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A Class of Gaussian Processes With Fractional Spectral Measures

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A CLASS OF GAUSSIAN PROCESSES WITH FRACTIONAL SPECTRAL MEASURES

DANIEL ALPAY, PALLE JORGENSEN, AND DAVID LEVANONY

ABSTRACT. We study a family of stationary increment Gaussian processes, indexed by time. These processes are determined by certain measures σ (generalized spectral measures), and our focus here is on the case when the measure σ is a singular measure. We characterize the processes arising from when σ is in one of the classes of affine self-similar measures. Our analysis makes use of Kondratiev-white noise spaces. With the use of *a priori* estimates and the Wick calculus, we extend and sharpen (see Theorem 7.1) earlier computations of Ito stochastic integration developed for the special case of stationary increment processes having absolutely continuous measures. We further obtain an associated Ito formula (see Theorem 8.1).

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1. INTRODUCTION

There are two ways of looking at stochastic processes, i.e., random variables indexed by a continuous parameter (for example time): (i) One starts with a probability space, i.e., a sample space, a set Ω with a sigma algebra \mathcal{B} of subsets, and a probability measure P on (Ω, \mathcal{B}) , and a system of random variables $\{X(t)\}$ on the (Ω, \mathcal{B}, P) . From this one may then compute quantities such as means, variances, co-variances, moments, etc, and then derive important **spectral data**. These in turn are used in various applications, such as in solving stochastic differential equations. Here we are concerned with the other direction: (ii) Given some *a priori* spectral data, how do we construct a suitable probability space (Ω, \mathcal{B}, P) and an associated process $\{X(t)\}$ such that the prescribed spectral data is recovered from the constructed process? In other words, this is a version of an *inverse spectral problem*. For a number of reasons, it is useful in the study of the inverse problem to focus on the case of Gaussian processes.

A zero mean Gaussian process $\{X(t)\}$ on a probability space is said to be *stationary increment* if the mean-square expectation of the increment $X(t) - X(s)$ is a function only of the time difference $t - s$. Then there is a measure σ such that the *covariance function* of such a process is of the form

$$(1.1) \quad E[X(t)X(s)^*] = K_\sigma(t, s) = \int_{\mathbb{R}} \chi_t(u)\chi_s(u)^* d\sigma(u), \quad t, s \in \mathbb{R},$$

where E is expectation, and we have set

$$(1.2) \quad \chi_t(u) = \frac{e^{itu} - 1}{u}.$$

The positive measure $d\sigma$ is called the *spectral measure*, and is subject to the restriction

$$(1.3) \quad \int_{\mathbb{R}} \frac{d\sigma(u)}{u^2 + 1} < \infty.$$

The covariance function $K_\sigma(t, s)$ can be rewritten as

$$K_\sigma(t, s) = r(t) + r(s)^* - r(t - s),$$

where

$$(1.4) \quad r(t) = - \int_{\mathbb{R}} \left\{ e^{itu} - 1 - \frac{itu}{u^2 + 1} \right\} \frac{d\sigma(u)}{u^2},$$

When σ is even, r is real and takes the simpler form

$$(1.5) \quad r(t) = \int_{\mathbb{R}} \frac{1 - \cos(tu)}{u^2} d\sigma(u).$$

We note that some authors call spectral measure instead the measure $u^2 d\sigma(u)$ rather than the measure $d\sigma(u)$. See [28, p. 25 (7)].

The literature contains a number of papers dealing with these processes, but our treatment here goes beyond this, offering two novelties: the inverse problem (see above), and an operator theory of singular measures. Both are motivated by the need to deal with families of singular measures σ (see (1.1) through (1.4)). Our focus is on families of purely singular measures σ with an intrinsic *spatial* selfsimilarity, typically with Cantor support, and with fractional scaling (and Hausdorff) dimension; see Section 2 below; these are measures with affine selfsimilarity. Note that this notion is different from self-similarity in the *time*-variable; the latter case includes fractional Brownian motion (fBm), studied in e.g., [1, 2, 3, 14, 26]. For the latter (fBm), it is known that the corresponding one-parameter family of measures σ consists of a scale of absolutely continuous measures.

The derivative of a stationary increment process is a (possibly generalized) stationary process, with covariance function

$$\widehat{\sigma}(t - s),$$

where $\widehat{\sigma}$ denotes the Fourier transform, possibly in the sense of distributions, of σ . For a function f , recall the Fourier transform

$$\widehat{f}(u) = \int_{\mathbb{R}} e^{iux} f(x) dx.$$

We note that second order stationary processes can be studied with the use of the theory of Hilbert spaces and of unitary one-parameter groups of operators in Hilbert space. One may then invoke the Stone-von Neumann spectral theorem, the spectral representation theorem, and a detailed multiplicity theory to study these processes. See for instance [27].

An important role in the theory is played by the space $\mathcal{M}(\sigma)$ of functions in $\mathbf{L}_2(\mathbb{R}, dx)$ such that

$$\int_{\mathbb{R}} |\widehat{f}(u)|^2 d\sigma(u) < \infty.$$

This space contains in particular the Schwartz space.

In the paper [3], see also [2], the case where $\sigma(u)$ is absolutely continuous with respect to Lebesgue measure, i.e., $d\sigma(u) = m(u)du$ (where the Radon-Nikodym derivative m satisfies moreover some growth conditions) was considered. The study of [3] included in particular the case of the Brownian motion and of the fractional Brownian motion. A key role in that paper was played by the (in general unbounded) operator T_m on $\mathbf{L}_2(\mathbb{R}, dx)$ defined by

$$(1.6) \quad \widehat{T_m f} = \sqrt{m} \widehat{f}.$$

So T_m is a convolution operator in $\mathbf{L}_2(\mathbb{R}, dx)$, i.e.,

$$T_m f = (\sqrt{m})^\vee \star f,$$

with \vee denoting the inverse Fourier transform, in the sense of distributions.

In this paper we focus on the case when the spectral measure is an *affine iterated function-system measure* (AIFSs). Among the AIFS-measures there is a subfamily which admits an orthonormal family of Fourier frequencies. These are lacunary Fourier series studied first in a paper by one of the authors and Steen Pedersen in 1998, see [24]. A lacunary Fourier series is one in which there are large gaps between consecutive nonzero coefficients. AIFS-measures may be visualized as fractals in the small, while their Fourier expansions as dual fractals in the large. The spectral measure of such a process $\{X(t)\}$ is important as it enters in a rigorous formulation of an associated Ito formula for functions $f(X(t))$ of the process.

The main results of the paper may be summarized as follows: We construct a densely defined operator Q_σ from $\mathbf{L}_2(d\sigma)$ into $\mathbf{L}_2(\mathbb{R}, dx)$ such that

$$(1.7) \quad \int_{\mathbb{R}} \chi_t(u) \chi_s(u)^* d\sigma(u) = \langle Q_\sigma(1_{[0,t]}), Q_\sigma(1_{[0,s]}) \rangle_{\mathbf{L}_2(\mathbb{R}, dx)}$$

This operator Q_σ is the counterpart of the operator T_m defined in (1.6) and introduced in [3]. We denote by $f \mapsto \widetilde{f}$ the natural isometric

imbedding of $L_2(\mathbb{R}, dx)$ into the white noise space; See Section 4 for details. The stochastic process $\{X_\sigma(t)\}$ defined by

$$X_\sigma(t) = \widetilde{Q_\sigma(1_{[0,t]})}, \quad t \in \mathbb{R},$$

has covariance function

$$E[X_\sigma(t)X_\sigma(s)^*] = \int_{\mathbb{R}} \chi_t(u)\chi_s(u)^* d\sigma(u)$$

Following [28], the measure σ in this expression will be called the *spectral measure* of the process. Its intuitive meaning is that of "spectral densities", not to be confused with "power spectral measure" traditionally used for the much more restrictive family of stochastic process, the stationary processes. In the case of stationary processes, and when the power spectral measure is absolutely continuous with respect to Lebesgue measure, one speaks of power spectral density (psd). It is then the Fourier transform of the covariance function, a function of a single variable, namely, the time difference. When the process can be differentiated, its derivative is stationary and σ is absolutely continuous with respect to Lebesgue measure, its derivative is the psd of the derivative process.

We show that $X_\sigma(t)$ admits a derivative $(X_\sigma(t))' \stackrel{\text{def.}}{=} W_\sigma(t)$ in the white noise space, which is moreover continuous in the white noise space norm. Furthermore

$$E[W_\sigma(t)(W_\sigma(s))^*] = \widehat{\sigma}(t - s).$$

It is found that both the processes $\{W_\sigma(t)\}$ and $\{X_\sigma(t)\}$ are in the white noise space. We define a stochastic integral with respect to X_σ , and prove results similar to those of [1], but for a different class of processes studied here.

The outline of the paper is as follows. The paper consists of nine sections besides the introduction. The first three small sections are of a review nature. In Section 2 we present some material on measures with affine selfsimilarity. In Section 3 we recall some properties of the associated $L_2(d\sigma)$ spaces. Hida's white noise space theory is based on a Hilbert space, and plays an important role in our work. Its main features are listed in Section 4. In Sections 4-9 we develop the new results of the paper. In Section 5 we construct an operator for which (1.7) holds. The corresponding process X_σ and its derivative are constructed in Section 6. The associated stochastic integral and a Ito formula are considered in Sections 7 and 8 respectively. To contrast

with the measures considered here, two examples of stationary increments Gaussian processes with measures with unbounded support are presented in Section 9. In Section 10 we briefly consider the case of a general measure σ . The last section is devoted to various concluding remarks.

2. MEASURES WITH AFFINE SELF-SIMILARITY

In understanding processes $\{X(t)\}$ with stationary increments, one must look at the variety possibilities for measures σ , representing spectral measures, in the sense outlined above. Each measure-type for σ entails properties of the associated Ito formulas for $\{X(t)\}$ as it enters into stochastic integration formulas.

While earlier literature on stationary-increment processes has been focused on the case when σ was assumed to be absolutely continuous with respect to Lebesgue measure, or perhaps the case when it is singular but purely atomic; in this section we will focus instead on a quite different family of measures: purely singular and non-atomic. They share the following four features:

- (i) they are given by explicit recursive formulas;
- (ii) they possess an intrinsic affine self-similarity; see (2.1),
- (iii) they admit a harmonic analysis based on a lacunary Fourier expansion; see (2.4), and finally,
- (iv) the Fourier transform of σ admits an explicit infinite-product formula; see (2.3).

Definition 2.1. *A Borel probability measure σ on \mathbb{R} is said to be an affine iterated function system measure (AIFS) if there is a finite family \mathcal{F} of (usually contractive) affine transformations on \mathbb{R} such that*

$$(2.1) \quad \sigma = \frac{1}{\text{card } \mathcal{F}} \sum_{\tau \in \mathcal{F}} \sigma \circ \tau^{-1}$$

holds i.e.

$$\int f(x) d\sigma(x) = \frac{1}{\text{card } \mathcal{F}} \sum_{\tau \in \mathcal{F}} \int f(\tau(x)) d\sigma(x),$$

for all bounded continuous functions on \mathbb{R} .

