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INTERPOLATION BY POLYNOMIALS WITH SYMMETRIES ON THE IMAGINARY AXIS

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ABSTRACT. We here specialize the standard matrix-valued polynomial interpolation to the case where on the imaginary axis the interpolating polynomials admit various symmetries: Positive semidefinite, Skew-Hermitian, J-Hermitian, Hamiltonian and others.

The procedure is comprized of three stages, illustrated through the case where on $i\mathbb{R}$ the interpolating polynomials are to be positive semidefinite. We first, on the expense of doubling the degree, obtain a minimal degree interpolating polynomial $P(s)$ which on $i\mathbb{R}$ is Hermitian. Then we find all polynomials $\Psi(s)$, vanishing at the interpolation points which are positive semidefinite on $i\mathbb{R}$. Finally, using the fact that the set of positive semidefinite matrices is a convex subcone of Hermitian matrices, one can compute the minimal scalar $\hat{\beta} \geq 0$ so that $P(s)+\beta\Psi(s)$ satisfies all interpolation constraints for all $\beta \geq \hat{\beta}$.

This approach is then adapted to cases when the family of interpolating polynomials is not convex. Whenever convex, we parameterize all minimal degree interpolating polynomials.

1. INTRODUCTION

Probably the simplest version of interpolation problem is as follows. Given a family of functions $\mathcal{F}$, nodes $x_1\ ,\ \dots\ ,\ x_p$ and image points $Y_1\ ,\ \dots\ ,\ Y_p$, search for $F \in \mathcal{F}$ so that

$$Y_j = F(x_j) \qquad j = 1\ ,\ \dots\ ,\ p. \tag{1.1}$$

More specifically, find out whether such $F(s)$ exists and if yes, search for all "simple" interpolating functions in $\mathcal{F}$. In the context of rational functions, "simple" means low degree, which here takes the form of the McMillan degree, see e.g. [15], [18], [20], [40].

There is a vast literature on this classical problem. For a comprehensive study see e.g. [18]. Additional relevant literature is presented in Section 2.

Using the framework in (1.1), in this work, the nodes x_j are in $\mathbb{C}$, the image points Y_j are $m \times m$ matrices and $\mathcal{F}$ are polynomials F of a complex variable s, i.e.

$$F(s) = \sum_{k=0}^{q} C_k s^k \qquad C_k \in \mathbb{C}^{m\times m}. \tag{1.2}$$

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Recall, that the McMillan degree of matrix-valued polynomials is well defined. In [20, Corollary 2.1.1] it is shown that for $F(s)$ in (1.2) it is equal to the rank of the block-triangular, block-Toeplitz matrix

$$\begin{pmatrix} C_q & C_{q-1} & C_{q-2} & \dots & C_1 & C_o \\ 0 & C_q & C_{q-1} & \dots & C_2 & C_1 \\ 0 & 0 & C_q & \dots & C_3 & C_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & C_q \end{pmatrix}. \tag{1.3}$$

In particular, if C_q is nonsingular, then the McMillan degree of $F(s)$ is equal to $m(q+1)$.

In this work, the polynomials in $\mathcal{F}$ are restricted to have various symmetries on the imaginary axis, described in the sequel. The importance of polynomial matrix interpolation with symmetries, was raised in [42, Subsection 2.1.48]. In the framework of *real* variable, this problem has already been treated in [35]. That work differs from ours in many ways.

Through Examples 1.1, 2.1, 2.2, 6.3 and 7.1 part A, we illustrate the fact that even in the scalar case, the question addressed here is not trivial. We start with the following.

Example 1.1. As a prototype example we shall seek a *minimal degree* interpolating function $F(s)$ so that

$$F(s) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 18 \\ 75 \\ 50 \end{pmatrix}.$$

Here and in Examples 2.1, 2.2 and 6.3, we shall consider various families of functions $\mathcal{F}$ with the same points.

Taking the family $\mathcal{F}$ to be *unstructured* polynomials a straightforward computation yields,

$$F_1(s) = -45s^2 + 180s - 121.$$

□

As mentioned, we here focus on *structured* polynomials. To formally set-up the problem addressed here, we need some background.

1.1. **Functions with symmetry on the imaginary axis.** Let F be $m \times m$-valued rational function in the sense that

$$F : \mathbb{C} \ \rightarrow \ \mathbb{C}^{m\times m}.$$

In the sequel we shall use the following notation,

$$F^{\#}(s) := F^*(-s^*).$$

We shall denote by $\mathbb{C}_+$ ($\overline{\mathbb{C}_+}$) the open (closed) right half plane and by $\overline{\mathbb{P}}_m$ ($\mathbb{P}_m$) the sets of $m \times m$ positive semidefinite (definite) matrices. Whenever clear from the context, the subscript will be omitted and we shall simply write $\overline{\mathbb{P}}$ or $\mathbb{P}$.

We call functions $F(s)$ *Positive*, denoted by $\mathcal{P}$, if they are analytic in $\mathbb{C}_+$ and

$$(F(s) + F^*(s)) \in \overline{\mathbb{P}} \qquad s \in \mathbb{C}_+ \ . \tag{1.4}$$

These functions has played an important in the theory of electrical networks from around 1930, see e.g. [15] and [22]. They also serve as the corner stone of the theory of linear dissipative systems (a.k.a absolutely stable), see e.g. [15, Theorem 2.7.1], [22, 3.18].

One can relax the condition and call functions $F(s)$ *Generalized Positive*, denoted by $\mathcal{GP}$, if

$$(F(s)+F^*(s))\in\overline{\mathbb{P}} \qquad \text{almost for all} \quad s\in i\mathbb{R}.$$

For an early study of rational $\mathcal{GP}$ functions see [14] and for recent references see e.g. [9], [10] and [11]. These functions were studied in different frameworks, for example when the upper half plane replaces $\mathbb{C}_+$, see e.g. [33], [36], [47] or when the unit disk replaces $\mathbb{C}_+$, either in the domain see e.g. [30] or the image, see e.g. [39].

Abusing terminology of real scalar functions, we call a function $F(s)$ *Odd* if

$$F^{\#}(s)=-F(s). \tag{1.5}$$

This implies that on the imaginary axis $F(s)$ is skew-Hermitian, i.e.

$$\left(F(s)_{|_{s\in i\mathbb{R}}}\right)^*=-F(s)_{|_{s\in i\mathbb{R}}} .$$

We shall denote by $\mathcal{O}dd$ the set of odd functions. Note that $\mathcal{O}dd\subset\mathcal{GP}$, for details, see [11, Proposition 4.2].

In a similar way we call $F(s)$ *Even* if

$$F^{\#}(s)=F(s). \tag{1.6}$$

This implies that on the imaginary axis $F(s)$ is Hermitian, i.e.

$$\left(F(s)_{|_{s\in i\mathbb{R}}}\right)^*=F(s)_{|_{s\in i\mathbb{R}}} .$$

The set of even functions, denote by $\mathcal{E}ven$, was studied in [11, Section 5].

Of particular interest is the class of *Generalized Positive Even*, denoted by $\mathcal{GPE}$,

$$\begin{matrix} F^{\#}(s)=F(s) & \\ F(s)\in\overline{\mathbb{P}} & \text{almost for all} \quad s\in i\mathbb{R}. \end{matrix} \tag{1.7}$$

For details see [11, Section 5]. Recall that $F\in\mathcal{GPE}$ if and only if there exist $G(s)$ so that

$$F(s)=G(s)G^{\#}(s). \tag{1.8}$$

If $F(s)$ is analytic on the imaginary axis (1.8) is called *spectral factorization*[1], see e.g. [15, Section 5.2], [21, Chapter 9], and [46, Section 19.3] Else, (1.8) is a *pseudo spectral factorization*, see e.g. [21, Chapter 10],

For future reference we recall that a convex cone which in addition is closed under inversion is called a Convex Invertible Cone, **cic** in short[2] , see e.g. [26], [27] and [28].

