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GENERALIZED Q-FUNCTIONS AND DIRICHLET-TO-NEUMANN MAPS FOR ELLIPTIC DIFFERENTIAL OPERATORS

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Abstract. The classical concept of Q-functions associated to symmetric and selfadjoint operators due to M.G. Krein and H. Langer is extended in such a way that the Dirichlet-to-Neumann map in the theory of elliptic differential equations can be interpreted as a generalized Q-function. For couplings of uniformly elliptic second order differential expression on bounded and unbounded domains explicit Krein type formulas for the difference of the resolvents and trace formulas in an H^2-framework are obtained.

1. Introduction

The notion of a Q-function associated to a pair $\{S, A\}$ consisting of a symmetric operator S and a selfadjoint extension A of S in a Hilbert or Pontryagin space was introduced by M.G. Krein and H. Langer in [37, 38]. A Q-function contains the spectral information of the selfadjoint extensions of the underlying symmetric operator and therefore these functions play a very important role in the spectral and perturbation theory of selfadjoint operators. Q-functions appear also naturally in the description of the resolvents of the selfadjoint extensions of a symmetric operator with the help of Krein's formula and they can be used to construct functional models for selfadjoint operators. In the theory of boundary triplets associated to symmetric operators Q-functions can be interpreted as so-called Weyl functions, cf. [16, 17, 18, 19, 29]. A prominent example for a Q-function is the classical Titchmarsh-Weyl coefficient in the theory of singular Sturm-Liouville operators.

The main objective of this paper is to extend the concept of Q-functions in such a way that the Dirichlet-to-Neumann map in the theory of elliptic differential equations can be identified as a generalized Q-function. In the abstract part of the paper we introduce the notion of generalized Q-functions and we show that these functions have similar properties as classical Q-functions. Besides a symmetric operator S and a selfadjoint extension A also an operator T whose closure coincides with S^* is used. Some of the ideas here parallel [9], where a more abstract approach with isometric and unitary relations in Krein spaces was used. The main result in the abstract part is Theorem 2.6 which states that an operator function is a generalized Q-function if and only if it coincides up to a possibly unbounded constant on a dense subspace with the restriction of a Nevanlinna function with an invertible imaginary part and a certain asymptotic behaviour.

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Section 3 and Section 4 deal with second order elliptic operators on bounded and unbounded domains, and with the coupling of such operators. Suppose first that the domain $\Omega \subset \mathbb{R}^n$, $n>1$, is bounded with a smooth boundary $\partial\Omega$. Let A_D and A_N be the selfadjoint realizations of an formally symmetric uniformly elliptic differential expression

$$\mathcal{L} = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} + a \tag{1}$$

in $L^2(\Omega)$ defined on $H^2(\Omega)$ and subject to Dirichlet and Neumann boundary conditions, respectively. If T denotes the realization of $\mathcal{L}$ on $H^2(\Omega)$, then the closure of T in $L^2(\Omega)$ coincides with the maximal operator associated to $\mathcal{L}$ in $L^2(\Omega)$, and A_D and A_N are both selfadjoint restrictions of T. For a function $f \in H^2(\Omega)$ denote the trace and the trace of the conormal derivative by $f|_{\partial\Omega}$ and $\frac{\partial f}{\partial \nu}|_{\partial\Omega}$, respectively. Then for each $\lambda \in \rho(A_D)$ the Dirichlet-to-Neumann map

$$Q(\lambda)(f_\lambda|_{\partial\Omega}) := -\frac{\partial f_\lambda}{\partial \nu}\Big|_{\partial\Omega}, \qquad \text{where} \quad Tf_\lambda = \lambda f_\lambda, \tag{2}$$

is well-defined and will be regarded as an operator in $L^2(\partial\Omega)$ defined on $H^{3/2}(\partial\Omega)$ with values in $H^{1/2}(\partial\Omega)$. The minus sign in (2) is used for technical reasons. It turns out that the operator function $\lambda \mapsto Q(\lambda)$ is a generalized Q-function in the sense of Definition 2.2 and an explicit variant of Krein's formula for the resolvents of A_D and A_N is obtained in Theorem 3.4, see also [9, 13, 25, 26, 47, 48, 49] for more general problems. In particular, in the case $n=2$ the difference of these resolvents is a trace class operator and we obtain the trace formula

$$\operatorname{tr}\big((A_D-\lambda)^{-1} - (A_N-\lambda)^{-1}\big) = \operatorname{tr}\left(\overline{Q(\lambda)^{-1}}\,\frac{d}{d\lambda}\,\widetilde{Q}(\lambda)\right) \tag{3}$$

for $\lambda \in \rho(A_D) \cap \rho(A_N)$. Here $\overline{Q(\lambda)^{-1}}$ is the closure of $Q(\lambda)^{-1}$ in $L^2(\partial\Omega)$ and $\widetilde{Q}$ is a Nevanlinna function which differs from the Dirichlet-to-Neumann map by a symmetric constant. Trace formulas for canonical differential expressions and in more abstract situations for the finite-dimensional case can be found in, e.g., [2, 3, 10].

In Section 4 we consider a so-called coupling of elliptic operators. Such couplings are of great interest in problems of mathematical physics, e.g., in the description of quantum networks; for more details and further references we refer the reader to the recent works [20, 21, 44, 45, 46]. Suppose that $\mathbb{R}^n$, $n>1$, is decomposed in a bounded domain Ω with smooth boundary $\mathcal{C}$ and the unbounded domain $\Omega' = \mathbb{R}^n \backslash \overline{\Omega}$. The orthogonal sum of the selfadjoint Dirichlet operators A_D and A_D' associated to $\mathcal{L}$ in $L^2(\Omega)$ and $L^2(\Omega')$, respectively, is regarded as a selfadjoint diagonal block operator matrix in $L^2(\mathbb{R}^n)$. The resolvent of $A_D \oplus A_D'$ is then compared with the resolvent of the usual selfadjoint realization $\widetilde{A}$ of $\mathcal{L}$ in $L^2(\mathbb{R}^n)$ defined on $H^2(\mathbb{R}^n)$. In order to express this difference in the Krein type formula

$$\big((A_D \oplus A_D') - \lambda\big)^{-1} - (\widetilde{A}-\lambda)^{-1} = \Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\bar{\lambda})^* \tag{4}$$

with a generalized Q-function an analogon of the Dirichlet-to-Neumann map is constructed which measures the jump of the conormal derivative of $L^2(\Omega)$ and $L^2(\Omega')$-solutions of $\mathcal{L}u = \lambda u$ on the boundary $\mathcal{C}$, see (52). The operator $\Gamma(\lambda):$ $L^2(\mathcal{C}) \to L^2(\mathbb{R}^n)$ in (4) is closely connected with the generalized Q-function and is here identified with a Poisson-type operator solving a certain Dirichlet problem. As a consequence of the representation (4) we also obtain a trace formula of the type (3) in the coupled case.

2. Generalized Q-functions

In this section we introduce the notion of generalized Q-functions associated to a symmetric operators in Hilbert spaces. The class of generalized Q-functions is characterized in Theorem 2.6, where it turns out that generalized Q-functions are closely connected with operator-valued Nevanlinna or Riesz-Herglotz functions. We also note in advance that for the case of finite deficiency indices of the underlying symmetric operator the concept of generalized Q-functions coincides with the classical notion of (ordinary) Q-functions studied by M.G. Krein and H. Langer in [37, 38], see also [35, 36].

Let $\mathcal{H}$ be a separable Hilbert space and let S be a densely defined closed symmetric operator with equal (in general infinite) deficiency indices

$$n_\pm(S) = \dim\ker(S^* \mp i) \leq \infty$$

in $\mathcal{H}$. It is well known that under this assumption S admits selfadjoint extensions in $\mathcal{H}$. In the following let A be a fixed selfadjoint extension of S in $\mathcal{H}$, so that, $S \subset A = A^* \subset S^*$. Furthermore, let T be a linear operator in $\mathcal{H}$ such that $A \subset T \subset S^*$ and $\overline{T} = S^*$ holds, i.e., the domain $\operatorname{dom} T$ of T is a core of $\operatorname{dom} S^*$ (see [34]), $\operatorname{dom} T$ contains $\operatorname{dom} A$ and $Af = Tf$ holds for all $f \in \operatorname{dom} A$.

For $\lambda \in \mathbb{C}$ belonging to the resolvent set $\rho(A)$ of the selfadjoint operator A define the defect spaces $\mathcal{N}_\lambda(T) = \ker(T-\lambda)$ and $\mathcal{N}_\lambda(S^*) = \ker(S^*-\lambda)$. Then the decompositions

$$\operatorname{dom} S^* = \operatorname{dom} A \dotplus \mathcal{N}_\lambda(S^*) \quad \text{and} \quad \operatorname{dom} T = \operatorname{dom} A \dotplus \mathcal{N}_\lambda(T) \tag{5}$$

hold for all $\lambda \in \rho(A)$ and the closure $\overline{\mathcal{N}_\lambda(T)}$ of $\mathcal{N}_\lambda(T)$ in $\mathcal{H}$ coincides with $\mathcal{N}_\lambda(S^*)$. Recall that the symmetric operator S is said to be *simple* if there exists no nontrivial subspace $\mathcal{D}$ in $\operatorname{dom} S$ such that S restricted to $\mathcal{D}$ is a selfadjoint operator in the Hilbert space $\overline{\mathcal{D}}$. It is important to note that S is simple if and only if

$$\mathcal{H} = \overline{\operatorname{span}}\,\big\{\mathcal{N}_\lambda(S^*) : \lambda \in \mathbb{C}\backslash\mathbb{R}\big\} \tag{6}$$

holds, cf. [36]. Here $\overline{\operatorname{span}}$ denotes the closed linear span. As $\overline{\mathcal{N}_\lambda(T)} = \mathcal{N}_\lambda(S^*)$ it is clear that the right hand side in (6) coincides with

$$\overline{\operatorname{span}}\,\big\{\mathcal{N}_\lambda(T) : \lambda \in \mathbb{C}\backslash\mathbb{R}\big\}.$$

Fix some $\lambda_0 \in \rho(A)$, let $\mathcal{G}$ be a Hilbert space with the same dimension as $\mathcal{N}_{\lambda_0}(T)$ and let Γ_{λ_0} be a densely defined bounded operator from $\mathcal{G}$ into $\mathcal{H}$ such that $\operatorname{ran}\Gamma_{\lambda_0} = \mathcal{N}_{\lambda_0}(T)$ and $\ker\Gamma_{\lambda_0} = \{0\}$ holds. The domain $\operatorname{dom}\Gamma_{\lambda_0}$ of Γ_{λ_0} will be denoted by $\mathcal{G}_0$. Observe that the closure $\overline{\Gamma}_{\lambda_0}$ of the operator Γ_{λ_0} is the bounded extension of Γ_{λ_0} which is defined on $\overline{\mathcal{G}}_0 = \mathcal{G}$. We write $\overline{\Gamma}_{\lambda_0} \in \mathcal{L}(\mathcal{G},\mathcal{H})$, where $\mathcal{L}(\mathcal{G},\mathcal{H})$ is the space of bounded linear operators defined on $\mathcal{G}$ with values in $\mathcal{H}$.

Lemma 2.1. *The operator function* $\lambda \mapsto \Gamma(\lambda) := (I + (\lambda-\lambda_0)(A-\lambda)^{-1})\Gamma_{\lambda_0}$ *satisfies* $\Gamma(\lambda_0) = \Gamma_{\lambda_0}$,

$$\Gamma(\lambda) = \big(I + (\lambda-\mu)(A-\lambda)^{-1}\big)\Gamma(\mu), \qquad \lambda,\mu \in \rho(A),$$

and $\Gamma(\lambda)$ *is a bounded operator from* $\mathcal{G}$ *into* $\mathcal{H}$ *which maps* $\operatorname{dom}\Gamma(\lambda)=\mathcal{G}_0$ *bijectively onto* $\mathcal{N}_\lambda(T)$ *for all* $\lambda\in\rho(A)$. *Moreover,* $\lambda\mapsto\Gamma(\lambda)g$ *is holomorphic on* $\rho(A)$ *for every* $g\in\mathcal{G}_0$.

Proof. Let us show that $\operatorname{ran}\Gamma(\lambda)=\mathcal{N}_\lambda(T)$ is true. The other assertions in the lemma are obvious or follow from a straightforward calculation. Since T is an extension of A we have $(T-\lambda)(A-\lambda)^{-1}=I$ for $\lambda\in\rho(A)$ and therefore

$$(T-\lambda)\Gamma(\lambda)h=(T-\lambda)\big(I+(\lambda-\lambda_0)(A-\lambda)^{-1}\big)\Gamma_{\lambda_0}h=(T-\lambda_0)\Gamma_{\lambda_0}h=0$$

shows that $\operatorname{ran}\Gamma(\lambda)\subset\mathcal{N}_\lambda(T)$ holds. Now let $f_\lambda\in\mathcal{N}_\lambda(T)$. Then it follows as above that

$$f_{\lambda_0}:=\big(I+(\lambda_0-\lambda)(A-\lambda_0)^{-1}\big)f_\lambda$$

is an element in $\mathcal{N}_{\lambda_0}(T)$ and hence there exists $h\in\mathcal{G}_0$ such that $f_{\lambda_0}=\Gamma_{\lambda_0}h$. Now a simple calculation shows $f_\lambda=\Gamma(\lambda)h$, thus $\operatorname{ran}\Gamma(\lambda)=\mathcal{N}_\lambda(T)$. □

In the following definition the concept of generalized Q-functions is introduced.

Definition 2.2. Let S, A, T, and $\Gamma(\cdot)$ be as above. An operator function Q defined on $\rho(A)$ whose values $Q(\lambda)$ are linear operators in $\mathcal{G}$ with $\operatorname{dom}Q(\lambda)=\mathcal{G}_0$ for all $\lambda\in\rho(A)$ is said to be a *generalized Q-function* of the triple $\{S,A,T\}$ if

$$Q(\lambda)-Q(\mu)^*=(\lambda-\bar\mu)\Gamma(\mu)^*\Gamma(\lambda) \tag{7}$$

holds for all $\lambda,\mu\in\rho(A)$. If, in addition, $\mathcal{G}_0=\mathcal{G}$ and $T=S^*$, then Q is called an *ordinary Q-function* of $\{S,A\}$.

We note that the values $Q(\lambda)$, $\lambda\in\rho(A)$, of a generalized Q-function can be unbounded non-closed operators. The adjoint $Q(\mu)^*$ in (7) is well defined since $\operatorname{dom}Q(\mu)$ is dense in $\mathcal{G}$ and by setting $\lambda=\bar\mu$ in (7) it follows $Q(\mu)\subset Q(\bar\mu)^*$. Hence the identity (7) holds on $\mathcal{G}_0$, the operators $Q(\lambda)$ are closable in $\mathcal{G}$ and symmetric for $\lambda\in\rho(A)\cap\mathbb{R}$. The real and imaginary parts of the operators $Q(\lambda)$ are defined as usual:

$$\operatorname{Re}Q(\lambda)=\frac{1}{2}\big(Q(\lambda)+Q(\lambda)^*\big)\quad\text{and}\quad\operatorname{Im}Q(\lambda)=\frac{1}{2i}\big(Q(\lambda)-Q(\lambda)^*\big).$$

Since $(\operatorname{Re}Q(\lambda)h,h)$ and $(\operatorname{Im}Q(\lambda)h,h)$ are real for all $h\in\mathcal{G}_0$ the operators $\operatorname{Re}Q(\lambda)$ and $\operatorname{Im}Q(\lambda)$ are symmetric.

