r) = \pi(I_r - \tilde{T}_r \tilde{T}_r^*) = \pi(I_r - \tilde{T}_r^* \tilde{T}_r).$$

(3) *If $\nu(I_n - T_n T_n^*) = \kappa$, then all of the matrices in (3.3) have at most κ negative eigenvalues.*

The condition $\nu(I_n - T_n T_n^*) = \kappa$ is necessary that $a_0, a_1, \ldots, a_{n-1}$ are the first n Taylor coefficients of a function in $\mathbf{S}_\kappa$. The point of statement (3) in the preceding corollary is that no stronger necessary condition can be obtained from the other matrices in (3.3).

Proof. (1) By the first equality in (3.6),

$$\nu(I_r - T_r T_r^*) = \nu(M_r),$$
$$\pi(I_r - T_r T_r^*) = \pi(M_r) - 1.$$

By (3.8),

$$\nu \begin{pmatrix} I_r - T_r T_r^* & Q_r \\ Q_r^* & I_r - \tilde{T}_r \tilde{T}_r^* \end{pmatrix} = \nu(M_{2r}),$$

$$\pi \begin{pmatrix} I_r - T_r T_r^* & Q_r \\ Q_r^* & I_r - \tilde{T}_r \tilde{T}_r^* \end{pmatrix} = \pi(M_{2r}) - 1.$$

If $s < r$, then M_s is a submatrix of M_r obtained by deleting a set of rows and corresponding columns, and therefore $\nu(M_s) \leq \nu(M_r)$, yielding (1).

(2) The first and third equalities hold by (3.6). Since $J_r T_r^* = \tilde{T}_r J_r$ and hence

$$I_r - \tilde{T}_r \tilde{T}_r^* = I_r - J_r T_r^* T_r J_r = J_r (I_r - T_r^* T_r) J_r,$$

the second equality also holds.

(3) By part (2), $\nu(I_n - \tilde{T}_n \tilde{T}_n^*) = \nu(I_n - T_n T_n^*) = \kappa$. By the proof of (1), if $1 \leq r \leq n/2$, then

$$\nu \begin{pmatrix} I_r - T_r T_r^* & Q_r \\ Q_r^* & I_r - \tilde{T}_r \tilde{T}_r^* \end{pmatrix} = \nu(M_{2r}) \leq \nu(M_n) = \nu(I_n - T_n T_n^*) = \kappa,$$

and this proves (3). $\qquad\square$

Corollary 3.3. *Let $a_0, a_1, a_2, \ldots$ be complex numbers and define $c_0 = 1, c_1, c_2, \ldots$ by (3.4). If one of the three nondecreasing sequences*

$$\{\nu(I_r - T_r T_r^*)\}_1^\infty, \quad \left\{\nu(I_r - \tilde{T}_r \tilde{T}_r^*)\right\}_1^\infty, \quad \left\{\nu \begin{pmatrix} I_r - T_r T_r^* & Q_r \\ Q_r^* & I_r - \tilde{T}_r \tilde{T}_r^* \end{pmatrix}\right\}_1^\infty$$

has constant value κ from some point on, then all do. If one of the three nondecreasing sequences

$$\{\pi(I_r - T_r T_r^*)\}_1^\infty, \quad \left\{\pi(I_r - \tilde{T}_r \tilde{T}_r^*)\right\}_1^\infty, \quad \left\{\pi \begin{pmatrix} I_r - T_r T_r^* & Q_r \\ Q_r^* & I_r - \tilde{T}_r \tilde{T}_r^* \end{pmatrix}\right\}_1^\infty$$

has constant value κ from some point on, then all do.

Proof. This follows on expressing all of the quantities in terms of the sequences $\{\nu(M_r)\}_{r=1}^{\infty}$ and $\{\pi(M_r)\}_{r=1}^{\infty}$. For example, for the negative eigenvalues, if one of the quantities has constant value κ from some point on, then $\nu(M_r) = \kappa$ for all sufficiently large r, and all have constant value κ from some point on. $\qquad\square$

We next recall a result from [10] on convergence of power series. We give a complete proof not only to keep the note selfcontained but also to show, as was done in Section 2, the role of realization theory: the coefficients of the power series are represented as Taylor coefficients of the transfer function of a colligation, which is holomorphic in a neighborhood of the origin.

Theorem 3.4. *Let $a_0, a_1, a_2, \ldots$ be complex numbers such that the matrices $I_j - T_j T_j^*$ have κ negative eigenvalues for all sufficiently large j. Then the power series $S(z) = \sum_{j=0}^{\infty} a_j z^j$ converges in some disk $|z| < \delta$ where $\delta > 0$.*

Proof. Let $\mathfrak{F} = \mathbf{C}$ be the complex numbers viewed as a Hilbert space in the Euclidean metric. Define $c_0, c_1, c_2, \ldots$ by (3.4). Then by (3.6), the matrices (3.5) have κ negative eigenvalues for all sufficiently large r, that is, the sequence $c_0, c_1, c_2, \ldots$ belongs to $\mathfrak{P}_{\kappa}$. As in Iokhvidov and Kreĭn [13, pp. 312-314], construct a Naimark dilation for $c_0, c_1, c_2, \ldots$; that is, we construct a Pontryagin space $\mathfrak{K}$ that contains $\mathfrak{F}$ isometrically as a regular subspace, and a unitary operator $U \in \mathfrak{L}(\mathfrak{K})$ such that

$$c_j = P_{\mathfrak{F}} U^j |_{\mathfrak{F}}, \qquad j = 0, 1, 2, \ldots,$$

where $P_{\mathfrak{F}}$ is the projection on $\mathfrak{K}$ with range $\mathfrak{F}$. Since $\mathfrak{F}$ is a regular subspace of $\mathfrak{K}$, we can write $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{F}$ where $\mathfrak{H}$ is a regular subspace of $\mathfrak{K}$. Let