The simplest examples are Bernoulli convolutions. Then $\text{card } \mathcal{F} = 2$ and there exists some fixed $\rho > 0$ such that

$$(2.2) \quad \tau_{\pm}(x) = \rho(x \pm 1).$$

In that case, the measure $d\sigma_\rho$ satisfying (2.1), has Fourier transform of the form

$$(2.3) \quad \widehat{\sigma}_\rho(t) = \prod_{k=1}^{\infty} \cos(\rho^k t).$$

Note that the function

$$\prod_{k=1}^{\infty} \cos(\rho^k(t-s))$$

is positive definite on the real line, since each of the functions in that product

$$\cos(\rho^k(t-s)) = \cos(\rho^k t) \cos(\rho^k s) + \sin(\rho^k t) \sin(\rho^k s)$$

is positive definite, and one can obtain σ from Bochner's theorem.

Cases with ρ of the form

$$\rho = \frac{1}{2m}, \quad m = 2, 3, 4, \dots$$

will be of special interest here. Fix m and let σ_m (that is with $\rho = \frac{1}{2m}$) be the corresponding Bernoulli measure. For $t \in \mathbb{R}$, set

$$e_t(u) = e^{itu}.$$

Let

$$(2.4) \quad \Lambda_m = 2\pi \left\{ \sum_0^N b_j (2m)^j, \text{ where } N \in \mathbb{N} \text{ and } b_j \in \left\{ 0, \frac{m}{2} \right\} \right\}.$$

For instance,

$$\Lambda_2 = 2\pi \{0, 1, 4, 5, 16, 17, 20, \dots\}$$

$$\Lambda_3 = 2\pi \left\{ 0, \frac{3}{2}, 9, \frac{21}{2}, 18, \dots \right\} \quad \text{and}$$

$$\Lambda_4 = 2\pi \{0, 2, 16, 18, 128, 130, \dots\}.$$

It is known that the set Λ_m makes

$$\{e_\lambda \mid \lambda \in \Lambda_m\}$$

into an orthonormal basis (ONB) in $\mathbf{L}_2(d\sigma_m)$; we say that (σ_m, Λ_m) is a *spectral pair*. Let us formalize this notion:

Definition 2.2. A Borel finite measure σ on \mathbb{R} is said to have a spectrum $\Lambda \subset \mathbb{R}$ if Λ is a discrete set and the set $\{e_\lambda, \lambda \in \Lambda\}$ is an orthonormal basis in $\mathbf{L}_2(d\sigma)$. Then (σ, Λ) is called a *spectral pair*.

By the Fourier basis property mentioned above, we refer to the presence of a Fourier orthonormal basis (ONB) in the Hilbert space $\mathbf{L}_2(d\sigma)$; and our discussion below is restricted to the case when σ is assumed to be a finite measure. The study of these singular measures was initiated by one of the authors in collaboration with co-authors, see [24, 36, 23, 22, 20, 21, 19, 5, 18, 6, 8, 7]. The Fourier expansion in $\mathbf{L}_2(d\sigma)$ for fractal measures σ differs from standard Fourier series (for periodic functions) in that the fractal Fourier expansion is local, much like wavelet expansions; see [37] for details. While the family of these singular measures is extensive, we found it helpful to focus our discussion below on one of the simplest cases, the first occurring in the literature, see [24]. It has Hausdorff dimension = scaling dimension = $1/2$, and its support is a Cantor-subset of the real axis.

It is proved in [24] that the support of $d\sigma_m$ is inside the closed interval $[-1/2, 1/2]$, and has Lebesgue measure 0. We note however, there are also spectral pairs (σ_m, Δ_m) where the measure $d\sigma_m$ is not compactly supported.

3. THE SPACES $\mathbf{L}_2(d\sigma)$

For later use, we review two results from [5, 6, 7, 8] and [24]. Recall that, for $t \in \mathbb{R}$, $e_t(u) = e^{itu}$. In general one does not assume that σ has compact support. When the support of σ is compact, it has a well defined Fourier transform, which is an entire function and not merely a distribution. We have the following result, proved in [22, 23, 8].

Theorem 3.1. *Let σ be a finite positive Borel measure and let $\Lambda \subset \mathbb{R}$ be a discrete set. Then, (σ, Λ) is a spectral pair if and only if*

$$(3.1) \quad \sum_{\lambda \in \Lambda} |\widehat{\sigma}(t - \lambda)|^2 = 1, \quad \forall t \in \mathbb{R}.$$

Lemma 3.2. *Let $t, s \in \mathbb{R}$. It holds that*

$$(3.2) \quad \|e_t - e_s\|_{\mathbf{L}_2(d\sigma)} \leq K|t - s|$$

where

$$K = \int_{[-\frac{1}{2}, \frac{1}{2}]} u^2 d\sigma(u).$$

Proof: We have

$$\begin{aligned}
\|e_t - e_s\|_{\mathbf{L}_2(d\sigma)}^2 &= \int_{[-1/2, 1/2]} |e^{itu} - e^{isu}|^2 d\sigma(u) \\
&= \int_{[-1/2, 1/2]} |1 - e^{iu(t-s)}|^2 d\sigma(u) \\
&= \left(\int_{[-1/2, 1/2]} 4u^2 \frac{\sin^2\left(\frac{u(t-s)}{2}\right)}{\left(\frac{u(t-s)}{2}\right)^2} d\sigma(u) \right) \cdot \frac{(t-s)^2}{4} \\
&\leq \left(\int_{[-1/2, 1/2]} u^2 d\sigma(u) \right) \cdot (t-s)^2.
\end{aligned}$$

□

As already mentioned, the measures σ we consider are such that an orthonormal basis of $\mathbf{L}_2(d\sigma)$ is of the form

$$e^{i\lambda_n u}, n = 0, 1, \dots,$$

where $\lambda_n \in \pi\mathbb{N}_0$ for all $n \in \mathbb{N}_0$.

4. A BRIEF SURVEY OF WHITE NOISE SPACE ANALYSIS

In this section we present some technical details required in the subsequent sections, taken from Hida's white noise space theory. We refer the reader to [12], [13], [14] for more information. The facts reviewed here are essential for our analysis of certain stochastic integrals (Section 7), and our Ito formula (Section 8). While convergence questions for stochastic integrals traditionally involve integration in probability spaces of paths, in our approach, the sample space will instead be a space of tempered distributions \mathcal{S}' derived from a Gelfand triple construction; but there is a second powerful tool involved, a completion called the Kondratiev-Wick algebra. We briefly explain the justification for this approach below.

The second system of duality spaces are called Kondratiev spaces, see Section 4 for details. Further, there is a particular Kondratiev space, endowed with a product, the Wick product and an algebra under this product. It serves as a powerful tool in building stochastic integrals because, as we show, the stochastic integral takes place in the Kondratiev-Wick algebra; and we can establish convergence there; see Theorem 7.1. Moreover (see Theorem 8.1), the stochastic integration making up our

Ito formula lives again in the Kondratiev-Wick algebra.

Let \mathcal{S} denote the Schwartz space of *real-valued* $C^\infty(\mathbb{R})$ functions such that

$$\forall p, q \in \mathbb{N}_0, \quad \lim_{x \rightarrow \pm\infty} x^p f^{(q)}(x) = 0$$

For $s \in \mathcal{S}$, let $\|s\|$ denote its $\mathbf{L}_2(\mathbb{R})$ norm. The function

$$K(s_1 - s_2) = e^{-\frac{\|s_1 - s_2\|^2}{2}}$$

is positive definite for s_1, s_2 running in \mathcal{S} . By the Bochner-Minlos theorem (see [32], [11, Théorème 3, p. 311]), there exists a probability measure P on \mathcal{S}' such that

$$(4.1) \quad K(s) = \int_{\mathcal{S}'} e^{-i\langle s', s \rangle} dP(s'),$$

where $\langle s', s \rangle$ denotes the duality between \mathcal{S} and \mathcal{S}' . Henceforth, we set $\Omega = \mathcal{S}'$. The real Hilbert space $\mathcal{W} = \mathbf{L}_2(\Omega, \mathcal{F}, dP)$, where \mathcal{F} is the Borelian σ -algebra, is called the white noise space.

For $s \in \mathcal{S}$ and $\omega \in \Omega$ we set

$$(4.2) \quad \tilde{s}(\omega) = \langle \omega, s \rangle$$

From (4.1) follows that the map $s \mapsto \tilde{s}$ is an isometry from \mathcal{S} endowed with the $\mathbf{L}_2(\mathbb{R}, dx)$ norm into \mathcal{W} . This isometry extends to all of $\mathbf{L}_2(\mathbb{R}, dx)$, and we will denote the extension by the same symbol.

We now present an orthogonal basis of \mathcal{W} . We set ℓ to be the space of sequences $(\alpha_1, \alpha_2, \dots)$, whose entries are in

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\},$$

where $\alpha_k \neq 0$ for only a finite number of indices k . Furthermore, we denote by $\mathbf{h}_0, \mathbf{h}_1, \dots$ the Hermite polynomials. The functions

$$H_\alpha = H_\alpha(\omega) = \prod_{k=1}^{\infty} \mathbf{h}_{\alpha_k}(\tilde{h}_k(\omega)), \quad \alpha \in \ell,$$

form an orthogonal base of the white noise space (the ω -dependence will be omitted throughout, unless specifically required). Furthermore, one has

$$(4.3) \quad \|H_\alpha\|_{\mathcal{W}}^2 = \alpha!,$$

where we have used the multi-index notation

$$\alpha! = \alpha_1! \alpha_2! \cdots,$$

The Wick product \diamond in \mathcal{W} is defined by the formula

$$H_\alpha \diamond H_\beta = H_{\alpha+\beta}, \quad \alpha, \beta \in \ell,$$

on the basis $(H_\alpha)_{\alpha \in \ell}$, and is extended by linearity to \mathcal{W} as

$$(4.4) \quad F \diamond G = \sum_{\gamma \in \ell} \left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \right) H_\gamma,$$

where $F = \sum_{\alpha \in \ell} f_\alpha H_\alpha$ and $G = \sum_{\alpha \in \ell} g_\alpha H_\alpha$. See [14, Definition 2.4.1, p. 39]. The Wick product $F \diamond G$ reduces to multiplication by a constant when one of the elements F or G is non random. The Wick product is not everywhere defined in \mathcal{W} , and one may remedy this by viewing \mathcal{W} as the middle part of a Gelfand triple. The first element in the triple is the Kondratiev space S_1 of stochastic test functions, defined as the intersection of the Hilbert spaces \mathcal{H}_k , $k = 1, 2, \dots$, of series $f = \sum_{\alpha \in \ell} f_\alpha H_\alpha$ such that

$$(4.5) \quad \|f\|_k^2 \stackrel{\text{def.}}{=} \sum_{\alpha \in \ell} (\alpha!)^2 |f_\alpha|^2 (2\mathbb{N})^{k\alpha} < \infty.$$

The third element in the Gelfand triple is the Kondratiev space S_{-1} of stochastic distributions. It is a nuclear space, and is defined as the inductive limit of the increasing family of Hilbert spaces \mathcal{H}'_k , $k = 1, 2, \dots$ of formal series $\sum_{\alpha \in \ell} f_\alpha H_\alpha$ such that

$$(4.6) \quad \|f\|_k^2 \stackrel{\text{def.}}{=} \sum_{\alpha \in \ell} |f_\alpha|^2 (2\mathbb{N})^{-k\alpha} < \infty,$$

where, for $\beta \in \ell$,

$$(2\mathbb{N})^{\pm\beta} = 2^{\pm\beta_1} (2 \times 2)^{\pm\beta_2} (2 \times 3)^{\pm\beta_3} \dots$$

See [14, §2.3, p. 28].