Remark 2.3. We note that the concept of generalized Q-functions is closely connected with the theory of boundary triplets and associated Weyl functions. The Weyl function of an ordinary or generalized boundary triplet (see [16, 18, 19, 29]) is also a generalized Q-function, but the converse is not true. The class of generalized Q-functions studied here coincides with the class of Weyl functions of so-called quasi boundary triplets introduced in [9]. Furthermore, we note that generalized Q-functions are no subclass of the Weyl families associated to boundary relations, see [17] and Theorem 2.6.

The concept of generalized Q-functions differs from the classical notion of ordinary Q-functions only in the case $n_\pm(S)=\infty$.

Proposition 2.4. *Let Q be a generalized Q-function of the triple $\{S,A,T\}$ and assume, in addition, that the deficiency indices $n_\pm(S)$ are finite. Then $T=S^*$ and Q is an ordinary Q-function of the pair $\{S,A\}$.*

Proof. If the deficiency indices of the closed operator S are finite, then T is a finite dimensional extension of S and hence also T is closed. Therefore $T = \overline{T} = S^*$. Moreover, in this case also $\dim \mathcal{G} = \dim \mathcal{N}_{\lambda_0}(T)$ is finite and hence $\mathcal{G}_0 = \operatorname{dom} \Gamma(\lambda) = \operatorname{dom} Q(\lambda) = \mathcal{G}$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$. □

The representation of a generalized Q-function with the help of the resolvent of A in the next proposition is formally the same as for ordinary Q-functions, see [37, 38, 39].

Proposition 2.5. *Let Q be a generalized Q-function of the triple $\{S, A, T\}$ and let $\lambda_0 \in \rho(A)$. Then Q can be written as the sum of the possibly unbounded operator $\operatorname{Re} Q(\lambda_0)$ and a bounded holomorphic operator function,*

$$Q(\lambda) = \operatorname{Re} Q(\lambda_0) + \Gamma_{\lambda_0}^* \big((\lambda - \operatorname{Re} \lambda_0) + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A - \lambda)^{-1} \big) \Gamma_{\lambda_0}, \tag{8}$$

and, in particular, any two generalized Q-functions of $\{S, A\}$ differ by a constant.

Proof. Let $h \in \mathcal{G}$ and set $\mu = \lambda_0$ in (7). Making use of the definition of $\Gamma(\lambda)$ in Lemma 2.1 we obtain

$$Q(\lambda)h = Q(\lambda_0)^* h + (\lambda - \bar{\lambda}_0)\Gamma_{\lambda_0}^* \big(I + (\lambda - \lambda_0)(A - \lambda)^{-1} \big) \Gamma_{\lambda_0} h.$$

As $Q(\lambda_0)h - Q(\lambda_0)^* h = (\lambda_0 - \bar{\lambda}_0)\Gamma_{\lambda_0}^* \Gamma_{\lambda_0} h$ we see that the above formula can be rewritten as

$$Q(\lambda)h = Q(\lambda_0)h + (\lambda - \lambda_0)\Gamma_{\lambda_0}^* \Gamma_{\lambda_0} h + \Gamma_{\lambda_0}^* (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A - \lambda)^{-1} \Gamma_{\lambda_0} h.$$

The representation (8) follows by inserting $Q(\lambda_0)h = \operatorname{Re} Q(\lambda_0)h + i \operatorname{Im} Q(\lambda_0)h$ and $\operatorname{Im} Q(\lambda_0)h = \operatorname{Im} \lambda_0 \Gamma_{\lambda_0}^* \Gamma_{\lambda_0} h$ into this expression. □

Generalized Q-functions are closely connected with the class of Nevanlinna functions, cf. Theorem 2.6 below. Let $\mathcal{L}(\mathcal{G})$ be the space of everywhere defined bounded linear operators in $\mathcal{G}$. Recall that an $\mathcal{L}(\mathcal{G})$-valued operator function $\widetilde{Q}$ which is holomorphic on $\mathbb{C}\backslash\mathbb{R}$ and satisfies

$$\frac{\operatorname{Im} \widetilde{Q}(\lambda)}{\operatorname{Im} \lambda} \geq 0 \qquad \text{and} \qquad \widetilde{Q}(\bar{\lambda}) = \widetilde{Q}(\lambda)^* \tag{9}$$

for $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is said to be an $\mathcal{L}(\mathcal{G})$-valued *Nevanlinna function*. We note that $\widetilde{Q}$ is an $\mathcal{L}(\mathcal{G})$-valued Nevanlinna function if and only if $\widetilde{Q}$ admits an integral representation of the form

$$\widetilde{Q}(\lambda) = \alpha + \lambda\beta + \int_{\mathbb{R}} \left(\frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\Sigma(t), \qquad \lambda \in \mathbb{C}\backslash\mathbb{R}, \tag{10}$$

where $\alpha = \alpha^* \in \mathcal{L}(\mathcal{G})$, $0 \leq \beta = \beta^* \in \mathcal{L}(\mathcal{G})$ and $t \mapsto \Sigma(t) \in \mathcal{L}(\mathcal{G})$ is a selfadjoint nondecreasing $\mathcal{L}(\mathcal{G})$-valued function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \frac{1}{1 + t^2} \, d\Sigma(t) \in \mathcal{L}(\mathcal{G}).$$

It is well known that Nevanlinna functions can be represented with the help of selfadjoint operators or relations in Hilbert spaces in a very similar form as in (8). Such operator and functional models for Nevanlinna functions can be found in, e.g., [1, 7, 12, 15, 19, 27, 33, 39, 41].

In the next theorem we characterize the class of generalized Q-functions. Roughly speaking, it turns out that up to a symmetric constant a generalized Q-function is a restrictions of an $\mathcal{L}(\mathcal{G})$-valued Nevanlinna function $\widetilde{Q}$ with invertible imaginary part on $\operatorname{dom} Q(\lambda)$ and $\widetilde{Q}$ satisfies certain limit properties at ∞.

Theorem 2.6. *Let $\mathcal{G}_0$ be a dense subspace of $\mathcal{G}$, $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$, and let Q be a function defined on $\mathbb{C}\backslash\mathbb{R}$ whose values $Q(\lambda)$ are linear operators in $\mathcal{G}$ with $\operatorname{dom} Q(\lambda) = \mathcal{G}_0$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Then the following is equivalent:*

(i) *Q is a generalized Q-function of a triple $\{S, A, T\}$, where S is a simple symmetric operator in some separable Hilbert space $\mathcal{H}$, A is a selfadjoint extension of S in $\mathcal{H}$ and $A \subset T \subset S^*$ with $\overline{T} = S^*$;*

(ii) *There exists an unique $\mathcal{L}(\mathcal{G})$-valued Nevanlinna function $\widetilde{Q}$ with the properties (α), (β) and (γ):*

(α) *The relations*
$$Q(\lambda)h - \operatorname{Re} Q(\lambda_0)h = \widetilde{Q}(\lambda)h$$
and
$$Q(\lambda)^*h - \operatorname{Re} Q(\lambda_0)h = \widetilde{Q}(\lambda)^*h$$
hold for all $h \in \mathcal{G}_0$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$;

(β) $\operatorname{Im} \widetilde{Q}(\lambda)h = 0$ *for some $h \in \mathcal{G}_0$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$ implies $h = 0$;*

(γ) *The conditions*
$$\lim_{\eta\to+\infty} \frac{1}{\eta}(\widetilde{Q}(i\eta)k, k) = 0 \quad \text{and} \quad \lim_{\eta\to+\infty} \eta \operatorname{Im}(\widetilde{Q}(i\eta)k, k) = \infty$$
are valid for all $k \in \mathcal{G}$, $k \neq 0$.

Proof. We start by showing that (i) implies (ii). For this, let Q be a generalized Q-function of the triple $\{S, A, T\}$ and suppose that S is simple. Let Γ_{λ_0} be a bounded operator defined on $\operatorname{dom} Q(\lambda) = \mathcal{G}_0$ such that $\operatorname{ran} \Gamma_{\lambda_0} = \mathcal{N}_{\lambda_0}(T)$ and $\ker \Gamma_{\lambda_0} = \{0\}$. According to Proposition 2.5 for each $\lambda \in \mathbb{C}\backslash\mathbb{R}$
$$Q(\lambda) - \operatorname{Re} Q(\lambda_0) = \Gamma_{\lambda_0}^*\big((\lambda - \operatorname{Re}\lambda_0) + (\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)(A-\lambda)^{-1}\big)\Gamma_{\lambda_0}$$
is a bounded operator in $\mathcal{G}$ defined on the dense subspace $\mathcal{G}_0$ and hence admits a unique bounded extension onto $\mathcal{G}$ which is given by
$$\widetilde{Q}(\lambda) := \Gamma_{\lambda_0}^*\big((\lambda - \operatorname{Re}\lambda_0) + (\lambda-\lambda_0)(\lambda-\bar{\lambda}_0)(A-\lambda)^{-1}\big)\overline{\Gamma}_{\lambda_0}, \tag{11}$$
where $\overline{\Gamma}_{\lambda_0} \in \mathcal{L}(\mathcal{G}, \mathcal{H})$ is the closure of Γ_{λ_0}. Obviously we have
$$Q(\lambda)h - \operatorname{Re} Q(\lambda_0)h = \widetilde{Q}(\lambda)h$$
for all $h \in \mathcal{G}_0$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$, which is the first relation in (α). Recall that for a generalized Q-function $Q(\bar{\lambda})^*$ is an extension of $Q(\lambda)$. This implies $\operatorname{Re} Q(\lambda_0) \subset (\operatorname{Re} Q(\lambda_0))^*$,
$$Q(\lambda)^* - \operatorname{Re} Q(\lambda_0) \subset \big(Q(\lambda) - \operatorname{Re} Q(\lambda_0)\big)^* = \widetilde{Q}(\lambda)^*$$
and therefore also $Q(\lambda)^*h - \operatorname{Re} Q(\lambda_0)h = \widetilde{Q}(\lambda)^*h$ is true for all $h \in \mathcal{G}_0$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Hence we have shown (α).

Clearly $\widetilde{Q}$ in (11) is a holomorphic $\mathcal{L}(\mathcal{G})$-valued function on $\mathbb{C}\backslash\mathbb{R}$. Denote by $\overline{\Gamma(\lambda)}$ the closure of $\Gamma(\lambda) = (I + (\lambda-\lambda_0)(A-\lambda)^{-1})\Gamma_{\lambda_0}$. Then
$$\overline{\Gamma(\lambda)} = \big(I + (\lambda-\lambda_0)(A-\lambda)^{-1}\big)\overline{\Gamma}_{\lambda_0}, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

and it is not difficult to see that (7) extends to

$$\widetilde{Q}(\lambda) - \widetilde{Q}(\mu)^* = (\lambda - \bar{\mu})\Gamma(\mu)^*\overline{\Gamma(\lambda)}.$$

Hence

$$\big(\operatorname{Im}\widetilde{Q}(\lambda)k, k\big) = (\operatorname{Im}\lambda)\big(\Gamma(\lambda)^*\overline{\Gamma(\lambda)}k, k\big) = (\operatorname{Im}\lambda)\|\overline{\Gamma(\lambda)}k\|^2$$

holds for all $k \in \mathcal{G}$ and this implies that $\widetilde{Q}$ is a Nevanlinna function, cf. (9). Furthermore, for $h \in \mathcal{G}_0$ we have

$$\operatorname{Im}\widetilde{Q}(\lambda)h = (\operatorname{Im}\lambda)\Gamma(\lambda)^*\Gamma(\lambda)h$$

and from the property $\ker\Gamma(\lambda) = \{0\}$, cf. Lemma 2.1, we conclude that $\operatorname{Im}\widetilde{Q}(\lambda)h = 0$ for $h \in \mathcal{G}_0$ implies $h = 0$, i.e., condition (β) holds. The same arguments as in [39, Theorem 2.4, Corollaries 2.5 and 2.6] together with the assumption that S is a densely defined closed simple symmetric operator show that $\widetilde{Q}$ satisfies the conditions in (γ).

Let us now verify the converse direction. If $\widetilde{Q}$ is a $\mathcal{L}(\mathcal{G})$-valued Nevanlinna function, $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$ and the first condition in (γ) holds, then it is well known that there exists a Hilbert space $\mathcal{H}$, a selfadjoint operator A in $\mathcal{H}$ and a mapping $\widetilde{\Gamma} \in \mathcal{L}(\mathcal{G},\mathcal{H})$ such that the representation

$$\widetilde{Q}(\lambda) = \operatorname{Re}\widetilde{Q}(\lambda_0) + \widetilde{\Gamma}^*\big((\lambda - \operatorname{Re}\lambda_0) + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A-\lambda)^{-1}\big)\widetilde{\Gamma} \tag{12}$$

is valid for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$, see, e.g., [33, 39]. Furthermore, the space $\mathcal{H}$ can be chosen minimal, i.e.,

$$\mathcal{H} = \overline{\operatorname{span}}\big\{\big(I + (\lambda-\lambda_0)(A-\lambda)^{-1}\big)\widetilde{\Gamma}k : k \in \mathcal{G},\, \lambda \in \mathbb{C}\backslash\mathbb{R}\big\}. \tag{13}$$

We define the mapping Γ_{λ_0} to be the restriction of $\widetilde{\Gamma}$ onto $\mathcal{G}_0$. As $\widetilde{\Gamma}$ is bounded the closure $\overline{\Gamma}_{\lambda_0}$ of Γ_{λ_0} coincides with $\widetilde{\Gamma}$. We claim that Γ_{λ_0} is injective. In fact, if $\Gamma_{\lambda_0}h = 0$ for some $h \in \mathcal{G}_0$ then $\widetilde{\Gamma}h = 0$ and by (12) we have $\widetilde{Q}(\lambda)h = \operatorname{Re}\widetilde{Q}(\lambda_0)h$. Therefore $\operatorname{Im}\widetilde{Q}(\lambda)h = 0$ and by assumption (β) this implies $h = 0$.

Define the operator S by

$$Sf = Af, \quad \operatorname{dom} S = \big\{f \in \operatorname{dom} A : ((A-\bar{\lambda}_0)f, \Gamma_{\lambda_0}h) = 0 \text{ for all } h \in \mathcal{G}_0\big\}.$$

Then S is a closed symmetric operator and the identities $\operatorname{ran}(S-\bar{\lambda}_0) = (\operatorname{ran}\Gamma_{\lambda_0})^\perp$ and $\ker(S^*-\lambda_0) = \overline{\operatorname{ran}\Gamma_{\lambda_0}}$ hold. Let

$$\Gamma(\lambda) = (I + (\lambda-\lambda_0)(A-\lambda)^{-1})\Gamma_{\lambda_0}, \qquad \lambda \in \mathbb{C}\backslash\mathbb{R}. \tag{14}$$

It is not difficult to check that $\operatorname{ran}(S-\bar{\lambda}) = (\operatorname{ran}\Gamma(\lambda))^\perp$ is true for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and the conditions in (γ) together with (13) now yield in the same way as in [39, Theorem 2.4, Corollaries 2.5 and 2.6] that S is densely defined and simple.