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

relative to this decomposition. We show that

(3.12) $$a_0 = D \qquad \text{and} \qquad a_m = CA^{m-1}B, \quad m \geq 1.$$

The cases $m = 0, 1$ are immediate. We prove the formula for a_m assuming it is known for $a_0, \ldots, a_{m-1}$. By (3.4),

$$c_{m+1} = c_0 a_m + c_1 a_{m-1} + \cdots + c_m a_0,$$

so it is the same thing to show that

(3.13) $$c_{m+1} = c_0 CA^{m-1}B + c_1 CA^{m-2}B + \cdots + c_{m-1}CB + c_m D.$$

Put

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}, \qquad j \geq 0.$$

Then

$$\begin{pmatrix} A_{m+1} & B_{m+1} \\ C_{m+1} & D_{m+1} \end{pmatrix} = \begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_m A + B_m C & A_m B + B_m D \\ C_m A + D_m C & C_m B + D_m D \end{pmatrix}.$$

Since $D_j = P_{\mathfrak{F}} U^j |_{\mathfrak{F}} = c_j$ for all $j \geq 0$, $c_{m+1} = C_m B + c_m D$. This allows us to bring (3.13) to the form

(3.14) $$C_m B = c_0 CA^{m-1}B + c_1 CA^{m-2}B + \cdots + c_{m-1}CB.$$

Dropping the factor B on the right in each term, we easily verify (3.14) by induction: the formula is evident for $m = 1$, and the inductive step follows from the identity $C_{m+1} = C_m A + c_m C$. This completes the proof of (3.12). The identity (3.12) implies that $|a_j| \leq K\rho^j$ for some positive constants K and ρ, and therefore the power series $\sum_{j=0}^{\infty} a_j z^j$ converges in a neighborhood of the origin. $\qquad\square$

We can now relate Problem I to an indefinite form of the trigonometric moment problem.

Let $\mathfrak{P}_\kappa$ ($\mathfrak{P}_{\nu,\pi}$) be the set of all sequences $\{c_j\}_{j=0}^{\infty}$ with $\bar{c}_0 = c_0$ such that the matrix M_r has κ negative (ν negative and π positive) eigenvalues for all sufficiently large r.

Problem II (Trigonometric moment problem). *Let $c_0, c_1, \ldots, c_{n-1}$ be n complex numbers with $\bar{c}_0 = c_0$. Determine for which nonnegative integers κ there is a sequence $\{c_p\}_{p=0}^{\infty}$ in $\mathfrak{P}_\kappa$ that extends the given numbers. Determine for which nonnegative integers ν and π there is a sequence $\{c_p\}_{p=0}^{\infty}$ in $\mathfrak{P}_{\nu,\pi}$ that extends the given numbers.*

This problem is an indefinite form of the trigonometric moment problem and it was considered by Iokhvidov and Kreǐn [13, §19]. In the classical case, this concerns the Fourier coefficients, or moments, $c_j = \int e^{-ijt}\, d\mu(t)$, $j = 0, \pm 1, \pm 2, \ldots$, of a nonnegative measure μ on $[0, 2\pi)$. In this case, the matrix $(c_{i-j})_{i,j=0}^{n}$ is nonnegative for every $n \geq 0$, since

$$\sum_{j,k=0}^{n} c_{k-j} \lambda_k \bar{\lambda}_j = \int_{[0,2\pi)} \sum_{j,k=0}^{n} \lambda_k \bar{\lambda}_j e^{-i(k-j)t}\, d\mu(t) = \int_{[0,2\pi)} \left| \sum_{j,k=0}^{n} \lambda_k e^{-ikt} \right|^2 d\mu(t) \geq 0$$

for arbitrary numbers $\lambda_0, \ldots, \lambda_n$. When μ is a probability measure, $c_0 = 1$. The classical trigonometric moment problem is to extend given numbers $c_0, c_1, \ldots, c_{n-1}$ with $\bar{c}_0 = c_0$ to such a moment sequence. In the indefinite extension, we still speak of the "trigonometric moment problem," but the underlying function theory is not the same.

We can show now that Problem I and Problem II are equivalent.

Theorem 3.5 (Equivalence of Problems I and II). *Assume that the numbers $a_0, \ldots, a_{n-1}$ and $c_0 = 1, c_1, \ldots, c_{n-1}, c_n$ are connected as in (3.4). Then Problem I is solvable with the data $a_0, \ldots, a_{n-1}$ if and only if Problem II is solvable with the data $c_0, \ldots, c_{n-1}, c_n$.*

Proof. Suppose that Problem I with the data $a_0, \ldots, a_{n-1}$ has a solution in $\mathbf{S}_\kappa$. Let

$$S(z) = \sum_{j=0}^{\infty} a_j z^j$$

be the Taylor expansion of this solution. By the necessary conditions for Problem I discussed above, $I_j - T_j T_j^*$ has κ negative eigenvalues for all sufficiently large j. Define $c_{n+1}, c_{n+2}, \ldots$ so that

$$c_j = c_0 a_{j-1} + c_1 a_{j-2} + \cdots c_{j-1} a_0$$

for all $j = 1, 2, \ldots$. Then (3.6) implies that M_j has κ negative eigenvalues for all $j = 0, 1, 2, \ldots$. Therefore $c_0, c_1, c_2, \ldots$ is a solution to Problem II with the data $c_0, \ldots, c_{n-1}, c_n$.