It is easy to see that the sets $\mathcal{O}dd$ and $\mathcal{E}ven$ are closed under positive scaling, summation and inversion, i.e. **cic**s.

The set $\mathcal{P}$ is a sub**cic** of $\mathcal{GP}$ functions. More precisely, $\mathcal{P}$ is a maximal **cic** of functions which are analytic in $\mathbb{C}_+$, see e.g. [28, Proposition 4.1.1]. The **cic** structure of $\mathcal{GP}$ functions was studied in [11].

Recall that the intersection of **cic**s is a **cic**, e.g. [26, Proposition 2.2]. In particular $\mathcal{GPE}$ is the intersection between the **cic**s of $\mathcal{GP}$ and $\mathcal{E}ven$. We now describe an

[1] $G(s)$ is analytic in $\overline{\mathbb{C}_+}$ sometimes also $G^{-1}(s)$ is analytic there.

[2] Strictly speaking, this means that whenever the inverse exists, it also belongs to the set, e.g. the set of positive semidefinite matrices is a **cic**. In contrast, the open upper half of $\mathbb{C}$ is not.

intermediate set between the $\mathcal{E}ven$ its sub**cic** $\mathcal{GPE}$. Recall that $\mathcal{E}ven$, see (1.6), means $F_{|_{s\in i\mathbb{R}}}$ is Hermitian and $\mathcal{GPE}$ means $F_{|_{s\in i\mathbb{R}}}$ is positive semidefinite, see (1.7).

Next consider the set where for all $\omega \in \mathbb{R}$, $F(i\omega)$ is Hermitian and there is no eigenvalues crossing from $\mathbb{C}_-$ to $\mathbb{C}_+$ (or vice versa). Roughly, along the imaginary axis the inertia is (almost) fixed. More precisely, we call a $m \times m$-valued rational function $F(s)$ *ν-Generalized Positive Even*, denoted by $\nu\mathcal{GPE}$ if it admits a factorization

$$F(s) = G(s)\text{diag}\{-I_\nu\ ,\ I_{m-\nu}\}G^{\#}(s) \qquad \nu \in [0,\ l],$$

for some $G(s)$. In particular, for $\nu = 0$ one returns to the $\mathcal{GPE}$ case in (1.7) and if $\nu = m$, then $-F \in \mathcal{GPE}$. Thus, to avoid triviality, in the sequel we shall focus on $\nu\mathcal{GPE}$ functions admitting factorization

$$F(s) = G(s)\text{diag}\{-I_\nu\ ,\ I_{m-\nu}\}G^{\#}(s) \qquad \nu \in [1,\ m-1]. \tag{1.9}$$

If $F(s)$ is analytic on the imaginary axis (1.9) is called *J-spectral factorization*. Else, this is J-pseudo spectral factorization, see e.g. [21, Part VII].

It should be pointed out that the technique we use in the sequel does not require factorization of $\nu\mathcal{GPE}$ (or $\mathcal{GPE}$) functions.

The aim of this work is to offer a way for solving the interpolation problem in (1.1) *where the family $\mathcal{F}$ is comprized of matrix-valued polynomials within: $\mathcal{O}dd$, $\mathcal{E}ven$, $\mathcal{GPE}$ and $\nu\mathcal{GPE}$. Moreover, whenever $\mathcal{F}$ is convex, all minimal degree interpolating polynomials are given.*

We now set up the main idea of this work in terms of (1.1), through the framework of $\mathcal{F} = \mathcal{GPE}$ polynomials. Other cases will turn to be (not necessarily small) variations on the same theme.

(i) In classical unstructured polynomial interpolation one obtains $P(s)$ ($P : \mathbb{C} \to \mathbb{C}^{m\times m}$)

$$P(s) = \sum_{k=0}^{q-1} C_k s^k \qquad C_k \in \mathbb{C}^{m\times m}, \tag{1.10}$$

an unstructured interpolating polynomial of minimal degree, namely it is at most mq with $p-1 \geq q$ (recall p is the number of interpolating nodes). On the expense of doubling its degree (i.e. $2m(n-1) \geq \deg(P)$), one can introduce partial structure to the interpolating polynomial $P(s)$ in (1.10), see Proposition 3.1. In particular, the resulting polynomial $P(s)$ may be in $\mathcal{E}ven$, see Subsection 5.1, or in $\mathcal{O}dd$, see Subsection 5.3.
If the obtained $P(s)$ is already in $\mathcal{GPE}$, we are done. Assume that this is not the case.

(ii) It is easy to obtain all minimal degree $\mathcal{GPE}$ polynomials vanishing at $x_1\ ,\ \ldots\ ,\ x_n$, denoted by $\Psi(s)$, ($2mn \geq \deg(\Psi)$), see Section 4 and Proposition 6.1.

(iii) Using the above $P(s)$ and $\Psi(s)$, for all $\beta \in \mathbb{C}$,

$$F(s) = P(s) + \beta\Psi(s), \tag{1.11}$$

is an interpolating polynomial. In fact, $F(s)$ is in $\mathcal{E}ven$ for all $\beta \in \mathbb{R}$.

(iv) The family of $\mathcal{GPE}$ polynomials is a convex subcone of $\mathcal{E}ven$ and by construction $\deg(\Psi) > \deg(P)$. Thus, one can find $\hat{\beta}$ so that for all $\beta \geq \hat{\beta}$ the interpolating polynomial $F(s)$ in (1.11) is in $\mathcal{GPE}$. See Proposition 6.2.

As mentioned, interpolation with $\mathcal{GPE}$ polynomials serves as a prototype of our technique. In Section 5 the same idea is extended to polynomials with various symmetries on the imaginary axis. In Section 7 refine the procedure by exploiting possible special structure of $P(s)$ obtained in stage (i) of the "recipe" to enlarge the family of polynomials $\Psi(s)$ obtained in stage (ii) of the "recipe" and to reduce the minimal degree of the interpolating polynomial $F(s)$. In Section 8 we modify the recipe to allow interpolation with non-convex set of polynomials, e.g. $\nu\mathcal{GPE}$ in (1.9). Finally, in Section 9 a sample of future research problems.

2. MOTIVATION AND BACKGROUND

In this section we review various aspects of the interpolation problem at hand. For completeness we start by presenting three of the popular variants of the interpolation problem not addressed in this work.

First, the *tangential interpolation*, i.e. given $x_1, \ldots, x_p \in \mathbb{C}$, $v_1, \ldots, v_p \in \mathbb{C}^m$ and $Y_1, \ldots, Y_p \in \mathbb{C}^l$ find $F(s)$ in $\mathcal{F}$ ($F : \mathbb{C} \to \mathbb{C}^{l\times m}$) so that

$$Y_j = F(x_j)v_j \qquad j = 1, \ldots, p. \tag{2.1}$$

Strictly speaking, this is the *right* tangential interpolation problem. The *left* tangential interpolation problem is: For given $x_1, \ldots, x_p \in \mathbb{C}$, $u_1, \ldots, u_n \in \mathbb{C}^l$ and $Z_1, \ldots, Z_p \in \mathbb{C}^m$ finding $F(s)$ ($F : \mathbb{C} \to \mathbb{C}^{l\times m}$) so that

$$Z_j^* = u_j^* F(x_j) \qquad j = 1, \ldots, p. \tag{2.2}$$

Combining both, one obtains the *bi-tangential* interpolation problem. For sample references see e.g. [20], [38] for the polynomial case, [16] for the unstructured rational case, and [4], [5], for the structured rational case.