Note that $\operatorname{dom} A \cap \operatorname{ran}\Gamma_{\lambda_0} = \{0\}$ since $\lambda_0 \in \rho(A)$ and $\operatorname{ran}\Gamma_{\lambda_0} \subset \mathcal{N}_{\lambda_0}(S^*)$. Let us define a linear operator T in $\mathcal{H}$ on $\operatorname{dom} T := \operatorname{dom} A \dotplus \operatorname{ran}\Gamma_{\lambda_0}$ by

$$T(f + f_{\lambda_0}) := Af + \lambda_0 f_{\lambda_0}, \qquad f \in \operatorname{dom} A,\ f_{\lambda_0} \in \operatorname{ran}\Gamma_{\lambda_0}.$$

Obviously T is an extension of A and since $\mathcal{N}_{\lambda_0}(T) = \operatorname{ran}\Gamma_{\lambda_0}$ and $\operatorname{ran}\Gamma_{\lambda_0}$ is dense in $\mathcal{N}_{\lambda_0}(S^*)$ we obtain from $\operatorname{dom} S^* = \operatorname{dom} A \dotplus \mathcal{N}_{\lambda_0}(S^*)$, cf. (5), that $T \subset S^*$ and $\overline{T} = S^*$ holds.

According to condition (α) the Nevanlinna function $\widetilde{Q}$ and the function Q are related by

$$Q(\lambda)h = \widetilde{Q}(\lambda)h + \operatorname{Re} Q(\lambda_0)h \quad \text{and} \quad Q(\lambda)^*h = \widetilde{Q}(\lambda)^*h + \operatorname{Re} Q(\lambda_0)h$$

for all $h \in \mathcal{G}_0$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$. It remains to show that Q satisfies (7). Observe first that for $\lambda, \mu \in \mathbb{C}\backslash\mathbb{R}$ we have

$$Q(\lambda)h - Q(\mu)^* h = \widetilde{Q}(\lambda)h - \widetilde{Q}(\mu)^* h. \tag{15}$$

Denote the closures of the operators $\Gamma(\lambda)$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, in (14) by $\widetilde{\Gamma}(\lambda)$. Then

$$\widetilde{\Gamma}(\lambda) = \overline{\Gamma(\lambda)} = \bigl(I + (\lambda - \lambda_0)(A-\lambda)^{-1}\bigr)\overline{\Gamma}_{\lambda_0} = \bigl(I + (\lambda - \lambda_0)(A-\lambda)^{-1}\bigr)\widetilde{\Gamma}$$

and it follows from (12) with a straightforward calculation that

$$\widetilde{Q}(\lambda) - \widetilde{Q}(\mu)^* = (\lambda - \bar{\mu})\widetilde{\Gamma}(\mu)^*\widetilde{\Gamma}(\lambda), \qquad \lambda, \mu \in \mathbb{C}\backslash\mathbb{R}, \tag{16}$$

holds. As $\widetilde{\Gamma}(\mu)^* = \overline{\Gamma(\mu)}^{\,*} = \Gamma(\mu)^*$ we conclude

$$Q(\lambda)h - Q(\mu)^* h = (\lambda - \bar{\mu})\Gamma(\mu)^*\Gamma(\lambda)h, \qquad h \in \mathcal{G}_0,$$

from (15). Therefore Q is a generalized Q-function of the triple $\{S, A, T\}$. □

Remark 2.7. The definition of a generalized Q-function can be extended to the case that A is a selfadjoint relation, S is a non-densely defined symmetric operator or relation and T is a linear relation which is dense in the relation S^*. We refer to [39] for ordinary Q-functions in this more general situation. In this case the condition (γ) in Theorem 2.6 can be dropped.

For ordinary Q-functions Theorem 2.6 reads as follows, cf. [39, Theorem 2.2 and Theorem 2.4].

Theorem 2.8. *A $\mathcal{L}(\mathcal{G})$-valued Nevanlinna function $\widetilde{Q}$ is an ordinary Q-function of some pair $\{S, A\}$, where S is a densely defined closed simple symmetric operator in some Hilbert space $\mathcal{H}$ and A is a selfadjoint extension of S in $\mathcal{H}$, if and only if condition (γ) in Theorem 2.6 and $0 \in \rho(\operatorname{Im}\widetilde{Q}(\lambda))$ holds for some, and hence for all, $\lambda \in \mathbb{C}\backslash\mathbb{R}$.*

Corollary 2.9. *Let Q be a generalized Q-function of $\{S, A, T\}$ and let $\widetilde{Q}$ be the $\mathcal{L}(\mathcal{G})$-valued Nevanlinna function in Theorem 2.6. Then for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and $h \in \mathcal{G}_0$ we have*

$$\frac{d}{d\lambda}\, Q(\lambda)h = \frac{d}{d\lambda}\, \widetilde{Q}(\lambda)h = \Gamma(\bar{\lambda})^*\Gamma(\lambda)h.$$

Proof. It follows from (16) that

$$\frac{d}{d\lambda}\, \widetilde{Q}(\lambda) = \lim_{\bar{\mu}\to\lambda} \frac{\widetilde{Q}(\lambda) - \widetilde{Q}(\mu)^*}{\lambda - \bar{\mu}} = \widetilde{\Gamma}(\bar{\lambda})^*\widetilde{\Gamma}(\lambda)$$

holds. Hence condition (α) in Theorem 2.6 and $\widetilde{\Gamma}(\lambda) = \overline{\Gamma(\lambda)}$ imply

$$\frac{d}{d\lambda}\, Q(\lambda)h = \lim_{\bar{\mu}\to\lambda} \frac{Q(\lambda)h - Q(\mu)^* h}{\lambda - \bar{\mu}} = \lim_{\bar{\mu}\to\lambda} \frac{\widetilde{Q}(\lambda)h - \widetilde{Q}(\mu)^* h}{\lambda - \bar{\mu}} = \Gamma(\bar{\lambda})^*\Gamma(\lambda)h$$

for $h \in \mathcal{G}_0$. □

3. Elliptic operators and the Dirichlet-to-Neumann map

Let $\Omega \subset \mathbb{R}^n$ be a bounded or unbounded domain with compact C^∞-boundary $\partial\Omega$. Let $\mathcal{L}$ be the "formally selfadjoint" uniformly elliptic second order differential expression

$$(\mathcal{L}f)(x) := -\sum_{j,k=1}^{n} \left(\frac{\partial}{\partial x_j} a_{jk} \frac{\partial f}{\partial x_k} \right)(x) + a(x)f(x), \tag{17}$$

$x \in \Omega$, with bounded infinitely differentiable coefficients $a_{jk} \in C^\infty(\overline{\Omega})$ satisfying $a_{jk}(x) = \overline{a_{kj}(x)}$ for all $x \in \overline{\Omega}$ and $j,k = 1,\dots,n$, the function $a \in L^\infty(\Omega)$ is real valued and

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq C \sum_{k=1}^{n} \xi_k^2 \tag{18}$$

holds for some $C > 0$, all $\xi = (\xi_1,\dots,\xi_n)^\top \in \mathbb{R}^n$ and $x \in \overline{\Omega}$. We note that the assumptions on the domain Ω and the coefficients of $\mathcal{L}$ can be relaxed but it is not our aim to treat the most general setting here. We refer the reader to e.g. [30, 40, 43, 51] for possible generalizations.

In the following we consider the selfadjoint realizations of $\mathcal{L}$ in $L^2(\Omega)$ subject to Dirichlet and Neumann boundary conditions. For a function f in the Sobolev space $H^2(\Omega)$ we denote the trace by $f|_{\partial\Omega}$ and the trace of the conormal derivative is defined by

$$\frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} := \sum_{j,k=1}^{n} a_{jk} n_j \frac{\partial f}{\partial x_k}\Big|_{\partial\Omega};$$

here $n(x) = (n_1(x),\dots,n_n(x))^\top$ is the unit vector at the point $x \in \partial\Omega$ pointing out of Ω. Recall that the mapping $C^\infty(\overline{\Omega}) \ni f \mapsto \left\{ f|_{\partial\Omega}, \frac{\partial f}{\partial \nu}\big|_{\partial\Omega} \right\}$ extends by continuity to a continuous surjective mapping

$$H^2(\Omega) \ni f \mapsto \left\{ f|_{\partial\Omega}, \frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} \right\} \in H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega). \tag{19}$$

The kernel of this map is

$$H_0^2(\Omega) = \left\{ f \in H^2(\Omega) : f|_{\partial\Omega} = \frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} = 0 \right\}$$

which coincides with the closure of $C_0^\infty(\Omega)$ in $H^2(\Omega)$. We refer the reader to the monographs [40, 43, 51] for more details. In the following the scalar products in $L^2(\Omega)$ and $L^2(\partial\Omega)$ are denoted by $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{\partial\Omega}$, respectively. Then Green's identity

$$(\mathcal{L}f, g)_\Omega - (f, \mathcal{L}g)_\Omega = \left(f|_{\partial\Omega}, \frac{\partial g}{\partial \nu}\Big|_{\partial\Omega} \right)_{\partial\Omega} - \left(\frac{\partial f}{\partial \nu}\Big|_{\partial\Omega}, g|_{\partial\Omega} \right)_{\partial\Omega} \tag{20}$$

holds for all functions $f, g \in H^2(\Omega)$. We note that (20) is even true for $f \in H^2(\Omega)$ and g belonging to the domain of the maximal operator associated to $\mathcal{L}$ in $L^2(\Omega)$ if the $(\cdot,\cdot)_{\partial\Omega}$ scalar product in $L^2(\partial\Omega)$ is extended by continuity to $H^{3/2}(\partial\Omega) \times H^{-3/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, respectively, see [40, 51]. However, we shall make use of (20) only for the case $f, g \in H^2(\Omega)$.

It is well known that the realizations A_D and A_N of $\mathcal{L}$ subject to Dirichlet and Neumann boundary conditions defined by

$$
\begin{aligned}
A_D f &= \mathcal{L}f, \quad \operatorname{dom} A_D = \bigl\{ f \in H^2(\Omega) : f|_{\partial\Omega} = 0 \bigr\}, \\
A_N f &= \mathcal{L}f, \quad \operatorname{dom} A_N = \Bigl\{ f \in H^2(\Omega) : \frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} = 0 \Bigr\},
\end{aligned}
\tag{21}
$$

are selfadjoint operators in $L^2(\Omega)$. The following statement is known and can be found in, e.g., [40]. It can be proved with similar methods as Theorem 4.1 in the next section.

Proposition 3.1. *Let $\mathcal{L}$ be the elliptic differential expression in* (17). *Then the operator*

$$
Sf = \mathcal{L}f, \qquad \operatorname{dom} S = H_0^2(\Omega),
\tag{22}
$$

is a densely defined closed symmetric operator in $L^2(\Omega)$ with infinite deficiency indices $n_\pm(S)$ and the adjoint S^* of S coincides with the maximal operator associated to $\mathcal{L}$,

$$
S^* f = \mathcal{L}f, \qquad \operatorname{dom} S^* = \bigl\{ f \in L^2(\Omega) : \mathcal{L}f \in L^2(\Omega) \bigr\}.
$$

The operator

$$
Tf = \mathcal{L}f, \qquad \operatorname{dom} T = H^2(\Omega),
$$

is not closed as an operator in $L^2(\Omega)$ and T satisfies $\overline{T} = S^*$ and $T^* = S$. Furthermore, the selfadjoint operators A_D and A_N in (21) *are extensions of S and restrictions of T.*

In order to define a mapping Γ_{λ_0} for the definition of a generalized Q-function associated to the triple $\{S, A_D, T\}$ we make use of the decomposition (5) in the present situation. More precisely, for all points λ in the resolvent set $\rho(A_D)$ of the selfadjoint Dirichlet operator A_D we have the direct sum decomposition of $\operatorname{dom} T = H^2(\Omega)$:

$$
H^2(\Omega) = \operatorname{dom} A_D \dotplus \mathcal{N}_\lambda(T) = \bigl\{ f \in H^2(\Omega) : f|_{\partial\Omega} = 0 \bigr\} \dotplus \mathcal{N}_\lambda(T),
\tag{23}
$$

where

$$
\mathcal{N}_\lambda(T) = \ker(T - \lambda) = \bigl\{ f_\lambda \in H^2(\Omega) : \mathcal{L}f_\lambda = \lambda f_\lambda \bigr\}.
$$

Let now φ be a function in $H^{3/2}(\partial\Omega)$ and let $\lambda_0 \in \rho(A_D)$. Then it follows from (19) and (23) that there exists a unique function $f_{\lambda_0} \in H^2(\Omega)$ which solves the equation $\mathcal{L}f_{\lambda_0} = \lambda_0 f_{\lambda_0}$, i.e., $f_{\lambda_0} \in \mathcal{N}_{\lambda_0}(T)$, and satisfies $f_{\lambda_0}|_{\partial\Omega} = \varphi$. We shall denote the mapping that assigns f_{λ_0} to φ by Γ_{λ_0},

$$
H^{3/2}(\partial\Omega) \ni \varphi \mapsto \Gamma_{\lambda_0}\varphi := f_{\lambda_0} \in \mathcal{N}_{\lambda_0}(T),
\tag{24}
$$

and we regard Γ_{λ_0} as an operator from $L^2(\partial\Omega)$ into $L^2(\Omega)$ with $\operatorname{dom}\Gamma_{\lambda_0} = H^{3/2}(\partial\Omega)$ and $\operatorname{ran}\Gamma_{\lambda_0} = \mathcal{N}_{\lambda_0}(T)$.

Proposition 3.2. *Let $\lambda_0 \in \rho(A_D)$, let Γ_{λ_0} be as in* (24) *and let $\lambda \in \rho(A_D)$. Then the following holds:*

(i) *Γ_{λ_0} is a bounded operator from $L^2(\partial\Omega)$ in $L^2(\Omega)$ with dense domain $H^{3/2}(\partial\Omega)$;*

(ii) *The operator $\Gamma(\lambda) = (I + (\lambda - \lambda_0)(A_D - \lambda)^{-1})\Gamma_{\lambda_0}$ is given by*

$$
\Gamma(\lambda)\varphi = f_\lambda, \quad \textit{where} \quad f_\lambda \in \mathcal{N}_\lambda(T) \ \textit{ and } \ f_\lambda|_{\partial\Omega} = \varphi;
$$

(iii) *The mapping* $\Gamma(\bar{\lambda})^* : L^2(\Omega) \to L^2(\partial\Omega)$ *satisfies*

$$\Gamma(\bar{\lambda})^*(A_D - \lambda)f = -\frac{\partial f}{\partial \nu}\Big|_{\partial\Omega}, \qquad f \in \operatorname{dom} A_D.$$

Proof. Statement (i) will be a consequence of (iii). We prove assertion (ii). Recall that by Lemma 2.1 the range of the operator $\Gamma(\lambda)$, $\lambda \in \rho(A_D)$, is $\mathcal{N}_\lambda(T)$. Let $\varphi \in \operatorname{dom}\Gamma(\lambda) = H^{3/2}(\partial\Omega)$ and choose elements $f_\lambda \in \mathcal{N}_\lambda(T)$ and $f_{\lambda_0} \in \mathcal{N}_{\lambda_0}(T)$ such that

$$f_\lambda|_{\partial\Omega} = \varphi = f_{\lambda_0}|_{\partial\Omega}$$

holds. According to (23) the functions f_λ and f_{λ_0} are unique. Then $\Gamma_{\lambda_0}\varphi = f_{\lambda_0}$ and hence we obtain

$$\Gamma(\lambda)\varphi = \Gamma_{\lambda_0}\varphi + (\lambda - \lambda_0)(A_D - \lambda)^{-1}\Gamma_{\lambda_0}\varphi = f_{\lambda_0} + (\lambda - \lambda_0)(A_D - \lambda)^{-1}\Gamma_{\lambda_0}\varphi.$$

Since $(\lambda - \lambda_0)(A_D - \lambda)^{-1}\Gamma_{\lambda_0}\varphi$ belongs to $\operatorname{dom} A_D$ it is clear that the trace of this element vanishes. Therefore, the traces of the functions $\Gamma(\lambda)\varphi \in \mathcal{N}_\lambda(T)$ and f_{λ_0} coincide,

$$(\Gamma(\lambda)\varphi)|_{\partial\Omega} = f_{\lambda_0}|_{\partial\Omega} = \varphi = f_\lambda|_{\partial\Omega}.$$

Thus we have that the traces of $\Gamma(\lambda)\varphi \in \mathcal{N}_\lambda(T)$ and $f_\lambda \in \mathcal{N}_\lambda(T)$ coincide and from (23) we conclude $\Gamma(\lambda)\varphi = f_\lambda$.