Conversely, assume that Problem II is solvable with the data $c_0, \ldots, c_{n-1}, c_n$, that is, the numbers can be extended to a sequence $c_0, c_1, c_2, \ldots$ in $\mathfrak{P}_\kappa$. Then the matrices (3.5) have κ negative eigenvalues for all sufficiently large r. Reversing the process above, we obtain a sequence $a_0, a_1, a_2, \ldots$ that extends $a_0, \ldots, a_{n-1}$ such that the matrices $I_j - T_j T_j^*$ have κ negative eigenvalues for all sufficiently large j. By Theorem 3.4, the series $S(z) = \sum_{j=0}^{\infty} a_j z^j$

converges in some disk $|z| < \delta$ where $\delta > 0$, and by a theorem of Kreĭn and Langer in [17, Theorem 6.3], the function $S(z)$ so defined belongs to $\mathbf{S}_\kappa$. Thus Problem I is solvable with the data $a_0, \ldots, a_{n-1}$. The argument for the classes $\mathfrak{P}_{\nu,\pi}$ and $\mathbf{S}_{\nu,\pi}$ is similar. $\qquad\square$

We use a series of propositions from [12]. The matrices $M_0, M_1, M_2, \ldots$ that appear in the list below are Hermitian matrices of the form (3.5) defined for appropriate numbers $c_0 = \bar{c}_0, c_1, c_2, \ldots$, and n is any positive integer. Recall that for any Hermitian matrix A we write $\pi(A)$ and $\nu(A)$ for the numbers of positive and negative eigenvalues of A counting multiplicity. The **signature** of A is $\sigma(A) = \pi(A) - \nu(A)$. Write $|A|$ for the determinant of A and $\rho(A) = \pi(A) + \nu(A)$ for the rank of A.

1°) The difference $\rho(M_n) - \rho(M_{n-1})$ is either 0, 1, or 2.

2°) If $\rho(M_n) - \rho(M_{n-1}) = 0$, then $\pi(M_n) = \pi(M_{n-1})$ and $\nu(M_n) = \nu(M_{n-1})$.

3°) If $\rho(M_n) - \rho(M_{n-1}) = 1$, then either $\pi(M_n) = \pi(M_{n-1}) + 1$ and $\nu(M_n) = \nu(M_{n-1})$, or $\pi(M_n) = \pi(M_{n-1})$ and $\nu(M_n) = \nu(M_{n-1}) + 1$.

4°) If $\rho(M_n) - \rho(M_{n-1}) = 2$, then $\pi(M_n) = \pi(M_{n-1}) + 1$ and $\nu(M_n) = \nu(M_{n-1}) + 1$.

5°) If $|M_{n-1}| \neq 0$, then there are infinitely many c_n such that $\rho(M_n) = \rho(M_{n-1})$.

6°) If $|M_{n-1}| = 0$ and $|M_{\rho(M_{n-1})-1}| \neq 0$, then there is a unique c_n such that $\rho(M_n) = \rho(M_{n-1})$.

7°) The assumptions in 6°) imply that there is a unique extension $(c_j)_{j=0}^\infty$ of $(c_j)_{j=0}^{n-1}$ such that $\rho(M_j) = \rho(M_{n-1})$, $j \geq n$.

8°) There exists a c_n with $\rho(M_n) = \rho(M_{n-1})$ if and only if $|M_{\rho(M_{n-1})-1}| \neq 0$.

9°) If $|M_{r-1}| \neq 0$ and $|M_{n-1}| = \cdots = |M_r| = 0$ for some $0 \leq r < \rho(M_{n-1})$ ($|M_{-1}| = 1$ by definition), then $\rho(M_n) = \rho(M_{n-1}) + 2$.

10°) If $|M_{n-1}| \neq 0$, then for each $k = 1, 2, \ldots$, there are infinitely many c_n, $\ldots$, c_{n+k-1} such that $\nu(M_{n+k-1}) = \nu(M_{n-1}) + k$ and $|M_{n+k-1}| \neq 0$.

11°) If $|M_{n-1}| \neq 0$, then for each $\ell = 1, 2, \ldots$, there are infinitely many c_n, $\ldots$, $c_{n+\ell-1}$ such that $\pi(M_{n+\ell-1}) = \pi(M_{n-1}) + \ell$ and $|M_{n+\ell-1}| \neq 0$.

12°) $\sigma(M_{n-1}) = \sum_{j=0}^{n-1} \operatorname{sign}(|M_{j-1}||M_j|)$, where, by definition, $|M_{-1}| = 1$ and $\operatorname{sign} 0 = 0$.

13°) If $\rho(M_j)$ is a constant ρ for all sufficiently large j, then $|M_{\rho-1}| \neq 0$.

Proofs. All of the citations below are from [12].

1°) Corollary on p. 34.

2°) Theorem 6.2, p. 36.

3°) Theorem 6.3, p. 36.

4°) Theorem 6.1, p. 35.

5°) Theorem 13.1, p. 97, and Remark 1, p. 98.