As a second popular variant we mention that in control theory there has been an interest in problems of the following form. Given $A(s), B(s)$ $m \times m$-valued polynomials where in addition $B \in \mathcal{E}ven$ one seeks all $m \times m$-valued polynomials $X(s)$ satisfying,

$$X^{\#}(s)A(s) + A^{\#}(s)X(s) = 2B(s).$$

See e.g. [41] and the survey in [31]

To present a third popular variant, we start by resorting to the notion of *reverse polynomial*. Recall that $\hat{F}(s)$ is said to be the reverse of a polynomial $F(s) = \sum_{j=0}^{q} s^j C_j$ in (1.2) if

$$\hat{F}(s) := s^q F(s^{-1}) = \sum_{j=0}^{q} s^j C_{q-j}, \tag{2.3}$$

see e.g. [13, Section 2], [37, Eq. (8.17)].

Next, we recall in the notion of *linearization*. For constant $mq \times mq$ matrices A, B we say that the pencil $sA - B$ is a *linearlization* of the polynomial $F(s)$ in (1.2) if there exist $mq \times mq$-valued unimodular polynomials $E(s), D(s), G(s), H(s)$ so that,

$$\begin{array}{rcl} \mathrm{diag}\{F(s), \quad I_{m(q-1)}\} & = & E(s)(sA-B)D(s) \\ \mathrm{diag}\{\hat{F}(s), \quad I_{m(q-1)}\} & = & G(s)(A-sB)H(s). \end{array}$$

For details see e.g. [13].

Roughly speaking, out of the many variants of interpolations, two of the better studied frameworks are (i) unstructured polynomials of the Lagrange type see e.g. [13], [23], [45], [48] and (ii) structured rational functions, of the Nevanlinna-Pick type, see e.g. [18, Section 18].

There are fundamental differences between these two problems: The Nevanlinna-Pick type problems is far more involved, but the underlying structure allows for the use of powerful tools.

Addressing interpolation through structured polynomials, as we do here, turn most of the classical arsenal, like linear fractional transformation, irrelevant.

Nevanlinna-Pick interpolation of positive functions $\mathcal{P}$ has been well studied, see e.g. [18, Chapter 18]. It was extended, not in the framework of $\mathcal{GP}$ but of (i) generalized Schur functions (contractive on the unit circle), in numerous works, see e.g. [1], [17], [19], [25], and [34] and (ii) generalized Nevanlinna functions (mapping the real axis to the upper half plane) [2], [12], [7], [24], [32, Section 3] and [6].

It should be pointed out that this extension of Nevanlinna-Pick interpolation from addressing $\mathcal{P}$ to $\mathcal{GP}$ is computationally involved. Moreover, the existing parameterization of all $\mathcal{GP}$ interpolating functions (after being translated from the generalized Schur framework) neither single out $\mathcal{GPE}$ functions nor polynomials. This is illustrated next.

Example 2.1. Consider the interpolation points in Example 1.1.

I. Assume now that the family $\mathcal{F}$ is the set of rational $\mathcal{GP}$ functions. Then, minimal degree interpolating function is

$$F_2(s) = \frac{150(574-451s)}{41(66-41s)} \ .$$

Clearly, $F_2(s)$ is neither in $\mathcal{E}ven$ nor a polynomial.

II. Consider the interpolating polynomial in Example 1.1. Then, $F_1(s)$ is neither in $\mathcal{E}ven$ nor in $\mathcal{GP}$. □

To summarize, from a practical point of view, the existing interpolation scheme for $\mathcal{GP}$ functions, is not very helpful for interpolation by $\mathcal{GPE}$ polynomials.

The Lagrange approach

A classical approach, see e.g. [37, Section 2.10], common to some interpolation problems of the form (1.1) is here illustrated

Take,

$$F(s) = \sum_{j=1}^{p} \tilde{F}_j(s) \qquad\qquad \tilde{F}_j \in \mathcal{F}, \quad j = 1, \ \dots \ , \ p$$

so that

	x_1	x_2	...	x_p
$\tilde{F}_1(s)$	Y_1	0	...	0
$\tilde{F}_2(s)$	0	Y_2	...	0
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
$\tilde{F}_p(s)$	0	0	...	Y_p.

In the framework of scalar (non-structured) polynomial interpolation, this approach probably preceded Lagrange [45] (in [48] it is attributed to [50]).

Computationally motivated, an interesting choice of $\tilde{F}_j(s)$ for unstructured scalar polynomials was presented in [23, Eq. (4.2)].

The above straightforward approach to interpolation problems has some limitations:
(i) It first assumes that it is easy to construct the elements $\tilde{F}_j$ in $\mathcal{F}$ (for example if $\mathcal{F}$ is the set of $\mathcal{GP}$ polynomials, this is not easy, see e.g. [11, Example 5.3b]).
(ii) It assumes that $\mathcal{F}$, the family of interpolating functions, is convex (in Section 8 we address a non-convex family $\mathcal{F}$).
(iii) In addition if for example $\mathcal{F}$ is a convex set of rational functions this scheme may yield high degree interpolating functions.

We now further scrutinize this interpolation scheme. To this end we here employ it to an example[3].

Example 2.2. We here show that adapting the Lagrange approach to the case where $\mathcal{F} = \mathcal{GPE}$ polynomials, enables us to construct *some* of the *minimal degree* interpolating polynomials in the problem addressed in Examples 1.1 and 2.1. Indeed, take

$$F_3(s) = \tilde{F}_1(s) + \tilde{F}_2(s) + \tilde{F}_3(s)$$

with

$$\begin{array}{lll} \tilde{F}_1(s) = & (4-s^2)(9-s^2)(\tilde{\alpha}(1-s^2)+\frac{3}{4}) & \tilde{\alpha}\geq 0, \\ \tilde{F}_2(s) = & (1-s^2)(9-s^2)(\tilde{\beta}(4-s^2)-\frac{5}{4}s^2) & \tilde{\beta}\geq 0, \\ \tilde{F}_3(s) = & (1-s^2)(4-s^2)(\tilde{\gamma}(9-s^2)+\frac{5}{4}) & \tilde{\gamma}\geq 0. \end{array}$$

This can be aggregated to a single parameter

$$F_3(s) = -3s^4 + 34s^2 - 13 + \beta(1-s^2)(4-s^2)(9-s^2) \qquad \beta \geq \frac{5}{4}\ . \tag{2.4}$$

□

In Example 6.3 below we show that in contrast to to (2.4), there is a $\mathcal{GPE}$ interpolating polynomial for all $\beta \geq \frac{1}{2}$. Thus, adapting the Lagrange approach to $\mathcal{GPE}$ polynomials is appealing due to its simplicity. However, even in the scalar case it turns out to provide conservative results. See also Part A of Example 7.1.

We conclude this section by further examining the structure of some the families $\mathcal{F}$ involved. Let $F(s)$ be a $m \times m$-valued polynomial as in (1.2). It is straightforward to verify that

$\mathcal{F}$	C_{2k}	C_{2k+1}
$\mathcal{E}ven$	Hermitian	skew−Hermitian
$\mathcal{O}dd$	skew−Hermitian	Hermitian.

However that there is no explicit way to characterize the sets $\mathcal{GP}$, $\mathcal{GPE}$ $\nu\mathcal{GPE}$ only through the structure of their coefficients. For example, scalar, second degree, $\mathcal{GPE}$

[3]We have already used it in the framework of $\mathcal{GPE}$ polynomials in [11, Example 5.3a].

polynomials are given by

$$a + b(r + is)^2 \qquad \text{with} \qquad a \geq 0 \quad b > 0 \quad r \in \mathbb{R}$$

(it is a strict subset of all $F(s)$ in (1.2) with $q = 2$ where $C_o \geq 0$, $C_1 \in i\mathbb{R}$ and $0 > C_2$). Hence, it is conceivable to presume that to solve the problem in (1.1) one needs to go beyond a simple modification of the classical polynomial interpolation.