(iii) Let $\varphi \in H^{3/2}(\partial\Omega)$ and choose the unique function $g_{\bar{\lambda}} \in \mathcal{N}_{\bar{\lambda}}(T)$ with the property $g_{\bar{\lambda}}|_{\partial\Omega} = \varphi$. Hence we have $\Gamma(\bar{\lambda})\varphi = g_{\bar{\lambda}}$ and for $f \in \operatorname{dom} A_D$ it follows

$$\big(\Gamma(\bar{\lambda})\varphi, (A_D - \lambda)f\big)_\Omega = (g_{\bar{\lambda}}, A_D f)_\Omega - (\bar{\lambda} g_{\bar{\lambda}}, f)_\Omega = (g_{\bar{\lambda}}, A_D f)_\Omega - (T g_{\bar{\lambda}}, f)_\Omega.$$

Making use of Green's identity (20) we find

$$(g_{\bar{\lambda}}, A_D f)_\Omega - (T g_{\bar{\lambda}}, f)_\Omega = \left(\frac{\partial g_{\bar{\lambda}}}{\partial \nu}\Big|_{\partial\Omega}, f|_{\partial\Omega}\right)_{\partial\Omega} - \left(g_{\bar{\lambda}}|_{\partial\Omega}, \frac{\partial f}{\partial \nu}\Big|_{\partial\Omega}\right)_{\partial\Omega}$$

and since the trace of $f \in \operatorname{dom} A_D$ vanishes the first summand on the right hand side is zero. Therefore

$$\big(\Gamma(\bar{\lambda})\varphi, (A_D - \lambda)f\big)_\Omega = -\left(g_{\bar{\lambda}}|_{\partial\Omega}, \frac{\partial f}{\partial \nu}\Big|_{\partial\Omega}\right)_{\partial\Omega} = \left(\varphi, -\frac{\partial f}{\partial \nu}\Big|_{\partial\Omega}\right)_{\partial\Omega}$$

holds for all $\varphi \in \operatorname{dom}\Gamma(\bar{\lambda}) = H^{3/2}(\partial\Omega)$. This gives $(A_D - \lambda)f \in \operatorname{dom}\Gamma(\bar{\lambda})^*$ and

$$\Gamma(\bar{\lambda})^*(A_D - \lambda)f = -\frac{\partial f}{\partial \nu}\Big|_{\partial\Omega}.$$

Moreover, as $\lambda \in \rho(A_D)$ and $f \in \operatorname{dom} A_D$ was arbitrary we see that $\Gamma(\bar{\lambda})^*$ is defined on the whole space $L^2(\Omega)$. This together with the fact that $\Gamma(\bar{\lambda})^*$ is closed implies

$$\Gamma(\bar{\lambda})^* \in \mathcal{L}\big(L^2(\Omega), L^2(\partial\Omega)\big)$$

for $\lambda \in \rho(A_D)$ and, in particular, $\Gamma(\bar{\lambda}) \subset \overline{\Gamma(\bar{\lambda})} = \Gamma(\bar{\lambda})^{**}$ is bounded. Inserting $\lambda_0 = \bar{\lambda}$ this yields assertion (i). □

In the study of elliptic differential operators the so-called Dirichlet-to-Neumann map plays an important role, we mention only [4, 14, 22, 23, 24, 25, 26, 31, 42, 44, 45, 46, 47, 48, 49, 50]. Roughly speaking this operator maps the Dirichlet boundary value $f_\lambda|_{\partial\Omega}$ of an $H^2(\Omega)$-solution of the equation $\mathcal{L}u = \lambda u$ onto the Neumann boundary value $\frac{\partial f_\lambda}{\partial \nu}|_{\partial\Omega}$ of this solution. In the following definition also a minus sign arises, which is needed to obtain a generalized Q-function in Theorem 3.4. Otherwise $-Q$ would turn out to be a generalized Q-function.

Definition 3.3. Let $\lambda \in \rho(A_D)$ and assign to $\varphi \in H^{3/2}(\partial\Omega)$ the unique function $f_\lambda \in \mathcal{N}_\lambda(T)$ such that $f_\lambda|_{\partial\Omega} = \varphi$, see (19) and (23). The operator $Q(\lambda)$ in $L^2(\partial\Omega)$ defined by

$$Q(\lambda)\varphi = Q(\lambda)(f_\lambda|_{\partial\Omega}) := -\frac{\partial f_\lambda}{\partial \nu}\Big|_{\partial\Omega}, \qquad \varphi \in \operatorname{dom} Q(\lambda) = H^{3/2}(\partial\Omega), \tag{25}$$

is called the *Dirichlet-to-Neumann map* associated to $\mathcal{L}$.

Note that by (19) the range of the Dirichlet-to-Neumann map $Q(\lambda)$, $\lambda \in \rho(A_D)$, lies in $H^{1/2}(\partial\Omega)$. We remark that the Dirichlet-to-Neumann map can be extended, e.g., to an operator from $H^1(\partial\Omega)$ in $L^2(\partial\Omega)$ if instead of $H^2(\Omega)$ the operator T is defined on a suitable subspace of $H^{3/2}(\Omega)$, cf. [4, 5, 6, 9, 32, 40]. However, for our purposes this is not necessary since A_D and A_N are defined on subspaces of $H^2(\Omega)$.

In the next theorem we show that the Dirichlet-to-Neumann map is a generalized Q-function and we illustrate the usefulness of this object in the representation of the difference of the resolvents of the Dirichlet and Neumann operators A_D and A_N in (21). Similar Krein type resolvent formulas can also be found in [9, 13, 25, 26, 47, 48, 49]. The fact that the difference of the resolvents belongs to some von Neumann-Schatten class depending on the dimension of the space is well-known and goes back to M.S. Birman, cf. [11].

Theorem 3.4. *Let $\mathcal{L}$ be the elliptic differential expression in* (17) *and let A_D and A_N be the selfadjoint realizations of $\mathcal{L}$ in* (21). *Denote by S the minimal operator associated to $\mathcal{L}$ and let $T = \mathcal{L} \upharpoonright H^2(\Omega)$ be as in Proposition 3.1. Define $\Gamma(\lambda)$ as in Proposition 3.2 and let $Q(\lambda)$, $\lambda \in \rho(A_D)$, be the Dirichlet-to-Neumann map. Then the following holds:*

(i) *Q is a generalized Q-function of the triple $\{S, A_D, T\}$;*
(ii) *The operator $Q(\lambda)$ is injective for all $\lambda \in \rho(A_D) \cap \rho(A_N)$ and the resolvent formula*

$$(A_D - \lambda)^{-1} - (A_N - \lambda)^{-1} = \Gamma(\lambda) Q(\lambda)^{-1} \Gamma(\bar{\lambda})^* \tag{26}$$

holds;
(iii) *For $p \in \mathbb{N}$ and $2p+1 > n$ the difference of the resolvents in* (26) *belongs to the von Neumann-Schatten class $\mathfrak{S}_p(L^2(\Omega))$.*

Proof. In order to proof assertion (i) we have to check the relation

$$Q(\lambda) - Q(\mu)^* = (\lambda - \bar{\mu})\Gamma(\mu)^*\Gamma(\lambda), \qquad \lambda, \mu \in \rho(A_D), \tag{27}$$

on $\operatorname{dom} Q(\lambda) \cap \operatorname{dom} Q(\mu)^*$. For this it will be first shown that $\operatorname{dom} Q(\lambda) = H^{3/2}(\partial\Omega)$ is a subset of $\operatorname{dom} Q(\mu)^*$ and that $Q(\mu)^*$ is an extension of $Q(\bar{\mu})$. Let $\psi \in H^{3/2}(\partial\Omega)$ and choose the unique function $f_{\bar{\mu}} \in \mathcal{N}_{\bar{\mu}}(T)$ such that $f_{\bar{\mu}}|_{\partial\Omega} = \psi$. For an arbitrary $\varphi \in \operatorname{dom} Q(\mu) = H^{3/2}(\partial\Omega)$ let $f_\mu \in \mathcal{N}_\mu(T)$ be the unique function that satisfies $f_\mu|_{\partial\Omega} = \varphi$. By the definition of the Dirichlet-to-Neumann map we have

$$Q(\mu)\varphi = -\frac{\partial f_\mu}{\partial \nu}\Big|_{\partial\Omega} \quad \text{and} \quad Q(\bar{\mu})\psi = -\frac{\partial f_{\bar{\mu}}}{\partial \nu}\Big|_{\partial\Omega}$$

and hence Green's identity (20) shows

$$\begin{aligned}(Q(\mu)\varphi,\psi)_{\partial\Omega} &= \left(-\frac{\partial f_\mu}{\partial\nu}\Big|_{\partial\Omega}, f_{\bar\mu}|_{\partial\Omega}\right)_{\partial\Omega} \\ &= \left(f_\mu|_{\partial\Omega}, \frac{\partial f_{\bar\mu}}{\partial\nu}\Big|_{\partial\Omega}\right)_{\partial\Omega} - \left(\frac{\partial f_\mu}{\partial\nu}\Big|_{\partial\Omega}, f_{\bar\mu}|_{\partial\Omega}\right)_{\partial\Omega} + \left(\varphi, -\frac{\partial f_{\bar\mu}}{\partial\nu}\Big|_{\partial\Omega}\right)_{\partial\Omega} \\ &= (Tf_\mu, f_{\bar\mu})_\Omega - (f_\mu, Tf_{\bar\mu})_\Omega + \left(\varphi, -\frac{\partial f_{\bar\mu}}{\partial\nu}\Big|_{\partial\Omega}\right)_{\partial\Omega}.\end{aligned}$$

Since $f_\mu \in \mathcal{N}_\mu(T)$ and $f_{\bar\mu} \in \mathcal{N}_{\bar\mu}(T)$ it is clear that $(Tf_\mu, f_{\bar\mu})_\Omega = (f_\mu, Tf_{\bar\mu})_\Omega$ holds and therefore we obtain

$$(Q(\mu)\varphi,\psi)_{\partial\Omega} = \left(\varphi, -\frac{\partial f_{\bar\mu}}{\partial\nu}\Big|_{\partial\Omega}\right)_{\partial\Omega}$$

for all $\varphi \in \operatorname{dom} Q(\mu)$. Thus $\psi \in \operatorname{dom} Q(\mu)^*$ and

$$Q(\mu)^*\psi = -\frac{\partial f_{\bar\mu}}{\partial\nu}\Big|_{\partial\Omega} = Q(\bar\mu)\psi.$$

Next we prove the relation (27). Let $\varphi, \psi \in H^{3/2}(\partial\Omega)$ and choose the functions $f_\lambda \in \mathcal{N}_\lambda(T)$ and $g_\mu \in \mathcal{N}_\mu(T)$ such that $f_\lambda|_{\partial\Omega} = \varphi$ and $g_\mu|_{\partial\Omega} = \psi$. Hence we have

$$Q(\lambda)\varphi = -\frac{\partial f_\lambda}{\partial\nu}\Big|_{\partial\Omega}, \quad Q(\mu)\psi = -\frac{\partial g_\mu}{\partial\nu}\Big|_{\partial\Omega}, \quad \Gamma(\lambda)\varphi = f_\lambda \quad \text{and} \quad \Gamma(\mu)\psi = g_\mu.$$

Note that $\varphi \in H^{3/2}(\Omega)$ belongs to $\operatorname{dom} Q(\mu)^*$ by the above considerations. With the help of Green's identity (20) we find

$$\begin{aligned}\big((Q(\lambda) - Q(\mu)^*)\varphi,\psi\big)_{\partial\Omega} &= -\left(\frac{\partial f_\lambda}{\partial\nu}\Big|_{\partial\Omega}, g_\mu|_{\partial\Omega}\right)_{\partial\Omega} + \left(f_\lambda|_{\partial\Omega}, \frac{\partial g_\mu}{\partial\nu}\Big|_{\partial\Omega}\right)_{\partial\Omega} \\ &= (Tf_\lambda, g_\mu)_\Omega - (f_\lambda, Tg_\mu)_\Omega = (\lambda - \bar\mu)(f_\lambda, g_\mu)_\Omega \\ &= (\lambda-\bar\mu)(\Gamma(\lambda)\varphi, \Gamma(\mu)\psi)_\Omega = \big((\lambda-\bar\mu)\Gamma(\mu)^*\Gamma(\lambda)\varphi,\psi\big)_{\partial\Omega}.\end{aligned}$$

This holds for all ψ in the dense subset $H^{3/2}(\partial\Omega)$ of $L^2(\partial\Omega)$ and therefore (27) is valid on $\operatorname{dom} Q(\lambda) = \operatorname{dom}\Gamma(\lambda) = H^{3/2}(\partial\Omega)$, i.e., the Dirichlet-to-Neumann map is a generalized Q-function of the triple $\{S, A_D, T\}$.

(ii) Let $\lambda \in \rho(A_D) \cap \rho(A_N)$ and suppose that we have $Q(\lambda)\varphi = 0$ for some $\varphi \in H^{3/2}(\partial\Omega)$. There exists a unique $f_\lambda \in \mathcal{N}_\lambda(T)$ such that $f_\lambda|_{\partial\Omega} = \varphi$ and for this f_λ by assumption we have $\frac{\partial f_\lambda}{\partial\nu}|_{\partial\Omega} = 0$. Hence $f_\lambda \in \operatorname{dom} A_N \cap \mathcal{N}_\lambda(T)$ and from $\lambda \in \rho(A_N)$ we conclude $f_\lambda = 0$, that is, $\varphi = f_\lambda|_{\partial\Omega} = 0$.