6°) Theorem 13.2, p. 100, and Remark 1, p. 102.

7°) Corollary on p. 101 and Remark 1, p. 102.

8°) The "if" part follows from 5°) and 6°), the "only if" part from Theorem 15.3, p. 119.

9°) Proposition 3°, p. 121.

10°) and 11°) It is enough to prove these statements for $k = 1$ in 10°) and $\ell = 1$ in 11°). To do this, we use the proof of Theorem 13.1, p. 97, and Remark 1, p. 98, to construct infinitely many extensions with $|M_n| > 0$ and infinitely many extensions with $|M_n| < 0$ (treat the

subcases $|M_{n-2}| \neq 0$ and $|M_{n-2}| = 0$ separately using the argument on p. 99). Then $10°$) and $11°$) follow from $3°$).

  $12°$) Theorem 16.1, p. 129.

  $13°$) Theorem 15.4, p. 119. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Our solution of Problem II is presented in Theorem 3.6. The first parts of the statements (a), (c), and (f) can be found in Iokhvidov's book as Excercise 8 on pp. 133–134; in the interest of completeness we prove these statements as well. It is clear that a given sequence $(c_j)_{j=0}^{n-1}$ does not have any extension $(c_j)_{j=0}^{\infty}$ in $\mathfrak{P}_{\nu}$ if $\nu < \nu(M_{n-1})$, and there is no extension in $\mathfrak{P}_{\nu,\pi}$ if either $\nu < \nu(M_{n-1})$ or $\pi < \pi(M_{n-1})$, because by $1°$)–$4°$), $\nu(M_j)$ and $\pi(M_j)$ are nondecreasing functions of j. If an extension $(c_j)_{j=0}^{\infty}$ belongs to the class $\mathfrak{P}_{\nu}$ then it is possible that $\rho(M_j)$ and hence also $\pi(M_j)$ tends to ∞ as $j \to \infty$. Such an extension does not belong to any of the classes $\mathfrak{P}_{\nu,\pi}$. According to $13°$) a necessary condition for $(c_j)_{j=0}^{\infty}$ to belong to $\mathfrak{P}_{\nu,\pi}$ is that $|M_{\nu+\pi-1}| \neq 0$.

Theorem 3.6. *Let* $c_0 = \bar{c}_0, c_1, \ldots, c_{n-1}$ *be given numbers, and define* $M_0, \ldots, M_{n-1}$ *as in* (3.5).

Assume $|M_{n-1}| \neq 0$.

  (a) *There exist infinitely many extensions in* $\mathfrak{P}_{\nu(M_{n-1})}$, *even infinitely many extensions in the smaller set* $\mathfrak{P}_{\nu(M_{n-1}),\pi(M_{n-1})}$.

  (b) *There exist infinitely many extensions in* $\mathfrak{P}_{\nu(M_{n-1})+\nu,\pi(M_{n-1})+\pi}$ *for all* $\nu \geq 0$ *and* $\pi \geq 0$.

Assume $|M_{n-1}| = 0$ *and* $|M_{\rho(M_{n-1})-1}| \neq 0$.

  (c) *There is a unique extension in* $\mathfrak{P}_{\nu(M_{n-1})}$; *it belongs to* $\mathfrak{P}_{\nu(M_{n-1}),\pi(M_{n-1})}$.

  (d) *There are no extensions in* $\mathfrak{P}_{\nu}$ *for* $\nu(M_{n-1}) < \nu < \nu(M_{n-1}) + \dim \ker M_{n-1}$; *there are no extensions in* $\mathfrak{P}_{\nu,\pi}$ *if*

$$\nu(M_{n-1}) < \nu < \nu(M_{n-1}) + \dim \ker M_{n-1}$$

  or if

$$\pi(M_{n-1}) < \pi < \pi(M_{n-1}) + \dim \ker M_{n-1}.$$

  (e) *There are infinitely many extensions in* $\mathfrak{P}_{\nu,\pi}$ *for all pairs* (ν, π) *with*

$$\nu \geq \nu(M_{n-1}) + \dim \ker M_{n-1} \quad and \quad \pi \geq \pi(M_{n-1}) + \dim \ker M_{n-1}.$$

Assume $|M_{n-1}| = 0$ *and* $|M_{\rho(M_{n-1})-1}| = 0$.

  (f) *There are no extensions in* $\mathfrak{P}_{\nu(M_{n-1})}$.

  (g) *There are no extensions in* $\mathfrak{P}_{\nu}$ *if* $\nu < \nu(M_{n-1}) + \dim \ker M_{n-1}$; *there are no extensions in* $\mathfrak{P}_{\nu,\pi}$ *if*

$$\nu < \nu(M_{n-1}) + \dim \ker M_{n-1}$$

  or if

$$\pi < \pi(M_{n-1}) + \dim \ker M_{n-1}.$$

  (h) *There are infinitely many extensions in* $\mathfrak{P}_{\nu,\pi}$ *for every pair* (ν, π) *with*

$$\nu \geq \nu(M_{n-1}) + \dim \ker M_{n-1} \quad and \quad \pi \geq \pi(M_{n-1}) + \dim \ker M_{n-1}.$$

Proof. For any extension of the given sequence by numbers $c_n, c_{n+1}, \ldots$, we assume that $M_n, M_{n+1}, \ldots$ are defined as in (3.5).