The Wick product is stable both in S_1 and in S_{-1} . Moreover, Våge's inequality (see [14, Proposition 3.3.2, p. 118]) makes precise the fact that $F \diamond G \in S_{-1}$ for every choice of F and G in S_{-1} : Let l and k be natural numbers such that $k > l + 1$. Let $h \in \mathcal{H}'_l$ and $u \in \mathcal{H}'_k$. Then,

$$(4.7) \quad \|h \diamond u\|_k \leq A(k-l) \|h\|_l \|u\|_k,$$

where

$$(4.8) \quad A(k-l) = \left(\sum_{\alpha \in \ell} (2\mathbb{N})^{(l-k)\alpha} \right)^{1/2}.$$

5. THE OPERATOR Q_σ

As we saw, the construction of a process $\{X(t)\}$ from a fixed spectral measure σ depends on properties of a certain operator in $\mathbf{L}_2(\mathbb{R}, dx)$. If σ is assumed absolutely continuous, with Radon-Nikodym derivative m , then this operator T_m was studied earlier and it is a convolution operator with the square root of m . See (1.6) and [3]. In this section we introduce the counterpart of the operator T_m in the present setting. Recall that h_0, h_1, \dots denote the Hermite functions. We define a unitary map W from $\mathbf{L}_2(d\sigma)$ onto $\mathbf{L}_2(\mathbf{R}, dx)$ via the formula

$$W(e_{\lambda_n}) = h_n, \quad n = 0, 1, \dots$$

Let M_u denote the operator of multiplication by the variable u in $\mathbf{L}_2(d\sigma)$. The formula

$$(5.1) \quad T = WM_uW^*$$

defines a bounded self-adjoint operator from $\mathbf{L}_2(\mathbf{R}, dx)$ into itself. Furthermore, for $\psi \in \mathcal{S}$ we set

$$Q_\sigma(\psi) = \widehat{\psi}(T)h_0.$$

Lemma 5.1. *Let $\psi \in \mathcal{S}$. Then, it holds that:*

$$(5.2) \quad (Q_\sigma\psi)(x) = \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} \widehat{\sigma}(y - \lambda_n)\psi(y)dy \right) h_n(x)$$

Proof: Using the functional calculus we have from (5.1) that,

$$\widehat{\psi}(T) = W\widehat{\psi}(M_u)W^*$$

and hence

$$\begin{aligned} \widehat{\psi}(T)h_0 &= W\widehat{\psi}(M_u)W^*h_0 \\ &= W\widehat{\psi}(M_u)1 \\ &= W\widehat{\psi}(u). \end{aligned}$$

Moreover, since $\widehat{\psi} \in \mathbf{L}_2(d\sigma)$ we have

$$\begin{aligned} \widehat{\psi}(u) &= \sum_{n=0}^{\infty} \langle \widehat{\psi}, e_{\lambda_n} \rangle_{\mathbf{L}_2(d\sigma)} e_{\lambda_n}(u) \\ &= \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} \widehat{\psi}(u) e^{-iu\lambda_n} d\sigma(u) \right) e_{\lambda_n}(u), \end{aligned}$$

so that $W^*\widehat{\psi}$ is given by (5.2). □

The operator Q_σ is typically an unbounded operator in the Hilbert space $\mathbf{L}_2(\mathbb{R}, dx)$, but it is well defined on a dense domain which consists of the Schwartz space $\mathcal{S} \subset \mathbf{L}_2(\mathbb{R}, dx)$. These facts are elaborated upon in the next theorem. In the statement, note that the Fréchet topology of \mathcal{S} is stronger than that of the $\mathbf{L}_2(\mathbb{R}, dx)$ -norm.

Theorem 5.2. *Let $\psi \in \mathcal{S}$. Then, it holds that,*

$$(5.3) \quad \|Q_\sigma(\psi)\|_{\mathbf{L}_2(\mathbb{R}, dx)}^2 = \int_{\mathbb{R}} |\widehat{\psi}(u)|^2 d\sigma(u).$$

In particular, Q_σ is a continuous operator from \mathcal{S} into $\mathbf{L}_2(\mathbb{R}, dx)$. More precisely,

$$(5.4) \quad \|Q_\sigma\psi\|_{\mathbf{L}_2(\mathbb{R}, dx)} \leq \sqrt{K} \left(\left(\int_{\mathbb{R}} |\psi(x)| dx \right)^2 + \left(\int_{\mathbb{R}} |\psi'(x)| dx \right)^2 \right)^{1/2},$$

where

$$(5.5) \quad K = \int_{\mathbb{R}} \frac{d\sigma(u)}{1+u^2} < \infty.$$

Remark 5.3. For some stochastic processes it is important to relax condition (5.5) to

$$(5.6) \quad K_p = \int_{\mathbb{R}} \frac{d\sigma(u)}{1+|u|^{2p}} < \infty$$

for some $p \in \mathbb{N}$. In this case the estimate in (5.4) becomes

$$(5.7) \quad \|Q_\sigma\psi\|_{\mathbf{L}_2(\mathbb{R}, dx)} \leq \sqrt{K_p} \left(\left(\int_{\mathbb{R}} |\psi(x)| dx \right)^2 + \left(\int_{\mathbb{R}} |\psi^{(p)}(x)| dx \right)^2 \right)^{1/2}.$$

Proof of Theorem 5.2: Let $\psi \in \mathcal{S}$. We have:

$$\begin{aligned}
\|Q_\sigma \psi\|_{\mathbf{L}_2(\mathbb{R}, dx)}^2 &= \sum_{n=0}^{\infty} \left| \int_{\mathbb{R}} \widehat{\sigma}(y - \lambda_n) \psi(y) dy \right|^2 \\
&= \sum_{n=0}^{\infty} \left| \iint e^{iu(y - \lambda_n)} \psi(y) d\sigma(u) dy \right|^2 \\
&= \sum_{n=0}^{\infty} \left| \int \left(\int \psi(y) e^{iuy} dy \right) e^{-iu\lambda_n} d\sigma(u) \right|^2 \\
&= \sum_{n=0}^{\infty} \left| \int \widehat{\psi}(u) e^{-iu\lambda_n} d\sigma(u) \right|^2 \\
&= \int_{\mathbb{R}} |\widehat{\psi}(u)|^2 d\sigma(u),
\end{aligned}$$

where we have used Fubini's theorem for the third equality, and Parseval's equality for the last equality.

We now prove (5.4). This will prove the continuity of Q_σ from \mathcal{S} endowed with its Fréchet topology into $\mathbf{L}_2(\mathbb{R}, dx)$. For $\psi \in \mathcal{S}$ we have

$$\begin{aligned}
\|Q_\sigma \psi\|_{\mathbf{L}_2(\mathbb{R}, dx)}^2 &= \int_{\mathbb{R}} |\widehat{\psi}(u)|^2 d\sigma(u) \\
&= \int_{\mathbb{R}} (1 + u^2) |\widehat{\psi}(u)|^2 \frac{d\sigma(u)}{1 + u^2} \\
&\leq K \max_{u \in \mathbb{R}} (1 + u^2) |\widehat{\psi}(u)|^2 \\
&\leq K \left(\int_{\mathbb{R}} |\psi \star \psi^\#|(x) dx + \int_{\mathbb{R}} |\psi' \star (\psi^\#)'|(x) dx \right) \\
&\leq K \left(\left(\int_{\mathbb{R}} |\psi(x)| dx \right)^2 + \left(\int_{\mathbb{R}} |\psi'(x)| dx \right)^2 \right),
\end{aligned}$$

where $\psi^\#(x) = \overline{\psi(-x)}$ and $\psi' = \frac{d\psi}{dx}$. Furthermore, we have used

$$\widehat{\psi \star \psi^\#}(u) = |\widehat{\psi}(u)|^2.$$

and

$$\begin{aligned}
\int_{\mathbb{R}} |\psi \star \psi^\#|(x) dx &\leq \left(\int_{\mathbb{R}} |\psi|(x) dx \right) \left(\int_{\mathbb{R}} |\psi^\#|(x) dx \right) \\
&= \left(\int_{\mathbb{R}} |\psi|(x) dx \right)^2.
\end{aligned}$$

Since the expression on the right hand side in this estimate is one of the Fréchet semi-norms of \mathcal{S} , the continuity assertion in Theorem 5.2 follows. The estimate (5.4) further gives an exact rate of continuity. \square

From equation (5.3) we can extend the domain of definition of Q_σ to a wider set, which in particular include the functions $1_{[0,t]}$. This is explicited in the following proposition. We remark that such a result may be extended to more general measures σ 's, see Section 10.

Proposition 5.4. *Let $f \in \mathbf{L}_2(d\sigma)$ be such that, for some sequence $(s_n)_{n \in \mathbb{N}}$ of Schwartz functions,*

$$(5.8) \quad \lim_{n \rightarrow \infty} |f - \widehat{s}_n|_\infty = 0.$$

Then the sequence $(Q_\sigma s_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbf{L}_2(\mathbb{R}, dx)$. Its limit is the same for all sequences which satisfy (5.8), and will be denoted by

$$Q_\sigma f \stackrel{\text{def.}}{=} \lim_{n \rightarrow \infty} Q_\sigma s_n.$$

Proof: From (5.3) follows that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n, m \geq N \implies |\widehat{s}_n - \widehat{s}_m|_\infty \leq \epsilon.$$

Thus for such n and m

$$(5.9) \quad \begin{aligned} \|Q_\sigma s_n - Q_\sigma s_m\|_{\mathbf{L}_2(\mathbb{R}, dx)}^2 &= \int_{\mathbb{R}} |\widehat{s}_n(u) - \widehat{s}_m(u)|^2 d\sigma(u) \\ &\leq \sigma(\mathbb{R}) \cdot |\widehat{s}_n - \widehat{s}_m|_\infty^2 \\ &\leq \sigma(\mathbb{R}) \cdot \epsilon^2. \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} Q_\sigma s_n$ in the norm of $\mathbf{L}_2(\mathbb{R}, dx)$. Call this limit q_1 , and assume that, for another sequence $(t_n)_{n \in \mathbb{N}}$ satisfying (5.8), we obtain another limit, say q_2 . Note that

$$(5.10) \quad \lim_{n \rightarrow \infty} |\widehat{s}_n - \widehat{t}_n|_\infty \leq \lim_{n \rightarrow \infty} |\widehat{s}_n - f|_\infty + \lim_{n \rightarrow \infty} |f - \widehat{t}_n|_\infty = 0.$$

Then,

$$\begin{aligned} \|q_1 - q_2\|_{\mathbf{L}_2(\mathbb{R}, dx)} &\leq \|q_1 - Q_\sigma s_n\|_{\mathbf{L}_2(\mathbb{R}, dx)} + \\ &\quad + \|Q_\sigma s_n - Q_\sigma t_n\|_{\mathbf{L}_2(\mathbb{R}, dx)} + \|Q_\sigma t_n - q_2\|_{\mathbf{L}_2(\mathbb{R}, dx)} \\ &\leq \|q_1 - Q_\sigma s_n\|_{\mathbf{L}_2(\mathbb{R}, dx)} + \\ &\quad + \sqrt{\sigma(\mathbb{R})} \cdot |\widehat{s}_n - \widehat{t}_n|_\infty + \|Q_\sigma t_n - q_2\|_{\mathbf{L}_2(\mathbb{R}, dx)}, \end{aligned}$$

which goes to 0 as $n \rightarrow \infty$ by definition of q_1 and q_2 and due to (5.10).