Therefore $Q(\lambda)^{-1}$, $\lambda \in \rho(A_D) \cap \rho(A_N)$ exists and, roughly speaking, $Q(\lambda)^{-1}$ maps the negative Neumann boundary values of $H^2(\Omega)$-solutions of $\mathcal{L}u = \lambda u$ onto their Dirichlet boundary values. Let us proof the formula (26) for the difference of the resolvents of A_D and A_N. Observe first, that the right hand side in (26) is well defined. In fact, by Proposition 3.2 (iii) and (19) the range of $\Gamma(\bar\lambda)^*$ lies in $H^{1/2}(\partial\Omega)$ and it follows from the surjectivity of the mapping in (19) that $Q(\lambda)^{-1}$ is defined on the whole space $H^{1/2}(\partial\Omega)$ and maps $H^{1/2}(\partial\Omega)$ onto $H^{3/2}(\partial\Omega)$, the domain of $\Gamma(\lambda)$.

Let now $f \in L^2(\Omega)$. We claim that the function

$$g = (A_D - \lambda)^{-1} f - \Gamma(\lambda) Q(\lambda)^{-1} \Gamma(\bar\lambda)^* f \tag{28}$$

belongs to $\operatorname{dom} A_N$. It is clear that g is in $H^2(\Omega)$ since $(A_D-\lambda)^{-1}f\in\operatorname{dom} A_D$ and the second term on the right hand side belongs to $\mathcal{N}_\lambda(T)$, the range of $\Gamma(\lambda)$. In order to verify $\frac{\partial g}{\partial\nu}|_{\partial\Omega}=0$ we choose $f_D\in\operatorname{dom} A_D$ such that $f=(A_D-\lambda)f_D$, so that (28) becomes

$$g=f_D-\Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\bar\lambda)^*(A_D-\lambda)f_D=f_D+\Gamma(\lambda)Q(\lambda)^{-1}\frac{\partial f_D}{\partial\nu}\Big|_{\partial\Omega},\tag{29}$$

where we have used Proposition 3.2 (iii). Let $f_\lambda:=\Gamma(\lambda)Q(\lambda)^{-1}\frac{\partial f_D}{\partial\nu}|_{\partial\Omega}$. Then $f_\lambda\in\mathcal{N}_\lambda(T)$ and the trace of f_λ is given by

$$f_\lambda|_{\partial\Omega}=Q(\lambda)^{-1}\frac{\partial f_D}{\partial\nu}\Big|_{\partial\Omega}.$$

Hence $Q(\lambda)f_\lambda|_{\partial\Omega}=\frac{\partial f_D}{\partial\nu}|_{\partial\Omega}$, but on the other hand, by the definition of the Dirichlet-to-Neumann map $Q(\lambda)f_\lambda|_{\partial\Omega}=-\frac{\partial f_\lambda}{\partial\nu}|_{\partial\Omega}$. Therefore, the sum of the Neumann boundary value of the function f_λ and the Neumann boundary value of f_D is zero and we conclude from (29)

$$\frac{\partial g}{\partial\nu}\Big|_{\partial\Omega}=\frac{\partial f_D}{\partial\nu}\Big|_{\partial\Omega}+\frac{\partial f_\lambda}{\partial\nu}\Big|_{\partial\Omega}=0.$$

We have shown that g in (28) belongs to $\operatorname{dom} A_N$. As T is an extension of A_N and A_D, and $\operatorname{ran}\Gamma(\lambda)=\ker(T-\lambda)$ we obtain

$$(A_N-\lambda)g=(T-\lambda)(A_D-\lambda)^{-1}f-(T-\lambda)\Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\bar\lambda)^*f=f.$$

Together with (28) we find

$$(A_N-\lambda)^{-1}f=(A_D-\lambda)^{-1}f-\Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\bar\lambda)^*f$$

for all $\lambda\in\rho(A_D)\cap\rho(A_N)$ and $f\in L^2(\Omega)$, and therefore the resolvent formula (26) is valid.

Up to some small modifications assertion (iii) was proved in [11]. □

We mention that for $\lambda,\lambda_0\in\rho(A_D)$ the Dirichlet-to-Neumann map is connected with the resolvent of A_D via

$$Q(\lambda)=\operatorname{Re}Q(\lambda_0)+\Gamma_{\lambda_0}\big((\lambda-\operatorname{Re}\lambda_0)+(\lambda-\lambda_0)(\lambda-\bar\lambda_0)(A_D-\lambda)^{-1}\big)\Gamma_{\lambda_0}.$$

This follows from the fact that Q is a generalized Q-function and Proposition 2.5. The following two corollaries collect some properties of the Dirichlet-to-Neumann map and its inverse.

Corollary 3.5. *For $\lambda,\lambda_0\in\rho(A_D)$ the Dirichlet-to-Neumann map $Q(\lambda)$ has the following properties.*

- (i) *$Q(\lambda)$ is a non-closed unbounded operator in $L^2(\partial\Omega)$ defined on $H^{3/2}(\partial\Omega)$ with $\operatorname{ran}Q(\lambda)\subset H^{1/2}(\partial\Omega)$;*
- (ii) *$Q(\lambda)-\operatorname{Re}Q(\lambda_0)$ is a non-closed bounded operator in $L^2(\partial\Omega)$ defined on $H^{3/2}(\partial\Omega)$;*
- (iii) *the closure $\widetilde{Q}(\lambda)$ of the operator $Q(\lambda)-\operatorname{Re}Q(\lambda_0)$ in $L^2(\partial\Omega)$ satisfies*

$$\frac{d}{d\lambda}\widetilde{Q}(\lambda)=\Gamma(\bar\lambda)^*\overline{\Gamma(\lambda)}$$

and $\widetilde{Q}$ is a $\mathcal{L}(L^2(\partial\Omega))$-valued Nevanlinna function.

Proof. Besides the statement that $Q(\lambda)$ is a non-closed unbounded operator the assertions follow from the fact that Q is a generalized Q-function and the results in Section 2. In Corollary 3.6 it will turn out that $\overline{Q(\lambda)^{-1}}$ is a compact operator and that $Q(\lambda)^{-1}$ is not closed. This implies that $\overline{Q(\lambda)}$ and $Q(\lambda)$ are unbounded and that $Q(\lambda)$ is not closed. □

Corollary 3.6. *For $\lambda \in \rho(A_D) \cap \rho(A_N)$ the inverse $Q(\lambda)^{-1}$ of the Dirichlet-to-Neumann map $Q(\lambda)$ has the following properties.*

(i) *$Q(\lambda)^{-1}$ is a non-closed bounded operator in $L^2(\partial\Omega)$ defined on $H^{1/2}(\partial\Omega)$ with $\operatorname{ran} Q(\lambda)^{-1} = H^{3/2}(\partial\Omega)$;*

(ii) *the closure $\overline{Q(\lambda)^{-1}}$ is a compact operator in $L^2(\partial\Omega)$;*

(iii) *the function $\lambda \mapsto -\overline{Q(\lambda)^{-1}}$ is a $\mathcal{L}(L^2(\partial\Omega))$-valued Nevanlinna function.*

Proof. It is clear that (i) is an immediate consequence of (ii). Statement (iii) follows from Theorem 2.6 and general properties of the Nevanlinna class. Assertion (ii) is essentially a consequence of the classical results in [40], see also [32, Theorem 2.1]. Namely, for $\lambda \in \rho(A_D) \cap \rho(A_N)$ the operator $Q(\lambda) : H^{3/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is an isomorphism and can be extended to an isomorphism $\widehat{Q}(\lambda) : H^1(\partial\Omega) \to L^2(\partial\Omega)$ which acts as in (25). Therefore $Q(\lambda)^{-1} \subset \widehat{Q}(\lambda)^{-1}$ is a densely defined operator in $L^2(\partial\Omega)$ which is bounded as an operator in $H^1(\partial\Omega)$ and hence also bounded when considered as an operator in $L^2(\partial\Omega)$. Its closure $\overline{Q(\lambda)^{-1}}$ in $L^2(\partial\Omega)$ is a bounded everywhere defined operator in $L^2(\partial\Omega)$ with values in $H^1(\partial\Omega)$ and coincides with $\widehat{Q}(\lambda)^{-1}$. As $H^1(\partial\Omega)$ is compactly embedded in $L^2(\partial\Omega)$ it follows that $\overline{Q(\lambda)^{-1}}$ is a compact operator in $L^2(\partial\Omega)$. □

The next corollary is a simple consequence of Theorem 3.4 for the case that the difference of the resolvents is a trace class operator.

Corollary 3.7. *Let the assumptions be as in Theorem 3.4, let $\widetilde{Q}$ be the Nevanlinna function from Corollary 3.5 and suppose, in addition, $n = 2$. Then*

$$\operatorname{tr}\big((A_D - \lambda)^{-1} - (A_N - \lambda)^{-1}\big) = \operatorname{tr}\left(\overline{Q(\lambda)^{-1}}\,\frac{d}{d\lambda}\,\widetilde{Q}(\lambda)\right) \tag{30}$$

holds for all $\lambda \in \rho(A_D) \cap \rho(A_N)$.

Proof. The resolvent formula (26) can be written in the form

$$(A_D - \lambda)^{-1} - (A_N - \lambda)^{-1} = \overline{\Gamma(\lambda)}\,\overline{Q(\lambda)^{-1}}\,\Gamma(\bar{\lambda})^*, \tag{31}$$

where the closures $\overline{\Gamma(\lambda)}$ and $\overline{Q(\lambda)^{-1}}$ are everywhere defined bounded operators, cf. Corollary 3.6 (ii). In the case $n = 2$ it follows from Theorem 3.4 (iii) that (31) is a trace class operator and from Corollaries 2.9, 3.5 (iii) and well known properties of the trace of bounded operators (see [28]) we conclude (30). □

4. Coupling of elliptic differential operators

In this section we study the uniformly elliptic second order differential expression $\mathcal{L}$ from (17) on two different domains and a coupling of the associated Dirichlet operators. More precisely, let $\Omega \subset \mathbb{R}^n$ be a simply connected bounded domain with C^∞-boundary $\mathcal{C} := \partial\Omega$ and let $\Omega' = \mathbb{R}^n \backslash \overline{\Omega}$ be the complement of the closure of Ω in

$\mathbb{R}^n$. Clearly, Ω' is an unbounded domain with the compact C^∞-boundary $\partial\Omega' = \mathcal{C}$. Let again $\mathcal{L}$ be given by

$$\mathcal{L}h = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial h}{\partial x_k} + ah \tag{32}$$

with bounded coefficients $a_{jk} \in C^\infty(\mathbb{R}^n)$ satisfying $a_{jk}(x) = \overline{a_{kj}(x)}$ for all $x \in \mathbb{R}^n$ and $j,k = 1,\dots,n$, the function $a \in L^\infty(\mathbb{R}^n)$ is real valued and suppose that $\mathcal{L}$ is uniformly elliptic, cf. (18). The restriction of $\mathcal{L}$ on functions f defined on Ω or functions f' defined on Ω' will be denoted by $\mathcal{L}_\Omega$ and $\mathcal{L}_{\Omega'}$, respectively. Then it is clear that the differential expressions $\mathcal{L}_\Omega$ and $\mathcal{L}_{\Omega'}$ are of the type as in Section 3.

In the following we will usually denote functions defined on $\mathbb{R}^n$ by h or k, and we denote functions defined on Ω or Ω' by f, g or f', g', respectively. The scalar products of $L^2(\Omega)$ and $L^2(\Omega')$ are indexed with Ω and Ω', respectively, whereas the scalar product of $L^2(\mathbb{R}^n)$ is just denoted by $(\cdot,\cdot)$. For the trace of a function $f \in H^2(\Omega)$ and $f' \in H^2(\Omega')$ we write $f|_\mathcal{C}$ and $f'|_\mathcal{C}$, and the trace of the conormal derivatives are

$$\frac{\partial f}{\partial \nu}\Big|_\mathcal{C} = \sum_{j,k=1}^{n} a_{jk} n_j \frac{\partial f}{\partial x_k}\Big|_\mathcal{C} \quad \text{and} \quad \frac{\partial f'}{\partial \nu'}\Big|_\mathcal{C} = \sum_{j,k=1}^{n} a_{jk} n'_j \frac{\partial f}{\partial x_k}\Big|_\mathcal{C}; \tag{33}$$

here $n(x) = (n_1(x),\dots,n_n(x))^\top$ and $n'(x) = -n(x)$ are the unit vectors at the point $x \in \mathcal{C} = \partial\Omega = \partial\Omega'$ pointing out of Ω and Ω', respectively. Note also that the coefficients a_{jk} in (33) are the restrictions of the coefficients in (32) onto Ω and Ω', respectively. The Dirichlet operators

$$\begin{aligned} A_\Omega f &= \mathcal{L}_\Omega f, & \operatorname{dom} A_\Omega &= \{ f \in H^2(\Omega) : f|_\mathcal{C} = 0 \}, \\ A_{\Omega'} f' &= \mathcal{L}_{\Omega'} f', & \operatorname{dom} A_{\Omega'} &= \{ f' \in H^2(\Omega') : f'|_\mathcal{C} = 0 \}, \end{aligned}$$

are selfadjoint operators in $L^2(\Omega)$ and $L^2(\Omega')$, respectively. Hence the orthogonal sum

$$A = \begin{pmatrix} A_\Omega & 0 \\ 0 & A_{\Omega'} \end{pmatrix}, \qquad \operatorname{dom} A = \operatorname{dom} A_\Omega \oplus \operatorname{dom} A_{\Omega'}, \tag{34}$$

is a selfadjoint operator in $L^2(\mathbb{R}^n) = L^2(\Omega) \oplus L^2(\Omega')$. Observe that

$$\begin{aligned} A(f \oplus f') &= \mathcal{L}(f \oplus f') = \mathcal{L}_\Omega f \oplus \mathcal{L}_{\Omega'} f', \\ \operatorname{dom} A &= \{ f \oplus f' \in H^2(\Omega) \oplus H^2(\Omega') : f|_\mathcal{C} = 0 = f'|_\mathcal{C} \}, \end{aligned} \tag{35}$$

and that A is not a usual second order elliptic differential operator on $\mathbb{R}^n$ since for a function $f \oplus f' \in \operatorname{dom} A$ the traces of the conormal derivatives $\frac{\partial f}{\partial \nu}|_\mathcal{C}$ and $-\frac{\partial f'}{\partial \nu'}|_\mathcal{C}$ at the boundary $\mathcal{C}$ of the domains Ω and Ω' in general do not coincide.

Besides the operator A we consider the usual selfadjoint operator associated to $\mathcal{L}$ in $L^2(\mathbb{R}^n)$ defined by

$$\widetilde{A}h = \mathcal{L}h, \qquad h \in \operatorname{dom} \widetilde{A} = H^2(\mathbb{R}^n), \tag{36}$$

and our aim is to prove a formula for the difference of the resolvents of $\widetilde{A}$ and A with the help of a generalized Q-function in a similar form as in the previous section.

The following theorem indicates how S and T in the triple $\{S, A, T\}$ for the definition of a generalized Q-function can be chosen.