3. PARTLY STRUCTURED POLYNOMIAL INTERPOLATION

The classical $m \times m$-valued unstructured polynomial interpolation can be formulated as follows. Given $x_1, \ \dots \ , \ x_p \in \mathbb{C}$ (distinct) and $Y_1 \ , \ \dots \ , \ Y_p \in \mathbb{C}^{m \times m}$ find $P(s)$, a minimal degree polynomial ($P \ : \ \mathbb{C} \ \rightarrow \ \mathbb{C}^{m \times m}$) see (1.10), so that

$$Y_j = P(x_j) \qquad\qquad j = 1, \ \dots \ , \ p.$$

It is known that the problem is solvable and in (1.10), $p \geq q$.

We here adapt[4] an idea from [4], [5, Section 2] enabling us, by roughly doubling q in (1.10), to impose structure on the matrical coefficients C_k. The problem can be formulated as follows.

Given: $x_1, \ \dots \ , \ x_p \in \mathbb{C}$ and $A, B, Y_1 \ , \ \dots \ , \ Y_p \in \mathbb{C}^{m \times m}$ find $P(s)$ a low degree $m \times m$-valued polynomial ($P \ : \ \mathbb{C} \ \rightarrow \ \mathbb{C}^{m \times m}$) so that

$$\begin{array}{lll} P^{\#}(s) = AP(s)B & A,B \in \mathbb{C}^{m \times m} & \text{non-singular} \\ P(x_j) = \quad Y_j & & j=1, \ \dots \ , \ p. \end{array} \tag{3.1}$$

To guarantee feasibility of the problem one needs to assume that the data satisfies

$$\begin{array}{lll} x_j - x_k = 0 & \Longrightarrow & Y_j = \quad Y_k \\ x_j + x_k^* = 0 & \Longrightarrow & Y_j = (AY_kB)^* \end{array} \qquad p \geq k > j \geq 1. \tag{3.2}$$

Next, we shall call the interpolation data set *reduced* if out of feasible points $x_1 \ , \ \dots \ , \ x_p$ we extract a maximal subset $x_1 \ , \ \dots \ , \ x_n$, i.e. $p \geq n$ (and the corresponding $Y_1, \ \dots \ , \ Y_n$) so that

$$\begin{array}{l} x_j - x_k \neq 0 \\ x_j + x_k^* \neq 0 \end{array} \qquad n \geq k > j \geq 1. \tag{3.3}$$

Note that the choice of $x_1 \ , \ \dots \ , \ x_n$ out of the p given nodes, is not unique (n is unique) but it will not affect the proposed procedure below[5].

Proposition 3.1. *Given a feasible data set* (3.2). *There always exists an interpolating polynomial* $P(s)$ (1.10) *satisfying* (3.1) *with*

$$2n \geq q,$$

where n *is the dimension of the reduced data set* (3.3).
The coefficients $C_o, \ \dots \ , \ C_{2n-1}$ *are obtained from the following matrix equation*

$$XC = Y \tag{3.4}$$

[4] Originally, it appeared in other frameworks: Carathéodory functions in [4] and the unit disk H_2 functions in [5].

[5] To summarize, $m \times m$ is the dimension of $F(s)$, the number of given interpolation points is p and it is then reduced to n. Finally q is the number of matrical coefficients in (1.10) and (1.11).

where the dimensions of both C and Y is $2nm \times m$,

$$C := \begin{pmatrix} C_o \\ \vdots \\ C_{2n-1} \end{pmatrix} \qquad Y := \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \\ (AY_1B)^* \\ \vdots \\ (AY_nB)^* \end{pmatrix}$$

and X is the $2nm \times 2nm$ block-Vandermonde matrix

$$X = \begin{pmatrix} I_m & x_1 I_m & x_1^2 I_m & \dots & x_1^{2n-1} I_m \\ I_m & x_2 I_m & x_2^2 I_m & \dots & x_2^{2n-1} I_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I_m & x_n I_m & x_n^2 I_m & \dots & x_n^{2n-1} I_m \\ I_m & -x_1^* I_m & (-x_1^*)^2 I_m & \dots & (-x_1^*)^{2n-1} I_m \\ I_m & -x_2^* I_m & (-x_2^*)^2 I_m & \dots & (-x_2^*)^{2n-1} I_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I_m & -x_n^* I_m & (-x_n^*)^2 I_m & \dots & (-x_n{}^*)^{2n-1} I_m \end{pmatrix}.$$

Proof Note that X can be written as

$$\mathrm{X} = \hat{X} \otimes I_m,$$

where $\hat{X}$ is an actual $2n \times 2n$ Vandermonde matrix (i.e. substitute $m = 1$ in X and $\otimes$ denotes the Kronecker product, see e.g. [44, Section 4.2]. Minimality of the data implies that the $2n$ points $x_1, \ \dots \ , \ x_n, \ -x_1^*, \ \dots \ , \ -x_n^*$ are distinct, so $\hat{X}$ is non-singular, see e.g. [44, Exercise 12]. This in turn implies that X is nonsingular, see e.g. [44, Corollary 4.2.11]. Hence, the matrical coefficients $C_o, \ \dots \ , \ C_{2n-1}$ in (1.10) are unique and explicitly obtained.

Finally, the structure guarantees that

$$C_k{}^* = (-1)^k A C_k B \qquad\qquad k = 0, \ \dots \ , \ 2n-1 \ .$$

□

In the sequel we focus on a special case of (3.1) where

$$B = (A^*)^{-1}$$

Namely $P(s)$ is so that,

$$\begin{array}{rlll} P^{\#}(s) = & AP(s)(A^*)^{-1} & A \in \mathbb{C}^{m \times m} & \text{non-singular} \\ P(x_j) = & Y_j & & j=1, \ \dots \ , \ n. \end{array} \tag{3.5}$$

4. CONSTRUCTING MINIMAL DEGREE SYMMETRIC NEUTRAL POLYNOMIALS

We now address the problem of constructing, within a prescribed family $\mathcal{F}$, polynomials $\Psi(s)$ of minimal degree, vanishing at the given nodes $x_1, \ \dots \ , \ x_p \in \mathbb{C}$.

Using (3.3) let now $x_1, \ \dots \ , \ x_n$ be a resulting reduced set, see (3.3). Next, take

$$\Psi(s) := \prod_{j=1}^{n} (x_j - s) M (x_j^* + s) \qquad\qquad M \in \mathbb{C}^{m \times m}, \tag{4.1}$$

with M parameter. Clearly, $\Psi(s)$ vanishes at the original points $x_1, \ \dots \ , \ x_p$.

Hence, taking M to be: Hermitian or Skew-Hermitian yields a minimal degree Ψ in: $\mathcal{E}ven$ or $\mathcal{O}dd$, respectively.

Note that for $s = i\omega$ with ω real, $\Psi(s)$ in (4.1) satisfies

$$\Psi(i\omega) = M \prod_{j=1}^{n} |x_j - i\omega|^2. \tag{4.2}$$

To guarantee that the sets of polynomials described in (3.5) and (4.1) indeed intersect, we need to further restrict the parameter M in (4.1) so that the product AM, with A from (3.5), is Hermitian, i.e.

$$AM = (AM)^*. \tag{4.3}$$

5. INTERESTING SPECIAL CASES

We here specialize the "recipe" from Section 1 to interesting classes of polynomials.

5.1. **Even.** To guarantee feasibility of the problem, one needs to substitute in (3.2) $A = B = I_m$ and thus obtain,

$$\begin{matrix} x_j - x_k = 0 & \Longrightarrow & Y_j = Y_k \\ x_j + x_k^* = 0 & \Longrightarrow & Y_j = Y_k^* \end{matrix} \qquad p \geq k \geq j \geq 1.$$

Taking in (3.5) $A = I_m$ results in $P(s)$ is in $\mathcal{E}ven$, i.e. $P = P^{\#}$.

5.2. $J - \mathcal{E}ven$**.** We shall find it convenient to denote by J an arbitrary $m \times m$ Hermitian involution, i.e.

$$J = J^* = J^{-1}. \tag{5.1}$$

Recall that J is unitarily similar to $\text{diag}\{-I_\nu \ , \ I_{m-\nu}\}$ with $\nu \in [0, \ m]$, see e.g. [43, Theorem 4.1.5].