We now verify that the function

$$\chi_t(u) = \frac{e^{itu} - 1}{u}$$

can be approximated in the supremum norm by Schwartz functions. The function χ_t vanishes at infinity, and hence can be approximated in the supremum norm by continuous functions with compact support; see [35, Theorem 3.17, p. 70]. These in turn can be approximated by functions in \mathcal{S} , using approximate identities, as for instance Step 6 in the proof of Theorem 6.1 in [1]. We sketch the argument for completeness. Let

$$(5.11) \quad k_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{x^2}{2\epsilon^2}\right).$$

k_ϵ is an $\mathcal{N}(0, \epsilon^2)$ density, and therefore

$$(5.12) \quad \int_{\mathbb{R}} k_\epsilon(x) dx = 1,$$

and, for every $r > 0$

$$(5.13) \quad \lim_{\epsilon \rightarrow 0} \int_{|x| > r} k_\epsilon(x) dx = 0.$$

Indeed, for $|x| > r > 0$,

$$\begin{aligned} \frac{1}{\epsilon\sqrt{2\pi}} \int_r^\infty e^{-\frac{x^2}{2\epsilon^2}} dx &= \frac{1}{\epsilon\sqrt{2\pi}} \int_r^\infty \frac{x}{\epsilon^2} e^{-\frac{x^2}{2\epsilon^2}} \frac{\epsilon^2}{x} dx \\ &\leq \frac{\epsilon}{r\sqrt{2\pi}} \int_r^\infty \frac{x}{\epsilon^2} e^{-\frac{x^2}{2\epsilon^2}} dx \\ &= \frac{\epsilon}{r\sqrt{2\pi}} e^{-\frac{r^2}{2\epsilon^2}} \\ &\rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \end{aligned}$$

These properties express the fact that k_ϵ is an approximate identity. Applying [10, Theorem 1.2.19, p. 25] we see that, for every continuous function with compact support,

$$\lim_{\epsilon \rightarrow 0} \|k_\epsilon * f - f\|_\infty = 0.$$

To conclude, one proves by induction on n that the n -th derivative

$$(k_\epsilon * f)^{(n)}(x)$$

is a finite sum of terms of the form

$$\frac{1}{\sqrt{2\pi\epsilon}} \int_{\mathbb{R}} \exp\left(-\frac{(u-x)^2}{2\epsilon^2}\right) p(x-u) f(u) du,$$

where p is a polynomial. All limits,

$$\lim_{|x| \rightarrow \infty} x^m (k_\epsilon * f)^{(n)}(x) = 0$$

are then shown, using the dominated convergence theorem, and all the functions

$$(k_\epsilon * f)(x) = \frac{1}{\sqrt{2\pi}\epsilon} \int_{\mathbb{R}} \exp\left(-\frac{(u-x)^2}{\epsilon^2}\right) f(u) du$$

are in the Schwartz space. \square

We now compute the adjoint operator Q_σ^* . Note that it is an operator from $\mathbf{L}_2(\mathbb{R}, dx)$ into \mathcal{S}' , and therefore lies outside $\mathbf{L}_2(\mathbb{R}, dx)$. We begin with a notation and a preliminary computation. For $\phi \in \mathbf{L}_2(\mathbb{R}, dx)$ set

$$c_n(\phi) = \int_{\mathbb{R}} \phi(y) h_n(y) dy.$$

Then $(c_n(\phi))_{n \in \mathbb{N}}$ is in ℓ_2 . Indeed

$$\|(c_n(\phi))_{n \in \mathbb{N}}\|_{\ell_2}^2 = \|\psi\|_{\mathbf{L}_2(\mathbb{R}, dx)}^2.$$

We introduce the operator from $\mathbf{L}_2(\mathbb{R}, dx)$ into $\mathbf{L}_2(d\sigma)$:

$$(T_\sigma \phi)(u) = \sum_{n=0}^{\infty} c_n(\phi) e^{i\lambda_n u}.$$

Clearly

$$\|\phi\|_{\mathbf{L}_2(\mathbb{R}, dx)} = \|T_\sigma \phi\|_{\mathbf{L}_2(d\sigma)}.$$

Theorem 5.5. *Let $\psi \in \mathcal{S}$ and $\phi \in \mathbf{L}_2(\mathbb{R}, dx)$. Then,*

$$(5.14) \quad \langle Q_\sigma \psi, \phi \rangle_{\mathbf{L}_2(\mathbb{R}, dx)} = \int_{\sup \sigma} \widehat{\psi}(u) \overline{(T_\sigma(\phi))} d\sigma(u)$$

$$(5.15) \quad = \int_{\mathbb{R}} \psi(y) \mathbf{X}(\phi)(y) dy,$$

where

$$(5.16) \quad (\mathbf{X}(\phi))(y) = \sum_{n=0}^{\infty} \langle h_n, \phi \rangle_{\mathbf{L}_2(\mathbb{R}, dx)} \widehat{\sigma}(y - \lambda_n).$$

Proof: We first prove (5.14). In view of the formula (5.2) for Q_σ , we have

$$\begin{aligned}
\langle Q_\sigma \psi, \phi \rangle_{\mathbf{L}_2(\mathbb{R}, dx)} &= \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} \widehat{\sigma}(y - \lambda_n) \psi(y) dy \right) \overline{\left(\int_{\mathbb{R}} \phi(x) h_n(x) dx \right)} \\
&= \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{iu(y - \lambda_n)} d\sigma(u) \right) \psi(y) dy \right) \overline{\left(\int_{\mathbb{R}} \phi(x) h_n(x) dx \right)} \\
&= \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{iuy} \psi(y) dy \right) e^{-iu\lambda_n} d\sigma(u) \right) \overline{\left(\int_{\mathbb{R}} \phi(x) h_n(x) dx \right)} \\
&= \int_{\mathbb{R}} \widehat{\psi}(u) \left(\sum_{n=0}^{\infty} e^{-i\lambda_n u} \overline{\left(\int_{\mathbb{R}} \phi(x) h_n(x) dx \right)} \right) d\sigma(u) \\
&= \int_{\mathbb{R}} \widehat{\psi}(u) \cdot \overline{T_\sigma(\phi)} d\sigma(u),
\end{aligned}$$

where we have used Fubini's theorem for the third equality, and the continuity of the inner product for the fourth equality.

We now turn to the second formula. The sequence $(\langle h_n, \phi \rangle_{\mathbf{L}_2(\mathbb{R}, dx)})_{n \in \mathbb{N}_0}$ is in ℓ_2 . In view of (3.1), the Cauchy-Schwarz inequality implies that $\mathbf{X}(\phi)$ in (5.16) converges pointwise for every real y . We note that, in general, $\mathbf{X}(\phi) \notin \mathbf{L}_2(\mathbb{R}, dx)$. For ϕ and ψ as in (5.14) we have:

$$\begin{aligned}
\langle Q_\sigma \psi, \phi \rangle_{\mathbf{L}_2(\mathbb{R}, dx)} &= \int_{\mathbb{R}} \left(\sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} \widehat{\sigma}(y - \lambda_n) \psi(y) dy \right) h_n(x) \right) \overline{\phi(x)} dx \\
&= \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} \widehat{\sigma}(y - \lambda_n) \psi(y) dy \right) \left(\int_{\mathbb{R}} h_n(x) \overline{\phi(x)} dx \right) \\
&= \int_{\mathbb{R}} \psi(y) \left(\sum_{n=0}^{\infty} \widehat{\sigma}(y - \lambda_n) \overline{\int_{\mathbb{R}} \phi(x) h_n(x) dx} \right) dy.
\end{aligned}$$

To obtain the second equality, we note the following: Write

$$\int_{\mathbb{R}} \widehat{\sigma}(y - \lambda_n) \psi(y) dy = \int_{\mathbb{R}} \frac{\widehat{\sigma}(y - \lambda_n)}{y^2 + 1} ((y^2 + 1)\psi(y)) dy.$$

Using the Cauchy-Schwarz inequality, we see that

$$\left| \int_{\mathbb{R}} \widehat{\sigma}(y - \lambda_n) \psi(y) dy \right|^2 \leq \left(\int_{\mathbb{R}} \frac{|\widehat{\sigma}(y - \lambda_n)|^2}{(y^2 + 1)^2} \right) \left(\int_{\mathbb{R}} (y^2 + 1)^2 |\psi(y)|^2 dy \right).$$

In view of (3.1),

$$\sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} \widehat{\sigma}(y - \lambda_n) \psi(y) dy \right) h_n(x)$$

belongs to $\mathbf{L}_2(\mathbb{R}, dx)$, and we use the continuity of the scalar product. Furthermore we have used the dominated convergence theorem to obtain the third equality. \square

Equations (5.14) and (5.15) allow to compute Q_σ^* :

Theorem 5.6. *The domain of Q_σ^* is the Lebesgue space $\mathbf{L}_2(\mathbb{R}, dx)$. For $\phi \in \mathbf{L}_2(\mathbb{R}, dx)$, $Q_\sigma^* \phi$ is the tempered distribution defined through*

$$\langle Q_\sigma^*(\phi), \psi \rangle_{\mathcal{S}', \mathcal{S}} = \int_{\sup \sigma} \overline{\widehat{\psi}(u)} (T_\sigma(\phi)) d\sigma(u).$$

Equivalently, $Q_\sigma^(\phi)$ is the tempered distribution defined by the function $\mathbf{X}(\phi)$, that is, with some abuse of notation*

$$Q_\sigma^*(\psi)(y) = \sum_{n=0}^{\infty} \langle h_n, \phi \rangle_{\mathbf{L}_2(\mathbb{R}, dx)} \widehat{\sigma}(y - \lambda_n).$$

The second representation for Q_σ^* has an important consequence:

Theorem 5.7. *It holds that*

$$\ker Q_\sigma^* = \{0\}.$$

It follows that the operator $Q_\sigma^* Q_\sigma$ is a continuous and bounded operator from \mathcal{S} into \mathcal{S}' . We now provide two formulas for this operator.