Theorem 4.1. *The operator*

$$Sh = \mathcal{L}h, \quad \operatorname{dom} S = \bigl\{ h = f \oplus f' \in H^2(\mathbb{R}^n) : f|_\mathcal{C} = 0 = f'|_\mathcal{C} \bigr\}, \tag{37}$$

is a densely defined closed symmetric operator in $L^2(\mathbb{R}^n)$ with infinite deficiency indices $n_\pm(S)$. The operator

$$\begin{aligned} T(f \oplus f') &= \mathcal{L}(f \oplus f'), \\ \operatorname{dom} T &= \bigl\{ f \oplus f' \in H^2(\Omega) \oplus H^2(\Omega') : f|_\mathcal{C} = f'|_\mathcal{C} \bigr\}, \end{aligned} \tag{38}$$

is not closed as an operator in $L^2(\mathbb{R}^n)$ and T satisfies $\overline{T} = S^*$ and $T^* = S$. Furthermore, the selfadjoint operators A and $\widetilde{A}$ in (34), (35) *and* (36) *are extensions of S and restrictions of T.*

Proof. The operator S is a restriction of the selfadjoint operator A and hence S is symmetric. The fact that $\operatorname{dom} S$ is dense follows, e.g., from the fact that $H^2_0(\Omega)$ and $H^2_0(\Omega')$ are dense subspaces of $L^2(\Omega)$ and $L^2(\Omega')$, respectively, cf. Proposition 3.1, and

$$H^2_0(\Omega) \oplus H^2_0(\Omega') \subset \operatorname{dom} S.$$

Since for any function $h \in H^2(\mathbb{R}^n)$ decomposed as $h = f \oplus f'$, where $f \in H^2(\Omega)$, $f' \in H^2(\Omega')$, we have $f|_\mathcal{C} = f'|_\mathcal{C} \in H^{3/2}(\mathcal{C})$ it follows that $\widetilde{A}$ is an extension of S and a restriction of the operator T. Moreover, $S \subset A \subset T$ is obvious.

Let us verify that $S = T^*$ holds. In particular this implies that S is closed and that $\overline{T} = S^*$ is true. We start with the inclusion $S \subset T^*$. Let $h = f \oplus f' \in \operatorname{dom} S$ and $k = g \oplus g' \in \operatorname{dom} T$, where $f, g \in H^2(\Omega)$ and $f', g' \in H^2(\Omega')$. First of all we have

$$(Tk, h) - (k, Sh) = (\mathcal{L}_\Omega g, f)_\Omega - (g, \mathcal{L}_\Omega f)_\Omega + (\mathcal{L}_{\Omega'} g', f')_{\Omega'} - (g', \mathcal{L}_{\Omega'} f')_{\Omega'}$$

and Green's identity (20) shows that this is equal to

$$\left(g|_\mathcal{C}, \frac{\partial f}{\partial \nu}\Big|_\mathcal{C} \right)_\mathcal{C} - \left(\frac{\partial g}{\partial \nu}\Big|_\mathcal{C}, f|_\mathcal{C} \right)_\mathcal{C} + \left(g'|_\mathcal{C}, \frac{\partial f'}{\partial \nu'}\Big|_\mathcal{C} \right)_\mathcal{C} - \left(\frac{\partial g'}{\partial \nu'}\Big|_\mathcal{C}, f'|_\mathcal{C} \right)_\mathcal{C}.$$

Since $h = f \oplus f' \in \operatorname{dom} S$ we have

$$f|_\mathcal{C} = f'|_\mathcal{C} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial \nu}\Big|_\mathcal{C} = -\frac{\partial f'}{\partial \nu'}\Big|_\mathcal{C},$$

and for $k = g \oplus g' \in \operatorname{dom} T$ by definition $g|_\mathcal{C} = g'|_\mathcal{C}$ holds. Hence we conclude

$$(Tk, h) - (k, Sh) = 0$$

and therefore every $h \in \operatorname{dom} S$ belongs to $\operatorname{dom} T^*$ and $T^* h = Sh$, i.e., $S \subset T^*$. Let us now prove the converse inclusion $T^* \subset S$. For this it is sufficient to check that every function $h \in \operatorname{dom} T^*$ belongs to $\operatorname{dom} S$. From the fact that T is an extension of the selfadjoint operators A and $\widetilde{A}$ we conclude

$$T^* \subset A^* = A \subset T \qquad \text{and} \qquad T^* \subset \widetilde{A}^* = \widetilde{A} \subset T,$$

so that T^* is a restriction of A and $\widetilde{A}$. Hence every function h in $\operatorname{dom} T^*$ belongs also to $\operatorname{dom} A$ and $\operatorname{dom} \widetilde{A}$. Thus $h = f \oplus f' \in H^2(\mathbb{R}^n)$ and $f \in H^2(\Omega)$ and $f' \in H^2(\Omega')$ satisfy $f|_\mathcal{C} = f'|_\mathcal{C} = 0$. Therefore $\operatorname{dom} T^* \subset \operatorname{dom} S$ and we have shown $T^* = S$.

Next it will be verified that T is not closed. The arguments are similar as in [8, Proof of Proposition 4.5] and could also be formulated in terms of unitary relations between Krein spaces, cf. [17]. Assume that T is closed, i.e., $T = \overline{T}$, and consider the subspace

$$\mathcal{M} = \left\{ \begin{bmatrix} f \oplus f' \\ T(f \oplus f') \\ f|_{\mathcal{C}} \\ \frac{\partial f}{\partial \nu}|_{\mathcal{C}} + \frac{\partial f'}{\partial \nu'}|_{\mathcal{C}} \end{bmatrix} : f \oplus f' \in \operatorname{dom} T \right\} \subset L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathcal{C}) \oplus L^2(\mathcal{C}).$$

Observe that by (19) and the definition of T the mapping

$$\operatorname{dom} T \ni f \oplus f' \;\mapsto\; \left\{ f|_{\mathcal{C}}, \frac{\partial f}{\partial \nu}\Big|_{\mathcal{C}} + \frac{\partial f'}{\partial \nu'}\Big|_{\mathcal{C}} \right\} \in H^{3/2}(\mathcal{C}) \times H^{1/2}(\mathcal{C}) \tag{39}$$

is onto. Setting $\mathcal{N} = L^2(\Omega) \oplus L^2(\Omega) \oplus \{0\} \oplus \{0\}$ it is clear that the sum of the subpaces $\mathcal{M}$ and $\mathcal{N}$ is

$$\mathcal{M} + \mathcal{N} = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus \big(H^{3/2}(\mathcal{C}) \times H^{1/2}(\mathcal{C})\big). \tag{40}$$

We will calculate the orthogonal complements of $\mathcal{M}$ and $\mathcal{N}$ in $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathcal{C}) \oplus L^2(\mathcal{C})$ and show that $\mathcal{M}^\perp + \mathcal{N}^\perp$ is closed. First of all we have

$$\mathcal{N}^\perp = \{0\} \oplus \{0\} \oplus L^2(\mathcal{C}) \oplus L^2(\mathcal{C}) \tag{41}$$

and in order to determine $\mathcal{M}^\perp$ suppose that

$$\begin{bmatrix} l \oplus l' \\ g \oplus g' \\ \varphi \\ \psi \end{bmatrix} \in \mathcal{M}^\perp, \qquad g, l \in L^2(\Omega),\; g', l' \in L^2(\Omega'),\; \varphi, \psi \in L^2(\mathcal{C}), \tag{42}$$

is an element in $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathcal{C}) \oplus L^2(\mathcal{C})$ which is orthogonal to $\mathcal{M}$. Then we have

$$\big(T(f \oplus f'), g \oplus g'\big) + \big(f \oplus f', l \oplus l'\big) = -\big(f|_{\mathcal{C}}, \varphi\big)_{\mathcal{C}} - \left(\frac{\partial f}{\partial \nu}\Big|_{\mathcal{C}} + \frac{\partial f'}{\partial \nu'}\Big|_{\mathcal{C}}, \psi \right)_{\mathcal{C}} \tag{43}$$

for all $f \oplus f' \in \operatorname{dom} T$. In particular, for $f \oplus f' \in \operatorname{dom} S$ we have

$$\frac{\partial f}{\partial \nu}\Big|_{\mathcal{C}} = -\frac{\partial f'}{\partial \nu'}\Big|_{\mathcal{C}} \quad \text{and} \quad f|_{\mathcal{C}} = f'|_{\mathcal{C}} = 0,$$

so that (43) becomes

$$\big(T(f \oplus f'), g \oplus g'\big) = \big(S(f \oplus f'), g \oplus g'\big) = -\big(f \oplus f', l \oplus l'\big)$$

and hence $g \oplus g' \in \operatorname{dom} S^*$ and $S^*(g \oplus g') = -l \oplus l'$. But we have assumed that T is closed and hence from $S = T^*$ we conclude $S^* = T^{**} = \overline{T} = T$, so that

$$g \oplus g' \in \operatorname{dom} T \qquad \text{and} \quad T(g \oplus g') = -l \oplus l'. \tag{44}$$

From Green's identity we then obtain

$$\begin{aligned}
&\big(T(f \oplus f'), g \oplus g'\big) - \big(f \oplus f', T(g \oplus g')\big) \\
&\quad = (\mathcal{L}_\Omega f, g)_\Omega - (f, \mathcal{L}_\Omega g)_\Omega + (\mathcal{L}_{\Omega'} f', g')_{\Omega'} - (f', \mathcal{L}_{\Omega'} g')_{\Omega'} \\
&\quad = \left(f|_{\mathcal{C}}, \frac{\partial g}{\partial \nu}\Big|_{\mathcal{C}} \right)_{\mathcal{C}} - \left(\frac{\partial f}{\partial \nu}\Big|_{\mathcal{C}}, g|_{\mathcal{C}} \right)_{\mathcal{C}} + \left(f'|_{\mathcal{C}}, \frac{\partial g'}{\partial \nu'}\Big|_{\mathcal{C}} \right)_{\mathcal{C}} - \left(\frac{\partial f'}{\partial \nu'}\Big|_{\mathcal{C}}, g'|_{\mathcal{C}} \right)_{\mathcal{C}} \\
&\quad = \left(f|_{\mathcal{C}}, \frac{\partial g}{\partial \nu}\Big|_{\mathcal{C}} + \frac{\partial g'}{\partial \nu'}\Big|_{\mathcal{C}} \right)_{\mathcal{C}} - \left(\frac{\partial f}{\partial \nu}\Big|_{\mathcal{C}} + \frac{\partial f'}{\partial \nu'}\Big|_{\mathcal{C}}, g|_{\mathcal{C}} \right)_{\mathcal{C}},
\end{aligned}$$

where we have used that $f\oplus f'$, $g\oplus g'\in\operatorname{dom} T$ satisfy $f|_{\mathcal C}=f'|_{\mathcal C}$ and $g|_{\mathcal C}=g'|_{\mathcal C}$. Inserting (44) in (43) and comparing this with the above relation shows that the identity

$$\left(f|_{\mathcal C},\frac{\partial g}{\partial\nu}\Big|_{\mathcal C}+\frac{\partial g'}{\partial\nu'}\Big|_{\mathcal C}+\varphi\right)_{\mathcal C}=\left(\frac{\partial f}{\partial\nu}\Big|_{\mathcal C}+\frac{\partial f'}{\partial\nu'}\Big|_{\mathcal C},g|_{\mathcal C}-\psi\right)_{\mathcal C}\tag{45}$$

holds for all $f\oplus f'\in\operatorname{dom} T$. As the mapping (39) is surjective and $H^{3/2}(\mathcal C)\times H^{1/2}(\mathcal C)$ is dense in $L^2(\mathcal C)\oplus L^2(\mathcal C)$ we conclude from (45) that

$$\varphi=-\left(\frac{\partial g}{\partial\nu}\Big|_{\mathcal C}+\frac{\partial g'}{\partial\nu'}\Big|_{\mathcal C}\right)\qquad\text{and}\qquad\psi=g|_{\mathcal C}$$

holds. Hence we have seen that the element (42) in $\mathcal M^\perp$ is of the form

$$\begin{bmatrix}-T(g\oplus g')\\ g\oplus g'\\ -\frac{\partial g}{\partial\nu}|_{\mathcal C}-\frac{\partial g'}{\partial\nu'}|_{\mathcal C}\\ g|_{\mathcal C}\end{bmatrix}\tag{46}$$

for some $g\oplus g'\in\operatorname{dom} T$. It is not difficult to check that conversely an element as in (46) belongs to $\mathcal M^\perp$. Therefore the orthogonal complement of $\mathcal M$ is given by

$$\mathcal M^\perp=\left\{\begin{bmatrix}-T(g\oplus g')\\ g\oplus g'\\ -\frac{\partial g}{\partial n}|_{\mathcal C}-\frac{\partial g'}{\partial\nu'}|_{\mathcal C}\\ g|_{\mathcal C}\end{bmatrix}:g\oplus g'\in\operatorname{dom} T\right\}\subset L^2(\mathbb R^n)\oplus L^2(\mathbb R^n)\oplus L^2(\mathcal C)\oplus L^2(\mathcal C)$$

and together with (41) we find that the sum of $\mathcal M^\perp$ and $\mathcal N^\perp$ is

$$\mathcal M^\perp+\mathcal N^\perp=\left\{\begin{bmatrix}-T(g\oplus g')\\ g\oplus g'\end{bmatrix}:g\oplus g'\in\operatorname{dom} T\right\}\oplus L^2(\mathcal C)\oplus L^2(\mathcal C).$$

The assumption that T is closed implies that $\mathcal M^\perp+\mathcal N^\perp$ is a closed subspace of $L^2(\mathbb R^n)\oplus L^2(\mathbb R^n)\oplus L^2(\mathcal C)\oplus L^2(\mathcal C)$. But then according to [34, IV Theorem 4.8] also $\mathcal M+\mathcal N$ is a closed subspace of $L^2(\mathbb R^n)\oplus L^2(\mathbb R^n)\oplus L^2(\mathcal C)\oplus L^2(\mathcal C)$ which is a contradiction to (40). Thus T can not be closed. □

The following lemma will be useful later in this section.