Taking in (3.5) $A = J$ with J as in (5.1), results in $P(s)$ in $J - \mathcal{E}ven$.

In order to have the problem feasible one needs to assume that the original data satisfies

$$x_j + x_k^* = 0 \quad \Longrightarrow \quad Y_j = JY_k^* J \qquad p \geq k \geq j \geq 1.$$

For example if m is even and $J = \begin{pmatrix} 0 & -I_{\frac{m}{2}} \\ I_{\frac{m}{2}} & 0 \end{pmatrix}$ then on the imaginary axis $F(s)$ has Hamiltonian structure, see e.g. [35, Section I], [40, Problem 3.21, Eq. (6.3.3), Theorem 11.5.1] or for a thorough treatment of this structure, [46, Section 7.2].

5.3. **Odd.** Taking in (3.5) $A = iI_m$ results in $P(s)$ in $\mathcal{O}dd$. In particular, in order to have the problem feasible one needs to assume that the original data satisfies

$$x_j + x_k^* = 0 \quad \Longrightarrow \quad Y_j = -Y_k^* \qquad p \geq k \geq j \geq 1.$$

In this case, the condition in (4.3) implies that M in (4.1) is skew-Hermitian.

6. MINIMAL DEGREE INTERPOLATING GENERALIZED POSITIVE EVEN POLYNOMIALS

Let $x_1\ ,\ \dots\ ,\ x_p \in \mathbb{C}$ and $Y_1, \dots\ ,\ Y_p \in \mathbb{C}^{m\times m}$ be a feasible data set i.e.

$$\begin{matrix} x_j+x_k^*=0 & \Longrightarrow & Y_j=Y_k & p\geq k>j\geq 1 \\ x_j\in i\mathbb{R} & \Longrightarrow & Y_j\in\overline{\mathbb{P}}_m & j=1,\ \dots\ ,\ p. \end{matrix}$$

One searches *all* minimal degree interpolating $\mathcal{GPE}$ polynomials, $F(s)$ in (1.1). To simplify presentation assume that the data is already *reduced*, i.e.

$$p = n.$$

Now take: (i) From Subsection 5.1 a minimal degree interpolating $P \in \mathcal{E}ven$. (ii) From (4.1) all minimal degree $\Psi \in \mathcal{GPE}$ vanishing at the interpolation points:

$$\Psi(s) := \prod_{j=1}^{n}(x_j - s)M(x_j^* + s) \qquad M \in \overline{\mathbb{P}}_m\ . \tag{6.1}$$

We now establish the minimality of the degree of $\Psi(s)$ in (6.1).

Proposition 6.1. *Let $\Psi(s)$ be a $m \times m$-valued, minimal degree $\mathcal{GPE}$ polynomial vanishing at a given set of distinct points $x_1,\ \dots\ ,\ x_p \in \mathbb{C}$. Let also $x_1,\ \dots\ ,\ x_n \in \mathbb{C}$ be a corresponding reduced set, see* (3.3). *Then $\Psi(s)$ is of the form* (6.1).

Proof : From a given data points $x_1,\ \dots\ ,\ x_p$ let us denote

$$g(s) := \prod_{j=1}^{p}(x_j - s).$$

Recall that every $\mathcal{GPE}$ function admits a factorization of the form (1.8). Thus, every $m \times m$-valued $\mathcal{GPE}$ polynomial $\Psi(s)$ vanishing at $x_1,\ \dots\ ,\ x_p$ is of the form

$$\Psi(s) = g(s)\tilde{\Psi}(s)g^{\#}(s) \qquad \tilde{\Psi} \in \mathcal{GPE}.$$

Let now $x_1,\ \dots\ ,\ x_n$ be a *reduced* subset of the data points and then

$$g_o(s) := \prod_{j=1}^{n}(x_j - s).$$

Hence, without loss of generality one can take

$$\Psi(s) = g_o(s)\tilde{\Psi}(s)g_o^{\#}(s) \qquad \tilde{\Psi} \in \mathcal{GPE}.$$

To guarantee minimality of the degree of $\tilde{\Psi}(s)$, take it to be of degree zero, i.e.

$$\tilde{\Psi}(s) \equiv M \in \overline{\mathbb{P}}_m\ ,$$

so the claim is established. □

Without loss of generality we shall find it convenient to normalize M in (6.1) so that

$$\|M\|_2 = 1.$$

Following the "recipe" from Section 1, the sought interpolating polynomial $F(s)$ is of the form

$$F(s) = P(s) + \beta\Psi(s), \tag{6.2}$$

with $P(s)$, $\Psi(s)$ from Propositions 3.1 6.1, respectively and β a parameter.

Assume that $P(s)$ in (1.10) is not in $\mathcal{GPE}$. First, we fix in (6.1) an arbitrary M in $\mathbb{P}_m$ (i.e. non-singular). By construction, for all $\beta\in\mathbb{R}$ in (6.2) is an interpolating polynomial (1.1) in $\mathcal{E}ven$.

Now, on the one hand in (1.10) $P\in\mathcal{E}ven$ is of degree of (at most) $(2n-1)m$. On the other hand, in (6.1) $\Psi(s)$ is of degree $2nm$ and in $\mathcal{GPE}$, a convex subset of $\mathcal{E}ven$. Thus, there exists $\hat{\beta}>0$ so that in (1.11), (6.2)

$$\Psi\in\mathcal{GPE}\qquad\qquad\forall\beta\geq\hat{\beta}.$$

We next show that here $\hat{\beta}$ can be explicitly obtained.

Proposition 6.2. *Assume that $P(s)$ in* (1.10) *is not in $\mathcal{GPE}$ and that in* (6.1) *$M\in\mathbb{P}_m$ is given. Then $\hat{\beta}$ in* (1.11), (6.2) *is given by*

$$\hat{\beta}=-\min_{\substack{\omega\in\mathbb{R}\\ \omega\neq-ix_j}}\left(\prod_{j=1}^{n}|x_j-i\omega|^{-2}\min_{i=1\ ,\ \dots\ ,\ m}\lambda_i\left(M^{-1}\sum_{k=0}^{2n-1}i^kC_k\omega^k\right)\right)\tag{6.3}$$

Proof Indeed, note that $F(s)$ in (1.11), (6.2) is in $\mathcal{GPE}$ if the following relations hold

$$\begin{array}{lcc}
\text{for almost all } s\in i\mathbb{R} & F(s) & \in\overline{\mathbb{P}}_m\\
\text{for almost all } \omega\in\mathbb{R} & F(i\omega) & \in\overline{\mathbb{P}}_m\\
\text{for almost all } \omega\in\mathbb{R} & \left(\sum_{k=0}^{2n-1}C_k(i\omega)^k+\beta\prod_{j=1}^{n}|x_j-i\omega|^2M\right) & \in\overline{\mathbb{P}}_m\\
\text{for almost all } \omega\in\mathbb{R} & \left(\sum_{k=0}^{2n-1}M^{-\frac12}C_kM^{-\frac12}i\omega^k+\beta\prod_{j=1}^{n}|x_j-i\omega|^2\right) & \in\overline{\mathbb{P}}_m\\
\text{for almost all } \omega\in\mathbb{R} & \left(\frac{\sum_{k=0}^{2n-1}M^{-\frac12}C_kM^{-\frac12}i\omega^k}{\prod_{j=1}^{n}|x_j-i\omega|^2}+\beta I_m\right) & \in\overline{\mathbb{P}}_m
\end{array}$$