Theorem 5.8. *Let ψ and ϕ be in \mathcal{S} . Then,*

$$\begin{aligned} \langle (Q_\sigma^* Q_\sigma) \phi, \psi \rangle_{\mathcal{S}, \mathcal{S}'} &= \\ &= \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} \widehat{\phi}(u) e^{-i\lambda_n u} d\sigma(u) \right) \overline{\left(\int_{\mathbb{R}} \widehat{\psi}(u) e^{-i\lambda_n u} d\sigma(u) \right)} \\ &= \int_{\mathbb{R}} \widehat{\phi}(u) \overline{\widehat{\psi}(u)} d\sigma(u). \end{aligned}$$

Proof: By definitions of $Q_\sigma\phi$ and of Q_σ^* we have

$$\begin{aligned} \langle (Q_\sigma^*Q_\sigma)(\phi), \psi \rangle_{\mathcal{S}, \mathcal{S}'} &= \int_{\mathbb{R}} \overline{\widehat{\psi}(u)} (T_\sigma(Q_\sigma\phi)) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{R}} \overline{\widehat{\psi}(u)} \left(\int_{\mathbb{R}} \widehat{\sigma}(y - \lambda_n)\phi(y)dy \right) e^{i\lambda_n u} d\sigma(u) \\ &= \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} \overline{\widehat{\psi}(u)} e^{-i\lambda_n u} d\sigma(u) \right) \left(\int_{\mathbb{R}} \widehat{\sigma}(y - \lambda_n)\phi(y)dy \right). \end{aligned}$$

But

$$\begin{aligned} \int_{\mathbb{R}} \widehat{\sigma}(y - \lambda_n)\phi(y)dy &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{i(y-\lambda_n)v} d\sigma(v) \right) \phi(y)dy \\ &= \int_{\mathbb{R}} \widehat{\phi}(y) e^{-i\lambda_n y} d\sigma(y) \end{aligned}$$

where we have used Fubini's theorem. This concludes the proof. \square

Remark 5.9. While the operator Q_σ (see Theorems 5.6 through 5.8) is well defined as an unbounded linear operator in the Hilbert space $\mathbf{L}_2(\mathbb{R}, dx)$, the other two operators Q_σ^* and $Q_\sigma^*Q_\sigma$ are not. The reason is that the range of Q_σ^* is not contained in $\mathbf{L}_2(\mathbb{R}, dx)$. In fact,

$$(Q_\sigma^*h_n)(x) = \widehat{\sigma}(x - \lambda_n),$$

where $\widehat{\sigma}$ is the infinite product expression (2.3). It can be shown that $x \mapsto \widehat{\sigma}(x)$ is not in $\mathbf{L}_2(\mathbb{R}, dx)$; so as an $\mathbf{L}_2(\mathbb{R}, dx)$ -operator, Q_σ is not closable (its adjoint, computed in $\mathbf{L}_2(\mathbb{R}, dx)$, does not have dense domain! Nonetheless Q_σ^* in the extended sense maps $\mathbf{L}_2(\mathbb{R}, dx)$ into \mathcal{S}'). The use of the ambient space \mathcal{S}' of tempered distributions is essential. We illustrate the above discussion with the following diagrams:

$$\begin{array}{ccccc} \mathcal{S} & \xrightarrow{i} & \mathbf{L}_2(\mathbb{R}, dx) & \xrightarrow{i^*} & \mathcal{S}' \\ & & \searrow Q_\sigma & & \nearrow Q_\sigma^* \\ \mathcal{S} & \xrightarrow{i} & \mathbf{L}_2(\mathbb{R}, dx) & \xrightarrow{i^*} & \mathcal{S}' \end{array} .$$

In the following diagram, $\text{dom } Q_\sigma^*$ is only a small subspace of $\mathbf{L}_2(\mathbb{R}, dx)$:

$$\begin{array}{ccc} & \mathbf{L}_2(\mathbb{R}, dx) & \xrightarrow{i^*} \mathcal{S}' \\ & \nearrow Q_\sigma^* \text{ restricted} & \nearrow Q_\sigma^* \text{ unbounded} \\ \text{Dom } Q_\sigma^* & \xrightarrow{i} & \mathbf{L}_2(\mathbb{R}, dx) \end{array} .$$

6. THE PROCESSES X_σ AND W_σ

In this section we build the Gaussian process with covariance function K_σ . First recall that, thanks to Proposition 5.4, the domain of the operator Q_σ has been extended to include the functions $1_{[0,t]}$. We begin with:

Theorem 6.1. *Let Q_σ be as (5). Then, for every $t, s \in \mathbb{R}$,*

$$\langle Q_\sigma 1_{[0,t]}, Q_\sigma 1_{[0,s]} \rangle_{\mathbf{L}_2(\mathbb{R}, dx)} = \int_{\mathbb{R}} \frac{e^{iut} - 1}{u} \frac{e^{-ius} - 1}{u} d\sigma(u).$$

Proof: We take $(s_n)_{n \in \mathbb{N}}$ and $(t_n)_{n \in \mathbb{N}}$ two sequences of elements of \mathcal{S} with Fourier transforms $(\widehat{s}_n)_{n \in \mathbb{N}}$ and $(\widehat{t}_n)_{n \in \mathbb{N}}$ converging in the supremum norm respectively to χ_t and χ_s^* . Then $Q1_{[0,t]}$ and $Q1_{[0,s]}$ are the limit in $\mathbf{L}_2(\mathbb{R}, dx)$ of the sequences $(Qs_n)_{n \in \mathbb{N}}$ and $(Qt_n)_{n \in \mathbb{N}}$ respectively. Hence,

$$\begin{aligned} \langle Q_\sigma 1_{[0,t]}, Q_\sigma 1_{[0,s]} \rangle_{\mathbf{L}_2(\mathbb{R}, dx)} &= \lim_{n, m \rightarrow \infty} \langle Q_\sigma s_n, Q_\sigma t_m \rangle_{\mathbf{L}_2(\mathbb{R}, dx)} \\ &= \lim_{n, m \rightarrow \infty} \int_{\mathbb{R}} \widehat{s}_n(u) \widehat{t}_m(u) d\sigma(u) \\ &= \int_{\mathbb{R}} \frac{e^{iut} - 1}{u} \frac{e^{-ius} - 1}{u} d\sigma(u). \end{aligned}$$

□

Recall that we have denoted by \widetilde{f} the natural isometric imbedding (4.2) of $\mathbf{L}_2(\mathbb{R}, dx)$ into the white noise space. We set

$$(6.1) \quad Z_n = \widetilde{h}_n, \quad n = 0, 1, \dots$$

The Z_n are independent, identically distributed $\mathcal{N}(0, 1)$ random variables, and represented in white noise space.

We arrive at the following decomposition:

$$(6.2) \quad \begin{aligned} X_\sigma(t) &= \widetilde{Q_\sigma(1_{[0,t]})} \\ &= \sum_{n=0}^{\infty} \left(\int_0^t \widehat{\sigma}(y - \lambda_n) dy \right) Z_n. \end{aligned}$$

Theorem 6.2. *It holds that*

$$(6.3) \quad \|X_\sigma(t) - X_\sigma(s)\|_{\mathcal{W}} \leq |t - s|, \quad t, s \in \mathbb{R}.$$

The function $t \mapsto X_\sigma(t)$ is differentiable in \mathcal{W} (white noise space), and its derivative is given by

$$(6.4) \quad W_\sigma(t) = \sum_{n=0}^{\infty} \widehat{\sigma}(t - \lambda_n) Z_n$$

Remark 6.3. Formulas (6.2) and (6.4) are analogous to, but different from Karhunen-Loève expansions; see e.g., [25].

Proof of Theorem 6.2: We first prove (6.3). We have

$$\begin{aligned} E_\sigma[|X_\sigma(t) - X_\sigma(s)|^2] &= \|X_\sigma(t) - X_\sigma(s)\|_{\mathcal{W}}^2 \\ &= \sum_{n=0}^{\infty} \left| \int_s^t \widehat{\sigma}(y - \lambda_n) dy \right|^2 \\ &\leq \sum_{n=0}^{\infty} \left(\int_s^t 1^2 dy \right) \left(\int_s^t |\widehat{\sigma}(y - \lambda_n)|^2 dy \right) \\ &= (t - s) \int_s^t \left(\sum_{n=0}^{\infty} |\widehat{\sigma}(y - \lambda_n)|^2 \right) dy \\ &= (t - s)^2, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality and (3.1) for the second and fourth equalities, respectively.

We remark that, in view of (3.1), $W_\sigma(t) \in \mathcal{W}$ for every real t . We have

$$\begin{aligned} \frac{X_\sigma(t) - X_\sigma(s)}{t - s} - W_\sigma(t) &= \frac{\sum_{n=0}^{\infty} \int_s^t \widehat{\sigma}(y - \lambda_n) dy Z_n}{t - s} - \\ &\quad - \sum_{n=0}^{\infty} \widehat{\sigma}(t - \lambda_n) dy Z_n \\ &= \frac{\sum_{n=0}^{\infty} \left\{ \int_s^t (\widehat{\sigma}(y - \lambda_n) - \widehat{\sigma}(t - \lambda_n)) dy \right\} Z_n}{t - s} \\ &= \frac{\sum_{n=0}^{\infty} \left\{ \int_s^t \langle e_y - e_t, e_{\lambda_n} \rangle_{\mathbf{L}_2(d\sigma)} dy \right\} Z_n}{t - s}. \end{aligned}$$

Hence, using Parseval's equality and (3.2), we obtain

$$\begin{aligned}
\left\| \frac{X_\sigma(t) - X_\sigma(s)}{t - s} - W_\sigma(t) \right\|_{\mathcal{W}}^2 &= \frac{\sum_{n=0}^{\infty} \left| \int_s^t \langle e_y - e_t, e_{\lambda_n} \rangle_{\mathbf{L}_2(d\sigma)} dy \right|^2}{(t - s)^2} \\
&\leq \sum_{n=0}^{\infty} \frac{1}{t - s} \int_s^t |\langle e_y - e_t, e_{\lambda_n} \rangle_{\mathbf{L}_2(d\sigma)}|^2 dy \\
&= \frac{1}{t - s} \int_s^t \|e_y - e_t\|_{\mathbf{L}_2(d\sigma)}^2 dy \\
&\leq \frac{1}{t - s} \int_s^t (y - t)^2 dy \\
&\leq \frac{(t - s)^2}{3}.
\end{aligned}$$