(a) According to 5°) there are infinitely many c_n such that $\rho(M_n) = \rho(M_{n-1}) = n$. For such M_n we have $|M_n| = 0$ and $|M_{\rho(M_n)-1}| \neq 0$. Hence by 7°) there is an extension $(c_j)_{j=0}^{\infty}$ of $(c_j)_{j=0}^{n-1}$ such that $\rho(M_j) = \rho(M_{n-1})$ for all $j \geq n-1$. Statement 2°) implies that

$$\nu(M_j) = \nu(M_{n-1}) \quad \text{and} \quad \pi(M_j) = \pi(M_{n-1}), \qquad j \geq n-1,$$

and hence $(c_j)_{j=0}^{\infty}$ belongs to $\mathfrak{P}_{\nu(M_{n-1}), \pi(M_{n-1})}$.

(b) By 10°) there are infinitely many numbers c_n such that $\nu(M_n) = \nu(M_{n-1}) + 1$ and $|M_n| \neq 0$. Therefore $\rho(M_n) = \rho(M_{n-1}) + 1$ and by 3°), $\pi(M_n) = \pi(M_{n-1})$. After ν steps, we obtain numbers $c_n, \ldots, c_{n+\nu-1}$ such that

$$|M_{n+\nu-1}| \neq 0, \qquad \nu(M_{n+\nu-1}) = \nu, \quad \text{and} \quad \pi(M_{n+\nu-1}) = \pi(M_{n-1}).$$

Using the same argument with 11°) instead of 10°), we obtain numbers $c_{n+\nu}, \ldots, c_{n+\nu+\pi-1}$ (each of which can be chosen in infinitely many ways) such that

$$|M_{n+\nu+\pi-1}| \neq 0, \qquad \nu(M_{n+\nu+\pi-1}) = \nu, \quad \text{and} \quad \pi(M_{n+\nu+\pi-1}) = \pi.$$

Now (b) follows from (a).

(c) According to 7°) there exists a unique extension $(c_j)_{j=0}^{\infty}$ of $(c_j)_{j=0}^{n-1}$ such that

$$\rho(M_j) = \rho(M_{n-1}), \qquad j \geq n.$$

It follows from 2°) that also $\pi(M_j) = \pi(M_{n-1})$ and $\nu(M_j) = \nu(M_{n-1})$ for $j \geq n$. Therefore there exists a unique extension of $(c_j)_{j=0}^{n-1}$ in the class $\mathfrak{P}_{\nu(M_{n-1})}$ and this extension belongs to $\mathfrak{P}_{\nu(M_{n-1}), \pi(M_{n-1})}$ (for the uniqueness part, note that by 3°) the equality $\nu(M_n) = \nu(M_{n-1})$ can only hold in the present situation when $\rho(M_n) = \rho(M_{n-1})$).

(d) and (e). By hypothesis

(3.15) $$|M_{n-1}| = 0 \quad \text{and} \quad |M_{\rho(M_{n-1})-1}| \neq 0.$$

The unique extension described in part (c) of the theorem cannot meet any of the conditions in parts (d) and (e); since for this extension $\rho(M_{n-1}) = \rho(M_n) = \rho(M_{n+1}) = \cdots$, in parts (d) and (e) we need only consider extensions such that

$$\rho(M_{n-1}) = \cdots = \rho(M_{n+k-1}) < \rho(M_{n+k})$$

for some $k \geq 0$. In this situation (3.15) holds with n replaced by $n+k$, and therefore we may restrict attention to extensions satisfying

(3.16) $$\rho(M_{n-1}) < \rho(M_n).$$

By 6°), (3.16) holds for all but one choice of c_n; in what follows, we assume that c_n is chosen so that (3.16) is satisfied. The question then is if the sequence $(c_j)_{j=0}^{n}$ can be further extended to an infinite sequence $(c_j)_{j=0}^{\infty}$ as required in (d) and (e).

Case (i): $\rho(M_n) = n+1$.

Since $\rho(M_{n-1}) < n$ by (3.15), by 1°) we must have $\rho(M_{n-1}) = n-1$. Thus $\dim \ker M_{n-1} = 1$, and hence part (d) holds vacuously. Part (e) also holds in this case. For by statement 4°), $\nu(M_n) = \nu(M_{n-1}) + 1$ and $\pi(M_n) = \pi(M_{n-1}) + 1$ and since M_n is invertible, part (e) follows from (a).

Case (ii): $\rho(M_n) < n + 1$.

Then with $r = \rho(M_{n-1})$, in view of (3.15) and (3.16),
$$|M_{r-1}| = |M_{\rho(M_{n-1})-1}| \neq 0, \quad |M_r| = \cdots = |M_{n-1}| = |M_n| = 0.$$

Consider any extension of $(c_j)_{j=0}^n$ by a number c_{n+1}. By (3.16), $r < \rho(M_{n+1})$. Applying 9°) with n replaced by $n + 1$, we obtain
$$\rho(M_{n+1}) = \rho(M_n) + 2,$$

and by 4°),
$$\nu(M_{n+1}) = \nu(M_n) + 1 \qquad \text{and} \qquad \pi(M_{n+1}) = \pi(M_n) + 1.$$