This can be written as,

$$\begin{array}{lccc}
\text{for almost all } \omega\in\mathbb{R} & \min_{i=1,\ \dots\ ,\ m}\lambda_i & \left(\frac{\sum_{k=0}^{2n-1}M^{-\frac12}C_kM^{-\frac12}i\omega^k}{\prod_{j=1}^{n}|x_j-i\omega|^2}+\beta I_m\right) & \geq0\\
\text{for almost all } \omega\in\mathbb{R} & \min_{i=1,\ \dots\ ,\ m}\lambda_i & \left(\frac{\sum_{k=0}^{2n-1}M^{-\frac12}C_kM^{-\frac12}i\omega^k}{\prod_{j=1}^{n}|x_j-i\omega|^2}\right) & \geq-\beta\\
\text{for almost all } \omega\in\mathbb{R} & \min_{i=1,\ \dots\ ,\ m}\lambda_i & \left(\frac{M^{-1}\sum_{k=0}^{2n-1}C_ki\omega^k}{\prod_{j=1}^{n}|x_j-i\omega|^2}\right) & \geq-\beta\\
\text{for almost all } \omega\in\mathbb{R}\left(\prod_{j=1}^{n}|x_j-i\omega|^{-2}\right. & \min_{i=1,\ \dots\ ,\ m}\lambda_i & \left.\left(M^{-1}\sum_{k=0}^{2n-1}C_ki\omega^k\right)\right) & \geq-\beta.
\end{array}$$

Thus, the claim is established. □

The search for minimum over all $\omega\in\mathbb{R}$ may in practice be confined to a small interval as

$$\lim_{\omega\to\ \pm\infty}\left(\frac{M^{-1}\sum_{k=0}^{2n-1}i^kC_k\omega^k}{\prod_{j=1}^{n}|x_j-i\omega|^2}\right)=0.$$

The above construction is illustrated by the following example.

Example 6.3. Consider the problem from Examples 1.1, 2.1 and 2.2 of finding $F(s)$ so that

$$F(s) \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 18 \\ 75 \\ 50 \end{pmatrix}.$$

From (1.10) we have that here

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -2 & 4 & -8 & 16 & -32 \\ 1 & -3 & 9 & -27 & 81 & -243 \end{pmatrix} \begin{pmatrix} C_o \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix} = \begin{pmatrix} 18 \\ 75 \\ 50 \\ 18 \\ 75 \\ 50 \end{pmatrix}$$

and thus

$$P_4(s) = -3s^4 + 34s^2 - 13$$

and from (6.1) $\Psi(s)$ is as in (7.2). Hence, for all $\beta \in \mathbb{R}$,

$$F_4(s) = P_4(s) + \beta\Psi(s) = -3s^4 + 34s^2 - 13 + \beta(1 - s^2)(4 - s^2)(9 - s^2), \tag{6.4}$$

is an interpolating polynomial in $\mathcal{E}ven$. Next note that $P_4(s)$ is not in $\mathcal{GPE}$,

$$P_4(s)_{|_{s=i\omega}} = -(3\omega^4 + 34\omega^2 + 13)$$

namely in fact $-P \in \mathcal{GPE}$. Next, from (6.4) one has that

$$F_4(s)_{|_{s=i\omega}} = (P_4(s) + \beta\Psi(s))_{|_{s=i\omega}} = \left(\beta - \frac{1}{2}\right)(\omega^6+14\omega^4+49\omega^2+36)+\frac{1}{2}(\omega^2-1)^2(\omega^2+10).$$

Thus, $F_4 \in \mathcal{GPE}$ for all $\beta \geq \frac{1}{2}$. Furthermore, on the boundary, i.e. for $\beta = \frac{1}{2}$

$$\min_{\omega \in \mathbb{R}} \text{Re } \left(F_4(s)_{|_{s=i\omega}}\right) = F_4(s)_{|_{s=\pm i}} = 0.$$

Thus indeed one obtains *all* interpolating $\mathcal{GPE}$ polynomials.

Obviously, formally applying (6.3) leads to the same conclusion.

Recall that in Example 2.2, through the simpler Lagrange approach we identified in (2.4) only a subset of interpolating functions with $\beta \geq \frac{5}{4}$. □

7. A REFINEMENT

We here address a case enabling us to refine the construction in the previous section by allowing in (6.1) $M \in \overline{\mathbb{P}}_m$ (singular) and consequently obtain lower degree interpolating polynomials $F(s)$.

Recall that in subsection 5.1 $P(s)$ obtained, is an interpolating polynomial in $\mathcal{E}ven$. Thus on $i\mathbb{R}$ it is Hermitian. Assume that there exists a (constant) nonsingular matrix T so that

$$TP(s)T^* = \text{diag}\{P_r(s),\ P_{m-r}(s)\} \qquad\qquad P_{m-r} \in \mathcal{GPE},\ r \in [0, m-1]. \tag{7.1}$$

(The case addressed in the previous section corresponds to $r = m$.)

If $r = 0$, $P(s)$ in (1.10) is already in $\mathcal{GPE}$, it is a minimal degree interpolating polynomial and one can now employ (6.2) with arbitrary $\Psi(s)$ from (6.1) and arbitrary $\beta \geq 0$. This is illustrated next.

Example 7.1. Consider the scalar problem of finding *all* $\mathcal{GPE}$ polynomials $F(s)$ so that

$$F(s) \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 \\ 1 \\ -4 \end{pmatrix}.$$

From (1.10) we have that here

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 3 & 9 & 27 & 81 & 243 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -2 & 4 & -8 & 16 & -32 \\ 1 & -3 & 9 & -27 & 81 & -243 \end{pmatrix} \begin{pmatrix} C_o \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -4 \\ 4 \\ 1 \\ -4 \end{pmatrix}$$

and thus,

$$P(s) = -s^2 + 5$$

and from (6.1)

$$\Psi(s) = (1 - s^2)(4 - s^2)(9 - s^2). \tag{7.2}$$

As we here have $P(i\omega) = \omega^2 + 5$ for $\omega \in \mathbb{R}$, $P \in \mathcal{GPE}$ and all interpolating polynomials are given by

$$F(s) = P(s) + \beta\Psi(s) = -s^2 + 5 + \beta(1 - s^2)(4 - s^2)(9 - s^2) \qquad \beta \geq 0. \tag{7.3}$$

It is of interest to mention that employing the simpler Lagrange approach described in Example 2.2 would identify the subset of interpolating polynomials $F(s)$ in (7.3) with $\beta \geq \frac{1}{36}$. In particular, it would have failed to find the minimal degree interpolating $\mathcal{GPE}$ polynomial corresponding to $\beta = 0$. □

Consider now the case where in (7.1)

$$r \in [1, m - 1].$$

Then, conforming to $P(s)$ in (7.1), $\Psi(s)$ in (6.1) can be constructed, with the same T, so that

$$T\Psi(s)T^* = \prod_{j=1}^{n}(x_j - s)\text{diag}\{M_r,\ M_{m-r}\}(x_j^* + s) \qquad M_r \in \mathbb{P}_r\ ,$$

where $M_{m-r} \in \overline{\mathbb{P}}_{m-r}$ is arbitrary, including zero.