□

Theorem 6.4. *The derivative process W_σ is continuous in the $\|\cdot\|_{\mathcal{W}}$ norm. It is furthermore stationary and of constant variance,*

$$(6.5) \quad E[|W_\sigma(t)|^2] \equiv 1.$$

Proof:

$$\begin{aligned}
\|W_\sigma(t) - W_\sigma(s)\|_{\mathcal{W}}^2 &= \sum_{n=0}^{\infty} |\widehat{\sigma}(t - \lambda_n) - \widehat{\sigma}(s - \lambda_n)|^2 \\
&= \sum_{n=0}^{\infty} |\langle e_t - e_s, e_{\lambda_n} \rangle_{\mathbf{L}_2(d\sigma)}|^2 \\
&= \|e_t - e_s\|_{\mathbf{L}_2(d\sigma)}^2 \\
&= 2(1 - \langle e_t, e_s \rangle_{\mathbf{L}_2(d\sigma)}) \\
&= 2(1 - \widehat{\sigma}(t - s)) \\
&= 2 \left(1 - \prod_{n=1}^{\infty} \cos \left(\frac{t - s}{4^n} \right) \right).
\end{aligned}$$

Continuity and stationarity follow from the above chain of inequalities, together with the fact that $\{W_\sigma(t)\}$ is a Gaussian process. Equation (6.5) follows from (3.1). □

We now turn to another type of representation for X_σ :

Theorem 6.5. *Let*

$$H(\omega, u) = \sum_{n=0}^{\infty} \widetilde{h}_n(\omega) e^{i\lambda_n u} \in \mathcal{W} \otimes \mathbf{L}_2(d\sigma).$$

Then

$$(X_\sigma(t))(\omega) = \int_{\mathbb{R}} H(\omega, u) \frac{e^{-iut} - 1}{u} d\sigma(u)$$

Proof: Using Fubini's theorem we have

$$\begin{aligned} \int_{\mathbb{R}} e^{i\lambda_n u} \frac{e^{-iut} - 1}{u} d\sigma(u) &= \int_{\mathbb{R}} e^{i\lambda_n u} \left(\int_0^t e^{-iuv} dv \right) d\sigma(u) \\ &= \int_0^t \int_{\mathbb{R}} e^{i(\lambda_n - v)u} d\sigma(u) \\ &= \int_0^t \widehat{\sigma}(\lambda_n - v). \end{aligned}$$

This together with (6.2) leads to the required conclusion. \square

7. THE WICK-ITO INTEGRAL

In this section we establish a stochastic integration formula for the general class of stationary increment processes considered here. An extension of Ito's formula is addressed in section 8. More precisely, see Theorem 7.1, with the use of *a priori* estimates and of a Wick calculus (from weighted symmetric Fock spaces), we extend and sharpen earlier computations of Ito-stochastic integration developed originally only for the special case of stationary increment processes having absolutely continuous spectral measures. We further obtain in the subsequent section an associated Ito formula (Theorem 8.2).

The main result of this section is the following theorem, which is the counterpart of [1, Theorem 5.1]. The fact that the derivative process $W_\sigma(t)$ is \mathcal{W} -valued (rather than lying in the larger Kondratiev space) allows to get a sharper statement. The proof follows the strategy of [1] and hence is only outlined.

Theorem 7.1. *Let $Y(t)$, $t \in [a, b]$ be an S_{-1} -valued function, continuous in the strong topology of S_{-1} . Then, there exists a $p \in \mathbb{N}$ such that the function $t \mapsto Y(t) \diamond W_\sigma(t)$ is \mathcal{H}'_p -valued, and*

$$\int_a^b Y(t, \omega) \diamond W_\sigma(t) dt = \lim_{|\Delta| \rightarrow 0} \sum_{k=0}^{n-1} Y(t_k, \omega) \diamond (X_\sigma(t_{k+1}) - X_\sigma(t_k)),$$

where the limit is in the \mathcal{H}'_p norm, with $\Delta : a = t_0 < t_1 < \dots < t_n = b$ a partition of the interval $[a, b]$ and $|\Delta| = \max_{0 \leq k \leq n-1} (t_{k+1} - t_k)$.

Proof: We proceed in a number of steps.

STEP 1: *There exists a $p \in \mathbb{N}$ such that $Y(t) \in \mathcal{H}'_p$ for all $t \in [a, b]$, being uniformly continuous from $[a, b]$ into \mathcal{H}'_p .*

This is proved in [1, STEP 2 of the Proof of Theorem 5.1].

STEP 2: *The function $t \mapsto Y(t) \diamond W_\sigma(t)$ is continuous over $[a, b]$.*

Here and in the sequel, we set $\|\cdot\|_p \stackrel{\text{def.}}{=} \|\cdot\|_{\mathcal{H}'_p}$ to simplify notation. Using Våge's inequality (4.7), it follows that, for $p > 1$,

$$\begin{aligned} \|Y(t) \diamond W_\sigma(t) - Y(s) \diamond W_\sigma(s)\|_p &\leq \\ &\leq \|(Y(t) - Y(s)) \diamond W_\sigma(t)\|_p + \|Y(s) \diamond (W_\sigma(t) - W_\sigma(s))\|_p \\ &\leq A(p) \|Y(t) - Y(s)\|_p \|W_\sigma(s)\|_0 + \\ &\quad + A(p) \|Y(s)\|_p \|W_\sigma(t) - W_\sigma(s)\|_0 \end{aligned}$$

where $A(p)$ is defined by (4.8). We conclude the proof of STEP 2 by observing that

$$\|\cdot\|_{\mathcal{H}_0} \leq \|\cdot\|_{\mathcal{W}},$$

which implies the continuity of the function $Y(t) \diamond W_\sigma(t)$ in the norm $\|\cdot\|_{\mathcal{W}}$.

In view of Step 2, the integral $\int_a^b Y(t) \diamond W_\sigma(t) dt$ makes sense as a Riemann integral of a continuous Hilbert space valued function.

STEP 3: *Let Δ be a partition of the interval $[a, b]$. We now compute an estimate for*

$$\begin{aligned} \int_a^b Y(t) \diamond W_\sigma(t) dt - \sum_{k=0}^{n-1} Y(t_k) \diamond (X_\sigma(t_{k+1}) - X_\sigma(t_k)) &= \\ &= \sum_{k=0}^{n-1} \left(\int_{t_k}^{t_{k+1}} (Y(t) - Y(t_k)) \diamond W_\sigma(t) dt \right). \end{aligned}$$

As for steps 1 and 2, we closely follow [1]. Let p be as in Step 2, and set $\epsilon > 0$. Since Y is uniformly continuous on $[a, b]$, there exists an $\eta > 0$ such that

$$|t - s| < \eta \implies \|Y(t) - Y(s)\|_p < \epsilon.$$

Set

$$\tilde{C} = \max_{s \in [a, b]} \|W_\sigma(s)\|_0 \quad \text{and} \quad A = A(p - N - 3).$$

Let Δ be a partition of $[a, b]$ with

$$|\Delta| = \max \{t_{k+1} - t_k\} < \eta.$$

We then have:

$$\begin{aligned} & \left\| \sum_{k=0}^{n-1} \left(\int_{t_k}^{t_{k+1}} (Y(t) - Y(t_k)) \diamond W_\sigma(t) dt \right) \right\|_p \leq \\ & \leq \sum_{k=0}^{n-1} \left(\int_{t_k}^{t_{k+1}} \|(Y(t) - Y(t_k)) \diamond W_\sigma(t)\|_p dt \right) \\ & \leq A \sum_{k=0}^{n-1} \left(\int_{t_k}^{t_{k+1}} \|Y(t) - Y(t_k)\|_p \|W_\sigma(t)\|_0 dt \right) \\ & \leq \tilde{C} A \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \|Y(t) - Y(t_k)\|_p dt \\ & \leq \epsilon \tilde{C} A (b - a), \end{aligned}$$

which completes the proof of Step 3 and the proof of the Theorem. \square

8. AN ITO FORMULA

We extend the classical Ito's formula to the present setting. Our present wider context for these stochastic processes entails important analytical points: Singular measures of fractal dimension, and singular operators, extending beyond the Hilbert space $\mathbf{L}_2(\mathbb{R}, dx)$. This in turn brings to light new aspects of Ito calculus which we detail below. In addition to the examples in Section 2 (affine IFS measures), we further offer two examples in sect 9 below: the periodic Brownian bridge, and the Orenstein-Uhlenbeck processes.

Lemma 8.1. *The function*

$$r(t) = \|Q_\sigma 1_{[0, t]}\|_{\mathbf{L}_2(\mathbb{R})}^2$$

is absolutely continuous with respect to the Lebesgue measure.

Proof: By (5.3) we have

$$\begin{aligned}\|Q_\sigma 1_{[0,t]}\|_{\mathbf{L}_2(\mathbb{R})}^2 &= \int_{\mathbb{R}} |\chi_t(u)|^2 d\sigma(u) \\ &= 2 \int_{\mathbb{R}} \frac{1 - \cos(tu)}{u^2} d\sigma(u).\end{aligned}$$

Since the support of $d\sigma$ is bounded, the dominated convergence theorem allows to show that $r(t)$ is differentiable and that its derivative is given by

$$r'(t) = 2 \int_{\mathbb{R}} \frac{\sin(tu)}{u} d\sigma(u).$$

□

Theorem 8.2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a $C^2(\mathbb{R})$ function. Then*

$$(8.1) \quad \begin{aligned}f(X_\sigma(t)) &= f(X_\sigma(t_0)) + \int_{t_0}^t f'(X_\sigma(s)) \diamond W_\sigma(s) ds + \\ &+ \frac{1}{2} \int_{t_0}^t f''(X_\sigma(s)) r'(s) ds, \quad t_0 < t \in \mathbb{R},\end{aligned}$$

where the equality is in the P -almost sure sense.

Proof: We prove for $t > t_0 = 0$. The proof for any other interval in \mathbb{R} is essentially the same. We divide the proof into a number of steps. Step 1-Step 5 are constructed so as to show that (8.1) holds, $\forall t > 0$, for Schwartz functions. This enables the extension to C^2 functions with compact support, with the equality holding in the \mathcal{H}'_p sense. This implies its validity in the P -a.s. sense (actually, holding $\forall \omega \in \Omega$), hence, setting the ground for the concluding step, in which the result is extended to hold for all C^2 functions f .

STEP 1: *For every $(u, t) \in \mathbb{R}^2$, it holds that*

$$e^{iuX_\sigma(t)} \in \mathcal{W},$$

and

$$(8.2) \quad e^{iuX_\sigma(t)} \diamond W_\sigma(t) \in \mathcal{H}'_2.$$

Indeed, since X_σ is real, we have

$$|e^{iuX_\sigma(t)}| \leq 1, \quad \forall u, t \in \mathbb{R},$$

and hence $e^{iuX_\sigma(t)} \in \mathcal{W}$. Since $\mathcal{W} \subset \mathcal{H}'_2$, we have in particular that $W_\sigma(t) \in \mathcal{H}'_2$ for all $t \in \mathbb{R}$, it follows from Våge's inequality (4.7) that

(8.2) holds.