If $\rho(M_{n+1}) < n + 2$, we can repeat this argument. We continue in this way for $k = 1, 2, \ldots$ and extend $(c_j)_{j=0}^n$ with any numbers $c_{n+1}, \ldots, c_{n+k}$, $k = 1, 2, \ldots$; by 9°) and 4°), we have $r < \rho(M_{n+k})$,

$$\rho(M_{n+k}) = \rho(M_n) + 2k,$$
$$\nu(M_{n+k}) = \nu(M_n) + k,$$
$$\pi(M_{n+k}) = \pi(M_n) + k,$$

and
$$|M_{r-1}| \neq 0, \quad |M_r| = \cdots = |M_{n-1}| = |M_n| = \cdots = |M_{n+k}| = 0,$$

provided $\rho(M_{n+k}) = \rho(M_n) + 2k < n + k + 1$. If equality holds, that is,
$$k = k_0 := n - \rho(M_n) + 1,$$

then M_{n+k_0} is invertible and the process stops. Hence if such an extension of $(c_j)_{j=0}^n$ can be continued to a sequence in a class $\mathfrak{P}_{\nu,\pi}$, then necessarily

$$\nu \geq \nu_0 := \nu(M_{n+k_0}) = \nu(M_n) + k_0 = \nu(M_n) + n - \rho(M_n) + 1,$$

$$\pi \geq \pi_0 := \pi(M_{n+k_0}) = \pi(M_n) + k_0 = \pi(M_n) + n - \rho(M_n) + 1,$$

and according to (a) and (b) each of the classes $\mathfrak{P}_\nu$ and $\mathfrak{P}_{\nu,\pi}$ contains infinitely many extensions. Thus the first part of (d) and (e) will follow once we show that

$$\nu_0 = \nu(M_{n-1}) + \dim \ker M_{n-1} \quad \text{and} \quad \pi_0 = \pi(M_{n-1}) + \dim \ker M_{n-1}.$$

Since $|M_{n-1}| = 0$, 12°) implies that
$$\sigma(M_n) - \sigma(M_{n-1}) = \text{sign} |M_{n-1}||M_n| = 0$$

and since $\rho(M_n) > \rho(M_{n-1})$, we therefore have $\rho(M_n) = \rho(M_{n-1}) + 2$, and by 4°), $\nu(M_n) = \nu(M_{n-1}) + 1$ and $\pi(M_n) = \pi(M_{n-1}) + 1$. This implies that $\nu_0 = \nu(M_{n-1}) + \dim \ker M_{n-1}$ and also that π_0 has the desired value.

From the first part of (d) it follows that there are no extensions in $\mathfrak{P}_{\nu,\pi}$ if
$$\nu(M_{n-1}) < \nu < \nu(M_{n-1}) + \dim \ker M_{n-1},$$

whatever the value of π. By considering the sequence $(-c_j)_{j=0}^{n-1}$ and its extensions $(-c_j)_{j=0}^\infty$ and applying the results just proved (together with $\nu(-M_j) = \pi(M_j)$) we find that there are no extensions in $\mathfrak{P}_{\nu,\pi}$ if
$$\pi(M_{n-1}) < \pi < \pi(M_{n-1}) + \dim \ker M_{n-1}$$

whatever the value of ν.

(f) is part of (g).

(g) and (h). By 8°), (3.16) holds for any choice of c_n. This allows us to proceed by an argument which is similar to the proof of (d) and (e) above; in case (ii) there, the exact value of r is unimportant in order to obtain the conclusion. $\square$

We can now deal with Problem I. According to Theorem 3.5, we must apply the previous result to the case where $c_0 = 1$, n is replaced by $n+1$, $|M_n| = |I_n - T_n T_n^*|$, $\dim \ker M_n = \dim \ker (I_n - T_n T_n^*)$, and

$$\rho(I_n - T_n T_n^*) = \rho(M_n) - 1, \ \pi(I_n - T_n T_n^*) = \pi(M_n) - 1, \ \nu(I_n - T_n T_n^*) = \nu(M_n).$$

Note that $\mathfrak{P}_{\nu,\pi}$ corresponds to the class $\mathbf{S}_{\nu,\pi'}$ with $\pi' = \pi - 1$. We obtain the following solution for the Carathéodory-Fejér problem.

Theorem 3.7. *Let $a_0, \ldots, a_{n-1}$ be given numbers, and define $T_1, \ldots, T_n$ as in (3.1). Assume $|I_n - T_n T_n^*| \neq 0$.*

(a′) *There exist infinitely many solutions of Problem I in $\mathbf{S}_{\nu(I_n - T_n T_n^*)}$, even in the smaller set $\mathbf{S}_{\nu(I_n - T_n T_n^*), \pi(I_n - T_n T_n^*)}$.*

(b′) *There exist infinitely many solutions in $\mathbf{S}_{\nu,\pi}$ for all pairs (ν, π) with $\nu \geq \nu(I_n - T_n T_n^*)$ and $\pi \geq \pi(I_n - T_n T_n^*)$.*

Assume $|I_n - T_n T_n^*| = 0$ and $|I_\rho - T_\rho T_\rho^*| \neq 0$, where $\rho = \rho(I_n - T_n T_n^*)$.