Lemma 4.2. *Let S and T be as in Theorem 4.1 and let $\widetilde A$ be the selfadjoint realization of $\mathcal L$ in $L^2(\mathbb R^n)$ defined on $H^2(\mathbb R^n)$. For a function $f\oplus f'\in\operatorname{dom} T$, where $f\in H^2(\Omega)$ and $f'\in H^2(\Omega')$, we have*

$$f\oplus f'\in\operatorname{dom}\widetilde A\qquad\textit{if and only if}\qquad\frac{\partial f}{\partial\nu}\Big|_{\mathcal C}=-\frac{\partial f'}{\partial\nu'}\Big|_{\mathcal C}.$$

Proof. For a function $f\oplus f'\in\operatorname{dom}\widetilde A=H^2(\mathbb R^n)$ it is clear that $\frac{\partial f}{\partial\nu}|_{\mathcal C}=-\frac{\partial f'}{\partial\nu'}|_{\mathcal C}$ holds. Conversely, let $f\oplus f'\in\operatorname{dom} T$ and assume

$$\frac{\partial f}{\partial\nu}\Big|_{\mathcal C}=-\frac{\partial f'}{\partial\nu'}\Big|_{\mathcal C}.\tag{47}$$

Then also $f|_{\mathcal C}=f'|_{\mathcal C}$ and since every $g\oplus g'\in\operatorname{dom}\widetilde A$ satisfies

$$g|_{\mathcal C}=g'|_{\mathcal C}\qquad\text{and}\qquad\frac{\partial g}{\partial\nu}\Big|_{\mathcal C}=-\frac{\partial g'}{\partial\nu'}\Big|_{\mathcal C}$$

Green's identity implies

$$\begin{aligned}&\big(\widetilde{A}(g\oplus g'), f\oplus f'\big) - \big(g\oplus g', T(f\oplus f')\big)\\ &= \left(g|_{\mathcal{C}}, \frac{\partial f}{\partial \nu}\Big|_{\mathcal{C}}\right)_{\mathcal{C}} - \left(\frac{\partial g}{\partial \nu}\Big|_{\mathcal{C}}, f|_{\mathcal{C}}\right)_{\mathcal{C}} + \left(g'|_{\mathcal{C}}, \frac{\partial f'}{\partial \nu}\Big|_{\mathcal{C}}\right)_{\mathcal{C}} - \left(\frac{\partial g'}{\partial \nu}\Big|_{\mathcal{C}}, f'|_{\mathcal{C}}\right)_{\mathcal{C}} = 0.\end{aligned}$$

Therefore $f\oplus f' \in \operatorname{dom}\widetilde{A}^* = \operatorname{dom}\widetilde{A}$. □

Next we define a mapping Γ_{λ_0} which satisfies the assumptions in the definition of a generalized Q-function. For this let A be the selfadjoint operator in $L^2(\mathbb{R}^n)$ in (34) and (35) which is the orthogonal sum of the Dirichlet operators A_Ω and $A_{\Omega'}$ in $L^2(\Omega)$ and $L^2(\Omega')$, respectively. For $\lambda\in\rho(A)$ the domain of the operator T in Theorem 4.1 can be decomposed in

$$\begin{aligned}\operatorname{dom} T &= \operatorname{dom} A \dotplus \mathcal{N}_\lambda(T)\\ &= \big\{f\oplus f' \in H^2(\Omega)\oplus H^2(\Omega') : f|_{\mathcal{C}} = f'|_{\mathcal{C}} = 0\big\} \dotplus \mathcal{N}_\lambda(T),\end{aligned} \tag{48}$$

cf. (5). Let us fix some $\lambda_0\in\rho(A)$. The decomposition (48) and the surjectivity of the map

$$\operatorname{dom} T \ni f\oplus f' \;\mapsto\; \left\{ f|_{\mathcal{C}}, \frac{\partial f}{\partial \nu}\Big|_{\mathcal{C}} + \frac{\partial f'}{\partial \nu'}\Big|_{\mathcal{C}} \right\} \in H^{3/2}(\mathcal{C}) \times H^{1/2}(\mathcal{C}), \tag{49}$$

cf. (19), (39) imply that for a given function $\varphi\in H^{3/2}(\mathcal{C})$ there exists a unique function $f_{\lambda_0}\oplus f'_{\lambda_0} \in \mathcal{N}_{\lambda_0}(T)$ such that $f_{\lambda_0}|_{\mathcal{C}} = f'_{\lambda_0}|_{\mathcal{C}} = \varphi$. Let Γ_{λ_0} be the mapping that assigns $f_{\lambda_0}\oplus f'_{\lambda_0}$ to φ,

$$H^{3/2}(\mathcal{C}) \ni \varphi \mapsto \Gamma_{\lambda_0}\varphi := f_{\lambda_0}\oplus f'_{\lambda_0}. \tag{50}$$

Similarly as in the previous section Γ_{λ_0} will be regarded as an operator from $L^2(\mathcal{C})$ to $L^2(\mathbb{R}^n)$ with $\operatorname{dom}\Gamma_{\lambda_0} = H^{3/2}(\mathcal{C})$ and $\operatorname{ran}\Gamma_{\lambda_0} = \mathcal{N}_{\lambda_0}(T)$. Observe that the function $\Gamma_{\lambda_0}\varphi = f_{\lambda_0}\oplus f'_{\lambda_0}$ consists of an $H^2(\Omega)$-solution f_{λ_0} of $\mathcal{L}_\Omega u = \lambda_0 u$ and an $H^2(\Omega')$-solution f'_{λ_0} of $\mathcal{L}_{\Omega'}u' = \lambda_0 u'$ satisfying the boundary conditions $\varphi = f_{\lambda_0}|_{\mathcal{C}} = f'_{\lambda_0}|_{\mathcal{C}}$.

The following proposition parallels Proposition 3.2.

Proposition 4.3. *Let $\lambda_0\in\rho(A)$, let Γ_{λ_0} be as in* (50) *and let $\lambda\in\rho(A)$. Then the following holds:*

(i) *Γ_{λ_0} is a bounded operator from $L^2(\mathcal{C})$ in $L^2(\mathbb{R}^n)$ with dense domain $H^{3/2}(\mathcal{C})$;*

(ii) *The operator $\Gamma(\lambda) = (I + (\lambda-\lambda_0)(A-\lambda)^{-1})\Gamma_{\lambda_0}$ is given by*

$$\Gamma(\lambda)\varphi = f_\lambda\oplus f'_\lambda, \quad \textit{where} \quad f_\lambda\oplus f'_\lambda\in\mathcal{N}_\lambda(T) \;\; \textit{and} \;\; f_\lambda|_{\mathcal{C}} = \varphi = f'_\lambda|_{\mathcal{C}};$$

(iii) *The mapping $\Gamma(\bar\lambda)^* : L^2(\mathbb{R}^n)\to L^2(\mathcal{C})$ satisfies*

$$\Gamma(\bar\lambda)^*(A-\lambda)h = -\frac{\partial f}{\partial \nu}\Big|_{\mathcal{C}} - \frac{\partial f'}{\partial \nu'}\Big|_{\mathcal{C}}, \qquad h = f\oplus f'\in\operatorname{dom} A.$$

Proof. We start with the proof (ii). Let $\varphi\in H^{3/2}(\mathcal{C})$ and choose the unique elements $f_\lambda\oplus f'_\lambda\in\mathcal{N}_\lambda(T)$ and $f_{\lambda_0}\oplus f'_{\lambda_0}\in\mathcal{N}_{\lambda_0}(T)$ such that

$$f_\lambda|_{\mathcal{C}} = f'_\lambda|_{\mathcal{C}} = \varphi = f_{\lambda_0}|_{\mathcal{C}} = f'_{\lambda_0}|_{\mathcal{C}}$$

holds. By definition $\Gamma_{\lambda_0}\varphi = f_{\lambda_0}\oplus f'_{\lambda_0}$ and therefore

$$\begin{aligned}\Gamma(\lambda)\varphi &= \Gamma_{\lambda_0}\varphi + (\lambda-\lambda_0)(A-\lambda)^{-1}\Gamma_{\lambda_0}\varphi\\ &= f_{\lambda_0}\oplus f'_{\lambda_0} + (\lambda-\lambda_0)(A-\lambda)^{-1}\Gamma_{\lambda_0}\varphi.\end{aligned}$$

Since $(\lambda-\lambda_0)(A-\lambda)^{-1}\Gamma_{\lambda_0}\varphi$ is a function belonging to $\operatorname{dom} A$ we have

$$\big((\lambda-\lambda_0)(A-\lambda)^{-1}\Gamma_{\lambda_0}\varphi\big)\big|_{\mathcal{C}}=0,$$

cf. (35). This implies

$$(\Gamma(\lambda)\varphi)|_{\mathcal{C}}=(\Gamma_{\lambda_0}\varphi)|_{\mathcal{C}}=\big(f_{\lambda_0}\oplus f'_{\lambda_0}\big)|_{\mathcal{C}}=f_{\lambda_0}|_{\mathcal{C}}=f'_{\lambda_0}|_{\mathcal{C}}=\varphi$$

and since $\operatorname{ran}\Gamma(\lambda)=\mathcal{N}_\lambda(T)$, see Lemma 2.1, and $f_\lambda\oplus f'_\lambda$ is the unique function in $\mathcal{N}_\lambda(T)$ with $f_\lambda|_{\mathcal{C}}=f'_\lambda|_{\mathcal{C}}=\varphi$ we conclude $\Gamma(\lambda)\varphi=f_\lambda\oplus f'_\lambda$.

Next we verify (iii). Observe that then $\Gamma(\bar\lambda)^*$, $\lambda\in\rho(A)$, is a closed operator which is defined on the whole space, i.e., $\Gamma(\bar\lambda)^*$ is bounded and hence assertion (i) follows by setting $\lambda_0=\bar\lambda$. Let $\varphi\in H^{3/2}(\mathcal{C})$ and choose the unique function $f_{\bar\lambda}\oplus f'_{\bar\lambda}\in\mathcal{N}_{\bar\lambda}(T)$ such that

$$f_{\bar\lambda}|_{\mathcal{C}}=f'_{\bar\lambda}|_{\mathcal{C}}=\varphi \tag{51}$$

holds. Then $\Gamma(\bar\lambda)\varphi=f_{\bar\lambda}\oplus f'_{\bar\lambda}$ and for each $h=f\oplus f'\in\operatorname{dom}A$, where $f\in H^2(\Omega)$, $f'\in H^2(\Omega')$, we have

$$\begin{aligned}\big(\Gamma(\bar\lambda)\varphi,(A-\lambda)h\big)&=\big(f_{\bar\lambda}\oplus f'_{\bar\lambda},A(f\oplus f')\big)-\big(T(f_{\bar\lambda}\oplus f'_{\bar\lambda}),f\oplus f'\big)\\&=(f_{\bar\lambda},\mathcal{L}_\Omega f)_\Omega-(\mathcal{L}_\Omega f_{\bar\lambda},f)_\Omega+(f'_{\bar\lambda},\mathcal{L}_{\Omega'}f')_{\Omega'}-(\mathcal{L}_{\Omega'}f'_{\bar\lambda},f')_{\Omega'}.\end{aligned}$$

With the help of Green's identity this can be rewritten as

$$\left(\frac{\partial f_{\bar\lambda}}{\partial\nu}\Big|_{\mathcal{C}},f|_{\mathcal{C}}\right)_{\mathcal{C}}-\left(f_{\bar\lambda}|_{\mathcal{C}},\frac{\partial f}{\partial\nu}\Big|_{\mathcal{C}}\right)_{\mathcal{C}}+\left(\frac{\partial f'_{\bar\lambda}}{\partial\nu'}\Big|_{\mathcal{C}},f'|_{\mathcal{C}}\right)_{\mathcal{C}}-\left(f'_{\bar\lambda}|_{\mathcal{C}},\frac{\partial f'}{\partial\nu'}\Big|_{\mathcal{C}}\right)_{\mathcal{C}}.$$

Since for $h=f\oplus f'\in\operatorname{dom}A$ we have $f|_{\mathcal{C}}=f'|_{\mathcal{C}}=0$ we conclude from the above calculation and (51) that

$$\big(\Gamma(\bar\lambda)\varphi,(A-\lambda)h\big)=-\left(\varphi,\frac{\partial f}{\partial\nu}\Big|_{\mathcal{C}}+\frac{\partial f'}{\partial\nu'}\Big|_{\mathcal{C}}\right)_{\mathcal{C}}$$

holds for every $\varphi\in H^{3/2}(\mathcal{C})=\operatorname{dom}\Gamma(\bar\lambda)$. Hence $(A-\lambda)h\in\operatorname{dom}\Gamma(\bar\lambda)^*$ and

$$\Gamma(\bar\lambda)^*(A-\lambda)h=-\frac{\partial f}{\partial\nu}\Big|_{\mathcal{C}}-\frac{\partial f'}{\partial\nu'}\Big|_{\mathcal{C}},\qquad h=f\oplus f'\in\operatorname{dom}A.$$

Furthermore, for $\lambda\in\rho(A)$ we have $\operatorname{ran}(A-\lambda)=L^2(\mathbb{R}^n)$, so that $\Gamma(\bar\lambda)^*$ is a bounded operator defined on $L^2(\mathbb{R}^n)$. □

Next we define a function Q in a similar way as the Dirichlet-to-Neumann map in Definition 3.3. For this we make use of the decomposition (48). Namely, for $\lambda\in\rho(A)$ and $\varphi\in H^{3/2}(\mathcal{C})$ there exists a unique function $f_\lambda\oplus f'_\lambda\in\mathcal{N}_\lambda(T)$ such that $f_\lambda|_{\mathcal{C}}=f'_\lambda|_{\mathcal{C}}=\varphi$. The operator $Q(\lambda)$ in $L^2(\mathcal{C})$ is now defined by

$$Q(\lambda)\varphi:=-\frac{\partial f_\lambda}{\partial\nu}\Big|_{\mathcal{C}}-\frac{\partial f'_\lambda}{\partial\nu'}\Big|_{\mathcal{C}},\qquad\varphi\in\operatorname{dom}Q(\lambda)=H^{3/2}(\mathcal{C}). \tag{52}$$

Observe that $\operatorname{ran}Q(\lambda)\subset H^{1/2}(\mathcal{C})$ holds. Roughly speaking, up to a minus sign $Q(\lambda)$ maps the Dirichlet boundary value of the H^2-solutions of $\mathcal{L}_\Omega u=\lambda u$ and $\mathcal{L}_{\Omega'}u'=\lambda u'$, $u|_{\mathcal{C}}=u'|_{\mathcal{C}}$, onto the sum of the Neumann boundary values of these solutions. We mention that in the analysis of so-called intermediate Hamiltonians a modified form of such a Dirichlet-to-Neumann map has been used in [44].

In the following theorem it turns out that Q can be interpreted as a generalized Q-function and the difference of the resolvents of A and $\widetilde{A}$ is expressed with the help of Q.