The aim of the following two steps is formula (8.1) for exponential functions. For $\alpha \in \mathbb{R}$ we set:

$$g(x) = \exp(i\alpha x).$$

The proofs are as in [1], taking into account that $r(t)$ is absolutely continuous with respect to Lebesgue measure and are omitted.

STEP 2: *It holds that*

$$(8.3) \quad g'(X_\sigma(t)) = i\alpha g(X_\sigma(t)) \diamond W_\sigma(t) + \frac{1}{2}(i\alpha)^2 g(X_\sigma(t)) r'(t).$$

STEP 3: *Equation (8.1) holds for exponentials.*

In the following two steps, we prove (8.1) to hold for Schwartz functions.

STEP 4: *The function $(u, t) \mapsto e^{iuX_\sigma(t)} \diamond W_\sigma(t)$ is continuous from \mathbb{R}^2 into \mathcal{H}'_2 .*

We first recall that the norm in \mathcal{H}'_p is denoted by $\|\cdot\|_p$. See (4.6). The particular case $p = 0$ in (4.6) gives in particular

$$\|f\|_0^2 = \sum_{\alpha \in \ell} |f_\alpha|^2.$$

The structure of \mathcal{H}'_0 has been studied in [4, Section 7].

Recall now that the function $t \mapsto X_\sigma(t)$ is continuous, and even uniformly continuous, from \mathbb{R} into \mathcal{W} , and hence from \mathbb{R} into \mathcal{H}'_p for any $p \in \mathbb{N}_0$ since

$$\|u\|_p \leq \|u\|_{\mathcal{W}}, \quad \text{for } u \in \mathcal{W}.$$

The function $(u, t) \mapsto e^{iuX_\sigma(t)}$ is in particular continuous from \mathbb{R}^2 into \mathcal{H}'_2 . Furthermore, using Våge's inequality (4.7) we have:

$$\begin{aligned} & \|e^{iu_1 X_\sigma(t_1)} \diamond W_\sigma(t_1) - e^{iu_2 X_\sigma(t_2)} \diamond W_\sigma(t_2)\|_{\mathcal{H}'_2} \leq \\ & \leq \|(e^{iu_1 X_\sigma(t_1)} - e^{iu_2 X_\sigma(t_2)}) \diamond W_\sigma(t_1)\|_{\mathcal{H}'_2} + \\ & \quad + \|e^{iu_1 X_\sigma(t_1)} \diamond (W_\sigma(t_2) - W_\sigma(t_1))\|_{\mathcal{H}'_2} \\ & \leq A(2) \|(e^{iu_1 X_\sigma(t_1)} - e^{iu_2 X_\sigma(t_2)})\|_{\mathcal{H}'_2} \cdot \|W_\sigma(t_1)\|_{\mathcal{H}'_0} + \\ & \quad + A(2) \|e^{iu_1 X_\sigma(t_1)}\|_{\mathcal{H}'_2} \cdot \|(W_\sigma(t_2) - W_\sigma(t_1))\|_{\mathcal{H}'_0}, \end{aligned}$$

where $A(2)$ is defined by (4.8). This completes the proof of STEP 4 since

$$\|\cdot\|_2 \leq \|\cdot\|_0 \leq \|\cdot\|_{\mathcal{W}}$$

and in particular $t \mapsto W_\sigma(t)$ is continuous in the norm of \mathcal{H}'_0 and $(u, t) \mapsto e^{iuX_\sigma(t)}$ is continuous in the norm of \mathcal{H}'_2 .

STEP 5: (8.1) holds for f in the Schwartz space.

The remainder of the proof is exactly as Steps 6-9 in the corresponding proof of [1, Theorem 6.1], and hence omitted. \square

9. TWO EXAMPLES

In this section, to contrast the measures considered above, we now consider two examples where the measure σ has an unbounded support.

Example (The periodic Brownian bridge). Take

$$(9.1) \quad \sigma(u) = \sum_{n=0}^{\infty} \delta(u - 2n),$$

that is the measure with support on the even positive integers with mass equal to 1 at each of these points. Then, it follows from the formula (1.5) on $r(t)$ that

$$r(t) = \pi \sum_{n=1}^{\infty} \frac{1 - \cos(2nt)}{(2n)^2}.$$

Note that,

$$t(\pi - t) = \pi \sum_{n=1}^{\infty} \frac{1 - \cos(2nt)}{(2n)^2}, \quad t \in [0, 2\pi].$$

In view of the preceding equality, we call the associated process $\{X_\sigma(t)\}$ the *periodic Brownian bridge over $[0, \pi]$* . We have

$$X_\sigma(t) = \sqrt{\frac{\pi}{2}} \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} Z_n,$$

where, as in (6.1), $Z_n = \widetilde{h}_n$.

We note that, by construction, for every t , $X_\sigma(t)$ belongs to the white noise space. On the other hand, the power series

$$W_\sigma(t) = \sqrt{\frac{\pi}{2}} \sum_{n=1}^{\infty} \cos(nt) Z_n$$

converges only in the Kondratiev space. More precisely,

Theorem 9.1. *Let σ be given by (9.1). For every t , we have that $W_\sigma(t) \in \mathcal{H}'_2$ and in the topology of \mathcal{H}'_4*

$$(9.2) \quad \lim_{s \rightarrow t} \frac{X_\sigma(t) - X_\sigma(s)}{t - s} = W_\sigma(t).$$

Proof: The fact that $W_\sigma(t)$ belongs to \mathcal{H}'_2 follows from definition (4.6) since

$$\|Z_n\|_2^2 = (2n)^{-2},$$

and since the Z_n are mutually orthogonal in \mathcal{H}'_2 . We now turn to (9.2). We have

$$\frac{X_\sigma(t) - X_\sigma(s)}{t - s} - W_\sigma(t) = \sum_{n=0}^{\infty} \left(\frac{\int_s^t (\cos(nu) - \cos(nt)) du}{t - s} \right) Z_n.$$

But

$$\begin{aligned} \left| \frac{\int_s^t (\cos(nu) - \cos(nt)) du}{t - s} \right| &\leq \left| \frac{\int_s^t (nu - ns) n \sin(nv) du}{t - s} \right|, \quad \text{for some } v \in (s, t) \\ &\leq n^2 \left| \frac{\int_s^t (u - s) du}{t - s} \right| \\ &= \frac{n^2 |t - s|}{2} \end{aligned}$$

and hence the limit goes to 0 in the \mathcal{H}'_4 norm since

$$\|Z_n\|_4^2 = (2n)^{-4},$$

and the Z_n are mutually orthogonal in \mathcal{H}'_4 . □

Example (The Ornstein-Uhlenbeck process). This is the solution of this stochastic differential equation

$$dX(t) = \theta(\mu - X(t))dt + \alpha dB(t), \quad t \geq 0,$$

where B is a Brownian motion and $\theta \neq 0, \mu$ and σ are parameters. We have

$$X(t) = \mu + \frac{\alpha}{\sqrt{2\theta}} W(e^{2\theta t}) e^{-\theta t},$$

and

$$E[X(t)] = e^{-\theta t} + \mu(1 - e^{-\mu t}),$$

$$E[(X(t) - E(X(t)))(X(s) - E(X(s)))] = \frac{\alpha^2}{2\theta} e^{-\theta(t+s)} (e^{2\theta s \wedge t} - 1).$$

Theorem 9.2. *The centered Ornstein-Uhlenbeck process $\{X(t) - E[X(t)]\}$ is a stationary increment Gaussian process. Furthermore*

$$E[|X(t) - E[X(t)]|^2] = \frac{\alpha^2}{2\theta} (1 - e^{-2\theta t})$$

$$= \int_{\mathbb{R}} \frac{1 - \cos(tu)}{u^2} d\sigma(u),$$

with

$$d\sigma(u) = \frac{\alpha^2}{2\pi\theta} \frac{\theta u^2 du}{\theta^2 + u^2}.$$

10. THE OPERATOR Q_σ FOR MORE GENERAL SPECTRAL MEASURES $d\sigma$

The purpose of this section is to contrast the difference between a singular measure or not. This is a crucial distinction in passing from the given measure σ to the associated process $X_\sigma(t)$; i.e., in solving the inverse problem. The two cases are: (i) σ is assumed absolutely continuous with respect to Lebesgue measure; versus: (ii) σ is singular. This distinction results in a dichotomy for the induced operators in $\mathbf{L}_2(\mathbb{R}, dx)$, so for the two operator questions. In case (i) we have a Radon-Nikodym derivative m , and the induced operator is T_m , a self-adjoint convolution operator in the Hilbert space $\mathbf{L}_2(\mathbb{R}, dx)$ with the Schwartz space \mathcal{S} as dense domain; see (1.6). In the second case (ii) it is not, even existence is subtle. Now rather the induced operator is our operator Q_σ from Section 5. But it is much more subtle, and below we address some of the technical points omitted in Section 5: (a) There is now more than one choice for Q_σ ; and (b) none of the choices will be closable operators, referring just to the Hilbert space $\mathbf{L}_2(\mathbb{R}, dx)$. (c) Nonetheless, by working within the environment of the Gelfand triples (of Gaussian random fields), we are still able to make precise the two operators Q_σ and the corresponding adjoint Q_σ^* . For the singular case, i.e., case (ii), the Gelfand triple is thus essential in justifying our construction of the process $\{X_\sigma(t)\}$, existence and related properties.

One common point between the previous work [3] and the present work is the construction of an operator Q_σ from a subspace of $\mathbf{L}_2(\mathbb{R}, dx)$ into

itself (denoted by T_m in [3]; see (1.6)) such that

$$(10.1) \quad K_\sigma(t, s) = \langle Q_\sigma(1_{[0,t]}), Q_\sigma(1_{[0,s]}) \rangle_{\mathbf{L}_2(\mathbb{R}, dx)} = E[X_\sigma(t)X_\sigma(s)^*].$$

The natural isometry from $\mathbf{L}_2(\mathbb{R}, dx)$ into the white noise space allows then to proceed by doing analysis in the Gelfand triple associated with the Kondratiev spaces, see Section 7. Although there are numerous possible other Gelfand triples, Vågø's inequality, see (4.7), is a feature which seems characteristic of this triple and is very useful in the computations. In the present section we present some general results on the existence and properties of the operator Q_σ . To develop the associated stochastic integral and Ito's formula, one has to relate the properties of σ and of Q . Not all the arguments go through for general σ 's. Details will be presented in forthcoming publications.