(c′) *There is a unique solution in $\mathbf{S}_{\nu(I_n - T_n T_n^*)}$; it belongs to $\mathbf{S}_{\nu(I_n - T_n T_n^*), \pi(I_n - T_n T_n^*)}$.*

(d′) *There are no solutions in $\mathbf{S}_\nu$ for $\nu(I_n - T_n T_n^*) < \nu < \nu(I_n - T_n T_n^*) + \dim \ker I_n - T_n T_n^*$; there are no solutions in $\mathbf{S}_{\nu,\pi}$ if*

$$\nu(I_n - T_n T_n^*) < \nu < \nu(I_n - T_n T_n^*) + \dim \ker (I_n - T_n T_n^*)$$

or if

$$\pi(I_n - T_n T_n^*) < \pi < \pi(I_n - T_n T_n^*) + \dim \ker (I_n - T_n T_n^*).$$

(e′) *There are infinitely many solutions in $\mathbf{S}_{\nu,\pi}$ for all pairs (ν, π) with*

$$\nu \geq \nu(I_n - T_n T_n^*) + \dim \ker (I_n - T_n T_n^*) \quad and \quad \pi \geq \pi(I_n - T_n T_n^*) + \dim \ker (I_n - T_n T_n^*).$$

Assume $|I_n - T_n T_n^*| = 0$ and $|I_\rho - T_\rho T_\rho^*| = 0$.

(f′) *There are no solutions in $\mathbf{S}_{\nu(I_n - T_n T_n^*)}$.*

(g′) *There are no solutions in $\mathbf{S}_\nu$ if $\nu < \nu(I_n - T_n T_n^*) + \dim \ker (I_n - T_n T_n^*)$; there are no solutions in $\mathbf{S}_{\nu,\pi}$ if*

$$\nu < \nu(I_n - T_n T_n^*) + \dim \ker (I_n - T_n T_n^*)$$

or if

$$\pi < \pi(I_n - T_n T_n^*) + \dim \ker (I_n - T_n T_n^*).$$

(h′) *There are infinitely many solutions in $\mathbf{S}_{\nu,\pi}$ for every pair (ν, π) with*

$$\nu \geq \nu(I_n - T_n T_n^*) + \dim \ker (I_n - T_n T_n^*) \quad and \quad \pi \geq \pi(I_n - T_n T_n^*) + \dim \ker (I_n - T_n T_n^*).$$

We mention a consequence of the solution of Problem I for the case $\nu = \nu(I_n - T_n T_n^*)$.

Corollary 3.8. *Let $a_0, a_1, \ldots, a_{n-1}$ be numbers such that $I_n - T_n T_n^*$ has ν negative eigenvalues.*

(1) *If $I_n - T_n T_n^*$ is invertible, Problem I has infinitely many solutions in $\mathbf{S}_\nu$.*

(2) *If $I_n - T_n T_n^*$ is singular and $\rho(I_{n-1} - T_{n-1} T_{n-1}^*) = \rho(I_n - T_n T_n^*)$, Problem I has a unique solution in $\mathbf{S}_\nu$.*

(3) *If $I_n - T_n T_n^*$ is singular and $\rho(I_{n-1} - T_{n-1} T_{n-1}^*) < \rho(I_n - T_n T_n^*)$, Problem I has no solution in $\mathbf{S}_\nu$.*

The results in [8] and [10] give a solution to the existence and uniqueness problems for the matrix versions of both the trigonometric moment problem and Carathéodory-Fejér coefficients problem in the indefinite case, but the question of obtaining a matrix analogue of (d')-(h') in Theorem 3.7 is open.

4. A REMARK ON HOLOMORPHIC KERNELS

The result below is used in Section 3 and is well known in particular cases. The general result is presumably also known, but we do not know a reference. For the convenience of the reader, we sketch a proof.

Let $K(w, z) = \sum_{m,n=0}^{\infty} C_{mn} z^m \bar{w}^n$ be a holomorphic Hermitian kernel defined for $|w| < R$ and $|z| < R$, with values in $\mathfrak{L}(\mathfrak{F})$ for some Kreĭn space $\mathfrak{F}$. For any nonnegative integer r, we may alternatively view the matrix $(C_{mn})_{m,n=0}^r$ as a selfadjoint operator on $\mathfrak{F}^{r+1} = \mathfrak{F} \oplus \cdots \oplus \mathfrak{F}$, where there are $r+1$ summands on the right side, or as a kernel on a finite set. The number of negative eigenvalues of $(C_{mn})_{m,n=0}^r$ as an operator and the number of negative squares of $(C_{mn})_{m,n=0}^r$ as a kernel coincide.

Theorem 4.1. *Let κ be a nonnegative integer. Then $\mathrm{sq}_- K = \kappa$ if and only if*

$$\nu(C_{mn})_{m,n=0}^r \leq \kappa$$

for all nonnegative integers r and equality holds for all sufficiently large r.

We can formulate this result in another way. Let $\mathbb{N}_0$ be the set of nonnegative integers. Define a kernel C on $\mathbb{N}_0 \times \mathbb{N}_0$ by

$$C(m, n) = C_{mn}, \qquad m, n \in \mathbb{N}_0.$$

Then $\mathrm{sq}_- K = \mathrm{sq}_- C$. The theory of Kolmogorov decompositions [9] gives a natural approach to this result, but we base our argument on similar notions for reproducing kernel Pontryagin spaces.

Proof. Since a holomorphic Hermitian kernel has the same number of negative squares on subregions [3, Theorem 1.1.4], by a change of scale we may assume that $R > 1$. We may also assume without loss of generality that $\mathfrak{F}$ is a Hilbert space. Let $H_{\mathfrak{F}}^2$ be the Hardy class of $\mathfrak{F}$-valued functions on the unit disk $\mathbf{D}$.