Theorem 4.4. *Let $\mathcal{L}$ be the elliptic differential expression in* (32) *and let A and $\widetilde{A}$ be the selfadjoint realizations of $\mathcal{L}$ in* (34)-(35) *and* (36)*, respectively. Let S and T be the operators in Theorem 4.1, define $\Gamma(\lambda)$ as in Proposition 4.3 and let $Q(\lambda)$, $\lambda \in \rho(A)$, be as in* (52)*. Then the following holds:*

(i) *Q is a generalized Q-function of the triple $\{S, A, T\}$;*

(ii) *The operator $Q(\lambda)$ is injective for all $\lambda \in \rho(A) \cap \rho(\widetilde{A})$ and the resolvent formula*

$$(A - \lambda)^{-1} - (\widetilde{A} - \lambda)^{-1} = \Gamma(\lambda) Q(\lambda)^{-1} \Gamma(\bar{\lambda})^* \tag{53}$$

holds;

(iii) *For $p \in \mathbb{N}$ and $2p + 1 > n$ the difference of the resolvents in* (53) *belongs to the von Neumann-Schatten class $\mathfrak{S}_p(L^2(\Omega))$.*

Proof. Let us prove assertion (i). Before the defining relation (7) for a generalized Q-function will be verified we show that the operator $Q(\mu)^*$ is an extension of $Q(\bar{\mu})$, $\mu \in \rho(A)$. For this let $\psi \in H^{3/2}(\mathcal{C})$ and choose the unique element $f_{\bar{\mu}} \oplus f'_{\bar{\mu}} \in \mathcal{N}_{\bar{\mu}}(T)$ with the property $f_{\bar{\mu}}|_{\mathcal{C}} = f'_{\bar{\mu}}|_{\mathcal{C}} = \psi$. For $\varphi \in H^{3/2}(\mathcal{C})$ let $f_\mu \oplus f'_\mu \in \mathcal{N}_\mu(T)$ be such that $f_\mu|_{\mathcal{C}} = f'_\mu|_{\mathcal{C}} = \varphi$ holds. By the definition of Q in (52) we have

$$Q(\mu)\varphi = -\frac{\partial f_\mu}{\partial \nu}\Big|_{\mathcal{C}} - \frac{\partial f'_\mu}{\partial \nu'}\Big|_{\mathcal{C}} \quad \text{and} \quad Q(\bar{\mu})\psi = -\frac{\partial f_{\bar{\mu}}}{\partial \nu}\Big|_{\mathcal{C}} - \frac{\partial f'_{\bar{\mu}}}{\partial \nu'}\Big|_{\mathcal{C}}.$$

This gives

$$(Q(\mu)\varphi, \psi) = -\left(\frac{\partial f_\mu}{\partial \nu}\Big|_{\mathcal{C}}, f_{\bar{\mu}}|_{\mathcal{C}}\right)_{\mathcal{C}} - \left(\frac{\partial f'_\mu}{\partial \nu'}\Big|_{\mathcal{C}}, f'_{\bar{\mu}}|_{\mathcal{C}}\right)_{\mathcal{C}} \tag{54}$$

and since

$$\begin{aligned}
\left(f_\mu|_{\mathcal{C}}, \frac{\partial f_{\bar{\mu}}}{\partial \nu}\Big|_{\mathcal{C}}\right)_{\mathcal{C}} - \left(\frac{\partial f_\mu}{\partial \nu}\Big|_{\mathcal{C}}, f_{\bar{\mu}}|_{\mathcal{C}}\right)_{\mathcal{C}} &= (\mathcal{L}_\Omega f_\mu, f_{\bar{\mu}})_\Omega - (f_\mu, \mathcal{L}_\Omega f_{\bar{\mu}})_\Omega = 0,\\
\left(f'_\mu|_{\mathcal{C}}, \frac{\partial f'_{\bar{\mu}}}{\partial \nu'}\Big|_{\mathcal{C}}\right)_{\mathcal{C}} - \left(\frac{\partial f'_\mu}{\partial \nu'}\Big|_{\mathcal{C}}, f'_{\bar{\mu}}|_{\mathcal{C}}\right)_{\mathcal{C}} &= (\mathcal{L}_{\Omega'} f'_\mu, f'_{\bar{\mu}})_{\Omega'} - (f'_\mu, \mathcal{L}_{\Omega'} f'_{\bar{\mu}})_{\Omega'} = 0
\end{aligned}$$

we can rewrite (54) in the form

$$(Q(\mu)\varphi, \psi) = -\left(f_\mu|_{\mathcal{C}}, \frac{\partial f_{\bar{\mu}}}{\partial \nu}\Big|_{\mathcal{C}}\right)_{\mathcal{C}} - \left(f'_\mu|_{\mathcal{C}}, \frac{\partial f'_{\bar{\mu}}}{\partial \nu'}\Big|_{\mathcal{C}}\right)_{\mathcal{C}} = -\left(\varphi, \frac{\partial f_{\bar{\mu}}}{\partial \nu}\Big|_{\mathcal{C}} + \frac{\partial f'_{\bar{\mu}}}{\partial \nu'}\Big|_{\mathcal{C}}\right)_{\mathcal{C}}.$$

This is true for every $\varphi \in \operatorname{dom} Q(\mu)$ and hence we conclude $\psi \in \operatorname{dom} Q(\mu)^*$ and

$$Q(\mu)^*\psi = -\frac{\partial f_{\bar{\mu}}}{\partial \nu}\Big|_{\mathcal{C}} - \frac{\partial f'_{\bar{\mu}}}{\partial \nu'}\Big|_{\mathcal{C}} = Q(\bar{\mu})\psi.$$

Let $\Gamma(\cdot)$ be as in Proposition 4.3. We prove now that

$$Q(\lambda) - Q(\mu)^* = (\lambda - \bar{\mu}) \Gamma(\mu)^* \Gamma(\lambda), \qquad \lambda, \mu \in \rho(A) \tag{55}$$

holds on $\operatorname{dom} \Gamma(\lambda) = H^{3/2}(\mathcal{C})$. For this let $\varphi, \psi \in H^{3/2}(\mathcal{C})$ and choose the unique elements $f_\lambda \oplus f'_\lambda \in \mathcal{N}_\lambda(T)$, $f_\mu \oplus f'_\mu \in \mathcal{N}_\mu(T)$ with the properties

$$f_\lambda|_{\mathcal{C}} = f'_\lambda|_{\mathcal{C}} = \varphi \quad \text{and} \quad f_\mu|_{\mathcal{C}} = f'_\mu|_{\mathcal{C}} = \psi. \tag{56}$$

Then according to Proposition 4.3 (ii) $\Gamma(\lambda)\varphi = f_\lambda \oplus f'_\lambda$ and $\Gamma(\mu)\psi = f_\mu \oplus f'_\mu$ and by the definition of $Q(\cdot)$ in (52) we have

$$Q(\lambda)\varphi = -\frac{\partial f_\lambda}{\partial \nu}\Big|_{\mathcal{C}} - \frac{\partial f'_\lambda}{\partial \nu'}\Big|_{\mathcal{C}} \quad \text{and} \quad Q(\mu)\psi = -\frac{\partial f_\mu}{\partial \nu}\Big|_{\mathcal{C}} - \frac{\partial f'_\mu}{\partial \nu'}\Big|_{\mathcal{C}}.$$

Therefore

$$\big((Q(\lambda)-Q(\mu)^*)\varphi,\psi\big)_{\mathcal C} = -\left(\frac{\partial f_\lambda}{\partial\nu}\Big|_{\mathcal C}+\frac{\partial f'_\lambda}{\partial\nu'}\Big|_{\mathcal C},\psi\right)_{\mathcal C} + \left(\varphi,\frac{\partial f_\mu}{\partial\nu}\Big|_{\mathcal C}+\frac{\partial f'_\mu}{\partial\nu'}\Big|_{\mathcal C}\right)_{\mathcal C}$$

and inserting (56) gives

$$-\left(\frac{\partial f_\lambda}{\partial\nu}\Big|_{\mathcal C}, f_\mu|_{\mathcal C}\right)_{\mathcal C} - \left(\frac{\partial f'_\lambda}{\partial\nu'}\Big|_{\mathcal C}, f'_\mu|_{\mathcal C}\right)_{\mathcal C} + \left(f_\lambda|_{\mathcal C},\frac{\partial f_\mu}{\partial\nu}\Big|_{\mathcal C}\right)_{\mathcal C} + \left(f'_\lambda|_{\mathcal C},\frac{\partial f'_\mu}{\partial\nu'}\Big|_{\mathcal C}\right)_{\mathcal C}.$$

Making use of Green's identity the above relations then become

$$\begin{aligned}
&\big((Q(\lambda)-Q(\mu)^*)\varphi,\psi\big)_{\mathcal C}\\
&\qquad= (\mathcal L_\Omega f_\lambda, f_\mu)_\Omega - (f_\lambda,\mathcal L_\Omega f_\mu)_\Omega + (\mathcal L_{\Omega'} f'_\lambda, f'_\mu)_{\Omega'} - (f'_\lambda,\mathcal L_{\Omega'} f'_\mu)_{\Omega'}\\
&\qquad= (\lambda-\bar\mu)\big((f_\lambda,f_\mu)_\Omega + (f'_\lambda,f'_\mu)_{\Omega'}\big) = (\lambda-\bar\mu)\big(f_\lambda\oplus f'_\lambda, f_\mu\oplus f'_\mu\big)\\
&\qquad= (\lambda-\bar\mu)(\Gamma(\lambda)\varphi,\Gamma(\mu)\psi) = \big((\lambda-\bar\mu)\Gamma(\mu)^*\Gamma(\lambda)\varphi,\psi\big)_{\mathcal C}.
\end{aligned}$$

Since this is true for any $\psi\in H^{3/2}(\mathcal C)$ we conclude that (55) holds on $H^{3/2}(\mathcal C)$. Thus Q in (52) is a generalized Q-function for the triple $\{S,A,T\}$.

(ii) We check first that $\ker Q(\lambda)=\{0\}$ holds for $\lambda\in\rho(A)\cap\rho(\widetilde A)$. Assume that $Q(\lambda)\varphi=0$ for some $\varphi\in H^{3/2}(\mathcal C)$ and let $f_\lambda\oplus f'_\lambda\in\mathcal N_\lambda(T)$ be the unique element with the property $f_\lambda|_{\mathcal C}=f'_\lambda|_{\mathcal C}=\varphi$. Then the definition of Q and the assumption $Q(\lambda)\varphi=0$ imply

$$\frac{\partial f_\lambda}{\partial\nu}\Big|_{\mathcal C} = -\frac{\partial f'_\lambda}{\partial\nu'}\Big|_{\mathcal C}.$$

According to Lemma 4.2 this yields $f_\lambda\oplus f'_\lambda\in\operatorname{dom}\widetilde A\cap\mathcal N_\lambda(T)$. But as $\lambda\in\rho(\widetilde A)$ we conclude $f_\lambda=0$ and $f'_\lambda=0$, and hence $\varphi=0$.

Now we prove the formula (53) for the difference of the resolvents of A and $\widetilde A$. By the above argument $Q(\lambda)^{-1}$ exists for $\lambda\in\rho(A)\cap\rho(\widetilde A)$. Furthermore, (49) implies $\operatorname{ran}Q(\lambda)=H^{1/2}(\mathcal C)$ and it follows from Proposition 4.3 that the right hand side in (53) is well defined.

Let $h\in L^2(\mathbb R^n)$ and define the function k as

$$k=(A-\lambda)^{-1}h-\Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\bar\lambda)^*h. \tag{57}$$

We show $k\in\operatorname{dom}\widetilde A$. First of all it is clear that $k\in\operatorname{dom}T$ since $(A-\lambda)^{-1}h\in\operatorname{dom}A\subset\operatorname{dom}T$ and $\Gamma(\lambda)$ maps into $\mathcal N_\lambda(T)$. Therefore $k=g\oplus g'$, where $g\in H^2(\Omega)$, $g'\in H^2(\Omega')$, and $g|_{\mathcal C}=g'|_{\mathcal C}$. According to Lemma 4.2 for $k\in\operatorname{dom}\widetilde A$ it is sufficient to check

$$\frac{\partial g}{\partial\nu}\Big|_{\mathcal C}+\frac{\partial g'}{\partial\nu'}\Big|_{\mathcal C}=0. \tag{58}$$

We proceed in a similar way as in the proof of Theorem 3.4. Let $h_A=f_A\oplus f'_A\in\operatorname{dom}A$ be such that $h=(A-\lambda)h_A$. Making use of Proposition 4.3 (iii) we obtain

$$k=h_A+\Gamma(\lambda)Q(\lambda)^{-1}\left(\frac{\partial f_A}{\partial\nu}\Big|_{\mathcal C}+\frac{\partial f'_A}{\partial\nu'}\Big|_{\mathcal C}\right) \tag{59}$$

from (57). Let

$$\mathcal N_\lambda(T)\ni f_\lambda\oplus f'_\lambda := \Gamma(\lambda)Q(\lambda)^{-1}\left(\frac{\partial f_A}{\partial\nu}\Big|_{\mathcal C}+\frac{\partial f'_A}{\partial\nu'}\Big|_{\mathcal C}\right).$$

Then by Proposition 4.3 (ii) we have

$$f_\lambda|_\mathcal{C} = f'_\lambda|_\mathcal{C} = Q(\lambda)^{-1}\left(\frac{\partial f_A}{\partial \nu}\Big|_\mathcal{C} + \frac{\partial f'_A}{\partial \nu'}\Big|_\mathcal{C}\right).$$

This together with the definition of $Q(\lambda)$ in (52) implies

$$\frac{\partial f_A}{\partial \nu}\Big|_\mathcal{C} + \frac{\partial f'_A}{\partial \nu'}\Big|_\mathcal{C} = Q(\lambda)(f_\lambda|_\mathcal{C}) = Q(\lambda)(f'_\lambda|_\mathcal{C}) = -\frac{\partial f_\lambda}{\partial \nu}\Big|_\mathcal{C} - \frac{\partial f'_\lambda}{\partial \nu'}\Big|_\mathcal{C}.$$

Hence we conclude that the function $k = g \oplus g'$ in (59) fulfils (58), i.e., $k \in \operatorname{dom} \widetilde{A}$. From (57) and $A, \widetilde{A} \subset T$ we obtain

$$(\widetilde{A} - \lambda)k = (T - \lambda)(A - \lambda)^{-1}h - (T - \lambda)\Gamma(\lambda)Q(\lambda)^{-1}\Gamma(\bar{\lambda})^* h = h$$

and now $k = (\widetilde{A} - \lambda)^{-1}h$ and (57) imply (53). □

The following corollaries can be proved in the same way as Corollary 3.5 and Corollary 3.6.

Corollary 4.5. *For $\lambda, \lambda_0 \in \rho(A)$ the following holds.*

(i) *$Q(\lambda)$ is a non-closed unbounded operator in $L^2(\mathcal{C})$ defined on $H^{3/2}(\mathcal{C})$ with* $\operatorname{ran} Q(\lambda) \subset H^{1/2}(\mathcal{C})$*;*
(ii) *$Q(\lambda) - \operatorname{Re} Q(\lambda_0)$ is a non-closed bounded operator in $L^2(\mathcal{C})$ defined on $H^{3/2}(\mathcal{C})$;*
(iii) *the closure $\widetilde{Q}(\lambda)$ of the operator $Q(\lambda) - \operatorname{Re} Q(\lambda_0)$ in $L^2(\mathcal{C})$ satisfies*

$$\frac{d}{d\lambda}\widetilde{Q}(\lambda) = \Gamma(\bar{\lambda})^*\overline{\Gamma(\lambda)}$$

and $\widetilde{Q}$ is a $\mathcal{L}(L^2(\mathcal{C}))$-valued Nevanlinna function.

Corollary 4.6. *For $\lambda \in \rho(A) \cap \rho(\widetilde{A})$ the following holds.*

(i) *$Q(\lambda)^{-1}$ is a non-closed bounded operator in $L^2(\mathcal{C})$ defined on $H^{1/2}(\mathcal{C})$ with* $\operatorname{ran} Q(\lambda)^{-1} = H^{3/2}(\mathcal{C})$*;*
(ii) *the closure $\overline{Q(\lambda)^{-1}}$ is a compact operator in $L^2(\mathcal{C})$;*
(iii) *the function $\lambda \mapsto -\overline{Q(\lambda)^{-1}}$ is a $\mathcal{L}(L^2(\mathcal{C}))$-valued Nevanlinna function.*

As a corollary of Theorem 4.4 we obtain a trace formula for the difference of the resolvents of A and $\widetilde{A}$.

Corollary 4.7. *Let the assumptions be as in Theorem 4.4, let $\widetilde{Q}$ be the Nevanlinna function from Corollary 4.5 and suppose, in addition, $n = 2$. Then*

$$\operatorname{tr}\big((A - \lambda)^{-1} - (\widetilde{A} - \lambda)^{-1}\big) = \operatorname{tr}\left(\overline{Q(\lambda)^{-1}}\frac{d}{d\lambda}\widetilde{Q}(\lambda)\right)$$

holds for all $\lambda \in \rho(A) \cap \rho(\widetilde{A})$.

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