One can now proceed as before. Namely, fix $M_r \in \mathbb{P}_r$, where $\|M_r\|_2 = 1$. Then $\hat{\beta}$ in is given by

$$\hat{\beta} = -\min_{\substack{\omega\in\mathbb{R} \\ \omega\neq ix_j}} \left(\prod_{j=1}^{n} |x_j - i\omega|^{-2} \min_{i=1\ ,\ \ldots\ ,\ r} \lambda_i \left(M_r^{-1} P_r(i\omega) \right) \right).$$

Example 7.2. Consider the following two dimensional problem: Find all $F(s)$ in $\mathcal{GPE}$ so that

$$F(1) = \text{diag}\{-35,\quad 9\}$$
$$F(2) = \text{diag}\{-20,\quad 0\}$$
$$F(3) = \text{diag}\{45,\quad 25\}$$

From (1.10) we have that here

$$\begin{pmatrix} I_2 & I_2 & I_2 & I_2 & I_2 & I_2 \\ I_2 & 2I_2 & 4I_2 & 8I_2 & 16I_2 & 32I_2 \\ I_2 & 3I_2 & 9I_2 & 27I_2 & 81I_2 & 243I_2 \\ I_2 & -I_2 & I_2 & -I_2 & I_2 & -I_2 \\ I_2 & -2I_2 & 4I_2 & -8I_2 & 16I_2 & -32I_2 \\ I_2 & -3I_2 & 9I_2 & -27I_2 & 81I_2 & -243I_2 \end{pmatrix} \begin{pmatrix} C_o \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{pmatrix} = \begin{pmatrix} \text{diag}\{-35 \quad 9\} \\ \text{diag}\{-20 \quad 0\} \\ \text{diag}\{45 \quad 25\} \\ \text{diag}\{-35 \quad 9\} \\ \text{diag}\{-20 \quad 0\} \\ \text{diag}\{45 \quad 25\} \end{pmatrix}.$$

Thus C_1, C_3, C_5 vanish and

$$C_o = \text{diag}\{-36 \quad 16\} \qquad C_2 = \text{diag}\{0 \quad -8\} \qquad C_4 = I_2\ .$$

Namely,

$$\begin{array}{rcl} P(s) &=& I_2s^4+\mathrm{diag}\{0,\ -8\}s^2+\mathrm{diag}\{-36,\ 16\}\\ &=& \mathrm{diag}\{s^4-36,\ (4-s^2)^2\}, \end{array}$$

is a minimal degree interpolating polynomial in $\mathcal{E}ven$. Using (1.3) the McMillan degree is 10.

Note now that $P(s)$ is of the form of (7.1) with $T = I_2$ and $r = 1$.

Now from (6.1) one can construct $\Psi(s)$ with $M \in \overline{\mathbb{P}}_2$. Indeed taking

$$\Psi(s) = (1-s^2)(4-s^2)(9-s^2)\,(\mathrm{diag}\{\beta,\ 0\} + \Delta) \qquad \Delta \in \overline{\mathbb{P}}_2\ ,$$

guarantees that

$$F(s) = P(s) + \Psi(s)$$

is a $\mathcal{GPE}$ interpolating polynomial for all

$$\beta \geq \hat{\beta} = 1.$$

Moreover for $\Delta = 0$ one obtains

$$\begin{array}{rcl} F(s) &=& \mathrm{diag}\{s^4-36+\beta(1-s^2)(4-s^2)(9-s^2),\ (4-s^2)^2\}\\ &=& \mathrm{diag}\{-\beta s^6+(14\beta+1)s^4-49\beta s^2+36(\beta-1),\ (4-s^2)^2\}\\ &=& s^6\mathrm{diag}\{-\beta,\ 0\}+s^4\mathrm{diag}\{14\beta+1,\ 1\}+s^2\mathrm{diag}\{-49\beta,\ -8\}+\mathrm{diag}\{36(\beta-1),\ 16\} \end{array}$$

where the McMillan degree is only 12 (see(1.3)). For $\beta \geq 1$ indeed $F \in \mathcal{GPE}$. □

8. A NON-CONVEX SET: ν-GENERALIZED POSITIVE EVEN POLYNOMIALS

We here address the the interpolation problem (1.1) where $\mathcal{F}$ is not convex. We modify the above "recipe" accordingly. As a test case, we take the set $\nu\mathcal{GPE}$ described in (1.9). First, we show that this set is indeed not convex.

As already mentioned, $\pm\mathcal{GPE}$ are sub**cic**s of $\mathcal{E}ven$, see e.g. [11, Section 5]. However, for $\nu \in [1,\ m-1]$ the set $\nu\mathcal{GPE}$ in (1.9) is an invertible cone, but not convex. This can be illustrated even by constant 2×2 matrices. Indeed if one takes

$$\begin{array}{llll} \mathrm{diag}\{-1,\ 4\} = A = R_a\mathrm{diag}\{-1,\ 1\}R_a^* & R_a = \mathrm{diag}\{1,\ 2\}\\ \mathrm{diag}\{4,\ -1\} = B = R_b\mathrm{diag}\{-1,\ 1\}R_b^* & R_b = \begin{pmatrix} 0 & 2\\ 1 & 0\end{pmatrix} \end{array}$$

then $A + B = 3I_2$ is not a $\nu\mathcal{GPE}$ function of the form of (1.9).

The fact that the set $\nu\mathcal{GPE}$ is not convex does not allow us to employ the Lagrange approach from Section 2. However, we can adapt the recipe from Section 1. Here are the details.

Given a reduced data set $x_1\ ,\ \ldots\ ,\ x_n \in \mathbb{C}$, the corresponding $Y_1\ ,\ \ldots\ ,\ Y_n \in \mathbb{C}^{m\times m}$ and ν, $\nu \in [1,\ m-1]$. Find (all) $F(s)$, low degree $m \times m$-valued interpolating polynomials $F\ :\ \mathbb{C}\ \to\ \mathbb{C}^{m\times m}$, i.e.

$$\begin{array}{lll} F(s) = G(s)\mathrm{diag}\{-I_\nu\ ,\ I_{m-\nu}\}G^{\#}(s) & \nu\in[1,\ m-1] & G(s)\ \text{polynomial}\\ F(x_j) = Y_j & & j=1,\ \ldots\ ,\ n. \end{array}$$

In order to have the problem feasible one needs to assume that the data satisfies

$$\begin{array}{llll} x_j+x_k^*=0 & \Longrightarrow & Y_j=Y_k & n\geq k>j\geq 1\\ x_j\in i\mathbb{R} & \Longrightarrow & Y_j=T_j\mathrm{diag}\{-I_\nu\ ,\ I_{m-\nu}\}T_j^* & j=1,\ \ldots\ ,\ n, \end{array}$$

for some[6] $\nu \in [1,\ m-1]$ and some $T_j \in \mathbb{C}^{m\times m}$.

[6]Recall, if $\nu = 0$ or $\nu = l$ we are essentially back to the $\mathcal{GPE}$ case of subsection 6.

Substituting in subsection 5.2 $A = J = I_m$ one obtains from (1.10) $P(s)$, the minimal degree interpolating polynomial in $\mathcal{E}ven$.

To construct the neutral polynomials $\Psi(s)$ substitute in (4.1) $M = R\text{diag}\{-I_\nu\ ,\ I_{m-\nu}\}R^*$ with $\nu \in [1,\ m-1]$ and x_j as above, to obtain

$$\Psi(s) := \prod_{j=1}^{n}(x_j - s)\,(R\text{diag}\{-I_\nu\ ,\ I_{m-\nu}\}R^*)\,(x_j^* + s) \qquad R \in \mathbb{C}^{m\times m}, \tag{8.1}$$

with R parameter.

By construction, with the above $P(s)$ and $\Psi(s)$, for all $\beta \in \mathbb{R}$

$$F(s) = P(s) + \beta\Psi(s), \tag{8.2}$$

is an interpolating polynomial (1.1) in $\mathcal{E}ven$. However, as we already remarked, the set $\nu\mathcal{GPE}$ not convex. Thus one needs to justify the existence of $\hat{\beta}$ so that $F(s)$ in (8.2) is a $\nu\mathcal{GPE}$ interpolating polynomial for all $\beta > \hat{\beta}$.

The idea relies on the following fact, here formulated in the framework of matrix theory.

Lemma 8.1. *Let $A_p, A_\Psi \in \mathbb{C}^{m\times m}$ Hermitian matrices. Assume that A_Ψ has π and $m-\pi$ eigenvalues in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively (A_Ψ is nonsingular). Then, for all*

$$\beta > \|A_p\|\|A_\Psi^{-1}\|$$

the matrix $(A_p + \beta A_\Psi)$ has π eigenvalues in $\mathbb{C}_+$ and $m-\pi$ eigenvalues in $\mathbb{C}_-$.

This may be deduced in several ways[7], e.g. Weyl's Theorem [43, Theorem 4.3.1]. A detailed proof in our framework, is given in Proposition 8.2 below where we estimate $\hat{\beta}$.