Assume that $\mathrm{sq}_- K = \kappa$. By a method of Alpay [2], we define a bounded selfadjoint operator P on $H_{\mathfrak{F}}^2$ such that

$$P \colon (1 - \bar{w}z)^{-1} f \to K(w, z) f, \qquad w \in \mathbf{D}, f \in \mathfrak{F},$$

and

$$P\colon z^n f \to A_n(z)f, \qquad f \in \mathfrak{F}, \; n = 0, 1, 2, \ldots.$$

where $K(w, z) = \sum_{n=0}^{\infty} A_n(z)\bar{w}^n$, that is, $A_n(z) = \sum_{m=0}^{\infty} C_{mn}z^m$ for all $n = 0, 1, 2, \ldots$. For another account of the construction of P, see [20, Theorem 8.4]. By the spectral theorem, we can write $P = P_+ + P_0 + P_-$, where $P_\pm$ and P_0 are selfadjoint operators corresponding to the spectral subspaces $\mathfrak{H}_+$, $\mathfrak{H}_-$, and $\mathfrak{H}_0 = \ker P$ for the sets $(0, \infty)$, $(-\infty, 0)$, and $\{0\}$. Since $\mathrm{sq}_- K = \kappa$, $\dim \mathfrak{H}_- = \kappa$. Let $\mathfrak{K}_0$ be $H^2_{\mathfrak{F}}/\ker P$. Write $[h] = h + \ker P$ for the coset determined by an element h of $H^2_{\mathfrak{F}}$. Define a nondegenerate inner product on $\mathfrak{K}_0$ by

$$\langle [h], [k] \rangle_{\mathfrak{K}_0} = \langle Ph, k \rangle_{H^2_{\mathfrak{F}}}, \qquad h, k \in H^2_{\mathfrak{F}}.$$

Using [14, Theorem 2.5, p. 20], complete $\mathfrak{K}_0$ to a Pontryagin space $\mathfrak{K}$ having negative index κ. The cosets determined by the polynomials are dense in $H^2_{\mathfrak{F}}/\ker P$ by [14, statement (i) on p. 20], and therefore $\{[z^n f]\colon f \in \mathfrak{F}, n = 0, 1, 2, \ldots\}$ is a total set in $\mathfrak{K}$. By construction,

$$\langle [z^m f_1], [z^n f_2] \rangle_{\mathfrak{K}} = \langle C_{mn} f_1, f_2 \rangle_{\mathfrak{F}}, \qquad f_1, f_2 \in \mathfrak{F}, \; m, n = 0, 1, 2, \ldots.$$

Hence by [3, Lemma 1.1.1], the matrix $(C_{mn})_{m,n=0}^{r}$ has at most κ negative eigenvalues for all $r = 0, 1, 2, \ldots$ and one such matrix has exactly κ negative eigenvalues. Since the number of negative eigenvalues of $(C_{mn})_{m,n=0}^{r}$ is a nondecreasing function of r, this number is κ for all sufficiently large r.

Conversely, assume that the matrix $(C_{mn})_{m,n=0}^{r}$ has at most κ negative eigenvalues for all $r = 0, 1, 2, \ldots$ and exactly κ negative eigenvalues for all sufficiently large r. By what we showed above, if we can only show that $\mathrm{sq}_- K \le \kappa$, it will follow that $\mathrm{sq}_- K = \kappa$. Let $\mathbb{N}_0$ be the set of nonnegative integers, and define a kernel C on $\mathbb{N}_0 \times \mathbb{N}_0$ by

$$C(m, n) = C_{mn}, \qquad m, n \in \mathbb{N}_0.$$

Our hypotheses imply that $\mathrm{sq}_- C = \kappa$. By [3, Theorem 1.1.3], there is a unique Pontryagin space $\mathfrak{H}_C$ of functions $h = \{h_n\}_{n=0}^{\infty}$ on $\mathbb{N}_0$ with reproducing kernel C. This means that for each $m \in \mathbb{N}_0$ and $f \in \mathfrak{F}$, the sequence $C(m, \cdot)f = \{C_{mn}f\}_{n=0}^{\infty}$ belongs to $\mathfrak{H}_C$, and for any element $h = \{h_n\}_{n=0}^{\infty}$ of $\mathfrak{H}_C$,

$$\langle \{h_n\}_{n=0}^{\infty}, \{C_{mn}f\}_{n=0}^{\infty} \rangle_{\mathfrak{H}_C} = \langle h_m, f \rangle_{\mathfrak{F}}.$$

By [3, Theorem 1.1.2], we can represent the kernel C in the form

$$C_{mn} = A_n^* A_m, \qquad m, n \in \mathbb{N}_0,$$

where for each $k \in \mathbb{N}_0$, A_k^* is the evaluation mapping on $\mathfrak{H}_C$ to $\mathfrak{F}$: $A_k^*(\{h_n\}_{n=0}^{\infty}) = h_k$. By the Cauchy representation, the operators C_{mn} are uniformly bounded, and therefore for w and z in a suitable neighborhood of the origin,

$$K(w, z) = \sum_{m,n=0}^{\infty} A_n^* A_m z^m \bar{w}^n = A(w)^* A(z),$$

where $A(z) = \sum_{m=0}^{\infty} A_m z^m$. The values of $A(z)$ lie in the Pontryagin space $\mathfrak{H}_C$, which has negative index κ. The restriction of $K(w, z)$ to a suitable neighborhood of the origin thus has at most κ negative squares, and since the number of negative squares is independent of the domain (see [3, Theorem 1.1.4]), $\mathrm{sq}_- K \le \kappa$. As noted above, this implies that $\mathrm{sq}_- K = \kappa$. $\quad\square$

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