Theorem 10.1. *Let σ be a positive measure subject to*

$$(10.2) \quad \int_{\mathbb{R}} \frac{d\sigma(u)}{1+u^2} < \infty,$$

and assume that $\dim \mathbf{L}_2(d\sigma) = \infty$. There exists a possibly unbounded operator Q from $\mathbf{L}_2(d\sigma)$ into $\mathbf{L}_2(\mathbb{R}, dx)$, with domain containing the Schwartz space, such that (10.1) holds:

$$K_\sigma(t, s) = \langle Q(1_{[0,t]}), Q(1_{[0,s]}) \rangle_{\mathbf{L}_2(\mathbb{R}, dx)},$$

and

$$X_\sigma(t) = \widetilde{Q1_{[0,t]}}.$$

The operator Q in the preceding theorem is not unique. In the previous cases, a specific choice of Q was made by recipe, which depended on the special structure of $\mathbf{L}_2(d\sigma)$. In general there seems no natural way to chose a specific Q . We have dropped therefore the index σ in the notation.

Proof of Theorem 10.1: We proceed in a number of step.

STEP 1: *Let W be a unitary map from $\mathbf{L}_2(d\sigma)$ onto $\mathbf{L}_2(\mathbb{R}, dx)$, and let*

$$f_1 = Wb_1,$$

where b_1 denotes the function

$$b_1(u) = \frac{1}{\sqrt{1+u^2}} \in \mathbf{L}_2(d\sigma).$$

Then:

$$\|f_1\|_{\mathbf{L}_2(\mathbb{R}, dx)}^2 = \|b_1\|_{\mathbf{L}_2(d\sigma)}^2 = \int_{\mathbb{R}} \frac{d\sigma(u)}{u^2+1} < \infty.$$

This is clear from the unitarity of W .

STEP 2: *The operator M_u of multiplication by the variable u is a priori an unbounded operator in $\mathbf{L}_2(d\sigma)$. It is densely defined and self-adjoint in $\mathbf{L}_2(d\sigma)$.*

This follows from (5.4), which is true for any measure σ satisfying (10.2).

It follows from the preceding step that the operator

$$T = WM_uW^*.$$

is a self-adjoint operator in $\mathbf{L}_2(\mathbb{R}, dx)$. The operator Q

$$(10.3) \quad Q\psi = \sqrt{1 + T^2}\widehat{\psi}(T)f_1$$

can therefore be computed using the spectral theorem. We claim Q satisfies (10.1) and (5.3). This is done in the next two steps.

STEP 3: (5.3) *holds.*

Indeed, using the spectral theorem we have

$$\begin{aligned} \|Q\psi\|_{\mathbf{L}(\mathbb{R}, dx)}^2 &= \|(\sqrt{1 + T^2}\widehat{\psi}(T)f_1)\|_{\mathbf{L}(\mathbb{R}, dx)}^2 \\ &= \|W^*\sqrt{1 + T^2}\widehat{\psi}(T)Wb_1\|_{\mathbf{L}_2(d\sigma)}^2 \\ &= \int_{\mathbb{R}} |\sqrt{1 + u^2}\widehat{\psi}(u)\frac{1}{\sqrt{1 + u^2}}|^2 d\sigma(u) \\ &= \int_{\mathbb{R}} |\widehat{\psi}(u)|^2 d\sigma(u). \end{aligned}$$

STEP 4: *Set*

$$X_\sigma(t) = \widetilde{Q1_{[0,t]}}.$$

Then

$$E[X_\sigma(t)X_\sigma(s)^*] = \langle Q1_{[0,t]}, Q1_{[0,s]} \rangle_{\mathbf{L}(\mathbb{R}, dx)}.$$

Indeed, the function $1_{[0,t]}$ belongs to the domain of Q and the claim is a direct consequence of the isometric imbedding of $\mathbf{L}(\mathbb{R}, dx)$ inside \mathcal{W} . \square

Corollary 10.2. *Let σ be a positive measure subject to*

$$(10.4) \quad \int_{\mathbb{R}} \frac{d\sigma(u)}{1+|u|} < \infty,$$

and let $\{X_\sigma(t)\}$ be the associated process as in Theorem 10.1. Then, $\{X_\sigma(t)\}$ has a continuous version. (By this we mean [34] that $\{X_\sigma(t)\}$ agrees a.e. with some time-continuous process.)

Proof: We will prove this as an application of Kolmogorov's test for the existence of a continuous version, [34, p. 14]. Because $\{X_\sigma(t)\}$ is Gaussian, there exists K independent of t, s such that

$$E[|X_\sigma(t) - X_\sigma(s)|^4] = K (E[|X_\sigma(t) - X_\sigma(s)|^2])^2.$$

On the other hand,

$$E[|X_\sigma(t) - X_\sigma(s)|^2] = 2\operatorname{Re} r(t-s) = 2 \int_{\mathbb{R}} \frac{1 - \cos((t-s)u)}{u^2} d\sigma(u).$$

Using (10.4) we now show that

$$(10.5) \quad \operatorname{Re} r(t) \leq C|t|, \quad |t| \in [0, 1].$$

The result will then follow from Kolmogorov's continuity criterion (see for instance [34, Theorem 2.2.3, p. 14] for the latter).

To prove (10.5) we proceed in a way similar as in [1] as follows. We compute first

$$\begin{aligned} \int_0^1 \frac{1 - \cos(tu)}{u^2} d\sigma(u) &= 2 \int_0^1 t^2 \frac{\sin\left(\frac{tu}{2}\right)^2}{t^2 u^2} d\sigma(u) \\ &\leq K_1 t^2 \quad (K_1 > 0 \text{ independent of } t) \\ &\leq K_1 |t| \quad \text{for } |t| \in [0, 1]. \end{aligned}$$

Furthermore, using the mean-value theorem for the function $u \mapsto \cos(tu)$ we have

$$1 - \cos(tu) = t^2 u \sin(t\xi_t), \quad \xi_t \in [0, u].$$

Thus

$$\begin{aligned} \int_1^\infty \frac{1 - \cos(tu)}{u^2} d\sigma(u) &= t^2 \int_1^\infty \sin(t\xi_t) \frac{d\sigma(u)}{u} \\ &\leq t^2 \int_1^\infty \frac{d\sigma(u)}{u} \\ &\leq K_2 t^2, \quad \text{where we use (10.4),} \\ &\leq K_2 |t| \quad \text{for } |t| \in [0, 1], \end{aligned}$$

where $K_2 > 0$ is independent of t . Inequality (10.5) follows and hence the result. \square

In Section 5 we defined a specific operator Q_σ on the Schwartz functions, and then defined $Q1_{[0,t]}$ by approximation. Here we have used the spectral theorem. Still it is possible to compute $Q1_{[0,t]}$ via approximating sequences. We note that, in view of (10.2), the measure

$$d\mu(u) = \frac{d\sigma(u)}{1+u^2}$$

satisfies the following property:

$$(10.6) \quad \forall \epsilon > 0, \exists \mathbb{K} \text{ compact and such that } \mu(\mathbb{R} \setminus \mathbb{K}) \leq \epsilon.$$

The arguments in Proposition 5.4 can be adapted as follow. We take as a special sequence

$$s_n = 1_{[0,t]} \star k_{1/n}$$

where $k_{1/n}$ is defined via (5.11). Then

$$\widehat{s}_n(u) = \chi_t(u) \cdot e^{-\frac{u^2}{n^2}}.$$

Instead of (5.9) we write (for the special sequence $(s_n)_{n \in \mathbb{N}}$ at hand)

$$\begin{aligned} \|Q_\sigma s_n - Q_\sigma s_m\|_{\mathbf{L}_2(\mathbb{R}, dx)}^2 &= \int_{\mathbb{R}} |\widehat{s}_n(u) - \widehat{s}_m(u)|^2 d\sigma(u) \\ &= \int_{\mathbb{R}} |\chi_t(u)|^2 (e^{-\frac{u^2}{n^2}} - e^{-\frac{u^2}{m^2}})^2 d\sigma(u) \\ &= \int_{\mathbb{K}} |\chi_t(u)|^2 (e^{-\frac{u^2}{n^2}} - e^{-\frac{u^2}{m^2}})^2 d\sigma(u) + \\ &\quad + \int_{\mathbb{R} \setminus \mathbb{K}} |\chi_t(u)|^2 (e^{-\frac{u^2}{n^2}} - e^{-\frac{u^2}{m^2}})^2 d\sigma(u). \end{aligned}$$

where \mathbb{K} is a compact to be determined.

We first focus on the second integral in the last equality above. Set

$$\sup_{m,n} |\chi_t(u)|^2 (1+u^2) (e^{-\frac{u^2}{n^2}} - e^{-\frac{u^2}{m^2}})^2 = M < \infty,$$

and recall that $d\mu(u) = \frac{d\sigma(u)}{1+u^2}$. Then

$$\begin{aligned} \int_{\mathbb{R} \setminus \mathbb{K}} |\chi_t(u)|^2 (e^{-\frac{u^2}{n^2}} - e^{-\frac{u^2}{m^2}})^2 d\sigma(u) &= \\ &= \int_{\mathbb{R} \setminus \mathbb{K}} |\chi_t(u)|^2 (1+u^2) (e^{-\frac{u^2}{n^2}} - e^{-\frac{u^2}{m^2}})^2 d\mu(u) \\ &\leq M\mu(\mathbb{R} \setminus \mathbb{K}). \end{aligned}$$

For a preassigned $\epsilon > 0$, chose now the compact \mathbb{K} such that

$$\mu(\mathbb{R} \setminus \mathbb{K}) \leq \frac{\epsilon^2}{2M}.$$

Then

$$(10.7) \quad \int_{\mathbb{R} \setminus \mathbb{K}} |\chi_t(u)|^2 (e^{-\frac{u^2}{n^2}} - e^{-\frac{u^2}{m^2}})^2 d\sigma(u) \leq \frac{\epsilon}{2}.$$

Since K is compact there exists N such that, for n, m larger than N , the integral

$$(10.8) \quad \int_{\mathbb{K}} |\chi_t(u)|^2 (e^{-\frac{u^2}{n^2}} - e^{-\frac{u^2}{m^2}})^2 d\sigma(u) \leq \frac{\epsilon}{2}.$$

Therefore, for $n, m \geq N$,

$$\|Q_\sigma s_n - Q_\sigma s_m\|_{\mathbf{L}_2(\mathbb{R}, dx)} \leq \epsilon,$$

and the sequence $(Q_\sigma s_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the norm of $\mathbf{L}_2(\mathbb{R}, dx)$. This provides a constructive way to compute $Q1_{[0,t]}$. To see that the obtained limit gives the same value as the one obtained from the spectral theorem, it suffices to let n go to infinity in (10.7) and (10.8).

11. CONCLUDING REMARKS

We conclude with comments comparing our approach with the literature.

1. As in [1], no adaptability of the integrand with respect to an underlying filtration has been made. In this sense, one may regard the integral defined here in fact as a Wick-Skorohod integral.

2. Motivated in part by questions in physics, e.g., [12] and [30, 33], there has been a recent increase in the use of operator theory in stochastic processes, as reflected in e.g. references [1, 2, 3, 12, 13, 14]. In addition, we call attention to the papers [25, 26, 15, 17, 29] and the papers cited there. In our present approach, we have been using tools

from the cross roads of harmonic analysis and stochastic process, as are covered in [9, 14, 16, 31].

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