Proposition 8.2. *Let $F(s)$ be as in* (8.2) *with $P(s)$ from* (1.10) *and $\Psi(s)$ as in* (8.1) *where R is given and nonsingular.*
Then, in (8.2) *$F \in \nu\mathcal{GPE}$ for all $\beta > \hat{\beta}$ where*

$$\hat{\beta} = \max_{\omega\in\mathbb{R}}\left(\prod_{j=1}^{n}|x_j - i\omega|^{-2}\left\|\sum_{k=0}^{2n-1}R^{-1}C_k(R^*)^{-1}(i\omega)^k\right\|\right). \tag{8.3}$$

If $x_j = i\omega_j$ with $\omega_j \in \mathbb{R}$ is an interpolation node, a whole neighborhood of ω_j is excluded from the above $\max\limits_{\omega\in\mathbb{R}}$.

Proof Indeed, note that $F(s)$ in (8.2) is in $\nu\mathcal{GPE}$ if there exist a polynomial $G(s)$ so that

$$\text{for almost all } s\in i\mathbb{R} \qquad (P(s)+\beta\Psi(s)) \qquad = G(s)\text{diag}\{-I_\nu\ ,\ I_{m-\nu}\}G^{\#}(s)$$

$$\text{for almost all } s\in i\mathbb{R} \left(\sum_{k=0}^{2n-1}C_k s^k + \beta\prod_{j=1}^{n}(x_j - s)R\text{diag}\{-I_\nu\ ,\ I_{m-\nu}\}R^*(x_j^*+s)\right) = G(s)\text{diag}\{-I_\nu\ ,\ I_{m-\nu}\}G^{\#}(s).$$

If $s = x_j \in i\mathbb{R}$ is an interpolation node, then

$$\sum_{k=0}^{2n-1}C_k s^k{}_{|s=x_j} = Y_j = T_j\text{diag}\{-I_\nu\ ,\ I_{m-\nu}\}T_j^*$$

[7] In operator theory this is formulated as having $A_p + A_\Psi$ invertible whenever A_Ψ is invertible and $\|A_\Psi^{-1}\|^{-1} > \|A_p\|$, see e.g. [49, Theorem 10.20].

and

$$\prod_{j=1}^{n}(x_j - s)R\text{diag}\{-I_\nu\ ,\ I_{m-\nu}\}R^*(x_j^* + s) = 0,$$

so the condition is satisfied in a neighborhood of x_j. Hence, assume hereafter that $s \in i\mathbb{R}$ is out of a neighborhood of interpolation points. Thus, the above condition may be written as having a rational function $\tilde{G}(s)$ so that

$$\text{for almost all } s\in i\mathbb{R} \quad \left(\frac{\sum\limits_{k=0}^{2n-1} R^{-1}C_k(R^*)^{-1}s^k}{\prod\limits_{j=1}^{n}(x_j-s)(x_j^*+s)} + \beta\text{diag}\{-I_\nu\ ,\ I_{m-\nu}\}\right) = \tilde{G}(s)\text{diag}\{-I_\nu\ ,\ I_{m-\nu}\}\tilde{G}^{\#}(s)$$

Now, this in turn is implied by,

$$\text{for almost all } s\in i\mathbb{R} \qquad \beta \geq \left\|\frac{\sum\limits_{k=0}^{2n-1} R^{-1}C_k(R^*)^{-1}s^k}{\prod\limits_{j=1}^{n}(x_j-s)(x_j^*+s)}\right\|.$$

Substituting $s = i\omega$, $\omega \in \mathbb{R}$, establishes the claim. □

Note that the value of $\hat{\beta}$ in (8.3) depends on the choices of R in (8.1) and of the norm in (8.3).

9. FUTURE RESEARCH

As it is often the case, this study opens the door for future research problem. We here mention a sample of them.

(i) Recall that in (1.8) we mentioned that $F \in \mathcal{GPE}$ if and only if it admits a factorization of the form $F(s) = G(s)G^{\#}(s)$.
In Sections 3 through 7 we presented a "factorization free" recipe for obtaining all minimal degree interpolating $\mathcal{GPE}$ polynomials.
It is now of interest, for a given minimal degree interpolating $\mathcal{GPE}$ polynomial $F(s)$, to explore properties of its (pseudo) spectral factors $G(s)$, see e.g. [15, Section 5.2], [21] and [46, Section 19.3].

(ii) An idea we have used throughout the work is as follows. If $P(s)$ maps $x_1,\ \dots\ ,\ x_n$ to $Y_1,\ \dots\ Y_n$ and $\Psi(s)$ vanishes at $x_1,\ \dots\ ,\ x_n$ taking

$$F(s) = P(s) + \beta\Psi(s) \tag{9.1}$$

yields $F(s)$ mapping $x_1,\ \dots\ ,\ x_n$ to $Y_1,\ \dots\ Y_n$, for all $\beta \in \mathbb{C}$.

Let now $\mathcal{F}_\Psi, \mathcal{F}_P$ be two families of functions where $\mathcal{F}_\Psi$ is a (convex) subcone of $\mathcal{F}_P$. Assuming $P \in \mathcal{F}_P$ and $\Psi \in \mathcal{F}_\Psi$, for $\beta \geq 0$ "sufficiently large" $F(s)$ in (9.1) is an interpolating function within the family $\mathcal{F}_\Psi$.

In [3] we adapt this idea to interpolation by scalar rational positive functions (as in the classical Nevanlinna-Pick Interpolation) but where the nodes are within $\mathbb{C}_-$. It is shown that there always exist interpolating functions of degree equal to the number of nodes. Moreover, an easy-to-compute recipe of constructing these functions is introduced.

(iii) Adapt the recipe in Section 1 for interpolation by $\mathcal{GP}$ (not necessarily even) polynomials. The framework of (9.1) still holds with $P(s)$ mapping x_j to Y_j and $\Psi(s)$ a minimal degree $\mathcal{GP}$ polynomial vanishing at x_j. However, there are two basic differences:

(a) There is no restriction on the structure of $P(s)$. Namely, it is no longer necessary to double its degree.
(b) $\Psi(s)$ is no longer of the form of (4.1). For example, for two points $x_1, x_2 \in \mathbb{C}_+$

$$\Psi(s) = (x_1 - s)(x_2 - s)(\theta x_1^* + (1-\theta)x_2^* + s)M$$

where the scalar θ, $\theta \in [0,\ 1]$ and the matrix M, $(M + M^*) \in \overline{\mathbb{P}}$ are parameters.

(iv) Adapt the interpolation scheme of this work to cope with *tangential* structured matrix-valued polynomial interpolation. Namely substitute (1.1) by (2.1) and (2.2).

(v) One may be interested in interpolation by polynomials whose symmetry on the imaginary axis is of a group type, e.g. unitary, J-unitary, contraction or J-contraction, in the spirit of e.g. [8], [39]. It is of interest to solve the interpolation problem in (1.1) where $\mathcal{F}$ is a family of matrix valued polynomials with this symmetry.

(vi) In Section 1 we pointed out that the set of $\mathcal{GPE}$ *functions* is a sub**cic** (Convex Invertible Cone) of $\mathcal{E}ven$. If one focuses, as in this work, on the respective subsets of $\mathcal{GPE}$ and $\mathcal{E}ven$ *polynomials*, invertibility is no longer relevant. Hence, out of the **cic** structure, it is only the Convex Cone part that can be used (as we indeed did).
In (2.3) we recalled the notion of *reverse* of a polynomial. It is then easy to verify that if a polynomial is in $\mathcal{GPE}$, so is the corresponding reverse polynomial, (2.3). In fact, it turns out that the set of $\mathcal{GPE}$ polynomials is a sub-Convex Reversible Cone of $\mathcal{E}ven$ polynomials.
It is of interest to explore the Convex Reversible Cone structure of $\mathcal{GPE}$ polynomials and then to try employ it to interpolation.

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